Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

Research Article

Maximal Product and Residue Product on Bipolar Picture Fuzzy Graphs

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Received: 25 February 2025; **Revised:** 7 May 2025; **Accepted:** 13 May 2025

Abstract: Fuzzy Graphs (FGs) are very useful for modeling systems with uncertain relationships between elements, allowing degrees of membership between edges and vertices. They are used in fields like decision-making, bioscience, information technology, and artificial intelligence. The Picture Fuzzy Graphs (PFGs) are a refined representation of uncertainty in decision-making environments, allowing for the inclusion of positive membership and non-membership degrees, making them useful in areas like medical diagnosis, pattern recognition, and risk assessment. While a Bipolar Picture Fuzzy Graph (BPFG) is a class of FG developed to describe six types of information: positive membership, positive neutrality, positive hesitancy, negative membership, negative neutrality, and negative hesitancy. The BPFGs are a vital approach to examining systems with coexisting support, resistance, neutrality, and reluctance. They are commonly used in domains like medical diagnosis, social networks, and multi-criteria decision-making. A BPFG is an extension of a PFG due to three additional negative functions. A BPFG is better for analyzing ambiguous and inconsistent data related to real-valued problems. In this paper, we define the Maximal Product (MP) and Residue Product (RP) of BPFG with the help of examples and related theorems. We discuss isomorphism and homomorphism of BPFG with the help of theorems. We define the ideas of weak isomorphism and co-weak isomorphism in BPFG. We prove that isomorphism between BPFG is an equivalence relation. At the end, we provide an application of BPFG.

***Keywords*:** BPFG, MP, RP, degree of vertex, total degree of vertex, homomorphism, isomorphism, application

MSC: 05C07, 05C72, 05C76

Abbreviation

FG	Fuzzy Graph
PFG	Picture Fuzzy Graph
IFG	Intuitionistic Fuzzy Graph
BFG	Bipolar Fuzzy Graph
MP	Maximal Product
RP	Residue Product
MF	Membership Function

DOI: https://doi.org/10.37256/cm.6420256633

NMF	Non Membership Function
neg-MF	negative of Membership Function
neg-NMF	negative of Non Membership Function
neg-NEMF	negative of Neutral Membership Function
Mv	Membership variable
NMv	Non Membership variable
NNv	Neutral Membership variable
neg-Mv	negative of Membership variable
neg-NMv	negative of Non Membership variable
neg-NEMv	negative of Neutral Membership variable

1. Introduction

Zadeh [1] proposed the remarkable concept of a fuzzy set characterized by a true membership function as a super set of a crisp set to address uncertainty and unclear information, which has many applications across various fields, such as computer science, chemical industry, telecommunications, decision-making, networking, and discrete mathematics. There is no doubt that fuzzy analysis needs flexible classes to decrease uncertainty in different fields of life. Atanassov [2] developed intuitionistic fuzzy sets, which provide both a degree of membership and non-membership and have many applications in many disciplines, such as computer science, engineering, mathematics, chemistry, medicine, and economics. Cuong and Kreinovich [3] introduced picture fuzzy sets, which allow membership degree, neutral membership degree, and non-membership degree to an element. Several mathematicians [4–14], contributed significantly to Picture fuzzy sets. Some researchers [15–17], are working on spherical fuzzy sets. Smarandache [18] proposed a new theory of the neutrosophic set involving indeterminacy and inconsistent data.

Euler [19] introduced the concept of graph theory. The first known theorem of graph theory in mathematical history is the well-known Koinigsberg bridge problem. Since then, combinatorics has added graph theory as a subfield. In the fields of geometry, algebra, number theory, topology, operations research, optimization, and computer science, among others, this theory is an extremely useful tool for solving combinatorial problems. A graph represents the relationships between vertices and edges in a network mathematically and has many applications in a wide range of fields, including data mining, image segmentation, networking, clustering, image segmentation, and scheduling. Gulzar et al. [20] presented the novel application of a complex intuitionistic fuzzy set. Gulzar et al. [21] studied the class of the t-intuitionistic fuzzy subgroup. Bhunia [22] deeply examined the algebraic characteristics of fuzzy sub-e-groups.

Rosenfeld [23] presented several ideas in the field of FG theory, which contains cycles, connectedness, and paths. FG finds many applications in the fields of topological spaces and algebra. Bhattacharya [24] explored the relationship between both fuzzy groups and FGs. Bhutani [25] presented automorphisms in FGs, while Gani and Latha [26] proposed the vital idea of irregularity in FGs. Also, Gani and Ahmad [27] investigated the degree and size of FGs. Mordeson and Peng [28] investigated some operations, including join, union, Cartesian product, and composition for fuzzy subgraphs. Mathew and Sunitha [29] explored the worthwhile applications of FGs.

Shannon and Atanassov [30] introduced both intuitionistic fuzzy relations and Intuitionistic Fuzzy Graphs (IFG). Several researchers [14, 31–34], contributed significantly to intuitionistic fuzzy graphs. Shao et al. [35] explored some modern ideas of the bondage number in the IFG. After that, Zuo et al. [36] proposed the definition of PFG by adding an additional neutral membership degree. PFG is a more generalized idea than IFG.

A bipolar fuzzy graph [37] is an innovative extension of an FG that covers negative membership functions. A bipolar fuzzy graph is an extension of traditional FGs, designed to represent both support and opposition within a graph structure. This dual approach allows for more complex interactions, particularly in decision-making problems where both positive and negative information coexist. Bipolar fuzzy graphs are important in fields like decision-making, social network analysis, and image processing. They provide advantages such as more accurate reflection of real-world problems. Rashmanlou et al. [37–39], investigated deeply in the field of a bipolar fuzzy graph. Additionally, Rashmanlou et al.

[40–43], also examined interval-valued fuzzy graphs. Hassan and Malik [44] explored the classification of the bipolar single-valued neutrosophic graph.

Motivation:

FG theory depends on the situation of uncertainty and ambiguity in relational data. PFG consists of a good model involving three types of degrees of membership: neutral membership degree and non-membership degree. Although the PFG model often does not handle bipolar-type knowledge. PFG covers three directional phenomena of uncertainty in an element, such as a membership function, a neutral function, and a non-membership function, but it does not cover negative points of view. There is a need for a model that covers negative points of view to deal with real-world problems like decision-making and networking.

This paper presents two novel operations on BPFG, the MP and RP. These operations extend algebraic and structural tools for characterizing uncertainty and dual-polarity information in complex systems. The paper also contributes to graph-theoretic analysis in BPFG by introducing isomorphism and homomorphism for BPFGs. These concepts enable exact structural comparison and mapping between BPFGs, opening up applications in network analysis, pattern recognition, and data fusion when directed trust and contradiction coexist. This is the first comprehensive study integrating these complex operations and morphisms within the BPFG structure, providing a unified and flexible platform for research and practical uses. The practical application of BPFG is more effective. This work expresses this gap by developing two operations, maximal product, and residue product, described for BPFGs. These operations are more needful to the generalized structure of FG algebraically. BPFG is another extension of PFG, having three additional negatives of all three functions of PFG. This research paper presents novel properties of the maximal product and residue product of BPFG, and several related properties are examined. We also investigate the application of BPFG in decision-making. BPFGs are an expanded version that provides a more flexible environment due to dual-sided uncertainty and gives a more accurate solution in decision-making. MP and RP provide logical implications, capture frequent interactions, dynamic updating, and real-time analysis.

The layout of this paper is as follows:

We discussed some basic fundamental definitions in Section 2 that facilitate the reader's understanding of this paper. Section 3 investigated the MP and RP of the BPFG. Additionally, we define the degree of a vertex and the total degree of a vertex with the help of examples. We have provided an application of BPFG in decision-making. Section 4 provided the concept of isomorphism and homomorphism of BPFG. In Section 5, we proposed an application of BPFG in decision-making. Finally, we provide concluding remarks and some future directions in Section 6.

The next section discusses some primary definitions, revisiting the main ideas to yield a comprehensive understanding of the research work.

2. Preliminaries

Definition 1 [45] A fuzzy set is defined as:

$$J = < j : \Upsilon_J(j),\ j \in X >$$

which follows: where $\Upsilon_J : V \to [0,\ 1]$ represents the degree of the Membership Function (MF).

Definition 2 [2] A intuitionistic fuzzy set is defined as:

$$J = < j : \Upsilon_J(j),\ \psi_J(j),\ j \in X >$$

which follows:

$$0 \leq \Upsilon_J(j) + \psi_J(j) \leq 1.$$

where $\Upsilon_J : V \to [0,\ 1]$ represents the degree of the MF, $\psi_J : V \to [0,\ 1]$ represents the degree of the Non Membership Function (NMF).

Definition 3 [3] A picture fuzzy set is defined as:

$$J = < j : \Upsilon_J(j),\ \upsilon_J(j),\ \psi_J(j),\ j \in X >$$

which follows:

$$0 \leq \Upsilon_J(j) + \upsilon_J(j) + \psi_J(j) \leq 1.$$

where $\Upsilon_J : V \to [0,\ 1]$ represents the degree of the MF, $\upsilon_J : V \to [0,\ 1]$ represents the degree of the NEMF, and $\psi_J : V \to [0,\ 1]$ represents the degree of the NMF.

Definition 4 [1] A bipolar picture fuzzy set is defined as:

$$J = < j : \Upsilon_J(j),\ \upsilon_J(j),\ \psi_J(j),\ \alpha_J(j),\ \beta_J(j),\ \gamma_J(j),\ j \in X >$$

which follows:

$$0 \leq \Upsilon_J(j) + \upsilon_J(j) + \psi_J(j) \leq 1 \text{ and } -1 \leq \alpha_J(j) + \beta_J(j) + \gamma_J(j) \leq 0.$$

where $\Upsilon_J : V \to [0,\ 1]$ represents the degree of the MF, $\upsilon_J : V \to [0,\ 1]$ represents the degree of the NEMF, and $\psi_J : V \to [0,\ 1]$ represents the degree of the NMF, $\alpha_J : V \to [-1,\ 0]$ represents the degree of the Negative of Membership Function (neg-MF), $\beta_J : V \to [-1,\ 0]$ represents the degree of the negative of the Negative of Neutral Membership Function (neg-NEMF), and $\gamma_J : V \to [-1,\ 0]$ represents the degree of the negative of Non Membership Function (neg-NMF).

Definition 5 [23] A FG on a non-empty set V is a pair $\mathbb{G} = (J,\ W)$, where J is a fuzzy set on V, and W is a fuzzy relation on V. It is expressed as follows:

$$\Upsilon_W(jq) \leq \wedge\{\Upsilon_J(j),\ \Upsilon_J(q)\}.$$

Definition 6 [30] A IFG on a non-empty set V is a pair $\mathbb{G} = (J,\ W)$, where J is a intuitionistic fuzzy set on V, and W is a intuitionistic fuzzy relation on V. It is expressed as follows:

$$\Upsilon_W(jq) \leq \wedge\{\Upsilon_J(j),\ \Upsilon_J(q)\},$$

$$\psi_W(jq) \geq \vee\{\psi_J(j),\ \psi_J(q)\}.$$

Definition 7 [36] A PFG on a non-empty set V is a pair $\mathbb{G} = (J,\ W)$, where J is a picture fuzzy set on V, and W is an picture fuzzy relation on V. It is expressed as follows:

$$\Upsilon_W(jq) \leq \wedge\{\Upsilon_J(j),\ \Upsilon_J(q)\},$$

$$\upsilon_W(jq) \leq \wedge\{\upsilon_J(j),\ \upsilon_J(q)\},$$

$$\psi_W(jq) \geq \vee\{\psi_J(j),\ \psi_J(q)\}.$$

Definition 8 [46] A BPFG on a non-empty set V is a pair $\mathbb{G} = (J,\ W)$, where J is a bipolar picture fuzzy set on V, and W is a bipolar picture fuzzy relation on V. It is expressed as follows:

$$\Upsilon_W(jq) \leq \wedge\{\Upsilon_J(j),\ \Upsilon_J(q)\},$$

$$\upsilon_W(jq) \leq \wedge\{\upsilon_J(j),\ \upsilon_J(q)\},$$

$$\psi_W(jq) \geq \vee\{\psi_J(j),\ \psi_J(q)\},$$

$$\alpha_W(jq) \geq \vee\{\alpha_J(j),\ \alpha_J(q)\},$$

$$\beta_W(jq) \geq \vee\{\beta_J(j),\ \beta_J(q)\},$$

$$\gamma_W(jq) \leq \wedge\{\gamma_J(j),\ \gamma_J(q)\}.$$

Definition 9 [45] A BPFG $\mathbb{G} = (J,\ W)$ on a crisp graph $G^* = (V,\ E)$ is said to be strong if

$$\Upsilon_W(jw) = \wedge\{\Upsilon_J(j),\ \Upsilon_J(w)\},$$

$$\upsilon_W(jw) = \wedge\{\upsilon_J(j),\ \upsilon_J(w)\},$$

$$\psi_W(jw) = \vee\{\psi_J(j),\ \psi_J(w)\},$$

$$\alpha_W(jw) = \vee\{\alpha_J(j),\ \alpha_J(w)\},$$

$$\beta_W(jw) = \vee\{\beta_J(j),\ \beta_J(w)\},$$

$$\gamma_W(jw) = \wedge\{\gamma_J(j),\ \gamma_J(w)\}.$$

$\forall jw \in E$.

Definition 10 [45] A BPFG $\mathbb{G} = (J,\ W)$ on a crisp graph $G^* = (V,\ E)$ is said to be complete if

$$\Upsilon_W(jw) = \wedge\{\Upsilon_J(j),\ \Upsilon_J(w)\},$$

$$\upsilon_W(jw) = \wedge\{\upsilon_J(j),\ \upsilon_J(w)\},$$

$$\psi_W(jw) = \vee\{\psi_J(j),\ \psi_J(w)\},$$

$$\alpha_W(jw) = \vee\{\alpha_J(j),\ \alpha_J(w)\},$$

$$\beta_W(jw) = \vee\{\beta_J(j),\ \beta_J(w)\},$$

$$\gamma_W(jw) = \wedge\{\gamma_J(j),\ \gamma_J(w)\},$$

$\forall j,\ w \in \mathrm{V}$.

3. BPFG

Definition 11 The Maximal Product (MP) $\mathbb{G}_1 * \mathbb{G}_2 = (J_1 * J_2,\ W_1 * W_2)$ of two BPFGs $\mathbb{G}_1 = (J_1,\ W_1)$ and $\mathbb{G}_2 = (J_2,\ W_2)$ is defined as:

(i)

$$(\Upsilon_{J_1} * \Upsilon_{J_2})((j_1,\ j_2)) = \vee\{\Upsilon_{J_1}(j_1),\ \Upsilon_{J_2}(j_2)\},$$

$$(\upsilon_{J_1} * \upsilon_{J_2})((j_1,\ j_2)) = \vee\{\upsilon_{J_1}(j_1),\ \upsilon_{J_2}(j_2)\},$$

$$(\psi_{J_1} * \psi_{J_2})((j_1,\ j_2)) = \wedge\{\psi_{J_1}(j_1),\ \psi_{J_2}(j_2)\}$$

$$(\alpha_{J_1} * \alpha_{J_2})((j_1,\ j_2)) = \wedge\{\alpha_{J_1}(j_1),\ \alpha_{J_2}(j_2)\},$$

$$(\beta_{J_1} * \beta_{J_2})((j_1,\ j_2)) = \wedge\{\beta_{J_1}(j_1),\ \beta_{J_2}(j_2)\},$$

$$(\gamma_{J_1} * \gamma_{J_2})((j_1,\ j_2)) = \vee\{\gamma_{J_1}(j_1),\ \gamma_{J_2}(j_2)\}$$

$\forall (j_1,\ j_2) \in (V_1 \times V_2)$.

(ii)

$$(\Upsilon_{J_1} * \Upsilon_{J_2})((m,\ j_2)(m,\ w_2)) = \vee\{\Upsilon_{J_1}(m),\ \Upsilon_{W_2}(j_2w_2)\},$$

$$(\upsilon_{J_1} * \upsilon_{J_2})((m,\ j_2)(m,\ w_2)) = \vee\{\upsilon_{J_1}(m),\ \upsilon_{W_2}(j_2w_2)\},$$

$$(\psi_{J_1} * \psi_{J_2})((m,\ j_2)(m,\ w_2)) = \wedge\{\psi_{J_1}(m),\ \psi_{W_2}(j_2w_2)\}.$$

$$(\alpha_{J_1} * \alpha_{J_2})((m,\ j_2)(m,\ w_2)) = \wedge\{\alpha_{J_1}(m),\ \alpha_{W_2}(j_2w_2)\},$$

$$(\beta_{J_1} * \beta_{J_2})((m,\ j_2)(m,\ w_2)) = \wedge\{\beta_{J_1}(m),\ \beta_{W_2}(j_2w_2)\},$$

$$(\gamma_{J_1} * \gamma_{J_2})((m,\ j_2)(m,\ w_2)) = \vee\{\gamma_{J_1}(m),\ \gamma_{W_2}(j_2w_2)\}.$$

$\forall m \in V_1$ and $j_2w_2 \in E_2$.

(iii)

$$(\Upsilon_{J_1} * \Upsilon_{J_2})((j_1,\ z)(w_1,\ z)) = \vee\{\Upsilon_{W_1}(j_1w_1),\ \Upsilon_{J_2}(z)\},$$

$$(\upsilon_{J_1} * \upsilon_{J_2})((j_1,\ z)(w_1,\ z)) = \vee\{\upsilon_{W_1}(j_1w_1),\ \upsilon_{J_2}(z)\},$$

$$(\psi_{J_1} * \psi_{J_2})((j_1,\ z)(w_1,\ z)) = \wedge\{\psi_{W_1}(j_1w_1),\ \psi_{J_2}(z)\}.$$

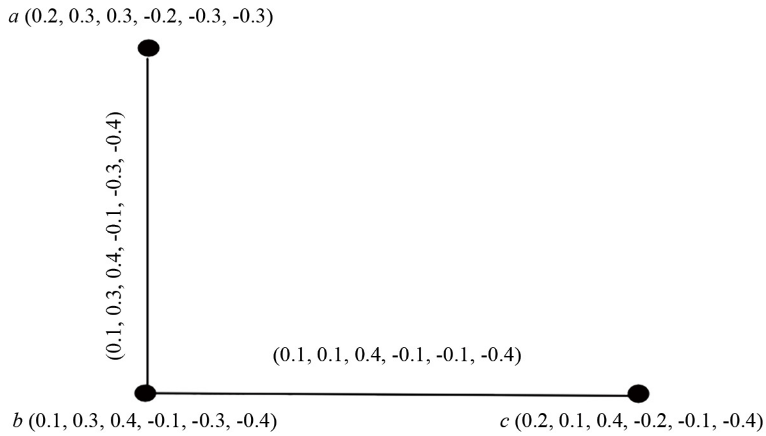
$$(\alpha_{J_1} * \alpha_{J_2})((j_1,\ z)(w_1,\ z)) = \wedge\{\alpha_{W_1}(j_1w_1),\ \alpha_{J_2}(z)\},$$

$$(\beta_{J_1} * \beta_{J_2})((j_1,\ z)(w_1,\ z)) = \wedge\{\beta_{W_1}(j_1w_1),\ \beta_{J_2}(z)\},$$

$$(\gamma_{J_1} * \gamma_{J_2})((j_1,\ z)(w_1,\ z)) = \vee\{\gamma_{W_1}(j_1w_1),\ \gamma_{J_2}(z)\}.$$

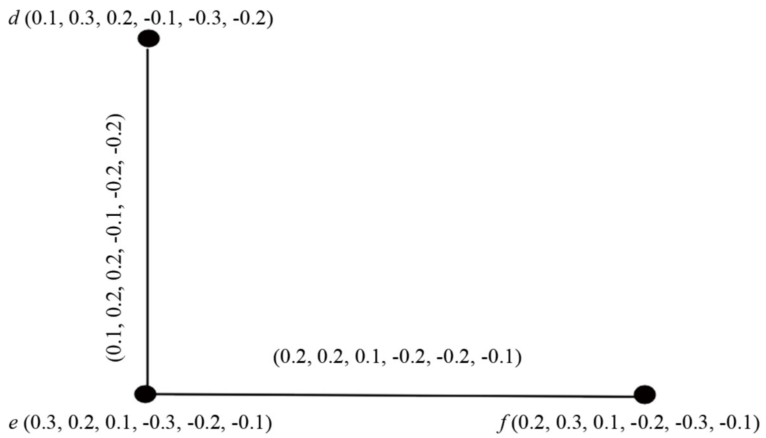
for all $z \in V_2$ and $j_1w_1 \in E_1$.

Example Suppose that $\mathbb{G}_1 = (J_1,\ W_1)$ and $\mathbb{G}_2 = (J_2,\ W_2)$ are two BPFGs, which are shown in Figures 1 and 2. Their MP $\mathbb{G}_1 * \mathbb{G}_2$ is shown in Figure 3.



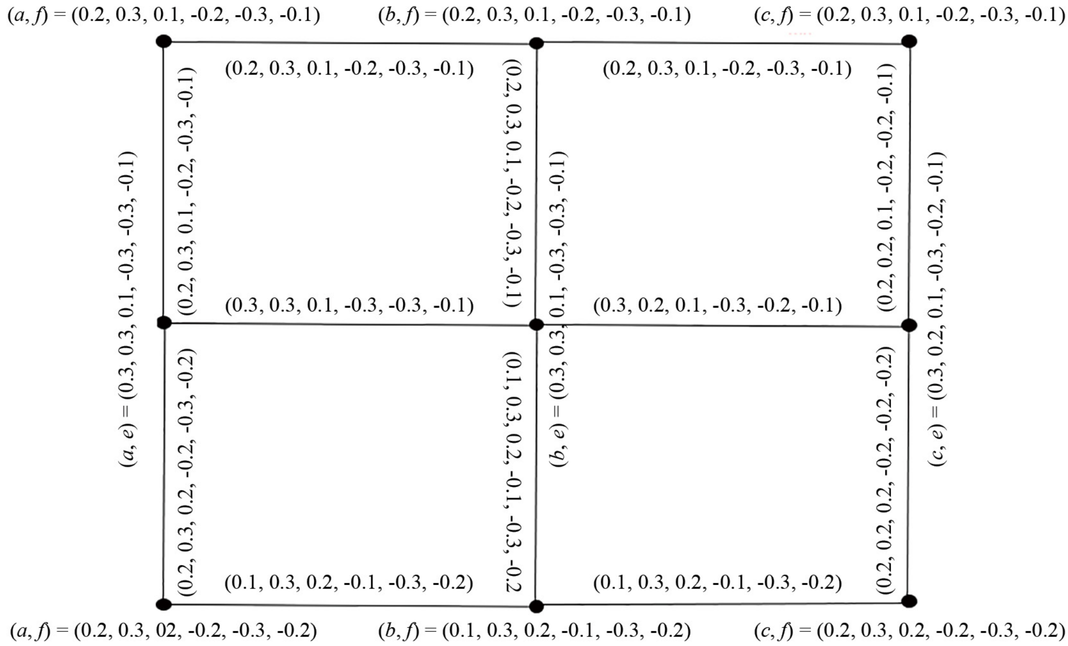
a = (0.2, 0.3, 0.3, -0.2, -0.3, -0.3); b = (0.1, 0.3, 0.4, -0.1, -0.3, -0.4)
c = (0.2, 0.1, 0.4, -0.2, -0.1, -0.4); ab = (0.1, 0.3, 0.4, -0.1, -0.3, -0.4)
bc = (0.1, 0.1, 0.4, -0.1, -0.1, -0.4)

Figure 1. $\mathbb{G}_1$



d = (0.1, 0.3, 0.2, -0.1, -0.3, -0.2); e = (0.3, 0.2, 0.1, -0.3, -0.2, -0.1)
f = (0.2, 0.3, 0.1, -0.2, -0.3, -0.1); de = (0.1, 0.2, 0.2, -0.1, -0.2, -0.2)
ef = (0.2, 0.2, 0.1, -0.2, -0.2, -0.1)

Figure 2. $\mathbb{G}_2$



(a, d) = (0.2, 0.3, 0.2, -0.2, -0.3, -0.2); (a, e) = (0.3, 0.3, 0.1, -0.3, -0.3, -0.1);
(a, f) = (0.2, 0.3, 0.1, -0.2, -0.3, -0.1); (b, d) = (0.1, 0.3, 0.2, -0.1, -0.3, -0.2);
(b, e) = (0.3, 0.3, 0.1, -0.3, -0.3, -0.1); (b, f) = (0.2, 0.3, 0.1, -0.2, -0.3, -0.1);
(c, d) = (0.2, 0.3, 0.2, -0.2, -0.3, -0.2); (c, e) = (0.3, 0.2, 0.1, -0.3, -0.2, -0.1);
(c, f) = (0.2, 0.3, 0.1, -0.2, -0.3, -0.1);
(a, d) (a, e) = (0.2, 0.3, 0.2, -0.2, -0.3, -0.2); (a, d) (b, d) = (0.1, 0.3, 0.2, -0.1, -0.3, -0.2);
(a, e) (a, f) = (0.2, 0.3, 0.1, -0.2, -0.3, -0.1); (a, e) (b, e) = (0.3, 0.3, 0.1, -0.3, -0.3, -0.1);
(a, f) (b, f) = (0.2, 0.3, 0.1, -0.2, -0.3, -0.1); (b, d) (b, e) = (0.1, 0.3, 0.2, -0.1, -0.3, -0.2);
(b, d) (c, d) = (0.1, 0.3, 0.2, -0.1, -0.3, -0.2); (b, e) (b, f) = (0.2, 0.3, 0.1, -0.2, -0.3, -0.1);
(b, e) (c, e) = (0.3, 0.2, 0.1, -0.3, -0.2, -0.1); (b, f) (c, f) = (0.2, 0.3, 0.1, -0.2, -0.3, -0.1);
(c, d) (c, e) = (0.2, 0.2, 0.2, -0.2, -0.2, -0.2); (c, e) (c, f) = (0.2, 0.2, 0.1, -0.2, -0.2, -0.1)

Figure 3. $\mathbb{G}_1 * \mathbb{G}_2$

For vertex $(c,\ f)$, we find Mv, NNv, NMv, neg-Mv, neg-NNv, and neg-NMv as follows:

$$(\Upsilon_{J_1} * \Upsilon_{J_2})((c,\ f)) = \vee\{\Upsilon_{J_1}(c),\ \Upsilon_{J_2}(f)\} = \vee\{0.2,\ 0.2\} = 0.2,$$

$$(\upsilon_{J_1} * \upsilon_{J_2})((c,\ f) = \vee\{\upsilon_{J_1}(c),\ \upsilon_{J_2}(f)\} = \vee\{0.1,\ 0.3\} = 0.3,$$

$$(\psi_{J_1} * \psi_{J_2})((c,\ f)) = \wedge\{\psi_{J_1}(c),\ \psi_{J_2}(f)\} = \wedge\{0.4,\ 0.1\} = 0.1,$$

$$(\alpha_{J_1} * \alpha_{J_2})((c,\ f)) = \wedge\{\alpha_{J_1}(c),\ \alpha_{J_2}(f)\} = \wedge\{-0.2,\ -0.2\} = -0.2,$$

$$(\beta_{J_1} * \beta_{J_2})((c,\ f)) = \wedge\{\beta_{J_1}(c),\ \beta_{J_2}(f)\} = \wedge\{-0.1,\ -0.3\} = -0.3,$$

$$(\gamma_{J_1} * \gamma_{J_2})((c,\ f)) = \vee\{\gamma_{J_1}(c),\ \gamma_{J_2}(f)\} = \vee\{-0.4,\ -0.1\} = -0.1$$

for $c \in V_1$ and $f \in V_2$.

For edge $(c,\ e)(c,\ f)$, we find Mv, NNv, NMv, neg-Mv, neg-NNv, and neg-NMv.

$$(\Upsilon_{J_1} * \Upsilon_{J_2})((c,\ e)(c,\ f)) = \vee\{\Upsilon_{J_1}(c),\ \Upsilon_{W_2}(ef)\} = \vee\{0.2,\ 0.2\} = 0.2,$$

$$(\upsilon_{J_1} * \upsilon_{J_2})((c,\ e)(c,\ f)) = \vee\{\upsilon_{J_1}(c),\ \upsilon_{W_2}(ef)\} = \vee\{0.1,\ 0.2\} = 0.2,$$

$$(\psi_{J_1} * \psi_{J_2})((c,\ e)(c,\ f)) = \wedge\{\psi_{J_1}(c),\ \psi_{W_2}(ef)\} = \wedge\{0.4,\ 0.1\} = 0.1$$

$$(\alpha_{J_1} * \alpha_{J_2})((c,\ e)(c,\ f)) = \wedge\{\alpha_{J_1}(c),\ \alpha_{W_2}(ef)\} = \wedge\{-0.2,\ -0.2\} = -0.2,$$

$$(\beta_{J_1} * \beta_{J_2})((c,\ e)(c,\ f)) = \wedge\{\beta_{J_1}(c),\ \beta_{W_2}(ef)\} = \wedge\{-0.1,\ -0.2\} = -0.2,$$

$$(\gamma_{J_1} * \gamma_{J_2})((c,\ e)(c,\ f)) = \vee\{\gamma_{J_1}(c),\ \gamma_{W_2}(ef)\} = \vee\{-0.4,\ -0.1\} = -0.1.$$

for $c \in V_1$ and $ef \in E_2$.

For edge $(a,\ f)(b,\ f)$:

$$(\Upsilon_{J_1} * \Upsilon_{J_2})((a,\ f)(b,\ f)) = \vee\{\Upsilon_{W_1}(ab),\ \Upsilon_{J_2}(f)\} = \vee\{0.1,\ 0.2\} = 0.2,$$

$$(\upsilon_{J_1} * \upsilon_{J_2})((a,\ f)(b,\ f)) = \vee\{\upsilon_{W_1}(ab),\ \upsilon_{J_2}(f)\} = \vee\{0.3,\ 0.3\} = 0.3,$$

$$(\psi_{J_1} * \psi_{J_2})((a,\ f)(b,\ f)) = \wedge\{\psi_{W_1}(ab),\ \psi_{J_2}(f)\} = \wedge\{0.4,\ 0.1\} = 0.1,$$

$$(\alpha_{J_1} * \alpha_{J_2})((a,\ f)(b,\ f)) = \wedge\{\alpha_{W_1}(ab),\ \alpha_{J_2}(f)\} = \wedge\{-0.1,\ -0.2\} = -0.2,$$

$$(\beta_{J_1} * \beta_{J_2})((a,\ f)(b,\ f)) = \wedge\{\beta_{W_1}(ab),\ \beta_{J_2}(f)\} = \wedge\{-0.3,\ -0.3\} = -0.3,$$

$$(\gamma_{J_1} * \gamma_{J_2})((a,\ f)(b,\ f)) = \vee\{\gamma_{W_1}(ab),\ \gamma_{J_2}(f)\} = \vee\{-0.4,\ -0.1\} = -0.1,$$

for $f \in V_2$ and $ab \in E_1$.

Mv, NNv, NMv, neg-Mv, neg-NNv, and neg-NMv can be obtain for all other vertices and edges.

Proposition 1 The MP of two BPFGs $\mathbb{G}_1$ and $\mathbb{G}_2$ is a BPFG.

Proof. Suppose that $\mathbb{G}_1 = (J_1,\ W_1)$ and $\mathbb{G}_2 = (J_2,\ W_2)$ are two BPFGs on crisp graphs $G_1 = (V_1,\ E_1)$ and $G_2 = (V_2,\ E_2)$, respectively, and $((j_1,\ j_2)(w_1,\ w_2)) \in E_1 \times E_2$.

By using Definition 11:

(i) If $j_1 = w_1 = m$,

$$
\begin{aligned}
(\Upsilon_{W_1} * \Upsilon_{W_2})((m,\ j_2)(m,\ w_2)) &= \vee\{\Upsilon_{J_1}(m),\ \Upsilon_{W_2}(j_2 w_2)\} \\
&\leq \vee\{\Upsilon_{J_1}(m),\ \wedge\{\Upsilon_{J_2}(j_2),\ \Upsilon_{J_2}(w_2)\}\} \\
&= \wedge\{\vee\{\Upsilon_{J_1}(m),\ \Upsilon_{J_2}(j_2)\},\ \vee\{\Upsilon_{J_1}(m),\ \Upsilon_{J_2}(w_2)\}\} \\
&= \wedge\{(\Upsilon_{J_1} * \Upsilon_{J_2})(m,\ j_2),\ (\Upsilon_{J_1} * \Upsilon_{J_2})(m,\ w_2)\},
\end{aligned}
$$

$$
\begin{aligned}
(\upsilon_{W_1} * \upsilon_{W_2})((m,\ j_2)(m,\ w_2)) &= \vee\{\upsilon_{J_1}(m),\ \upsilon_{W_2}(j_2 w_2)\} \\
&\leq \vee\{\upsilon_{J_1}(m),\ \wedge\{\upsilon_{J_2}(j_2),\ \upsilon_{J_2}(w_2)\}\} \\
&= \wedge\{\vee\{\upsilon_{J_1}(m),\ \upsilon_{J_2}(j_2)\},\ \vee\{\upsilon_{J_1}(m),\ \upsilon_{J_2}(w_2)\}\} \\
&= \wedge\{(\upsilon_{J_1} * \upsilon_{J_2})(m,\ j_2),\ (\upsilon_{J_1} * \upsilon_{J_2})(m,\ w_2)\},
\end{aligned}
$$

$$
\begin{aligned}
(\psi_{W_1} * \psi_{W_2})((m,\ j_2)(m,\ w_2)) &= \wedge\{\psi_{J_1}(m),\ \psi_{W_2}(j_2 w_2)\} \\
&\geq \wedge\{\psi_{J_1}(m),\ \vee\{\psi_{J_2}(j_2),\ \psi_{J_2}(w_2)\}\} \\
&= \vee\{\wedge\{\psi_{J_1}(m),\ \psi_{J_2}(j_2)\},\ \wedge\{\psi_{J_1}(m),\ \psi_{J_2}(w_2)\}\} \\
&= \vee\{(\psi_{J_1} * \psi_{J_2})(m,\ j_2),\ (\psi_{J_1} * \psi_{J_2})(m,\ w_2)\}.
\end{aligned}
$$

$$
\begin{aligned}
(\alpha_{W_1} * \alpha_{W_2})((m,\ j_2)(m,\ w_2)) &= \wedge\{\alpha_{J_1}(m),\ \alpha_{W_2}(j_2 w_2)\} \\
&\geq \wedge\{\alpha_{J_1}(m),\ \vee\{\alpha_{J_2}(j_2),\ \alpha_{J_2}(w_2)\}\} \\
&= \vee\{\wedge\{\alpha_{J_1}(m),\ \alpha_{J_2}(j_2)\},\ \wedge\{\alpha_{J_1}(m),\ \alpha_{J_2}(w_2)\}\} \\
&= \vee\{(\alpha_{J_1} * \alpha_{J_2})(m,\ j_2),\ (\alpha_{J_1} * \alpha_{J_2})(m,\ w_2)\},
\end{aligned}
$$

$$\begin{aligned}(\beta_{W_1}*\beta_{W_2})((m,\ j_2)(m,\ w_2)) &= \wedge\{\beta_{J_1}(m),\ \beta_{W_2}(j_2w_2)\}\\ &\geq \wedge\{\beta_{J_1}(m),\ \vee\{\beta_{J_2}(j_2),\ \beta_{J_2}(w_2)\}\}\\ &= \vee\{\wedge\{\beta_{J_1}(m),\ \beta_{J_2}(j_2)\},\ \wedge\{\beta_{J_1}(m),\ \beta_{J_2}(w_2)\}\}\\ &= \vee\{(\beta_{J_1}*\beta_{J_2})(m,\ j_2),\ (\beta_{J_1}*\beta_{J_2})(m,\ w_2)\},\\ (\gamma_{W_1}*\gamma_{W_2})((m,\ j_2)(m,\ w_2)) &= \vee\{\gamma_{J_1}(m),\ \gamma_{W_2}(j_2w_2)\}\\ &\leq \vee\{\gamma_{J_1}(m),\ \wedge\{\gamma_{J_2}(j_2),\ \gamma_{J_2}(w_2)\}\}\\ &= \wedge\{\vee\{\gamma_{J_1}(m),\ \gamma_{J_2}(j_2)\},\ \vee\{\gamma_{J_1}(m),\ \gamma_{J_2}(w_2)\}\}\\ &= \wedge\{(\gamma_{J_1}*\gamma_{J_2})(m,\ j_2),\ (\gamma_{J_1}*\gamma_{J_2})(m,\ w_2)\}.\end{aligned}$$

(ii) If $j_2 = w_2 = z$,

$$\begin{aligned}(\Upsilon_{W_1}*\Upsilon_{W_2})((j_1,\ z)(w_1,\ z)) &= \vee\{\Upsilon_{W_1}(j_1w_1),\ \Upsilon_{J_2}(z)\}\\ &\leq \vee\{\wedge\{\Upsilon_{J_1}(j_1),\ \Upsilon_{J_1}(w_1)\},\ \Upsilon_{J_2}(z)\}\\ &= \wedge\{\vee\{\Upsilon_{J_1}(j_1),\ \Upsilon_{J_2}(z)\},\ \vee\{\Upsilon_{J_1}(w_1),\ \Upsilon_{J_2}(z)\}\}\\ &= \wedge\{(\Upsilon_{J_1}*\Upsilon_{J_2})(j_1,\ z),\ (\Upsilon_{J_1}*\Upsilon_{J_2})(w_1,\ z)\},\\ (\upsilon_{W_1}*\upsilon_{W_2})((j_1,\ z)(w_1,\ z)) &= \vee\{\upsilon_{W_1}(j_1w_1),\ \upsilon_{J_2}(z)\}\\ &\leq \vee\{\wedge\{\upsilon_{J_1}(j_1),\ \upsilon_{J_1}(w_1)\},\ \upsilon_{J_2}(z)\}\\ &= \wedge\{\vee\{\upsilon_{J_1}(j_1),\ \upsilon_{J_2}(z)\},\ \vee\{\upsilon_{J_1}(w_1),\ \upsilon_{J_2}(z)\}\}\\ &= \wedge\{(\upsilon_{J_1}*\upsilon_{J_2})(j_1,\ z),\ (\upsilon_{J_1}*\upsilon_{J_2})(w_1,\ z)\},\end{aligned}$$

$$(\psi_{W_1} * \psi_{W_2})((j_1, z)(w_1, z)) = \wedge\{\psi_{W_1}(j_1 w_1), \psi_{J_2}(z)\}$$

$$\geq \wedge\{\vee\{\psi_{J_1}(j_1), \psi_{J_1}(w_1)\}, \psi_{J_2}(z)\}$$

$$= \vee\{\wedge\{\psi_{J_1}(j_1), \psi_{J_2}(z)\}, \wedge\{\psi_{J_1}(w_1), \psi_{J_2}(z)\}\}$$

$$= \vee\{(\psi_{J_1} * \psi_{J_2})(j_1, z), (\psi_{J_1} * \psi_{J_2})(w_1, z)\}.$$

$$(\alpha_{W_1} * \alpha_{W_2})((j_1, z)(w_1, z)) = \wedge\{\alpha_{W_1}(j_1 w_1), \alpha_{J_2}(z)\}$$

$$\geq \wedge\{\vee\{\alpha_{J_1}(j_1), \alpha_{J_1}(w_1)\}, \alpha_{J_2}(z)\}$$

$$= \vee\{\wedge\{\alpha_{J_1}(j_1), \alpha_{J_2}(z)\}, \wedge\{\alpha_{J_1}(w_1), \alpha_{J_2}(z)\}\}$$

$$= \vee\{(\alpha_{J_1} * \alpha_{J_2})(j_1, z), (\alpha_{J_1} * \alpha_{J_2})(w_1, z)\}.$$

$$(\beta_{W_1} * \beta_{W_2})((j_1, z)(w_1, z)) = \wedge\{\beta_{W_1}(j_1 w_1), \beta_{J_2}(z)\}$$

$$\geq \wedge\{\vee\{\beta_{J_1}(j_1), \beta_{J_1}(w_1)\}, \beta_{J_2}(z)\}$$

$$= \vee\{\wedge\{\beta_{J_1}(j_1), \beta_{J_2}(z)\}, \wedge\{\beta_{J_1}(w_1), \beta_{J_2}(z)\}\}$$

$$= \vee\{(\beta_{J_1} * \beta_{J_2})(j_1, z), (\beta_{J_1} * \beta_{J_2})(w_1, z)\}.$$

$$(\gamma_{W_1} * \gamma_{W_2})((j_1, z)(w_1, z)) = \vee\{\gamma_{W_1}(j_1 w_1), \gamma_{J_2}(z)\} \leq \vee\{\wedge\{\gamma_{J_1}(j_1), \gamma_{J_1}(w_1)\}, \gamma_{J_2}(z)\}$$

$$= \wedge\{\vee\{\gamma_{J_1}(j_1), \gamma_{J_2}(z)\}, \vee\{\gamma_{J_1}(w_1), \gamma_{J_2}(z)\}\}$$

$$= \wedge\{(\gamma_{J_1} * \gamma_{J_2})(j_1, z), (\gamma_{J_1} * \gamma_{J_2})(w_1, z)\},$$

We conclude that $\mathbb{G}_1 * \mathbb{G}_2$ is a BPFG. □

Theorem 1 The MP of two strong BPFGs $\mathbb{G}_1$ and $\mathbb{G}_2$ is a strong BPFG.

Proof. Suppose that $\mathbb{G}_1 = (J_1, W_1)$ and $\mathbb{G}_2 = (J_2, W_2)$ are two strong BPFGs on two crisp graphs and $((j_1, j_2)(w_1, w_2)) \in E_1 \times E_2$.

By using Proposition 1, we obtain:

(i) If $j_1 = w_1 = m$,

$$
\begin{aligned}
(\Upsilon_{W_1} * \Upsilon_{W_2})((m,\ j_2)(m,\ w_2)) &= \vee\{\Upsilon_{J_1}(m),\ \Upsilon_{W_2}(j_2w_2)\}\\
&= \vee\{\Upsilon_{J_1}(m),\ \wedge\{\Upsilon_{J_2}(j_2),\ \Upsilon_{J_2}(w_2)\}\}\\
&= \wedge\{\vee\{\Upsilon_{J_1}(m),\ \Upsilon_{J_2}(j_2)\},\ \vee\{\Upsilon_{J_1}(m),\ \Upsilon_{J_2}(w_2)\}\}\\
&= \wedge\{(\Upsilon_{J_1} * \Upsilon_{J_2})(m,\ j_2),\ (\Upsilon_{J_1} * \Upsilon_{J_2})(m,\ w_2)\},\\
(\upsilon_{W_1} * \upsilon_{W_2})((m,\ j_2)(m,\ w_2)) &= \vee\{\upsilon_{J_1}(m),\ \upsilon_{W_2}(j_2w_2)\}\\
&= \vee\{\upsilon_{J_1}(m),\ \wedge\{\upsilon_{J_2}(j_2),\ \upsilon_{J_2}(w_2)\}\}\\
&= \wedge\{\vee\{\upsilon_{J_1}(m),\ \upsilon_{J_2}(j_2)\},\ \vee\{\upsilon_{J_1}(m),\ \upsilon_{J_2}(w_2)\}\}\\
&= \wedge\{(\upsilon_{J_1} * \upsilon_{J_2})(m,\ j_2),\ (\upsilon_{J_1} * \upsilon_{J_2})(m,\ w_2)\},\\
(\psi_{W_1} * \psi_{W_2})((m,\ j_2)(m,\ w_2)) &= \wedge\{\psi_{J_1}(m),\ \psi_{W_2}(j_2w_2)\}\\
&= \wedge\{\psi_{J_1}(m),\ \vee\{\psi_{J_2}(j_2),\ \psi_{J_2}(w_2)\}\}\\
&= \vee\{\wedge\{\psi_{J_1}(m),\ \psi_{J_2}(j_2)\},\ \wedge\{\psi_{J_1}(m),\ \psi_{J_2}(w_2)\}\}\\
&= \vee\{(\psi_{J_1} * \psi_{J_2})(m,\ j_2),\ (\psi_{J_1} * \psi_{J_2})(m,\ w_2)\}.\\
(\alpha_{W_1} * \alpha_{W_2})((m,\ j_2)(m,\ w_2)) &= \wedge\{\alpha_{J_1}(m),\ \alpha_{W_2}(j_2w_2)\}\\
&= \wedge\{\alpha_{J_1}(m),\ \vee\{\alpha_{J_2}(j_2),\ \alpha_{J_2}(w_2)\}\}\\
&= \vee\{\wedge\{\alpha_{J_1}(m),\ \alpha_{J_2}(j_2)\},\ \wedge\{\alpha_{J_1}(m),\ \alpha_{J_2}(w_2)\}\}\\
&= \vee\{(\alpha_{J_1} * \alpha_{J_2})(m,\ j_2),\ (\alpha_{J_1} * \alpha_{J_2})(m,\ w_2)\}.
\end{aligned}
$$

$$(\beta_{W_1} * \beta_{W_2})((m,\ j_2)(m,\ w_2)) = \wedge\{\beta_{J_1}(m),\ \beta_{W_2}(j_2w_2)\}$$

$$= \wedge\{\beta_{J_1}(m),\ \vee\{\beta_{J_2}(j_2),\ \beta_{J_2}(w_2)\}\}$$

$$= \vee\{\wedge\{\beta_{J_1}(m),\ \beta_{J_2}(j_2)\},\ \wedge\{\beta_{J_1}(m),\ \beta_{J_2}(w_2)\}\}$$

$$= \vee\{(\beta_{J_1} * \beta_{J_2})(m,\ j_2),\ (\beta_{J_1} * \beta_{J_2})(m,\ w_2)\}.$$

$$(\gamma_{W_1} * \gamma_{W_2})((m,\ j_2)(m,\ w_2)) = \vee\{\gamma_{J_1}(m),\ \gamma_{W_2}(j_2w_2)\}$$

$$= \vee\{\gamma_{J_1}(m),\ \wedge\{\gamma_{J_2}(j_2),\ \gamma_{J_2}(w_2)\}\}$$

$$= \wedge\{\vee\{\gamma_{J_1}(m),\ \gamma_{J_2}(j_2)\},\ \vee\{\gamma_{J_1}(m),\ \gamma_{J_2}(w_2)\}\}$$

$$= \wedge\{(\gamma_{J_1} * \gamma_{J_2})(m,\ j_2),\ (\gamma_{J_1} * \gamma_{J_2})(m,\ w_2)\},$$

(ii) If $j_2 = w_2 = z$,

$$(\Upsilon_{W_1} * \Upsilon_{W_2})((j_1,\ z)(w_1,\ z)) = \vee\{\Upsilon_{W_1}(j_1w_1),\ \Upsilon_{J_2}(z)\}$$

$$= \vee\{\wedge\{\Upsilon_{J_1}(j_1),\ \Upsilon_{J_1}(w_1)\},\ \Upsilon_{J_2}(z)\}$$

$$= \wedge\{\vee\{\Upsilon_{J_1}(j_1),\ \Upsilon_{J_2}(z)\},\ \vee\{\Upsilon_{J_1}(w_1),\ \Upsilon_{J_2}(z)\}\}$$

$$= \wedge\{(\Upsilon_{J_1} * \Upsilon_{J_2})(j_1,\ z),\ (\Upsilon_{J_1} * \Upsilon_{J_2})(w_1,\ z)\},$$

$$(\upsilon_{W_1} * \upsilon_{W_2})((j_1,\ z)(w_1,\ z)) = \vee\{\upsilon_{W_1}(j_1w_1),\ \upsilon_{J_2}(z)\}$$

$$= \vee\{\wedge\{\upsilon_{J_1}(j_1),\ \upsilon_{J_1}(w_1)\},\ \upsilon_{J_2}(z)\}$$

$$= \wedge\{\vee\{\upsilon_{J_1}(j_1),\ \upsilon_{J_2}(z)\},\ \vee\{\upsilon_{J_1}(w_1),\ \upsilon_{J_2}(z)\}\}$$

$$= \wedge\{(\upsilon_{J_1} * \upsilon_{J_2})(j_1,\ z),\ (\upsilon_{J_1} * \upsilon_{J_2})(w_1,\ z)\},$$

$$
\begin{aligned}
(\psi_{W_1} * \psi_{W_2})((j_1,\, z)(w_1,\, z)) &= \wedge\{\psi_{W_1}(j_1 w_1),\ \psi_{J_2}(z)\} \\
&= \wedge\{\vee\{\psi_{J_1}(j_1),\ \psi_{J_1}(/w_1)\},\ \psi_{J_2}(z)\} \\
&= \vee\{\wedge\{\psi_{J_1}(j_1),\ \psi_{J_2}(z)\},\ \wedge\{\psi_{J_1}(w_1),\ \psi_{J_2}(z)\}\} \\
&= \vee\{(\psi_{J_1} * \psi_{J_2})(j_1,\, z),\ (\psi_{J_1} * \psi_{J_2})(w_1,\, z)\}.
\end{aligned}
$$

$$
\begin{aligned}
(\alpha_{W_1} * \alpha_{W_2})((j_1,\, z)(w_1,\, z)) &= \wedge\{\alpha_{W_1}(j_1 w_1),\ \alpha_{J_2}(z)\} \\
&= \wedge\{\vee\{\alpha_{J_1}(j_1),\ \alpha_{J_1}(w_1)\},\ \alpha_{J_2}(z)\} \\
&= \vee\{\wedge\{\alpha_{J_1}(j_1),\ \alpha_{J_2}(z)\},\ \wedge\{\alpha_{J_1}(w_1),\ \alpha_{J_2}(z)\}\} \\
&= \vee\{(\alpha_{J_1} * \alpha_{J_2})(j_1,\, z),\ (\alpha_{J_1} * \alpha_{J_2})(w_1,\, z)\}.
\end{aligned}
$$

$$
\begin{aligned}
(\beta_{W_1} * \beta_{W_2})((j_1,\, z)(w_1,\, z)) &= \wedge\{\beta_{W_1}(j_1 w_1),\ \beta_{J_2}(z)\} \\
&= \wedge\{\vee\{\beta_{J_1}(j_1),\ \beta_{J_1} w_1)\},\ \beta_{J_2}(z)\} \\
&= \vee\{\wedge\{\beta_{J_1}(j_1),\ \beta_{J_2}(z)\},\ \wedge\{\beta_{J_1}(w_1),\ \beta_{J_2}(z)\}\} \\
&= \vee\{(\beta_{J_1} * \beta_{J_2})(j_1,\, z),\ (\beta_{J_1} * \beta_{J_2})(w_1,\, z)\}.
\end{aligned}
$$

$$
\begin{aligned}
(\gamma_{W_1} * \gamma_{W_2})((j_1,\, z)(w_1,\, z)) &= \vee\{\gamma_{W_1}(j_1 w_1),\ \gamma_{J_2}(z)\} \\
&= \vee\{\wedge\{\gamma_{J_1}(j_1),\ \gamma_{J_1}(w_1)\},\ \gamma_{J_2}(z)\} \\
&= \wedge\{\vee\{\gamma_{J_1}(j_1),\ \gamma_{J_2}(z)\},\ \vee\{\gamma_{J_1}(w_1),\ \gamma_{J_2}(z)\}\} \\
&= \wedge\{(\gamma_{J_1} * \gamma_{J_2})(j_1,\, z),\ (\gamma_{J_1} * \gamma_{J_2})(w_1,\, z)\},
\end{aligned}
$$

Hence, $\mathbb{G}_1 * \mathbb{G}_2$ is a strong BPFG. □

Definition 12 Let $\mathbb{G}_1 = (J_1,\, W_1)$ and $\mathbb{G}_2 = (J_2,\, W_2)$ be two BPFGs. $\forall (j_1,\ j_2) \in\ V_1 \times V_2$,

$$(d_\Upsilon)_{\mathbb{G}_1*\mathbb{G}_2}(j_1,\ j_2) = \sum_{(j_1,\ j_2)(w_1,\ w_2)\in E_1\times E_2.} (\Upsilon_{W_1} * \Upsilon_{W_2})((j_1,\ j_2)(w_1,\ w_2))$$

$$= \sum_{j_1=w_1,\ j_2w_2\in E_2} \vee\{\Upsilon_{J_1}(j_1),\ \Upsilon_{W_2}(j_2w_2)\} + \sum_{j_1w_1\in E_1,\ j_2=w_2} \vee\{\Upsilon_{W_1}(j_1w_1),\ \Upsilon_{J_2}(j_2)\},$$

$$(d_\upsilon)_{\mathbb{G}_1*\mathbb{G}_2}(j_1,\ j_2) = \sum_{(j_1,\ j_2)(w_1,\ w_2)\in E_1\times E_2.} (\upsilon_{W_1} * \upsilon_{W_2})((j_1,\ j_2)(w_1,\ w_2))$$

$$= \sum_{j_1=w_1,\ j_2w_2\in E_2} \vee\{\upsilon_{J_1}(j_1),\ \upsilon_{W_2}(j_2w_2)\} + \sum_{j_1w_1\in E_1,\ j_2=w_2} \vee\{\upsilon_{W_1}(j_1w_1),\ \upsilon_{J_2}(j_2)\},$$

$$(d_\psi)_{\mathbb{G}_1*\mathbb{G}_2}(j_1,\ j_2) = \sum_{(j_1,\ j_2)(w_1,\ w_2)\in E_1\times E_2.} (\psi_{W_1} * \psi_{W_2})((j_1,\ j_2)(w_1,\ w_2))$$

$$= \sum_{j_1=w_1,\ j_2w_2\in E_2} \wedge\{\psi_{J_1}(j_1),\ \psi_{W_2}(j_2w_2)\} + \sum_{j_1w_1\in E_1,\ j_2=w_2} \wedge\{\psi_{W_1}(j_1w_1),\ \psi_{J_2}(j_2)\}.$$

$$(d_\alpha)_{\mathbb{G}_1*\mathbb{G}_2}(j_1,\ j_2) = \sum_{(j_1,\ j_2)(w_1,\ w_2)\in E_1\times E_2.} (\alpha_{W_1} * \alpha_{W_2})((j_1,\ j_2)(w_1,\ w_2))$$

$$= \sum_{j_1=w_1,\ j_2w_2\in E_2} \wedge\{\alpha_{J_1}(j_1),\ \alpha_{W_2}(j_2w_2)\} + \sum_{j_1w_1\in E_1,\ j_2=w_2} \wedge\{\alpha_{W_1}(j_1w_1),\ \alpha_{J_2}(j_2)\}.$$

$$(d_\beta)_{\mathbb{G}_1*\mathbb{G}_2}(j_1,\ j_2) = \sum_{(j_1,\ j_2)(w_1,\ w_2)\in E_1\times E_2.} (\beta_{W_1} * \beta_{W_2})((j_1,\ j_2)(w_1,\ w_2))$$

$$= \sum_{j_1=w_1,\ j_2w_2\in E_2} \wedge\{\beta_{J_1}(j_1),\ \beta_{W_2}(j_2w_2)\} + \sum_{j_1w_1\in E_1,\ j_2=w_2} \wedge\{\beta_{W_1}(j_1w_1),\ \beta_{J_2}(j_2)\}.$$

$$(d_\gamma)_{\mathbb{G}_1*\mathbb{G}_2}(j_1,\ j_2) = \sum_{(j_1,\ j_2)(w_1,\ w_2)\in E_1\times E_2.} (\gamma_{W_1} * \gamma_{W_2})((j_1,\ j_2)(w_1,\ w_2))$$

$$= \sum_{j_1=w_1,\ j_2w_2\in E_2} \vee\{\gamma_{J_1}(j_1),\ \gamma_{W_2}(j_2w_2)\} + \sum_{j_1w_1\in E_1,\ j_2=w_2} \vee\{\gamma_{W_1}(j_1w_1),\ \gamma_{J_2}(j_2)\},$$

Theorem 2 Suppose that $\mathbb{G}_1 = (J_1,\ W_1)$ and $\mathbb{G}_2 = (J_2,\ W_2)$ are two BPFGs. If $\Upsilon_{J_1} \geq \Upsilon_{W_2},\ \upsilon_{J_1} \geq \upsilon_{W_2},\ \psi_{J_1} \leq \psi_{W_2},\ \alpha_{J_1} \leq \alpha_{W_2},\ \beta_{J_1} \leq \beta_{W_2},\ \gamma_{J_1} \geq \gamma_{W_2}$ and $\Upsilon_{J_2} \geq \Upsilon_{W_1},\ \upsilon_{J_2} \geq \upsilon_{W_1},\ \psi_{J_2} \leq \psi_{W_1},\ \alpha_{J_2} \leq \alpha_{W_1},\ \beta_{J_2} \leq \beta_{W_1},\ \gamma_{J_2} \geq \gamma_{W_1}$, then $\forall (j_1,\ j_2) \in\ V_1 \times V_2$.

$$(d_{\Upsilon})_{\mathbb{G}_1 * \mathbb{G}_2}(j_1,\ j_2) = (d)_{G_2}(j_2)\Upsilon_{J_1}(j_1) + (d)_{G_1}(j_1)\Upsilon_{J_2}(j_2),$$

$$(d_{\upsilon})_{\mathbb{G}_1 * \mathbb{G}_2}(j_1,\ j_2) = (d)_{G_2}(j_2)\upsilon_{J_1}(j_1) + (d)_{G_1}(j_1)\upsilon_{J_2}(j_2),$$

$$(d_{\psi})_{\mathbb{G}_1 * \mathbb{G}_2}(j_1,\ j_2) = (d)_{G_2}(j_2)\psi_{J_1}(j_1) + (d)_{G_1}(j_1)\psi_{J_2}(j_2),$$

$$(d_{\alpha})_{\mathbb{G}_1 * \mathbb{G}_2}(j_1,\ j_2) = (d)_{G_2}(j_2)\alpha_{J_1}(j_1) + (d)_{G_1}(j_1)\alpha_{J_2}(j_2),$$

$$(d_{\beta})_{\mathbb{G}_1 * \mathbb{G}_2}(j_1,\ j_2) = (d)_{G_2}(j_2)\beta_{J_1}(j_1) + (d)_{G_1}(j_1)\beta_{J_2}(j_2),$$

$$(d_{\gamma})_{\mathbb{G}_1 * \mathbb{G}_2}(j_1,\ j_2) = (d)_{G_2}(j_2)\gamma_{J_1}(j_1) + (d)_{G_1}(j_1)\gamma_{J_2}(j_2).$$

Proof.

$$\begin{aligned}
(d_{\Upsilon})_{\mathbb{G}_1 * \mathbb{G}_2}(j_1,\ j_2) &= \sum_{(j_1,\ j_2)(w_1,\ w_2) \in E_1 \times E_2.} (\Upsilon_{W_1} * \Upsilon_{W_2})((j_1,\ j_2)(w_1,\ w_2)) \\
&= \sum_{j_1 = w_1,\ j_2 w_2 \in E_2} \vee\{\Upsilon_{J_1}(j_1),\ \Upsilon_{W_2}(j_2 w_2)\} + \sum_{j_1 w_1 \in E_1,\ j_2 = w_2} \vee\{\Upsilon_{W_1}(j_1 w_1),\ \Upsilon_{J_2}(j_2)\} \\
&= \sum_{j_2 w_2 \in E_2,\ j_1 = w_1} \Upsilon_{J_1}(j_1) + \sum_{j_1 w_1 \in E_1,\ j_2 = w_2} \Upsilon_{J_2}(j_2) \\
&= (d)_{G_2}(j_2)\Upsilon_{J_1}(j_1) + (d)_{G_1}(j_1)\Upsilon_{J_2}(j_2), \\
(d_{\upsilon})_{\mathbb{G}_1 * \mathbb{G}_2}(j_1,\ j_2) &= \sum_{(j_1,\ j_2)(w_1,\ w_2) \in E_1 \times E_2.} (\upsilon_{W_1} * \upsilon_{W_2})((j_1,\ j_2)(w_1,\ w_2)) \\
&= \sum_{j_1 = w_1,\ j_2 w_2 \in E_2} \vee\{\upsilon_{J_1}(j_1),\ \upsilon_{W_2}(j_2 w_2)\} + \sum_{j_1 w_1 \in E_1,\ j_2 = w_2} \vee\{\upsilon_{W_1}(j_1 w_1),\ \upsilon_{J_2}(j_2)\} \\
&= \sum_{j_2 w_2 \in E_2,\ j_1 = w_1} \upsilon_{J_1}(j_1) + \sum_{j_1 w_1 \in E_1,\ j_2 = w_2} \upsilon_{J_2}(j_2) \\
&= (d)_{G_2}(j_2)\upsilon_{J_1}(j_1) + (d)_{G_1}(j_1)\upsilon_{J_2}(j_2),
\end{aligned}$$

$$(d_\psi)_{\mathbb{G}_1*\mathbb{G}_2}(j_1,\ j_2)=\sum_{(j_1,\ j_2)(w_1,\ w_2)\in E_1\times E_2.}(\psi_{W_1}*\psi_{W_2})((j_1,\ j_2)(w_1,\ w_2))$$

$$=\sum_{j_1=w_1,\ j_2w_2\in E_2}\wedge\{\psi_{J_1}(j_1),\ \psi_{W_2}(j_2w_2)\}+\sum_{j_1w_1\in E_1,\ j_2=w_2}\wedge\{\psi_{W_1}(j_1w_1),\ \psi_{J_2}(j_2)\}$$

$$=\sum_{j_2w_2\in E_2,\ j_1=w_1}\psi_{J_1}(j_1)+\sum_{j_1w_1\in E_1,\ j_2=w_2}\psi_{J_2}(j_2)$$

$$=(d)_{G_2}(j_2)\psi_{J_1}(j_1)+(d)_{G_1}(j_1)\psi_{J_2}(j_2).$$

$$(d_\alpha)_{\mathbb{G}_1*\mathbb{G}_2}(j_1,\ j_2)=\sum_{(j_1,\ j_2)(w_1,\ w_2)\in E_1\times E_2.}(\alpha_{W_1}*\alpha_{W_2})((j_1,\ j_2)(w_1,\ w_2))$$

$$=\sum_{j_1=w_1,\ j_2w_2\in E_2}\wedge\{\alpha_{J_1}(j_1),\ \alpha_{W_2}(j_2w_2)\}+\sum_{j_1w_1\in E_1,\ j_2=w_2}\wedge\{\alpha_{W_1}(j_1w_1),\ \alpha_{J_2}(j_2)\}$$

$$=\sum_{j_2w_2\in E_2,\ j_1=w_1}\alpha_{J_1}(j_1)+\sum_{j_1w_1\in E_1,\ j_2=w_2}\alpha_{J_2}(j_2)$$

$$=(d)_{G_2}(j_2)\alpha_{J_1}(j_1)+(d)_{G_1}(j_1)\alpha_{J_2}(j_2).$$

$$(d_\beta)_{\mathbb{G}_1*\mathbb{G}_2}(j_1,\ j_2)=\sum_{(j_1,\ j_2)(w_1,\ w_2)\in E_1\times E_2.}(\beta_{W_1}*\beta_{W_2})((j_1,\ j_2)(w_1,\ w_2))$$

$$=\sum_{j_1=w_1,\ j_2w_2\in E_2}\wedge\{\beta_{J_1}(j_1),\ \beta_{W_2}(j_2w_2)\}+\sum_{j_1w_1\in E_1,\ j_2=w_2}\wedge\{\beta_{W_1}(j_1w_1),\ \beta_{J_2}(j_2)\}$$

$$=\sum_{j_2w_2\in E_2,\ j_1=w_1}\beta_{J_1}(j_1)+\sum_{j_1w_1\in E_1,\ j_2=w_2}\beta_{J_2}(j_2)$$

$$=(d)_{G_2}(j_2)\beta_{J_1}(j_1)+(d)_{G_1}(j_1)\beta_{J_2}(j_2).$$

$$(d_\gamma)_{\mathbb{G}_1*\mathbb{G}_2}(j_1,\ j_2)=\sum_{(j_1,\ j_2)(w_1,\ w_2)\in E_1\times E_2.}(\gamma_{W_1}*\gamma_{W_2})((j_1,\ j_2)(w_1,\ w_2))$$

$$=\sum_{j_1=w_1,\ j_2w_2\in E_2}\vee\{\gamma_{J_1}(j_1),\ \gamma_{W_2}(j_2w_2)\}+\sum_{j_1w_1\in E_1,\ j_2=w_2}\vee\{\gamma_{W_1}(j_1w_1),\ \gamma_{J_2}(j_2)\}$$

$$=\sum_{j_2w_2\in E_2,\ j_1=w_1}\gamma_{J_1}(j_1)+\sum_{j_1w_1\in E_1,\ j_2=w_2}\gamma_{J_2}(j_2)=(d)_{G_2}(j_2)\gamma_{J_1}(j_1)+(d)_{G_1}(j_1)\gamma_{J_2}(j_2),$$

□

Definition 13 Let $\mathbb{G}_1=(J_1,\ W_1)$ and $\mathbb{G}_2=(J_2,\ W_2)$ be two BPFGs. $\forall(j_1,\ j_2)\in\ V_1\times V_2$,

$$(td_{\Upsilon})_{\mathbb{G}_1 * \mathbb{G}_2}(j_1,\ j_2) = \sum_{(j_1,\ j_2)(w_1,\ w_2) \in E_1 \times E_2.} (\Upsilon_{W_1} * \Upsilon_{W_2})((j_1,\ j_2)(w_1,\ w_2)) + (\Upsilon_{J_1} * \Upsilon_{J_2})(j_1,\ j_2)$$

$$= \sum_{j_1 = w_1,\ j_2 w_2 \in E_2} \vee \{\Upsilon_{J_1}(j_1),\ \Upsilon_{W_2}(j_2 w_2)\}$$

$$+ \sum_{j_1 w_1 \in E_1,\ j_2 = w_2} \vee \{\Upsilon_{W_1}(j_1 w_1),\ \Upsilon_{J_2}(j_2)\} + \vee \{\Upsilon_{J_1}(j_1),\ \Upsilon_{J_2}(j_2)\},$$

$$(td_{\upsilon})_{\mathbb{G}_1 * \mathbb{G}_2}(j_1,\ j_2) = \sum_{(j_1,\ j_2)(w_1,\ w_2) \in E_1 \times E_2.} (\upsilon_{W_1} * \upsilon_{W_2})((j_1,\ j_2)(w_1,\ w_2)) + (\upsilon_{J_1} * \upsilon_{J_2})(j_1,\ j_2)$$

$$= \sum_{j_1 = w_1,\ j_2 w_2 \in E_2} \vee \{\upsilon_{J_1}(j_1),\ \upsilon_{W_2}(j_2 w_2)\}$$

$$+ \sum_{j_1 w_1 \in E_1,\ j_2 = w_2} \vee \{\upsilon_{W_1}(j_1 w_1),\ \upsilon_{J_2}(j_2)\} + \vee \{\upsilon_{J_1}(j_1),\ \upsilon_{J_2}(j_2)\},$$

$$(td_{\psi})_{\mathbb{G}_1 * \mathbb{G}_2}(j_1,\ j_2) = \sum_{(j_1,\ j_2)(w_1,\ w_2) \in E_1 \times E_2.} (\psi_{W_1} * \psi_{W_2})((j_1,\ j_2)(w_1,\ w_2)) + (\psi_{J_1} * \psi_{J_2})(j_1,\ j_2)$$

$$= \sum_{j_1 = w_1,\ j_2 w_2 \in E_2} \wedge \{\psi_{J_1}(j_1),\ \psi_{W_2}(j_2 w_2)\}$$

$$+ \sum_{j_1 w_1 \in E_1,\ j_2 = w_2} \wedge \{\psi_{W_1}(j_1 w_1),\ \psi_{J_2}(j_2)\} + \wedge \{\psi_{J_1}(j_1),\ \psi_{J_2}(j_2)\}.$$

$$(td_{\alpha})_{\mathbb{G}_1 * \mathbb{G}_2}(j_1,\ j_2) = \sum_{(j_1,\ j_2)(w_1,\ w_2) \in E_1 \times E_2.} (\alpha_{W_1} * \alpha_{W_2})((j_1,\ j_2)(w_1,\ w_2)) + (\alpha_{J_1} * \alpha_{J_2})(j_1,\ j_2)$$

$$= \sum_{j_1 = w_1,\ j_2 w_2 \in E_2} \wedge \{\alpha_{J_1}(j_1),\ \alpha_{W_2}(j_2 w_2)\}$$

$$+ \sum_{j_1 w_1 \in E_1,\ j_2 = w_2} \wedge \{\alpha_{W_1}(j_1 w_1),\ \alpha_{J_2}(j_2)\} + \wedge \{\alpha_{J_1}(j_1),\ \alpha_{J_2}(j_2)\}.$$

$$(td_{\beta})_{\mathbb{G}_1*\mathbb{G}_2}(j_1,\ j_2)=\sum_{(j_1,\ j_2)(w_1,\ w_2)\in E_1\times E_2.}(\beta_{W_1}*\beta_{W_2})((j_1,\ j_2)(w_1,\ w_2))+(\beta_{J_1}*\beta_{J_2})(j_1,\ j_2)$$

$$=\sum_{j_1=w_1,\ j_2w_2\in E_2}\wedge\{\beta_{J_1}(j_1),\ \beta_{W_2}(j_2w_2)\}$$

$$+\sum_{j_1w_1\in E_1,\ j_2=w_2}\wedge\{\beta_{W_1}(j_1w_1),\ \beta_{J_2}(j_2)\}+\wedge\{\beta_{J_1}(j_1),\ \beta_{J_2}(j_2)\}.$$

$$(td_{\gamma})_{\mathbb{G}_1*\mathbb{G}_2}(j_1,\ j_2)=\sum_{(j_1,\ j_2)(w_1,\ w_2)\in E_1\times E_2.}(\gamma_{W_1}*\gamma_{W_2})((j_1,\ j_2)(w_1,\ w_2))+(\gamma_{J_1}*\gamma_{J_2})(j_1,\ j_2)$$

$$=\sum_{j_1=w_1,\ j_2w_2\in E_2}\vee\{\gamma_{J_1}(j_1),\ \gamma_{W_2}(j_2w_2)\}$$

$$+\sum_{j_1w_1\in E_1,\ j_2=w_2}\vee\{\gamma_{W_1}(j_1w_1),\ \gamma_{J_2}(j_2)\}+\vee\{\gamma_{J_1}(j_1),\ \gamma_{J_2}(j_2)\},$$

Theorem 3 Suppose that $\mathbb{G}_1=(J_1,\ W_1)$ and $\mathbb{G}_2=(J_2,\ W_2)$ are two BPFGs. If $\Upsilon_{J_1}\geq\Upsilon_{W_2},\ \upsilon_{J_1}\geq\upsilon_{W_2},\ \psi_{J_1}\leq\psi_{W_2},\ \alpha_{J_1}\leq\alpha_{W_2},\ \beta_{J_1}\leq\beta_{W_2},\ \gamma_{J_1}\geq\gamma_{W_2}$ and $\Upsilon_{J_2}\geq\Upsilon_{W_1},\ \upsilon_{J_2}\geq\upsilon_{W_1},\ \psi_{J_2}\leq\psi_{W_1},\ \alpha_{J_2}\leq\alpha_{W_1},\ \beta_{J_2}\leq\beta_{W_1},\ \gamma_{J_2}\geq\gamma_{W_1}$, then $\forall(j_1,\ j_2)\in V_1\times V_2$.

$$(td_{\Upsilon})_{\mathbb{G}_1*\mathbb{G}_2}(j_1,\ j_2)=(d)_{G_2}(j_2)\Upsilon_{J_1}(j_1)+(d)_{G_1}(j_1)\Upsilon_{J_2}(j_2)+\vee\{\Upsilon_{J_1}(j_1),\ \Upsilon_{J_2}(j_2)\},$$

$$(td_{\upsilon})_{\mathbb{G}_1*\mathbb{G}_2}(j_1,\ j_2)=(d)_{G_2}(j_2)\upsilon_{J_1}(j_1)+(d)_{G_1}(j_1)\upsilon_{J_2}(j_2)+\vee\{\upsilon_{J_1}(j_1),\ \upsilon_{J_2}(j_2)\},$$

$$(td_{\psi})_{\mathbb{G}_1*\mathbb{G}_2}(j_1,\ j_2)=(d)_{G_2}(j_2)\psi_{J_1}(j_1)+(d)_{G_1}(j_1)\psi_{J_2}(j_2)+\wedge\{\psi_{J_1}(j_1),\ \psi_{J_2}(j_2)\},$$

$$(td_{\alpha})_{\mathbb{G}_1*\mathbb{G}_2}(j_1,\ j_2)=(d)_{G_2}(j_2)\alpha_{J_1}(j_1)+(d)_{G_1}(j_1)\alpha_{J_2}(j_2)+\wedge\{\alpha_{J_1}(j_1),\ \alpha_{J_2}(j_2)\},$$

$$(td_{\beta})_{\mathbb{G}_1*\mathbb{G}_2}(j_1,\ j_2)=(d)_{G_2}(j_2)\beta_{J_1}(j_1)+(d)_{G_1}(j_1)\beta_{J_2}(j_2)+\wedge\{\beta_{J_1}(j_1),\ \beta_{J_2}(j_2)\},$$

$$(td_{\gamma})_{\mathbb{G}_1*\mathbb{G}_2}(j_1,\ j_2)=(d)_{G_2}(j_2)\gamma_{J_1}(j_1)+(d)_{G_1}(j_1)\gamma_{J_2}(j_2)+\vee\{\gamma_{J_1}(j_1),\ \gamma_{J_2}(j_2)\}.$$

Proof.

$$(td_{\Upsilon})_{\mathbb{G}_1*\mathbb{G}_2}(j_1,\ j_2) = \sum_{(j_1,\ j_2)(w_1,\ w_2)\in E_1\times E_2.} (\Upsilon_{W_1}*\Upsilon_{W_2})((j_1,\ j_2)(w_1,\ w_2)) + (\Upsilon_{J_1}*\Upsilon_{J_2})(j_1,\ j_2)$$

$$= \sum_{j_1=w_1,\ j_2w_2\in E_2} \vee\{\Upsilon_{J_1}(j_1),\ \Upsilon_{W_2}(j_2w_2)\}$$

$$+ \sum_{j_1w_1\in E_1,\ j_2=w_2} \vee\{\Upsilon_{W_1}(j_1w_1),\ \Upsilon_{J_2}(j_2)\} + \vee\{\Upsilon_{J_1}(j_1),\ \Upsilon_{J_2}(j_2)\}$$

$$= \sum_{j_2w_2\in E_2,\ j_1=w_1} \Upsilon_{J_1}(j_1) + \sum_{j_1w_1\in E_1,\ j_2=w_2} \Upsilon_{J_2}(j_2) + \vee\{\Upsilon_{J_1}(j_1),\ \Upsilon_{J_2}(j_2)\}$$

$$= (d)_{G_2}(j_2)\Upsilon_{J_1}(j_1) + (d)_{G_1}(j_1)\Upsilon_{J_2}(j_2) + \vee\{\Upsilon_{J_1}(j_1),\ \Upsilon_{J_2}(j_2)\}$$

$$(td_{\upsilon})_{\mathbb{G}_1*\mathbb{G}_2}(j_1,\ j_2) = \sum_{(j_1,\ j_2)(w_1,\ w_2)\in E_1\times E_2.} (\upsilon_{W_1}*\upsilon_{W_2})((j_1,\ j_2)(w_1,\ w_2)) + (\upsilon_{J_1}*\upsilon_{J_2})(j_1,\ j_2)$$

$$= \sum_{j_1=w_1,\ j_2w_2\in E_2} \vee\{\upsilon_{J_1}(j_1),\ \upsilon_{W_2}(j_2w_2)\}$$

$$+ \sum_{j_1w_1\in E_1,\ j_2=w_2} \vee\{\upsilon_{W_1}(j_1w_1),\ \upsilon_{J_2}(j_2)\} + \vee\{\upsilon_{J_1}(j_1),\ \upsilon_{J_2}(j_2)\}$$

$$= \sum_{j_2w_2\in E_2,\ j_1=w_1} \upsilon_{J_1}(j_1) + \sum_{j_1w_1\in E_1,\ j_2=w_2} \upsilon_{J_1}(j_2) + \vee\{\upsilon_{J_1}(j_1),\ \upsilon_{J_2}(j_2)\}$$

$$= (d)_{G_2}(j_2)\upsilon_{J_1}(j_1) + (d)_{G_1}(j_1)\upsilon_{J_2}(j_2) + \vee\{\upsilon_{J_1}(j_1),\ \upsilon_{J_2}(j_2)\}$$

$$(td_{\psi})_{\mathbb{G}_1*\mathbb{G}_2}(j_1,\ j_2) = \sum_{(j_1,\ j_2)(w_1,\ w_2)\in E_1\times E_2.} (\psi_{W_1}*\psi_{W_2})((j_1,\ j_2)(w_1,\ w_2)) + (\psi_{J_1}*\psi_{J_2})(j_1,\ j_2)$$

$$= \sum_{j_1=w_1,\ j_2w_2\in E_2} \wedge\{\psi_{J_1}(j_1),\ \psi_{W_2}(j_2w_2)\}$$

$$+ \sum_{j_1w_1\in E_1,\ j_2=w_2} \wedge\{\psi_{W_1}(j_1w_1),\ \psi_{J_2}(j_2)\} + \wedge\{\psi_{J_1}(j_1),\ \psi_{J_2}(j_2)\}$$

$$= \sum_{j_2w_2\in E_2,\ j_1=w_1} \psi_{J_1}(j_1) + \sum_{j_1w_1\in E_1,\ j_2=w_2} \psi_{J_2}(j_2) + \wedge\{\psi_{J_1}(j_1),\ \psi_{J_2}(j_2)\}$$

$$= (d)_{G_2}(j_2)\psi_{J_1}(j_1) + (d)_{G_1}(j_1)\psi_{J_2}(j_2) + \wedge\{\psi_{J_1}(j_1),\ \psi_{J_2}(j_2)\}$$

$$
\begin{aligned}
(td_\alpha)_{\mathbb{G}_1*\mathbb{G}_2}(j_1,\ j_2) &= \sum_{(j_1,\ j_2)(w_1,\ w_2)\in E_1\times E_2.} (\alpha_{W_1}*\alpha_{W_2})((j_1,\ j_2)(w_1,\ w_2)) + (\alpha_{J_1}*\alpha_{J_2})(j_1,\ j_2)\\
&= \sum_{j_1=w_1,\ j_2w_2\in E_2} \wedge\{\alpha_{J_1}(j_1),\ \alpha_{W_2}(j_2w_2)\}\\
&\quad + \sum_{j_1w_1\in E_1,\ j_2=w_2} \wedge\{\alpha_{W_1}(j_1w_1),\ \alpha_{J_2}(j_2)\} + \wedge\{\alpha_{J_1}(j_1),\ \alpha_{J_2}(j_2)\}\\
&= \sum_{j_2w_2\in E_2,\ j_1=w_1} \alpha_{J_1}(j_1) + \sum_{j_1w_1\in E_1,\ j_2=w_2} \alpha_{J_2}(j_2) + \wedge\{\alpha_{J_1}(j_1),\ \alpha_{J_2}(j_2)\}\\
&= (d)_{G_2}(j_2)\alpha_{J_1}(j_1) + (d)_{G_1}(j_1)\alpha_{J_2}(j_2) + \wedge\{\alpha_{J_1}(j_1),\ \alpha_{J_2}(j_2)\}\\
(td_\beta)_{\mathbb{G}_1*\mathbb{G}_2}(j_1,\ j_2) &= \sum_{(j_1,\ j_2)(w_1,\ w_2)\in E_1\times E_2.} (\beta_{W_1}*\beta_{W_2})((j_1,\ j_2)(w_1,\ w_2)) + (\beta_{J_1}*\beta_{J_2})(j_1,\ j_2)\\
&= \sum_{j_1=w_1,\ j_2w_2\in E_2} \wedge\{\beta_{J_1}(j_1),\ \beta_{W_2}(j_2w_2)\}\\
&\quad + \sum_{j_1w_1\in E_1,\ j_2=w_2} \wedge\{\beta_{W_1}(j_1w_1),\ \beta_{J_2}(j_2)\} + \wedge\{\beta_{J_1}(j_1),\ \beta_{J_2}(j_2)\}\\
&= \sum_{j_2w_2\in E_2,\ j_1=w_1} \beta_{J_1}(j_1) + \sum_{j_1w_1\in E_1,\ j_2=w_2} \beta_{J_2}(j_2) + \wedge\{\beta_{J_1}(j_1),\ \beta_{J_2}(j_2)\}\\
&= (d)_{G_2}(j_2)\beta_{J_1}(j_1) + (d)_{G_1}(j_1)\beta_{J_2}(j_2) + \wedge\{\beta_{J_1}(j_1),\ \beta_{J_2}(j_2)\}\\
(td_\gamma)_{\mathbb{G}_1*\mathbb{G}_2}(j_1,\ j_2) &= \sum_{(j_1,\ j_2)(w_1,\ w_2)\in E_1\times E_2.} (\gamma_{W_1}*\gamma_{W_2})((j_1,\ j_2)(w_1,\ w_2)) + (\gamma_{J_1}*\gamma_{J_2})(j_1,\ j_2)\\
&= \sum_{j_1=w_1,\ j_2w_2\in E_2} \vee\{\gamma_{J_1}(j_1),\ \gamma_{W_2}(j_2w_2)\}\\
&\quad + \sum_{j_1w_1\in E_1,\ j_2=w_2} \vee\{\gamma_{W_1}(j_1w_1),\ \gamma_{J_2}(j_2)\} + \vee\{\gamma_{J_1}(j_1),\ \gamma_{J_2}(j_2)\}\\
&= \sum_{j_2w_2\in E_2,\ j_1=w_1} \gamma_{J_1}(j_1) + \sum_{j_1w_1\in E_1,\ j_2=w_2} \gamma_{J_2}(j_2) + \vee\{\gamma_{J_1}(j_1),\ \gamma_{J_2}(j_2)\}\\
&= (d)_{G_2}(j_2)\gamma_{J_1}(j_1) + (d)_{G_1}(j_1)\gamma_{J_2}(j_2) + \vee\{\gamma_{J_1}(j_1),\ \gamma_{J_2}(j_2)\}
\end{aligned}
$$

□

Example Let $\mathbb{G}_1 = (J_1,\ W_1)$ and $\mathbb{G}_2 = (J_2,\ W_2)$ be two BPFGs, with $\Upsilon_{J_1} \geq \Upsilon_{W_2},\ \upsilon_{J_1} \geq \upsilon_{W_2},\ \psi_{J_1} \leq \psi_{W_2},\ \alpha_{J_1} \leq \alpha_{W_2},\ \beta_{J_1} \leq \beta_{W_2},\ \gamma_{J_1} \geq \gamma_{W_2}$ and $\Upsilon_{J_2} \geq \Upsilon_{W_1},\ \upsilon_{J_2} \geq \upsilon_{W_1},\ \psi_{J_2} \leq \psi_{W_1},\ \alpha_{J_2} \leq \alpha_{W_1},\ \beta_{J_2} \leq \beta_{W_1},\ \gamma_{J_2} \geq \gamma_{W_1}$. In Example 3, we calculate the total degree of nodes of $\mathbb{G}_1 * \mathbb{G}_2$ by using Figures 1-3. We calculate the degree and total degree of nodes in MP for node $(b,\ f)$.

$$(d_\Upsilon)_{\mathbb{G}_1*\mathbb{G}_2}(b,\ f) = (d)_{G_2}(f)\Upsilon_{J_1}(b) + (d)_{G_1}(b)\Upsilon_{J_2}(f)$$

$$= 1(0.1) + 2(0.2) = 0.1 + 0.4 = 0.5,$$

$$(d_\upsilon)_{\mathbb{G}_1*\mathbb{G}_2}(b,\ f) = (d)_{G_2}(f)\upsilon_{J_1}(b) + (d)_{G_1}(b)\upsilon_{J_2}(f)$$

$$= 1(0.3) + 2(0.3) = 0.3 + 0.6 = 0.9,$$

$$(d_\psi)_{\mathbb{G}_1*\mathbb{G}_2}(b,\ f) = (d)_{G_2}(f)\psi_{J_1}(b) + (d)_{G_1}(b)\psi_{J_2}(f)$$

$$= 1(0.4) + 2(0.1) = 0.4 + 0.2 = 0.6,$$

$$(d_\alpha)_{\mathbb{G}_1*\mathbb{G}_2}(b,\ f) = (d)_{G_2}(f)\alpha_{J_1}(b) + (d)_{G_1}(b)\alpha_{J_2}(f)$$

$$= 1(-0.1) + 2(-0.2) = -0.1 - 0.4 = -0.5,$$

$$(d_\beta)_{\mathbb{G}_1*\mathbb{G}_2}(b,\ f) = (d)_{G_2}(f)\beta_{J_1}(b) + (d)_{G_1}(b)\beta_{J_2}(f)$$

$$= 1(-0.3) + 2(-0.3) = -0.3 - 0.6 = -0.9,$$

$$(d_\gamma)_{\mathbb{G}_1*\mathbb{G}_2}(b,\ f) = (d)_{G_2}(f)\gamma_{J_1}(b) + (d)_{G_1}(b)\gamma_{J_2}(f)$$

$$= 1(-0.4) + 2(-0.1) = -0.4 - 0.2 = -0.6,$$

$$(td_\Upsilon)_{\mathbb{G}_1*\mathbb{G}_2}(b,\ f) = (d)_{G_2}(f)\Upsilon_{J_1}(b) + (d)_{G_1}(b)\Upsilon_{J_2}(f) + \vee\{\Upsilon_{J_1}(b),\ \Upsilon_{J_2}(f)\}$$

$$= 1(0.1) + 2(0.2) + \vee\{0.1,\ 0.2\} = 0.1 + 0.4 + 0.2 = 0.7,$$

$$(td_\upsilon)_{\mathbb{G}_1*\mathbb{G}_2}(b,\ f) = (d)_{G_2}(f)\upsilon_{J_1}(b) + (d)_{G_1}(b)\upsilon_{J_2}(f) + \vee\{\upsilon_{J_1}(b),\ \upsilon_{J_2}(f)\}$$

$$= 1(0.3) + 2(0.3) + \vee\{0.3,\ 0.3\} = 0.3 + 0.6 + 0.3 = 1.2,$$

$$(td_\psi)_{\mathbb{G}_1*\mathbb{G}_2}(b,\ f) = (d)_{G_2}(f)\psi_{J_1}(b) + (d)_{G_1}(b)\psi_{J_2}(f) + \wedge\{\psi_{J_1}(b),\ \psi_{J_2}(f)\}$$

$$= 1(0.4) + 2(0.1) + \wedge\{0.4,\ 0.1\} = 0.4 + 0.2 + 0.1 = 0.7,$$

$$(td_\alpha)_{\mathbb{G}_1*\mathbb{G}_2}(b,\ f) = (d)_{G_2}(f)\alpha_{J_1}(b) + (d)_{G_1}(b)\alpha_{J_2}(f) + \wedge\{\alpha_{J_1}(b),\ \alpha_{J_2}(f)\}$$

$$= 1(-0.1) + 2(-0.2) + \wedge\{-0.1,\ -0.2\} = -0.1 - 0.4 - 0.2 = -0.7,$$

$$(td_\beta)_{\mathbb{G}_1*\mathbb{G}_2}(b,\ f) = (d)_{G_2}(f)\beta_{J_1}(b) + (d)_{G_1}(b)\beta_{J_2}(f) + \wedge\{\beta_{J_1}(b),\ \beta_{J_2}(f)\}$$

$$= 1(-0.3) + 2(-0.3) + \wedge\{-0.3,\ -0.3\} = -0.3 - 0.6 - 0.3 = -1.2,$$

$$(td_\gamma)_{\mathbb{G}_1*\mathbb{G}_2}(b,\ f) = (d)_{G_2}(f)\gamma_{J_1}(b) + (d)_{G_1}(b)\gamma_{J_2}(f) + \vee\{\gamma_{J_1}(b),\ \gamma_{J_2}(f)\}$$

$$= 1(-0.4) + 2(-0.1) + \vee\{-0.4,\ -0.1\} = -0.4 - 0.2 - 0.1 = -0.7,$$

We can calculate it similarly for other nodes.

Definition 14 The RP $\mathbb{G}_1 \bullet \mathbb{G}_2 = (J_1 \bullet J_2,\ W_1 \bullet W_2)$ of two BPFGs $\mathbb{G}_1 = (J_1,\ W_1)$ and $\mathbb{G}_2 = (J_2,\ W_2)$ is defined as:
(i)

$$(\Upsilon_{J_1} \bullet \Upsilon_{J_2})((j_1,\ j_2)) = \vee\{\Upsilon_{J_1}(j_1),\ \Upsilon_{J_2}(j_2)\},$$

$$(\upsilon_{J_1} \bullet \upsilon_{J_2})((j_1,\ j_2)) = \vee\{\upsilon_{J_1}(j_1),\ \upsilon_{J_2}(j_2)\},$$

$$(\psi_{J_1} \bullet \psi_{J_2})((j_1,\ j_2)) = \wedge\{\psi_{J_1}(j_1),\ \psi_{J_2}(j_2)\},$$

$$(\alpha_{J_1} \bullet \alpha_{J_2})((j_1,\ j_2)) = \wedge\{\alpha_{J_1}(j_1),\ \alpha_{J_2}(j_2)\},$$

$$(\beta_{J_1} \bullet \beta_{J_2})((j_1,\ j_2)) = \wedge\{\beta_{J_1}(j_1),\ \beta_{J_2}(j_2)\},$$

$$(\gamma_{J_1} \bullet \gamma_{J_2})((j_1,\ j_2)) = \vee\{\gamma_{J_1}(j_1),\ \gamma_{J_2}(j_2)\},$$

$\forall (j_1,\ j_2) \in (V_1 \times V_2)$.

(ii)

$$(\Upsilon_{W_1} \bullet \Upsilon_{W_2})((j_1,\ j_2)(w_1,\ w_2)) = \Upsilon_{W_1}(j_1w_1),$$

$$(\upsilon_{W_1} \bullet \upsilon_{W_2})((j_1,\ j_2)(w_1,\ w_2)) = \upsilon_{W_1}(j_1w_1),$$

$$(\psi_{W_1} \bullet \psi_{W_2})((j_1,\ j_2)(w_1,\ w_2)) = \psi_{W_1}(j_1w_1),$$

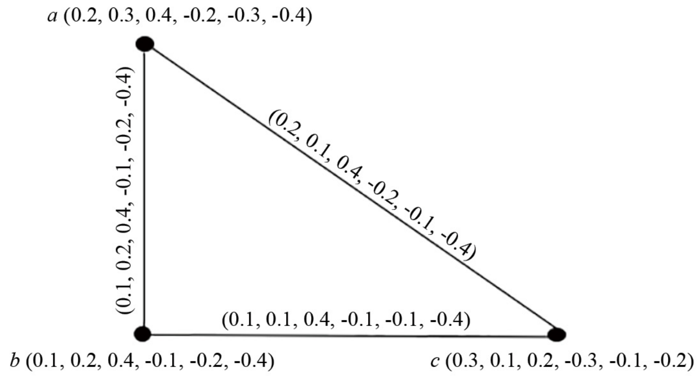
$$(\alpha_{W_1} \bullet \alpha_{W_2})((j_1,\ j_2)(w_1,\ w_2)) = \alpha_{W_1}(j_1w_1),$$

$$(\beta_{W_1} \bullet \beta_{W_2})((j_1,\ j_2)(w_1,\ w_2)) = \beta_{W_1}(j_1w_1),$$

$$(\gamma_{W_1} \bullet \gamma_{W_2})((j_1,\ j_2)(w_1,\ w_2)) = \gamma_{W_1}(j_1w_1),$$

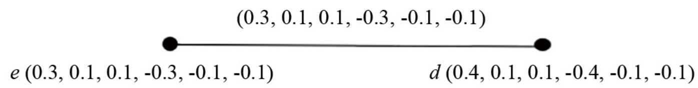
$\forall j_1w_1 \in E_1,\ j_2 \neq w_2$.

Example Taking two BPFGs $\mathbb{G}_1$ and $\mathbb{G}_2$, as in Figures 4 and 5, we can see the RP of two BPFGs $\mathbb{G}_1$ and $\mathbb{G}_2$, that is, $\mathbb{G}_1 \bullet \mathbb{G}_2$, in Figure 6.



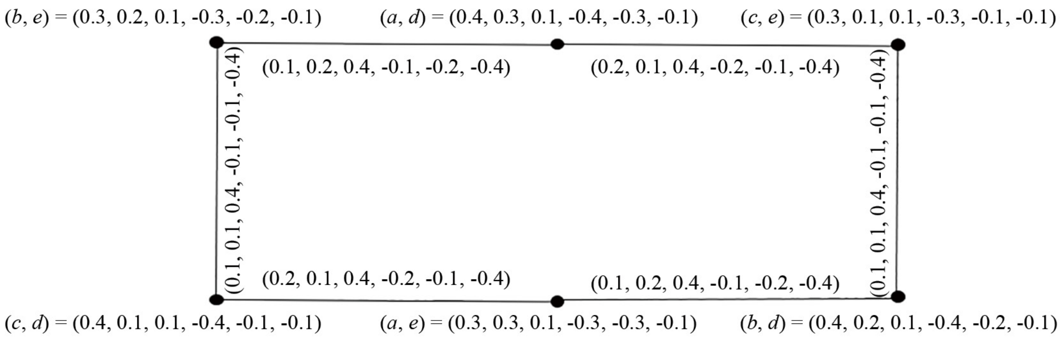
a = (0.2, 0.3, 0.4, -0.2, -0.3, -0.4); *b* = (0.1, 0.2, 0.4, -0.1, -0.2, -0.4);
c = (0.3, 0.1, 0.2, -0.3, -0.1, -0.2); *ab* = (0.1, 0.2, 0.4, -0.1, -0.2, -0.4);
bc = (0.1, 0.1, 0.4, -0.1, -0.1, -0.4); *ac* = (0.2, 0.1, 0.4, -0.2, -0.1, -0.4);

Figure 4. $\mathbb{G}_1$



d = (0.4, 0.1, 0.1, -0.4, -0.1, -0.1); *e* = (0.3, 0.1, 0.1, -0.3, -0.1, -0.1)
de = (0.3, 0.1, 0.1, -0.3, -0.1, -0.1)

Figure 5. $\mathbb{G}_2$



$(a, d) = (0.4, 0.3, 0.1, -0.4, -0.3, -0.1)$; $(b, e) = (0.3, 0.2, 0.1, -0.3, -0.2, -0.1)$
$(c, d) = (0.4, 0.1, 0.1, -0.4, -0.1, -0.1)$; $(a, e) = (0.3, 0.3, 0.1, -0.3, -0.3, -0.1)$
$(c, e) = (0.3, 0.1, 0.1, -0.3, -0.1, -0.1)$; $(b, d) = (0.4, 0.2, 0.1, -0.4, -0.2, -0.1)$
$(a, d)(b, e) = (0.1, 0.2, 0.4, -0.1, -0.2, -0.4)$; $(b, e)(c, d) = (0.1, 0.1, 0.4, -0.1, -0.1, -0.4)$
$(c, d)(a, e) = (0.2, 0.1, 0.4, -0.2, -0.1, -0.4)$; $(a, d)(c, e) = (0.2, 0.1, 0.4, -0.2, -0.1, -0.4)$
$(c, e)(b, d) = (0.1, 0.1, 0.4, -0.1, -0.1, -0.4)$; $(b, d)(a, e) = (0.1, 0.2, 0.4, -0.1, -0.2, -0.4)$

Figure 6. $\mathbb{G}_1 \bullet \mathbb{G}_2$

For node (c, e), we find Mv, NNv, NMv, neg-Mv, neg-NNv, and neg-NMv as follows:

$$(\Upsilon_{J_1} \bullet \Upsilon_{J_2})((c, e)) = \vee\{\Upsilon_{J_1}(c), \Upsilon_{J_2}(e)\} = \vee\{0.3, 0.3\} = 0.3,$$

$$(\upsilon_{J_1} \bullet \upsilon_{J_2})((c, e)) = \vee\{\upsilon_{J_1}(c), \upsilon_{J_2}(e)\} = \vee\{0.1, 0.1\} = 0.1,$$

$$(\psi_{J_1} \bullet \psi_{J_2})((c, e)) = \wedge\{\psi_{J_1}(c), \psi_{J_2}(e)\} = \wedge\{0.2, 0.1\} = 0.1,$$

$$(\alpha_{J_1} \bullet \alpha_{J_2})((c, e)) = \wedge\{\alpha_{J_1}(c), \alpha_{J_2}(e)\} = \wedge\{-0.3, -0.3\} = -0.3,$$

$$(\beta_{J_1} \bullet \beta_{J_2})((c, e)) = \wedge\{\beta_{J_1}(c), \beta_{J_2}(e)\} = \wedge\{-0.1, -0.1\} = -0.1,$$

$$(\gamma_{J_1} \bullet \gamma_{J_2})((c, e)) = \vee\{\gamma_{J_1}(c), \gamma_{J_2}(e)\} = \vee\{-0.2, -0.1\} = -0.1.$$

for $c \in V_1$ and $e \in V_2$.

For arc $(c, d)(a, e)$, we find Mv, NNv, NMv, neg-Mv, neg-NNv, and neg-NMv respectively.

$$(\Upsilon_{W_1} \bullet \Upsilon_{W_2})((c, d)(a, e)) = \Upsilon_{W_1}(ac) = 0.2,$$

$$(\upsilon_{W_1} \bullet \upsilon_{W_2})((c, d)(a, e)) = \upsilon_{W_1}(ac) = 0.1,$$

$$(\psi_{W_1} \bullet \psi_{W_2})((c, d)(a, e)) = \psi_{W_1}(ac) = 0.4,$$

$$(\alpha_{W_1} \bullet \alpha_{W_2})((c,\ d)(a,\ e)) = \alpha_{W_1}(ac) = -0.2,$$

$$(\beta_{W_1} \bullet \beta_{W_2})((c,\ d)(a,\ e)) = \beta_{W_1}(ac) = -0.1,$$

$$(\gamma_{W_1} \bullet \gamma_{W_2})((c,\ d)(a,\ e)) = \gamma_{W_1}(ac) = -0.4,$$

for $ac \in E_1$ and $e \neq d$.

Hence, we can calculate Mv, NNv, NMv, neg-Mv, neg-NNv, and neg-NMv for other nodes and arcs.

Proposition 2 The RP of two BPFGs $\mathbb{G}_1$ and $\mathbb{G}_2$ is a BPFG.

Proof. Suppose that $\mathbb{G}_1 = (J_1,\ W_1)$ and $\mathbb{G}_2 = (J_2,\ W_2)$ are two BPFGs on crisp graphs $G_1 = (V_1,\ E_1)$ and $G_2 = (V_2,\ E_2)$, respectively, and $((j_1,\ j_2)(w_1,\ w_2)) \in E_1 \times E_2$. If $j_1 w_1 \in E_1$ *and* $j_2 \neq w_2$, then we have:

$$\begin{aligned}
(\Upsilon_{W_1} \bullet \Upsilon_{W_2})((j_1,\ j_2)(w_1,\ w_2)) &= \Upsilon_{W_1}(j_1 w_1) \leq \wedge\{\Upsilon_{J_1}(j_1),\ \Upsilon_{J_1}(w_1)\} \\
&\leq \vee\{\wedge\{\Upsilon_{J_1}(j_1),\ \Upsilon_{J_1}(w_1)\},\ \wedge\{\Upsilon_{J_2}(j_2),\ \Upsilon_{J_2}(w_2)\}\} \\
&= \wedge\{\vee\{\Upsilon_{J_1}(j_1),\ \Upsilon_{J_2}(j_2)\},\ \vee\{\Upsilon_{J_1}(w_1),\ \Upsilon_{J_2}(w_2)\}\} \\
&= \wedge\{(\Upsilon_{J_1} \bullet \Upsilon_{J_2})(j_1,\ j_2),\ (\Upsilon_{J_1} \bullet \Upsilon_{J_2})(w_1,\ w_2)\},
\end{aligned}$$

$$\begin{aligned}
(\upsilon_{W_1} \bullet \upsilon_{W_2})((j_1,\ j_2)(w_1,\ w_2)) &= \upsilon_{W_1}(j_1 w_1) \leq \wedge\{\upsilon_{J_1}(j_1),\ \upsilon_{J_1}(w_1)\} \\
&\leq \vee\{\wedge\{\upsilon_{J_1}(j_1),\ \upsilon_{J_1}(w_1)\},\ \wedge\{\upsilon_{J_2}(j_2),\ \upsilon_{J_2}(w_2)\}\} \\
&= \wedge\{\vee\{\upsilon_{J_1}(j_1),\ \upsilon_{J_2}(j_2)\},\ \vee\{\upsilon_{J_1}(w_1),\ \upsilon_{J_2}(w_2)\}\} \\
&= \wedge\{(\upsilon_{J_1} \bullet \upsilon_{J_2})(j_1,\ j_2),\ (\upsilon_{J_1} \bullet \upsilon_{J_2})(w_1,\ w_2)\},
\end{aligned}$$

$$\begin{aligned}
(\psi_{W_1} \bullet \psi_{W_2})((j_1,\ j_2)(w_1,\ w_2)) &= \psi_{W_1}(j_1 w_1) \geq \vee\{\psi_{J_1}(j_1),\ \psi_{J_1}(w_1)\} \\
&\geq \wedge\{\vee\{\psi_{J_1}(j_1),\ \psi_{J_1}(w_1)\},\ \vee\{\psi_{J_2}(j_2),\ \psi_{J_2}(w_2)\}\} \\
&= \vee\{\wedge\{\psi_{J_1}(j_1),\ \psi_{J_2}(j_2)\},\ \wedge\{\psi_{J_1}(w_1),\ \psi_{J_2}(w_2)\}\} \\
&= \vee\{(\psi_{J_1} \bullet \psi_{J_2})(j_1,\ j_2),\ (\psi_{J_1} \bullet \psi_{J_2})(w_1,\ w_2)\}.
\end{aligned}$$

$$(\alpha_{W_1} \bullet \alpha_{W_2})((j_1,\ j_2)(w_1,\ w_2)) = \alpha_{W_1}(j_1 w_1) \geq \vee\{\alpha_{J_1}(j_1),\ \alpha_{J_1}(w_1)\}$$

$$\geq \wedge\{\vee\{\alpha_{J_1}(j_1),\ \alpha_{J_1}(w_1)\},\ \vee\{\alpha_{J_2}(j_2),\ \alpha_{J_2}(w_2)\}\}$$

$$= \vee\{\wedge\{\alpha_{J_1}(j_1),\ \alpha_{J_2}(j_2)\},\ \wedge\{\alpha_{J_1}(w_1),\ \alpha_{J_2}(w_2)\}\}$$

$$= \vee\{(\alpha_{J_1} \bullet \alpha_{J_2})(j_1,\ j_2),\ (\alpha_{J_1} \bullet \alpha_{J_2})(w_1,\ w_2)\}.$$

$$(\beta_{W_1} \bullet \beta_{W_2})((j_1,\ j_2)(w_1,\ w_2)) = \beta_{W_1}(j_1 w_1)$$

$$\geq \vee\{\beta_{J_1}(j_1),\ \beta_{J_1}(w_1)\}$$

$$\geq \wedge\{\vee\{\beta_{J_1}(j_1),\ \beta_{J_1}(w_1)\},\ \vee\{\beta_{J_2}(j_2),\ \beta_{J_2}(w_2)\}\}$$

$$= \vee\{\wedge\{\beta_{J_1}(j_1),\ \beta_{J_2}(j_2)\},\ \wedge\{\beta_{J_1}(w_1),\ \beta_{J_2}(w_2)\}\}$$

$$= \vee\{(\beta_{J_1} \bullet \beta_{J_2})(j_1,\ j_2),\ (\beta_{J_1} \bullet \beta_{J_2})(w_1,\ w_2)\}.$$

$$(\gamma_{W_1} \bullet \gamma_{W_2})((j_1,\ j_2)(w_1,\ w_2)) = \gamma_{W_1}(j_1 w_1)$$

$$\leq \wedge\{\gamma_{J_1}(j_1),\ \gamma_{J_1}(w_1)\}$$

$$\leq \vee\{\wedge\{\gamma_{J_1}(j_1),\ \gamma_{J_1}(w_1)\},\ \wedge\{\gamma_{J_2}(j_2),\ \gamma_{J_2}(w_2)\}\}$$

$$= \wedge\{\vee\{\gamma_{J_1}(j_1),\ \gamma_{J_2}(j_2)\},\ \vee\{\gamma_{J_1}(w_1),\ \gamma_{J_2}(w_2)\}\}$$

$$= \wedge\{(\gamma_{J_1} \bullet \gamma_{J_2})(j_1,\ j_2),\ (\gamma_{J_1} \bullet \gamma_{J_2})(w_1,\ w_2)\},$$

□

Definition 15 Suppose that $\mathbb{G}_1 = (J_1,\ W_1)$ and $\mathbb{G}_2 = (J_2,\ W_2)$ are two BPFGs. For any node $(j_1,\ j_2) \in V_1 \times V_2$, we have:

$$(d_\Upsilon)_{\mathbb{G}_1 \bullet \mathbb{G}_2}(j_1,\ j_2) = \sum_{(j_1,\ j_2)(w_1,\ w_2) \in E_1 \times E_2.} (\Upsilon_{W_1} \bullet \Upsilon_{W_2})((j_1,\ j_2)(w_1,\ w_2))$$

$$= \sum_{j_1 w_1 \in E_1,\ j_2 \neq w_2} \Upsilon_{W_1}(j_1 w_1) = (d_\Upsilon)_{\mathbb{G}_1}(j_1),$$

$$(d_\upsilon)_{\mathbb{G}_1 \bullet \mathbb{G}_2}(j_1,\ j_2) = \sum_{(j_1,\ j_2)(w_1,\ w_2) \in E_1 \times E_2.} (\upsilon_{W_1} \bullet \upsilon_{W_2})((j_1,\ j_2)(w_1,\ w_2))$$

$$= \sum_{j_1 w_1 \in E_1,\ j_2 \neq w_2} \upsilon_{W_1}(j_1 w_1)$$

$$= (d_\upsilon)_{\mathbb{G}_1}(j_1),$$

$$(d_\psi)_{\mathbb{G}_1 \bullet \mathbb{G}_2}(j_1,\ j_2) = \sum_{(j_1,\ j_2)(w_1,\ w_2) \in E_1 \times E_2.} (\psi_{W_1} \bullet \psi_{W_2})((j_1,\ j_2)(w_1,\ w_2))$$

$$= \sum_{j_1 w_1 \in E_1,\ j_2 \neq w_2} \psi_{W_1}(j_1 w_1)$$

$$= (d_\psi)_{\mathbb{G}_1}(j_1).$$

$$(d_\alpha)_{\mathbb{G}_1 \bullet \mathbb{G}_2}(j_1,\ j_2) = \sum_{(j_1,\ j_2)(w_1,\ w_2) \in E_1 \times E_2.} (\alpha_{W_1} \bullet \alpha_{W_2})((j_1,\ j_2)(w_1,\ w_2))$$

$$= \sum_{j_1 w_1 \in E_1,\ j_2 \neq w_2} \alpha_{W_1}(j_1 w_1)$$

$$= (d_\alpha)_{\mathbb{G}_1}(j_1).$$

$$(d_\beta)_{\mathbb{G}_1 \bullet \mathbb{G}_2}(j_1,\ j_2) = \sum_{(j_1,\ j_2)(w_1,\ w_2) \in E_1 \times E_2.} (\beta_{W_1} \bullet \beta_{W_2})((j_1,\ j_2)(w_1,\ w_2))$$

$$= \sum_{j_1 w_1 \in E_1,\ j_2 \neq w_2} \beta_{W_1}(j_1 w_1)$$

$$= (d_\beta)_{\mathbb{G}_1}(j_1).$$

$$(d_\gamma)_{\mathbb{G}_1 \bullet \mathbb{G}_2}(j_1,\ j_2) = \sum_{(j_1,\ j_2)(w_1,\ w_2) \in E_1 \times E_2.} (\gamma_{W_1} \bullet \gamma_{W_2})((j_1,\ j_2)(w_1,\ w_2))$$

$$= \sum_{j_1 w_1 \in E_1,\ j_2 \neq w_2} \gamma_{W_1}(j_1 w_1)$$

$$= (d_\gamma)_{\mathbb{G}_1}(j_1).$$

Definition 16 Suppose that $\mathbb{G}_1 = (J_1,\ W_1)$ and $\mathbb{G}_2 = (J_2,\ W_2)$ are two BPFGs. For any node $(j_1,\ j_2) \in V_1 \times V_2$, we have:

$$
\begin{aligned}
(td_{\Upsilon})_{\mathbb{G}_1 \bullet \mathbb{G}_2}(j_1,\ j_2) &= \sum_{(j_1,\ j_2)(w_1,\ w_2)\in E_1\times E_2.} (\Upsilon_{W_1} \bullet \Upsilon_{W_2})((j_1,\ j_2)(w_1,\ w_2)) + (\Upsilon_{J_1} \bullet \Upsilon_{J_2})(j_1,\ j_2) \\
&= \sum_{j_1 w_1 \in E_1,\ j_2 \neq w_2} \Upsilon_{W_1}(j_1 w_1) + \vee\{\Upsilon_{J_1}(j_1),\ \Upsilon_{J_2}(j_2)\} \\
&= \sum_{j_1 w_1 \in E_1,\ j_2 \neq w_2} \Upsilon_{W_1}(j_1 w_1) + \Upsilon_{J_1}(j_1) + \Upsilon_{J_2}(j_2) - \wedge\{\Upsilon_{J_1}(j_1),\ \Upsilon_{J_2}(j_2)\} \\
&= (td_{\Upsilon})_{\mathbb{G}_1}(j_1) + \Upsilon_{J_2}(j_2) - \wedge\{\Upsilon_{J_1}(j_1),\ \Upsilon_{J_2}(j_2)\}, \\
(td_{\upsilon})_{\mathbb{G}_1 \bullet \mathbb{G}_2}(j_1,\ j_2) &= \sum_{(j_1,\ j_2)(w_1,\ w_2)\in E_1\times E_2.} (\upsilon_{W_1} \bullet \upsilon_{W_2})((j_1,\ j_2)(w_1,\ w_2)) + (\upsilon_{J_1} \bullet \upsilon_{J_2})(j_1,\ j_2) \\
&= \sum_{j_1 w_1 \in E_1,\ j_2 \neq w_2} \upsilon_{W_1}(j_1 w_1) + \vee\{\upsilon_{J_1}(j_1),\ \upsilon_{J_2}(j_2)\} \\
&= \sum_{j_1 w_1 \in E_1,\ j_2 \neq w_2} \upsilon_{W_1}(j_1 w_1) + \upsilon_{J_1}(j_1) + \upsilon_{J_2}(j_2) - \wedge\{\upsilon_{J_1}(j_1),\ \upsilon_{J_2}(j_2)\} \\
&= (td_{\upsilon})_{\mathbb{G}_1}(j_1) + \upsilon_{J_2}(j_2) - \wedge\{\upsilon_{J_1}(j_1),\ \upsilon_{J_2}(j_2)\}, \\
(td_{\psi})_{\mathbb{G}_1 \bullet \mathbb{G}_2}(j_1,\ j_2) &= \sum_{(j_1,\ j_2)(w_1,\ w_2)\in E_1\times E_2.} (\psi_{W_1} \bullet \psi_{W_2})((j_1,\ j_2)(w_1,\ w_2)) + (\psi_{J_1} \bullet \psi_{J_2})(j_1,\ j_2) \\
&= \sum_{j_1 w_1 \in E_1,\ j_2 \neq w_2} \psi_{W_1}(j_1 w_1) + \wedge\{\psi_{J_1}(j_1),\ \psi_{J_2}(j_2)\} \\
&= \sum_{j_1 w_1 \in E_1,\ j_2 \neq w_2} \psi_{W_1}(j_1 w_1) + \psi_{J_1}(j_1) + \psi_{J_2}(j_2) - \vee\{\psi_{J_1}(j_1),\ \psi_{J_2}(j_2)\} \\
&= (td_{\psi})_{\mathbb{G}_1}(j_1) + \psi_{J_2}(j_2) - \vee\{\psi_{J_1}(j_1),\ \psi_{J_2}(j_2)\}.
\end{aligned}
$$

$$(td_\alpha)_{\mathbb{G}_1 \bullet \mathbb{G}_2}(j_1,\ j_2) = \sum_{(j_1,\ j_2)(w_1,\ w_2) \in E_1 \times E_2.} (\alpha_{W_1} \bullet \alpha_{W_2})((j_1,\ j_2)(w_1,\ w_2)) + (\alpha_{J_1} \bullet \alpha_{J_2})(j_1,\ j_2)$$

$$= \sum_{j_1 w_1 \in E_1,\ j_2 \neq w_2} \alpha_{W_1}(j_1 w_1) + \wedge\{\alpha_{J_1}(j_1),\ \alpha_{J_2}(j_2)\}$$

$$= \sum_{j_1 w_1 \in E_1,\ j_2 \neq w_2} \alpha_{W_1}(j_1 w_1) + \alpha_{J_1}(j_1) + \alpha_{J_2}(j_2) - \vee\{\alpha_{J_1}(j_1),\ \alpha_{J_2}(j_2)\}$$

$$= (td_\alpha)_{\mathbb{G}_1}(j_1) + \alpha_{J_2}(j_2) - \vee\{\alpha_{J_1}(j_1),\ \alpha_{J_2}(j_2)\}.$$

$$(td_\beta)_{\mathbb{G}_1 \bullet \mathbb{G}_2}(j_1,\ j_2) = \sum_{(j_1,\ j_2)(w_1,\ w_2) \in E_1 \times E_2.} (\beta_{W_1} \bullet \beta_{W_2})((j_1,\ j_2)(w_1,\ w_2)) + (\beta_{J_1} \bullet \beta_{J_2})(j_1,\ j_2)$$

$$= \sum_{j_1 w_1 \in E_1,\ j_2 \neq w_2} \beta_{W_1}(j_1 w_1) + \wedge\{\beta_{J_1}(j_1),\ \beta_{J_2}(j_2)\}$$

$$= \sum_{j_1 w_1 \in E_1,\ j_2 \neq w_2} \beta_{W_1}(j_1 w_1) + \beta_{J_1}(j_1) + \beta_{J_2}(j_2) - \vee\{\beta_{J_1}(j_1),\ \beta_{J_2}(j_2)\}$$

$$= (td_\beta)_{\mathbb{G}_1}(j_1) + \beta_{J_2}(j_2) - \vee\{\beta_{J_1}(j_1),\ \beta_{J_2}(j_2)\}.$$

$$(td_\gamma)_{\mathbb{G}_1 \bullet \mathbb{G}_2}(j_1,\ j_2) = \sum_{(j_1,\ j_2)(w_1,\ w_2) \in E_1 \times E_2.} (\gamma_{W_1} \bullet \gamma_{W_2})((j_1,\ j_2)(w_1,\ w_2)) + (\gamma_{J_1} \bullet \gamma_{J_2})(j_1,\ j_2)$$

$$= \sum_{j_1 w_1 \in E_1,\ j_2 \neq w_2} \gamma_{W_1}(j_1 w_1) + \vee\{\gamma_{J_1}(j_1),\ \gamma_{J_2}(j_2)\}$$

$$= \sum_{j_1 w_1 \in E_1,\ j_2 \neq w_2} \gamma_{W_1}(j_1 w_1) + \gamma_{J_1}(j_1) + \gamma_{J_2}(j_2) - \wedge\{\gamma_{J_1}(j_1),\ \gamma_{J_2}(j_2)\}$$

$$= (td_\gamma)_{\mathbb{G}_1}(j_1) + \gamma_{J_2}(j_2) - \wedge\{\gamma_{J_1}(j_1),\ \gamma_{J_2}(j_2)\}.$$

Example We calculate the degree and the total degree of node $(c,\ d)$ by using Example 3.

$$(d_\Upsilon)_{\mathbb{G}_1 \bullet \mathbb{G}_2}(c,\ d) = (d_\Upsilon)_{\mathbb{G}_1}(c) = 0.3,$$

$$(d_\upsilon)_{\mathbb{G}_1 \bullet \mathbb{G}_2}(c,\ d) = (d_\upsilon)_{\mathbb{G}_1}(c) = 0.1,$$

$$(d_\psi)_{\mathbb{G}_1 \bullet \mathbb{G}_2}(c,\ d) = (d_\psi)_{\mathbb{G}_1}(c) = 0.2,$$

$$(d_\alpha)_{\mathbb{G}_1 \bullet \mathbb{G}_2}(c,\ d) = (d_\alpha)_{\mathbb{G}_1}(c) = -0.3,$$

$$(d_\beta)_{\mathbb{G}_1 \bullet \mathbb{G}_2}(c,\ d) = (d_\beta)_{\mathbb{G}_1}(c) = -0.1,$$

$$(d_\gamma)_{\mathbb{G}_1 \bullet \mathbb{G}_2}(c,\ d) = (d_\gamma)_{\mathbb{G}_1}(c) = -0.2.$$

Therefore,

$$(d)_{\mathbb{G}_1 \bullet \mathbb{G}_2}(c,\ d) = (0.3,\ 0.1,\ 0.2,\ -0.3,\ -0.1,\ -0.2)$$

Additionally, the total degree of vertex $(c,\ d)$ can be determined as follows:

$$(td_\Upsilon)_{\mathbb{G}_1 \bullet \mathbb{G}_2}(c,\ d) = (td_\Upsilon)_{\mathbb{G}_1}(c) + \Upsilon_{J_2}(d) - \wedge\{\Upsilon_{J_1}(c),\ \Upsilon_{J_2}(d)\} = 0.7,$$

$$(td_\upsilon)_{\mathbb{G}_1 \bullet \mathbb{G}_2}(c,\ d) = (td_\upsilon)_{\mathbb{G}_1}(c) + \upsilon_{J_2}(d) - \wedge\{\upsilon_{J_1}(c),\ \upsilon_{J_2}(d)\} = 0.3,$$

$$(td_\psi)_{\mathbb{G}_1 \bullet \mathbb{G}_2}(c,\ d) = (td_\psi)_{\mathbb{G}_1}(c) + \psi_{J_2}(d) - \vee\{\psi_{J_1}(c),\ \psi_{J_2}(d)\} = 0.9,$$

$$(td_\alpha)_{\mathbb{G}_1 \bullet \mathbb{G}_2}(c,\ d) = (td_\alpha)_{\mathbb{G}_1}(c) + \alpha_{J_2}(d) - \vee\{\alpha_{J_1}(c),\ \alpha_{J_2}(d)\} = -0.7.$$

$$(td_\beta)_{\mathbb{G}_1 \bullet \mathbb{G}_2}(c,\ d) = (td_\beta)_{\mathbb{G}_1}(c) + \beta_{J_2}(d) - \vee\{\beta_{J_1}(c),\ \beta_{J_2}(d)\} = -0.3,$$

$$(td_\gamma)_{\mathbb{G}_1 \bullet \mathbb{G}_2}(c,\ d) = (td_\gamma)_{\mathbb{G}_1}(c) + \gamma_{J_2}(d) - \wedge\{\gamma_{J_1}(c),\ \gamma_{J_2}(d)\} = -0.9.$$

Thus,

$$(td)_{\mathbb{G}_1 \bullet \mathbb{G}_2}(a,\ e) = (0.7,\ 0.3,\ 0.9,\ -0.7,\ -0.3,\ -0.9)$$

We can calculate these for all other nodes.

4. Isomorphism and homomorphism of BPFG

Definition 17 Let $\mathbb{G}_1 = (J_1,\ W_1)$ and $\mathbb{G}_2 = (J_2,\ W_2)$ be two BPFGs.
1. A homomorphism h from a BPFG $\mathbb{G}_1$ and $\mathbb{G}_2$ is a mapping function $h: V_1 \to V_2$ which always satisfy followings:
(i) $\forall j_1 \in V_1,\ j_1 m_1 \in E_1,$

$$\Upsilon_{J_1}(j_1) \leq \Upsilon_{J_2}(h(j_1)),$$

$$\upsilon_{J_1}(j_1) \leq \upsilon_{J_2}(h(j_1)),$$

$$\psi_{J_1}(j_1) \geq \psi_{J_2}(h(j_1)),$$

$$\alpha_{J_1}(j_1) \geq \alpha_{J_2}(h(j_1)),$$

$$\beta_{J_1}(j_1) \geq \beta_{J_2}(h(j_1)),$$

$$\gamma_{J_1}(j_1) \leq \gamma_{J_2}(h(j_1)).$$

(ii)

$$\Upsilon_{W_1}(j_1m_1) \leq \Upsilon_{W_2}(h(j_1)h(m_1)),$$

$$\upsilon_{W_1}(j_1m_1) \leq \upsilon_{W_2}(h(j_1)h(m_1)),$$

$$\psi_{W_1}(j_1m_1) \geq \psi_{W_2}(h(j_1)h(m_1)),$$

$$\alpha_{W_1}(j_1m_1) \geq \alpha_{W_2}(h(j_1)h(m_1)),$$

$$\beta_{W_1}(j_1m_1) \geq \beta_{W_2}(h(j_1)h(m_1)),$$

$$\gamma_{W_1}(j_1m_1) \leq \gamma_{W_2}(h(j_1)h(m_1)).$$

2. A isomorphism h from a BPFG $\mathbb{G}_1$ and $\mathbb{G}_2$ is a bijective mapping function $h: V_1 \to V_2$ which always satisfy followings:

(i) $\forall j_1 \in V_1,\ j_1m_1 \in E_1$

$$\Upsilon_{J_1}(j_1) = \Upsilon_{J_2}(h(j_1)),$$

$$\upsilon_{J_1}(j_1) = \upsilon_{J_2}(h(j_1)),$$

$$\psi_{J_1}(j_1) = \psi_{J_2}(h(j_1)),$$

$$\alpha_{J_1}(j_1) = \alpha_{J_2}(h(j_1)),$$

$$\beta_{J_1}(j_1) = \beta_{J_2}(h(j_1)),$$

$$\gamma_{J_1}(j_1) = \gamma_{J_2}(h(j_1)).$$

(ii)

$$\Upsilon_{W_1}(j_1 m_1) = \Upsilon_{W_2}(h(j_1)h(m_1)),$$

$$\upsilon_{W_1}(j_1 m_1) = \upsilon_{W_2}(h(j_1)h(m_1)),$$

$$\psi_{W_1}(j_1 m_1) = \psi_{W_2}(h(j_1)h(m_1)),$$

$$\alpha_{W_1}(j_1 m_1) = \alpha_{W_2}(h(j_1)h(m_1)),$$

$$\beta_{W_1}(j_1 m_1) = \beta_{W_2}(h(j_1)h(m_1)),$$

$$\gamma_{W_1}(j_1 m_1) = \gamma_{W_2}(h(j_1)h(m_1)).$$

3. A weak isomorphism h from a BPFG $\mathbb{G}_1$ and $\mathbb{G}_2$ is a bijective mapping function $h : V_1 \to V_2$ which always satisfy followings:

(i) $\forall j_1 \in V_1$

(a) h is homomorphism

(b)

$$\Upsilon_{J_1}(j_1) = \Upsilon_{J_2}(h(j_1)),$$

$$\upsilon_{J_1}(j_1) = \upsilon_{J_2}(h(j_1)),$$

$$\psi_{J_1}(j_1) = \psi_{J_2}(h(j_1)),$$

$$\alpha_{J_1}(j_1) = \alpha_{J_2}(h(j_1)),$$

$$\beta_{J_1}(j_1) = \beta_{J_2}(h(j_1)),$$

$$\gamma_{J_1}(j_1) = \gamma_{J_2}(h(j_1)).$$

Thus a weak isomorphism maintains the costs of the vertices but not necessarily the costs of the edges.

4. A co-weak isomorphism h from a BPFG $\mathbb{G}_1$ and $\mathbb{G}_2$ is a bijective mapping function $h : V_1 \to V_2$ which always satisfy followings:

(i) $\forall j_1 m_1 \in E_1$

(a) h is homomorphism

(b)

$$\Upsilon_{W_1}(j_1 m_1) = \Upsilon_{W_2}(h(j_1)h(m_1)),$$

$$\upsilon_{W_1}(j_1 m_1) = \upsilon_{W_2}(h(j_1)h(m_1)),$$

$$\psi_{W_1}(j_1 m_1) = \psi_{W_2}(h(j_1)h(m_1)),$$

$$\alpha_{W_1}(j_1 m_1) = \alpha_{W_2}(h(j_1)h(m_1)),$$

$$\beta_{W_1}(j_1 m_1) = \beta_{W_2}(h(j_1)h(m_1)),$$

$$\gamma_{W_1}(j_1 m_1) = \gamma_{W_2}(h(j_1)h(m_1)).$$

Thus a weak isomorphism maintains the costs of the edges but not necessarily the costs of the vertices.

Theorem 4 If $\mathbb{G}_1$ and $\mathbb{G}_2$ isomorphic BPFG then the degrees of their nodes are preserved.

Proof. Let $h : \mathrm{S}_1 \to S_2$ be an isomorphism of $\mathbb{G}_1$ onto $\mathbb{G}_2$. By the definition of isomorphism

$$\Upsilon_{W_1}(jm) = \Upsilon_{W_2}(h(j)h(m)),$$

$$\upsilon_{W_1}(jm) = \upsilon_{W_2}(h(j)h(m)),$$

$$\psi_{W_1}(jm) = \psi_{W_2}(h(j)h(m)),$$

$$\alpha_{W_1}(jm) = \alpha_{W_2}(h(j)h(m)),$$

$$\beta_{W_1}(jm) = \beta_{W_2}(h(j)h(m)),$$

$$\gamma_{W_1}(jm) = \gamma_{W_2}(h(j)h(m)).$$

$\forall j,\ m \in S.$

$$d(u) = \left(\sum_{u \neq v} \Upsilon_{W_1}(uv),\ \sum_{u \neq v} \upsilon_{W_1}(uv),\ \sum_{u \neq v} \psi_{W_1}(uv),\ \sum_{u \neq v} \alpha_{W_1}(uv),\ \sum_{u \neq v} \beta_{W_1}(uv),\ \sum_{u \neq v} \gamma_{W_1}(uv) \right)$$

$$= (\Upsilon_{W_2}(h(u)h(v)),\ \upsilon_{W_2}(h(u)h(v)),\ \psi_{W_2}(h(u)h(v)),\ \alpha_{W_2}(h(u)h(v)),\ \beta_{W_2}(h(u)h(v)),\ \gamma_{W_2}(h(u)h(v)))$$

$$= d(h(u)).$$

□

Theorem 5 Isomorphism between BPFG is an equivalence relation
Proof. Let $\mathbb{G}_1$, $\mathbb{G}_2$ and $\mathbb{G}_3$ be BPFG with underlying sets S, S_1 and S_2 respectively
(i) Reflexive:
Consider the identity map $h : S \to Sh(j) = j$ for all j in S. This h is a bijective map satisfying

$$\Upsilon_{J_1}(j) = \Upsilon_{J_2}(h(j)),$$

$$\upsilon_{J_1}(j) = \upsilon_{J_2}(h(j)),$$

$$\psi_{J_1}(j) = \psi_{J_2}(h(j)),$$

$$\alpha_{J_1}(j) = \alpha_{J_2}(h(j)),$$

$$\beta_{J_1}(j) = \beta_{J_2}(h(j)),$$

$$\gamma_{J_1}(j) = \gamma_{J_2}(h(j)).$$

$\forall\ j \in S.$

$$\mu_{W_1}(jm) = \mu_{W_2}(h(j)h(m)) \quad \forall j,\ m \in S.$$

Hence h is an isomorphism of the BPFG to itself.
Therefore it satisfies reflexive relation.
(ii) Symmetric:
Let $h : S \to S$ be an isomorphism of $\mathbb{G}_1$ onto $\mathbb{G}_2$ then h is a bijective map.
(i)

$$h(j) = j^{'},\ j \in S, \text{ satisfying.}$$

(ii)

$$\Upsilon_{J_1}(j) = \Upsilon_{J_2}(h(j)),$$

$$\upsilon_{J_1}(j) = \upsilon_{J_2}(h(j)),$$

$$\psi_{J_1}(j) = \psi_{J_2}(h(j)),$$

$$\alpha_{J_1}(j) = \alpha_{J_2}(h(j)),$$

$$\beta_{J_1}(j) = \beta_{J_2}(h(j)),$$

$$\gamma_{J_1}(j) = \gamma_{J_2}(h(j)),$$

$$\Upsilon_{W_1}(jm) = \Upsilon_{W_2}(h(j)h(m)),$$

$$\upsilon_{W_1}(jm) = \upsilon_{W_2}(h(j)h(m)),$$

$$\psi_{W_1}(jm) = \psi_{W_2}(h(j)h(m)),$$

$$\alpha_{W_1}(jm) = \alpha_{W_2}(h(j)h(m)),$$

$$\beta_{W_1}(jm) = \beta_{W_2}(h(j)h(m)),$$

$$\gamma_{W_1}(jm) = \gamma_{W_2}(h(j)h(m)),$$

$\forall j,\ m \in S$. As h is bijective, by (i) $h^{-1}(j^{'}) = j,\ j^{'} \in S_1$, using (ii)

$$\Upsilon_{J_1}(h^{-1}(j)) = \Upsilon_{J_2}(j^{'}),$$

$$\upsilon_{J_1}(h^{-1}(j)) = \upsilon_{J_2}(j^{'}),$$

$$\psi_{J_1}(h^{-1}(j)) = \psi_{J_2}(j^{'}),$$

$$\alpha_{J_1}(h^{-1}(j)) = \alpha_{J_2}(j^{'}),$$

$$\beta_{J_1}(h^{-1}(j)) = \beta_{J_2}(j^{'}),$$

$$\gamma_{J_1}(h^{-1}(j)) = \gamma_{J_2}(j^{'}),$$

$\forall j^{'} \in S_1$.

$$\Upsilon_{W_1}(h^{-1}(j^{'})h^{-1}(m^{'}) = \Upsilon_{W_2}(j^{'}m^{'}),$$

$$\upsilon_{W_1}(h^{-1}(j^{'})h^{-1}(m^{'}) = \upsilon_{W_2}(j^{'}m^{'}),$$

$$\psi_{W_1}(h^{-1}(j^{'})h^{-1}(m^{'}) = \psi_{W_2}(j^{'}m^{'}),$$

$$\alpha_{W_1}(h^{-1}(j^{'})h^{-1}(m^{'}) = \alpha_{W_2}(j^{'}m^{'}),$$

$$\beta_{W_1}(h^{-1}(j^{'})h^{-1}(m^{'}) = \beta_{W_2}(j^{'}m^{'}),$$

$$\gamma_{W_1}(h^{-1}(j^{'})h^{-1}(m^{'}) = \gamma_{W_2}(j^{'}m^{'}),$$

$\forall j^{'},\ m^{'} \in S_1$.

Hence we get 1-1, onto map $h^{'} : S_1 \to S$, which is an isomorphism from $\mathbb{G}_2$ to $\mathbb{G}_1$. i.e, $\mathbb{G}_2 \cong \mathbb{G}_1$.

(iii) Transitive:

Let $h : S \to S_1$ and $g : S \to S_1$ be isomorphisms of the BPFG $\mathbb{G}_1$ onto $\mathbb{G}_2$ and $\mathbb{G}_2$ onto $\mathbb{G}_3$ respectively.

Then $g \circ h$ is a 1-1 onto map from S to S_2 where, $(g \circ h)(j) = g(h(j))\ \forall j \in S$

As $h : S \to S_1$ is an isomorphism $h(j) = j^{'},\ j \in S$

$$\Upsilon_{J_1}(j) = \Upsilon_{J_2}(h(j)),$$

$$\upsilon_{J_1}(j) = \upsilon_{J_2}(h(j)),$$

$$\psi_{J_1}(j) = \psi_{J_2}(h(j)),$$

$$\alpha_{J_1}(j) = \alpha_{J_2}(h(j)),$$

$$\beta_{J_1}(j) = \beta_{J_2}(h(j)),$$

$$\gamma_{J_1}(j) = \gamma_{J_2}(h(j)),\ \ \forall j \in S.$$

$$\Upsilon_{W_1}(jm) = \Upsilon_{W_2}(h(j)h(m)),$$

$$\upsilon_{W_1}(jm) = \upsilon_{W_2}(h(j)h(m)),$$

$$\psi_{W_1}(jm) = \psi_{W_2}(h(j)h(m)),$$

$$\alpha_{W_1}(jm) = \alpha W_2(h(j)h(m)),$$

$$\beta_{W_1}(jm) = \beta_{W_2}(h(j)h(m)),$$

$$\gamma_{W_1}(jm) = \gamma_{W_2}(h(j)h(m)), \ \ \forall j, \ m \in S.$$

i.e,

$$\Upsilon_{J_1}(j) = \Upsilon_{J_2}(j^{'}),$$

$$\upsilon_{J_1}(j) = \upsilon_{J_2}(j^{'}),$$

$$\psi_{J_1}(j) = \psi_{J_2}(j^{'}),$$

$$\alpha_{J_1}(j) = \alpha_{J_2}(j^{'}),$$

$$\beta_{J_1}(j) = \beta_{J_2}(j^{'}),$$

$$\gamma_{J_1}(j) = \gamma_{J_2}(j^{'}), \ \ \forall j \in S.$$

$$\Upsilon_{W_1}(jm) = \Upsilon_{W_2}(j^{'}m^{'}),$$

$$\upsilon_{W_1}(jm) = \upsilon_{W_2}(j^{'}m^{'}),$$

$$\psi_{W_1}(jm) = \psi_{W_2}(j^{'}m^{'}),$$

$$\alpha_{W_1}(jm) = \alpha_{W_2}(j^{'}m^{'}),$$

$$\beta_{W_1}(jm) = \beta_{W_2}(j^{'}m^{'}),$$

$$\gamma_{W_1}(jm) = \gamma_{W_2}(j^{'}m^{'}), \ \ \forall j, \ m \in S.$$

As g is an isomorphism from S_1 to S_2 we have:

$$g(j^{'}) = j^{''},\ j^{'} \in S_1\ \&$$

$$\Upsilon_{J_2}(j^{'}) = \Upsilon_{J_3}(g(j^{'})),$$

$$\upsilon_{J_2}(j^{'}) = \upsilon_{J_3}(g(j^{'})),$$

$$\psi_{J_2}(j^{'}) = \psi_{J_3}(g(j^{'})),$$

$$\alpha_{J_2}(j^{'}) = \alpha_{J_3}(g(j^{'})),$$

$$\beta_{J_2}(j^{'}) = \beta_{J_3}(g(j^{'})),$$

$$\gamma_{J_2}(j^{'}) = \gamma_{J_3}(g(j^{'})),\ \forall j^{'} \in S_1.$$

$$\Upsilon_{W_2}(j^{'}m^{'}) = \Upsilon_{W_3}(g(j^{'})g(m^{'})),$$

$$\upsilon_{W_2}(j^{'}m^{'}) = \upsilon_{W_3}(g(j^{'})g(m^{'})),$$

$$\psi_{W_2}(j^{'}m^{'}) = \psi_{W_3}(g(j^{'})g(m^{'})),$$

$$\alpha_{W_2}(j^{'}m^{'}) = \alpha_{W_3}(g(j^{'})g(m^{'})),$$

$$\beta_{W_2}(j^{'}m^{'}) = \beta_{W_3}(g(j^{'})g(m^{'})),$$

$$\gamma_{W_2}(j^{'}m^{'}) = \gamma_{W_3}(g(j^{'})g(m^{'})),\ \forall j^{'},\ m^{'} \in S_1.$$

By using $h(j) = j^{'},\ j \in S$

$$\Upsilon_{J_1}(j) = \Upsilon_{J_2}(j^{'}) = \Upsilon_{J_3}(g(j^{'})),\quad \forall j^{'} \in S_1 = \Upsilon_{J_3}(g(h(j)))\ \ \forall j \in S.$$

$$\upsilon_{J_1}(j) = \upsilon_{J_2}(j^{'}) = \upsilon_{J_3}(g(j^{'})),\quad \forall j^{'} \in S_1 = \upsilon_{J_3}(g(h(j)))\ \ \forall j \in S.$$

$$\psi_{J_1}(j) = \psi_{J_2}(j^{'}) = \psi_{J_3}(g(j^{'})),\quad \forall j^{'} \in S_1 = \psi_{J_3}(g(h(j)))\ \ \forall j \in S.$$

$$\alpha_{J_1}(j) = \alpha_{J_2}(j^{'}) = \alpha_{J_3}(g(j^{'})), \quad \forall j^{'} \in S_1 = \alpha_{J_3}(g(h(j))) \;\; \forall j \in S.$$

$$\beta_{J_1}(j) = \beta_{J_2}(j^{'}) = \beta_{J_3}(g(j^{'})), \quad \forall j^{'} \in S_1 = \beta_{J_3}(g(h(j))) \;\; \forall j \in S.$$

$$\gamma_{J_1}(j) = \gamma_{J_2}(j^{'}) = \gamma_{J_3}(g(j^{'})), \quad \forall j^{'} \in S_1 = \gamma_{J_3}(g(h(j))) \;\; \forall j \in S.$$

We have:

$$\begin{aligned} \Upsilon_{W_1}(jm) &= \Upsilon_{W_2}(j^{'}m^{'}) \;\; \forall j,\, m \in S \\ &= \Upsilon_{W_3}(g(j^{'})g(m^{'})) \;\; \forall j^{'},\, m^{'} \in S_1 \\ &= \Upsilon_{W_3}(g(h(j))g(h(m))) \;\; \forall j,\, m \in S. \end{aligned}$$

$$\begin{aligned} \upsilon_{W_1}(jm) &= \upsilon_{W_2}(j^{'}m^{'}) \;\; \forall j,\, m \in S \\ &= \upsilon_{W_3}(g(j^{'})g(m^{'})) \;\; \forall j^{'},\, m^{'} \in S_1 \\ &= \upsilon_{W_3}(g(h(j))g(h(m))) \;\; \forall j,\, m \in S. \end{aligned}$$

$$\begin{aligned} \psi_{W_1}(jm) &= \psi_{W_2}(j^{'}m^{'}) \;\; \forall j,\, m \in S \\ &= \psi_{W_3}(g(j^{'})g(m^{'})) \;\; \forall j^{'},\, m^{'} \in S_1 \\ &= \psi_{W_3}(g(h(j))g(h(m))) \;\; \forall j,\, m \in S. \end{aligned}$$

$$\begin{aligned} \alpha_{W_1}(jm) &= \alpha_{W_2}(j^{'}m^{'}) \;\; \forall j,\, m \in S \\ &= \alpha_{W_3}(g(j^{'})g(m^{'})) \;\; \forall j^{'},\, m^{'} \in S_1 \\ &= \alpha_{W_3}(g(h(j))g(h(m))) \;\; \forall j,\, m \in S. \end{aligned}$$

$$\beta_{W_1}(jm) = \beta_{W_2}(j^{'}m^{'}) \;\; \forall j,\, m \in S$$

$$= \beta_{W_3}(g(j^{'})g(m^{'})) \;\; \forall j^{'},\, m^{'} \in S_1$$

$$= \beta_{W_3}(g(h(j))g(h(m))) \;\; \forall j,\, m \in S.$$

$$\gamma_{W_1}(jm) = \gamma_{W_2}(j^{'}m^{'}) \;\; \forall j,\, m \in S$$

$$= \gamma_{W_3}(g(j^{'})g(m^{'})) \;\; \forall j^{'},\, m^{'} \in S_1$$

$$= \gamma_{W_3}(g(h(j))g(h(m))) \;\; \forall j,\, m \in S.$$

Therefore $g \circ h$ is an isomorphism between $\mathbb{G}_1$ and $\mathbb{G}_3$.
Hence isomorphism between BPFG is an equivalence relation. □

Theorem 6 Weak isomorphism between BPFG satisfies the partial order relation.

Proof. It is obvious. □

5. Application of BPFG

Algorithm

In this algorithm these are the steps

Step No. 1: Begin.

Step No. 2: Input $\Upsilon(z)$, $\upsilon(z)$ and $\psi(z)$ for all p applicants.

Step No. 3: For any two nodes y_i and yj $\Upsilon(y_iy_j)$, $\upsilon(y_iy_j)$ and $\psi(y_iy_j)$ are positive but $\alpha(y_iy_j)$, $\beta(y_iy_j)$ *and* $\gamma(y_iy_j)$ are negative. Then we have, $(y_i,\, \Upsilon(y_iy_j),\, \upsilon(y_iy_j),\, \psi(y_iy_j),\, \alpha(y_iy_j),\, \beta(y_iy_j),\, \gamma(y_iy_j))$.

Step 4: To obtain bipolar picture fuzzy out-neighborhoods $N(y_i)$. Revise step 3 for all nodes y_i *and* y_j.

Step 5: Calculation for $N(y_i) \cap N(y_j)$.

Step 6: Find out the height $h(N(y_i) \cap N(y_j))$.

Step 7: Consider all edge where $N(y_i) \cap N(y_j)$ is non empty.

Step 8: Give a membership value to each arc y_iy_j by using the some axioms.
$\Upsilon(y_iy_j = (\wedge\{y_i \cap y_j\})[N(y_i \cap N(y_j)]$, $\upsilon(y_iy_j = (\vee\{y_i \cap y_j\})[N(y_i \cap N(y_j)]$,
$\psi(y_iy_j = (\vee\{y_i \cap y_j\})[N(y_i \cap N(y_j)]$, $\alpha(y_iy_j = (\vee\{y_i \cap y_j\})[N(y_i \cap N(y_j)]$,
$\beta(y_iy_j = (\wedge\{y_i \cap y_j\})[N(y_i \cap N(y_j)]$, $\gamma(y_iy_j = (\wedge\{y_i \cap y_j\})[N(y_i \cap N(y_j)]$.

Step 9: If y, $z_1,\, z_2,\, z_3,\, \ldots,\, z_p$ are applicants for designations d, then strength of applicants competition is $R(y,\, d) = (\Upsilon(y,\, d),\, \upsilon(y,\, d),\, \psi(y,\, d),\, \alpha(y,\, d),\, \beta(y,\, d),\, \gamma(y,\, d))$ of every applicants y and designation d is taken as:
$R(y,\, d) = \left(\dfrac{\Upsilon(xz_1) + \ldots\Upsilon(xz_p)}{p},\, \dfrac{\upsilon(xz_1) + \ldots\upsilon(xz_p)}{p},\, \dfrac{\psi(xz_1) + \ldots\psi(xz_p)}{p},\, \dfrac{\alpha(xz_1) + \ldots\alpha(xz_p)}{p},\, \dfrac{\beta(xz_1) + \ldots\beta(xz_p)}{p},\, \dfrac{\gamma(xz_1) + \ldots\gamma(xz_p)}{p}\right).$

Step 10: Calculate $S(y,\, d) = 1 + \alpha(y,\, d) + \delta(y,\, d) - (\beta(y,\, d) + \gamma(y,\, d) + \eta(y,\, d) + \theta(y,\, d))$.

Step 11: End.

5.1 *Agricultural nomination designation*

Suppose that {Bella, Aria, Sophie, Iram} is a set of four applicants for the following designations {Deputy Director Officer (DDO), Director Officer (DO), and Agriculture Officer (AO)}. For this purpose, we suppose that p the number of applicants and the number of designations is denoted by d. Take the bipolar picture fuzzy digraph as shown in Figure 7, depicting the competition among all four applicants for the given designation win field of an agriculture organization. $\Upsilon(z)$ be the degree of membership for every applicant, depending on their level of suitability for the purpose of the agriculture field. $\upsilon(z)$ and $\psi(z)$ represnt the levels of neutrality and falsity, respectively, represented as percentages. On the other hand, $\alpha(z)$ represents the negative of membership degree for each applicant but $\beta(z)$ and $\gamma(z)$ representing neutrality and falsity percentages. $\Upsilon(z)$ of every directed edge between both designations and applicants denote the eligibility or positive response from designation in organization, $\upsilon(z)$ and $\psi(z)$ are neutrality and false in this percentage. $\alpha(z)$ of every directed edge between both designations and applicants denote the non-eligibility or negative response from designation in organization, $\beta(z)$ and $\gamma(z)$ are neutral and false in this percentage. The edge membership degree is given as Table 1:

Table 1. The edge membership degree

$z \in Y$	$N(z)$
Bella	$\{(AO, 0.4, 0.2, 0.3, -0.4, -0.3, -0.3), (DDO, 0.3, 0.2, 0.1, -0.3, -0.2, -0.3)\}$
Aria	$\{(AO, 0.3, 0.2, 0.2, -0.1, -0.2, -0.3), (DDO, 0.2, 0.1, 0.3, -0.2, -0.3, -0.4), (DO, 0.2, 0.3, 0.3, -0.1, -0.2, -0.1)\}$
Sophie	$\{(DO, 0.3, 0.2, 0.1, -0.2, -0.3, -0.2)\})$
Iram	$\{(DDO, 0.3, 0.2, 0.1, -0.2, -0.3, -0.3), (DO, 0.2, 0.3, 0.2, -0.2, -0.2, -0.3)\})$

$$N(Bella) \cap N(Aria) = \{(AO, 0.3, 0.2, 0.3, -0.1, -0.2, -0.3), (DDO, 0.2, 0.1, 0.3, -0.2, -0.2, -0.4)\}$$

$$N(Bella) \cap N(Sophie) = \{\}$$

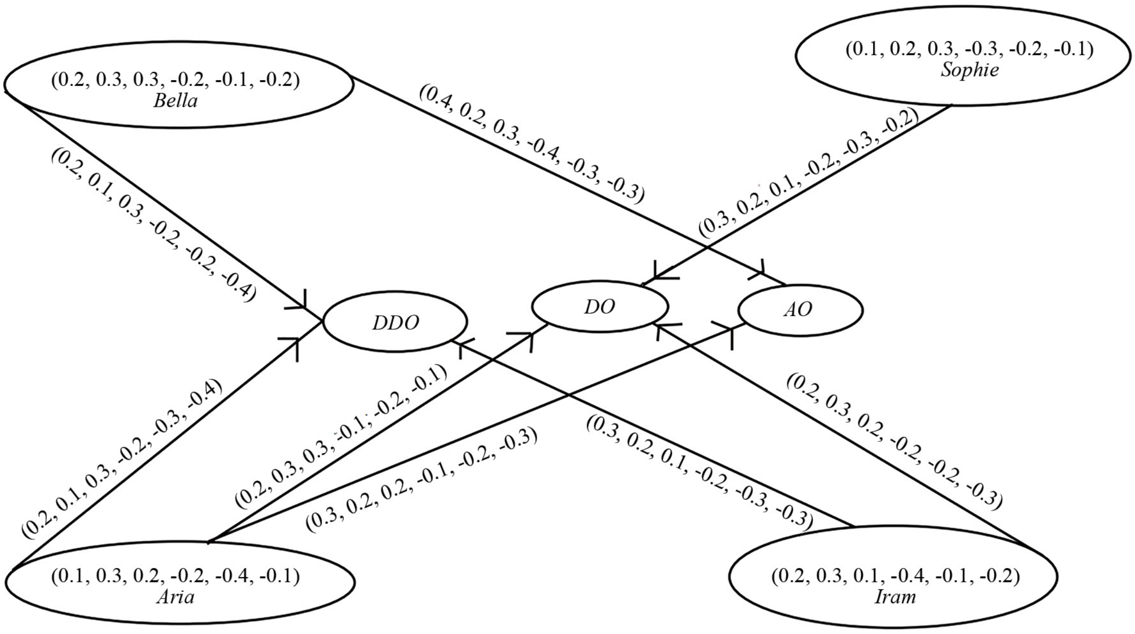
$$N(Bella) \cap N(Iram) = \{(DDO, 0.3, 0.2, 0.1, -0.2, -0.2, -0.3)\}$$

$$N(Aria) \cap N(Sophie) = \{(DO, 0.2, 0.2, 0.3, -0.1, -0.2, -0.2)\}$$

$$N(Aria) \cap N(Iram) = \{(DDO, 0.2, 0.1, 0.3, -0.2, -0.3, -0.4), (DO, 0.2, 0.3, 0.3, -0.1, -0.2, -0.3)\}$$

$$N(Sophie) \cap N(Iram) = \{(DO, 0.2, 0.2, 0.2, -0.2, -0.2, -0.3)\}$$

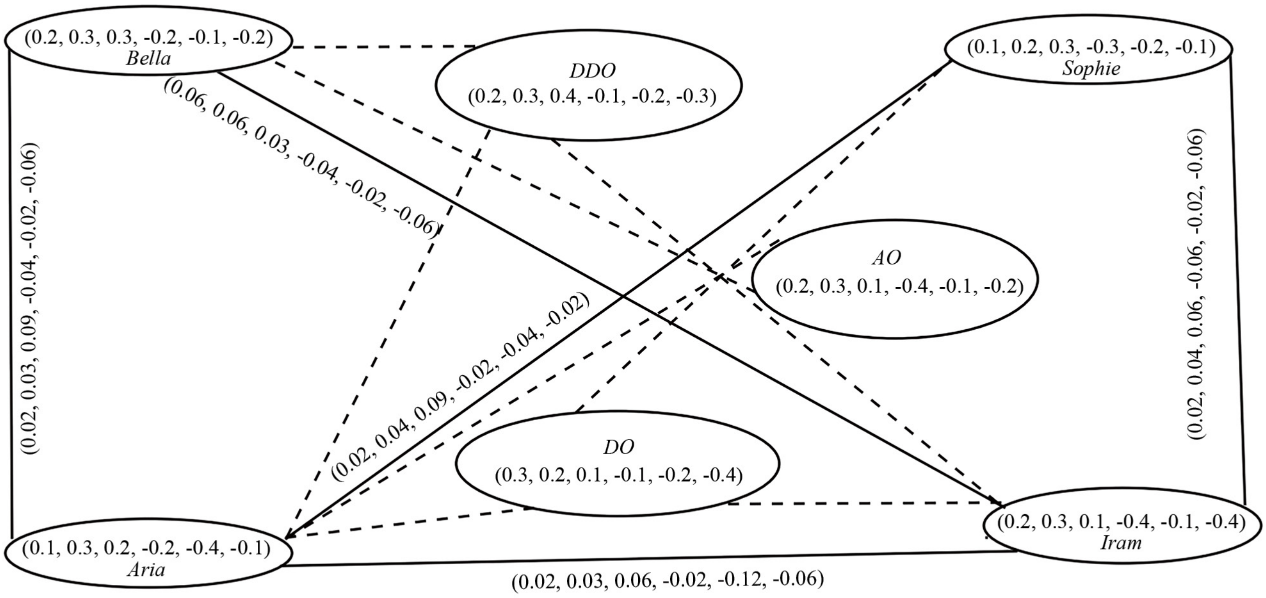
There is no common designation between Sophie and Bella.



(*Bella*, *Aria*) = (0.1, 0.3, 0.3, -0.2, -0.1, -0.2)(0.2, 0.1, 0.3, 0.2, 0.2, 0.3)
= (0.02, 0.03, 0.09, -0.04, -0.02, -0.06)
(*Bella*, *Sophie*) = {}
(*Bella*, *Iram*) = (0.2, 0.3, 0.3, -0.2, -0.1, -0.2)(0.3, 0.2, 0.1, 0.2, 0.2, 0.3)
= (0.06, 0.06, 0.03, -0.04, -0.02, -0.06)
(*Aria*, *Sophie*) = (0.1, 0.2, 0.3, -0.2, -0.2, -0.1)(0.2, 0.2, 0.3, 0.1, 0.2, 0.2)
= (0.02, 0.04, 0.09, -0.02, -0.04, -0.02)
(*Aria*, *Iram*) = (0.1, 0.3, 0.2, -0.1, -0.4, -0.2)(0.2, 0.1, 0.3, 0.2, 0.3, 0.3)
= (0.02, 0.03, 0.06, -0.02, -0.12, -0.06)
(*Sophie*, *Iram*) = (0.1, 0.2, 0.3, -0.3, -0.1, -0.2)(0.2, 0.2, 0.2, 0.2, 0.2, 0.3)
= (0.02, 0.04, 0.06, -0.06, -0.02, -0.06)

Figure 7. Bipolar picture fuzzy diagraph

A BPFG is shown in Figure 8 which represent the competition of all participant. Also, The competition between the two individually applicants and when applicants compete for designation is shown in graph 8.



R (applicant, Nomination) is given as:

$$R\,(Bella,\, DDO) = \left(\frac{0.02+0.06}{2}, \frac{0.03+0.06}{2}, \frac{0.09+0.03}{2}, \frac{-0.04-0.04}{2}, \frac{-0.02-0.02}{2}, \frac{-0.06-0.06}{2}\right)$$

$$= (0.04, 0.0225, 0.06, -0.04, -0.02, -0.06)$$

$S\,(Bella, DDO) = 1 + 0.04 - 0.04 - (0.0225 + 0.06 - 0.02 - 0.06) = 0.8375$

$S\,(Aria, DDO) = 1.14$

$S\,(Iram, DDO) = 0.93$

$S\,(Bella, AO) = 1.30$

$S\,(Aria, AO) = 1.30$

$S\,(Aria, DO) = 0.98$

$S\,(Sophie, DO) = 0.985$

$S\,(Iram, DO) = 0.985$

Figure 8. Bipolar picture fuzzy competition graph

There are two types of lines from which one is solid lines show the comparison between two applicants and dot lines represent applicant competes for the required designation.

Table 2. Comparaison among two applicants

(Applicant, designation)	in competition	*R* (applicant, Nomination)	*S* (applicant, Nomination)
(*Bella*, *DDO*)	*Aria*, *Iram*	(0.04, 0.0225, 0.06, -0.04, -0.02, -0.06)	0.795
(*Aria*, *DDO*)	*Bella*, *Iram*	(0.02, 0.03, 0.075, -0.03, -0.07, -0.06)	1.015
(*Iram*, *DDO*)	*Bella*, *Aria*	(0.04, 0.045, 0.045, -0.03, -0.07, -0.06)	1.05
(*Bella*, *AO*)	*Aria*	(0.02, 0.03, 0.09, -0.04, -0.02, -0.06)	0.94
(*Aria*, *AO*)	*Bella*	(0.02, 0.03, 0.09, -0.04, -0.02, -0.06)	0.94
(*Aria*, *DO*)	*Sophie*, *Iram*	(0.02, 0.035, 0.075, -0.02, -0.08, -0.04)	1.01
(*Sophie*, *DO*)	*Aria*, *Iram*	(0.02, 0.04, 0.075, -0.04, -0.03, -0.04)	0.935
(*Iram*, *DO*)	*Aria*, *Sophie*	(0.02, 0.035, 0.06, -0.04, -0.07, -0.06)	1.015

From the Table 2, all four applicants are superior if it has greater strength than the other. We have seen that, in *DDO* designation the strength value of Aria is better than all the applicants. Therefore, Its eligibility is better than another participant. In *AO* designation Aria and Bella are in the same position due to having the same strength. In *DO* designation, Iram compete with the other participants.

6. Conclusion

BPFG has a broader idea due to its significantly enhanced flexibility and comparability compared to FGs and IFGs. This research study includes the concepts of MP and RP in BPFG, along with an examination of vertex degree and total degree. We develop some theorems related to these operations. We define the isomorphism and homomorphism of BPFG. We define the idea of weak isomorphism and co-weak isomorphism of BPFG. We have proved that isomorphism between BPFG is an equivalence relation.

Limitations:

There are some limitations of our work provided:

• The presence of dual memberships creates difficulties in mathematical operations and collecting data.

• The lack of software tools and methods specifically designed for dealing with BPFGs makes it a challenge for practical implementation.

• The concept is new and so limited due to its novelty and ongoing theoretical development.

Future directions:

We will introduce the bondage number and non-bondage number of BPFG. Our aim is to demonstrate the application of BPFG in machine learning. In future work, we will introduce some new operations on BPFG and try to utilize them in decision-making problems.

Comparative analysis:

We explore the properties of the maximal product and residue product of BPFG, a flexible and accurate decision-making tool. It examines its application in decision-making, highlighting its ability to capture frequent interactions, dynamic updating, and real-time analysis, while also providing logical implications. PFG deals with just positive evaluation whenever BPFG deals with dual nature.

Acknowledgment

The authors would like to thank Deanship of scientific research in King Saud University for funding and supporting this research through the initiative of DSR Graduate Students Research Support (GSR).

Conflict of interest

The authors declare that they have no relevant or material financial interests that relate to the research described in this paper.

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