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Research Article

Asymptotic Behavior of Positive Solutions to a Nonlinear Elliptic Coupled System on an Exterior Domain

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Abstract: In this article, we study the existence and the asymptotic behavior of positive continuous solutions for the following elliptic coupled system

$$\begin{cases} -\Delta u = p(x)u^{\alpha}v^{a} \text{ in D}, \\ -\Delta v = q(x)u^{b}v^{\beta} \text{ in } D, \\ u_{/\partial D} = v_{/\partial D} = 0, \\ \lim\limits_{|x|\to\infty} u(x) = \lim\limits_{|x|\to\infty} v(x) = 0, \end{cases}$$

where D is an unbounded regular domain in $\mathbb{R}^n$, $n \geq 3$, with a compact boundary. The exponents $\alpha, \beta \in (-1, 1)$, $a, b \in \mathbb{R}$ such that $(1 - |\alpha|)(1-|\beta|) - |ab| > 0$ and p, q are positive continuous functions on D satisfying some suitable assumptions with reference to Karamata regular variation theory.

***Keywords*:** positive solutions, asymptotic behavior, Shäuder's fixed point theorem, Karamata classes

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1. Introduction and preliminaries

The study of nonlinear elliptic systems has a strong motivation and important research efforts have been made recently for these systems intending to use the results of existence and asymptotic behavior of positive solutions in applied fields. Coupled nonlinear elliptic systems occur in various nonlinear phenomena, such as pattern formation, population evolution, chemical reaction where for instance, u and v correspond to the concentrations of two species in the process. Accordingly, positive solutions of such systems are attractive. Existence, uniqueness and boundary behavior of positive solutions of nonlinear elliptic systems in both bounded and unbounded domains with various boundary conditions, have been extensively investigated in the literature with various methods [1-10].

In [1], Ghergu considered the following elliptic system

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$$\begin{cases} -\Delta u = u^{\alpha} v^{a} \ \text{ in } \ \Omega, \\ -\Delta v = u^{b} v^{\beta} \ \text{ in } \ \Omega, \\ u_{/\partial\Omega} = v_{/\partial\Omega} = 0, \end{cases} \tag{1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) with a smooth boundary, $\alpha, \beta \leq 0$ and $a, b < 0$. The author showed that system (1) has at least one solution if,

$$(1-\alpha)(1-\beta) - ab > 0$$

and one of the following conditions is satisfied:

(i) $\alpha + a \min(1, \frac{2+b}{1-\beta}) \geq -1$ and $b > -2$.

(ii) $b + \beta \min(1, \frac{2+a}{1-\alpha}) \geq -1$ and $a > -2$.

(iii) $\alpha, \beta \leq -1$ and $a, b > -2$.

Later, Zhang [2] derived the existence, boundary behavior and uniqueness of solutions for system (1) for a different range of exponents to those in [1]. He assumed that $\alpha, \beta \leq 0$, $a, b < 0$ satisfying one of the following conditions:

(i) $\beta - 1 < a$, $\alpha + \frac{a(2+b)}{1-\beta} < -1$ and $\alpha - 1 < b$, $\beta + \frac{b(2+a)}{1-\alpha} < -1$.

(ii) $\beta - 1 > a$, $\alpha + \frac{a(2+b)}{1-\beta} > -1$ and $\alpha - 1 > b$, $\beta + \frac{b(2+a)}{1-\alpha} > -1$.

Then, the author proved that system (1) has at least one classical solution (u, v) satisfying for $x \in \overline{\Omega}$,

$$m d(x) \leq u(x) \leq M (d(x))^{\frac{2(1-\beta+a)}{(1-\alpha)(1-\beta)-ab}}$$

and

$$m d(x) \leq v(x) \leq M (d(x))^{\frac{2(1-\alpha+b)}{(1-\alpha)(1-\beta)-ab}},$$

where m and M are positive constants and $d(x)$ denotes the Euclidean distance from $x \in \overline{\Omega}$ to the boundary $\partial\Omega$.

In [3], Kawano and Kusano considered the elliptic system

$$\begin{cases} -\Delta u = p(x) u^{\alpha} v^{a} \ \text{ in } \ \mathbb{R}^n, \\ -\Delta v = q(x) u^{b} v^{\beta} \ \text{ in } \ \mathbb{R}^n, \\ \lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} v(x) = 0, \end{cases} \tag{2}$$

where $n \geq 3$, $\alpha, \beta < 0$ and p, q are nonnegative locally Hölder continuous functions in $\mathbb{R}^n$. The authors assumed that there exist locally Hölder continuous functions $\phi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$p(x) \leq \phi(|x|) \text{ and } q(x) \leq \psi(|x|), \ \ x \in \mathbb{R}^n$$

and

$$\int_0^\infty r\phi(r)dr < \infty \text{ and } \int_0^\infty r\psi(r)dr < \infty.$$

Then, they proved by using the method of sub- and super-solutions that system (2) has entire positive solutions either if $\alpha + a < 1$, $\beta + b < 1$ or if $\alpha + a > 1$, $\beta + b > 1$.

Noussair and Swanson [4] discussed a class of coupled systems of semilinear elliptic partial differential equations in an exterior domain in $\mathbb{R}^n$, $n \geq 3$. They established necessary and sufficient conditions for the existence of positive solution dominated by $|x|^{2-n}$ when $|x| \to \infty$.

In this article, we are concerned with the investigation of the following nonlinear elliptic system

$$\begin{cases} -\Delta u = p(x)u^{\alpha}v^{a} \text{ in } D, \\ -\Delta v = q(x)u^{b}v^{\beta} \text{ in } D, \\ u_{/\partial D} = v_{/\partial D} = 0, \\ \lim\limits_{|x|\to\infty} u(x) = \lim\limits_{|x|\to\infty} v(x) = 0, \end{cases} \tag{3}$$

where $\alpha, \beta \in (-1, 1)$, $a, b \in \mathbb{R}$ such that $\chi := (1 - |\alpha|)(1 - |\beta|) - |ab| > 0$ and D is an unbounded regular domain in $\mathbb{R}^n$, $n \geq 3$, with a compact boundary. The positive weight functions p and q are required to be continuous on D that may be singular at the boundary ∂D or unbounded near ∞ and satisfying some assumptions with reference to the Karamata classes $\mathcal{K}_0$ and $\mathcal{K}_\infty$ (see Definition 1.4 below).

Our intention is to prove the existence of positive continuous solutions with an exact asymptotic behavior for system (3).

We point out that there are two main features of this work. The first one is the fact that we consider system (3) in D, which is an exterior domain. In this sense, system (3) can be considered as a natural extension of the following elliptic boundary value problem

$$\begin{cases} -\Delta u = p(x)u^{\sigma} \text{ in } D, \\ u_{/\partial D} = 0, \\ \lim\limits_{|x|\to\infty} u(x) = 0, \end{cases} \tag{4}$$

studied in [11] with $\sigma < 1$. But dealing with system (3) presents some difference because of, as far as we know, the lack of a meaningful maximum principle for systems in exterior domains. Indeed, to obtain an existence result for problem (4), the authors in [11] applied the sub- and super-solution method which is based on the maximum principle; see [12]. Hence, it seems that the method employed in the study of problem (4) does not carry over naturally to system (3). Therefore, we have to work around this difficulty and we shall apply the Schäuder fixed point theorem which requires invariance of a convex set under an appropriate integral operator. This restricts us to dealing with only the cases α, $\beta \in (-1, 1)$. The second one is that our paper deals with a large class of nonlinearities that may be singular at ∂D or unbounded near ∞. Moreover, we do not make a restriction on the sign of the exponents.

Throughout this paper, we will use the following notations and definitions:

(i) Let E be a domain of $\mathbb{R}^n$, $n \geq 3$.

- For $x, y \in E$, $G_E(x, y)$ denotes the Green function of the Dirichlet Laplacian.
- For $x \in E$, $\delta_E(x)$ denotes the Euclidean distance from $x \in E$ to ∂E, the boundary of E.

(ii) Let $x_0 \in \mathbb{R}^n \backslash \overline{D}$ and $r > 0$ such that $\overline{B}(x_0, r) := \{x \in \mathbb{R}^n : |x - x_0| \leq r\} \subset \mathbb{R}^n \backslash \overline{D}$. Then we have

$$G_D(x, y) = r^{2-n} G_{\frac{D-x_0}{r}}\left(\frac{x-x_0}{r}, \frac{y-x_0}{r}\right), \text{ for } x, y \in D$$

and

$$\delta_D(x) = r\,\delta_{\frac{D-x_0}{r}}\left(\frac{x-x_0}{r}\right), \text{ for } x \in D.$$

Hence, we may suppose without loss of generality, that $\overline{B}(0, 1) \subset \mathbb{R}^n \backslash \overline{D}$.

(iii) For $x \in D$, we denote by $\delta(x) = \delta_D(x)$ and $\rho(x) = \dfrac{\delta(x)}{1+\delta(x)}$.

(iv) Let f and g be two nonnegative functions defined on a set S. Then, we write $f(x) \approx g(x)$ in S, if there exists a constant $c > 0$ such that for each $x \in S$, $\frac{1}{c} g(x) \le f(x) \le c g(x)$.

(v) $\mathcal{B}(D)$ is the collection of Borel measurable functions in D and $\mathcal{B}^+(D)$ is the collection of nonnegative ones.

(vi) $C_0(D)$ is the space of continuous functions f in $\overline{D}$ vanishing at ∂D and satisfying $f(x) \to 0$ as $|x| \to \infty$, within D.

(vii) For a function $f \in \mathcal{B}^+(D)$, we denote by Vf the potential of f defined on D by

$$Vf(x) = \int_D G_D(x, y) f(y) dy.$$

We point out that if f is a nonnegative function in $L^1_{loc}(D)$ such that $Vf \in L^1_{loc}(D)$, then we have in the distributional sense $-\Delta(Vf) = f$ in D; see ([13], p.52).

Definition 1.1 A function $q \in \mathcal{B}(D)$ is in the class $K(D)$ if q verifies the following assumptions:

$$\lim_{r\to 0} \sup_{x\in D} \int_{(|x-y|\le r)\cap D} \frac{\rho(y)}{\rho(x)} G_D(x,y) |q(y)| dy = 0$$

and

$$\lim_{M\to +\infty} \sup_{x\in D} \int_{(|y|\ge M)\cap D} \frac{\rho(y)}{\rho(x)} G_D(x,y) |q(y)| dy = 0.$$

Remark 1.2 ([14], Proposition 3.4)

$$\text{The map } q : x \mapsto |x|^{\lambda-\mu} (\delta(x))^{-\lambda} \in K(D) \Leftrightarrow \lambda < 2 < \mu.$$

Proposition 1.3 Let $q \in K(D)$ be a nonnegative function. Then, we have

(i) $Vq \in C_0(D)$.

(ii) The family $\mathfrak{F} = \{V(f), f \in \mathcal{B}(D); |f| \le q\}$ is relatively compact in $C_0(D)$.

Proof. (i) See ([14], Proposition 3.7).

(ii) Let $f \in \mathcal{B}(D)$ satisfying $|f| \le q$.

For $x \in D$, we have

$$|Vf(x)| \le Vq(x).$$

Using (i), we have

$$|Vf(x)| \le \|Vq\|_\infty < \infty.$$

Thus the family $\mathfrak{F}$ is uniformly bounded. On the other hand, as in the proof of Proposition 3.7 in [14], we prove that the family $\mathfrak{F}$ is equicontinuous in $C_0(D)$.

Consequently, Ascoli's theorem implies that $\mathfrak{F}$ is relatively compact in $C_0(D)$.

Now, we introduce the Karamata classes of regularly varying functions.

Definition 1.4 [15]

(i) The class $\mathcal{K}_0$ is the collection of all Karamata functions L defined on $(0, \eta]$ by

$$L(t) := c \exp\left(\int_t^\eta \frac{z(s)}{s} ds\right),$$

for some $\eta > 0$, where $c > 0$ and z is a continuous function on $[0, \eta]$, with $z(0) = 0$.

(ii) The class $\mathcal{K}_\infty$ is the set of Karamta functions L defined on $[1, \infty)$ by

$$L(t) := c\exp(\int_1^t \frac{z(s)}{s} ds),$$

where $c > 0$ and z is a continuous function on $[1, \infty)$ such that $\lim_{t\to\infty} z(t) = 0$.

It is easy to check the next result.

Proposition 1.5 (i) A function L defined on $(0, \eta]$, $\eta > 0$, belongs to $\mathcal{K}_0$ if and only if L is a positive function in $C^1((0, \eta])$, such that

$$\lim_{t\to 0^+} \frac{tL'(t)}{L(t)} = 0.$$

(ii) A function L belongs to $\mathcal{K}_\infty$ if and only if L is a positive function in C^1 ($[1, \infty)$), such that

$$\lim_{t\to\infty} \frac{tL'(t)}{L(t)} = 0.$$

Remark 1.6 [16] If $L \in \mathcal{K}_\infty$ then there exists $m \geq 0$ such that for every $r > 0$ and $t \geq 1$, we have

$$(1 + r)^{-m} L(t) \leq L(r + t) \leq (1 + r)^m L(t).$$

As a typical example of a function belonging to the class $\mathcal{K}_0$ (resp. $\mathcal{K}_\infty$), we give

$$L(t) = \prod_{k=1}^{m} (log_k(\frac{d}{t}))^{\xi_k} \quad (\text{resp. } L(t) = \exp\left(\prod_{k=1}^{m} (log_k(dt))^{\tau_k}\right)),$$

where

$$log_k x = \underbrace{log \circ log \circ \ldots \circ log}_{k \text{ times}} x,$$

$\xi_k \in \mathbb{R}$ (resp. $\tau_k \in (0, 1)$) and d is a sufficiently large positive real number such that the function L is defined and positive on $(0, \eta]$, for $\eta > 1$ (resp. on $[1, \infty)$).

Let us consider the following hypothesis:

(H) p and q are positive continuous functions on D satisfying for $x \in D$

$$p(x) \approx (\rho(x))^{-\lambda} M(\rho(x)) |x|^{-\mu} N(|x|),$$

$$q(x) \approx (\rho(x))^{-\sigma} K(\rho(x)) |x|^{-\gamma} L(|x|),$$

where $\lambda, \mu, \sigma, \gamma \in \mathbb{R}$ and $M, K \in \mathcal{K}_0$ defined on $(0, \eta]$, $(\eta > 1)$, $N, L \in \mathcal{K}_\infty$.

Additional assumptions are needed to establish our main result, we recall that we assume that the exponents $\alpha, \beta \in (-1, 1)$, $a, b \in \mathbb{R}$ satisfy the hypothesis:

$$\chi = (1 - |\alpha|)(1 - |\beta|) - |ab| > 0.$$

Hence the constant

$$\omega = (1 - \alpha)(1 - \beta) - ab > 0.$$

For simplicity, we set:

$$\delta_1 = \min\left(1, \frac{2-\sigma+b}{1-\beta}\right), \ \delta_2 = \frac{(1-\beta)(2-\lambda)+a(2-\sigma)}{\omega},$$

$$\delta_3 = \frac{(1-\alpha)(2-\sigma)+b(2-\lambda)}{\omega}, \ \delta_4 = \min\left(1, \frac{2-\sigma}{1-\beta}\right),$$

$$\nu_1 = \min\left(n-2, \frac{\gamma-2+b(n-2)}{1-\beta}\right), \ \nu_2 = \frac{(1-\beta)(\mu-2)+a(\gamma-2)}{\omega},$$

$$\nu_3 = \frac{(1-\alpha)(\gamma-2)+b(\mu-2)}{\omega}, \ \nu_4 = \min\left(n-2, \frac{\gamma-2}{1-\beta}\right).$$

Remark 1.7 We point out that:
(i) $\delta_2 = \frac{2-\lambda+a\delta_3}{1-\alpha}$ and $\delta_3 = \frac{2-\sigma+b\delta_2}{1-\beta}$.
(ii) $\nu_2 = \frac{\mu-2+a\nu_3}{1-\alpha}$ and $\nu_3 = \frac{\gamma-2+b\nu_2}{1-\beta}$.

Now, we give sufficient hypotheses which permit us to show our main result for the existence and the asymptotic behavior of solutions for system (3).

(A) One of the following assumptions is satisfied
(A_1) $\lambda < 1 + \alpha + a\delta_1$, $\sigma \leq 2 + b$ and K satisfies $\int_0^\eta \frac{K(t)}{t} dt < \infty$ if $\sigma = 2 + b$.
(A_2) $\lambda = 1 + \alpha + a\delta_1$, $\sigma < 2 + b$, $\sigma \neq 1 + b + \beta$.
(A_3) $0 < \delta_2 < 1$ and $0 < \delta_3 < 1$.
(A_4) $\lambda = 2 + a\delta_4$, $\sigma < 2$, $\sigma \neq 1 + \beta$ and M, K satisfy
(i) $\int_0^\eta \frac{M(t)}{t} dt < \infty$ if $\sigma < 1 + \beta$,
(ii) $\int_0^\eta \frac{(MK^{\frac{a}{1-\beta}})(t)}{t} dt < \infty$ if $1 + \beta < \sigma < 2$.

(B) One of the following assumptions is satisfied
(B_1) $\mu > n - \alpha(n - 2)\ a\nu_1$, $\gamma \geq 2 - b(n - 2)$ and L satisfies $\int_1^\infty \frac{L(t)}{t} dt < \infty$ if $\gamma = 2 - b(n - 2)$.
(B_2) $\mu = n - \alpha(n - 2) - a\nu_1$, $\gamma > 2 - b(n - 2)$, $\gamma \neq n - (\beta + b)(n - 2)$.
(B_3) $0 < \nu_2 < n - 2$ and $0 < \nu_3 < n - 2$.
(B_4) $\mu = 2 - a\nu_4$, $\gamma > 2$, $\gamma \neq n - \beta(n - 2)$ and N, L satisfy
(i) $\int_1^\infty \frac{N(t)}{t} dt < \infty$ if $\gamma > n - \beta(n - 2)$,
(ii) $\int_1^\infty \frac{(NL^{\frac{a}{1-\beta}})(t)}{t} dt < \infty$ if $2 < \gamma < n - \beta(n - 2)$.

Our main result is the following.

Theorem 1.8 Assume (H), (A) and (B). Then the nonlinear elliptic system (3) has a positive solution $(u, v) \in C_0(D) \times C_0(D)$ satisfying for $x \in D$,

$$u(x) \approx \frac{\rho(x)^{\tilde{\lambda}}}{|x|^{\tilde{\mu}}} \tilde{M}(\rho(x))\tilde{N}(|x|)$$

and

$$v(x) \approx \frac{\rho(x)^{\tilde{\sigma}}}{|x|^{\tilde{\gamma}}} \tilde{K}(\rho(x)) \tilde{L}(|x|).$$

Here $\tilde{\lambda}$, $\tilde{\mu}$, $\tilde{\sigma}$, $\tilde{\gamma}$ are nonnegative real numbers, $\tilde{M}$, $\tilde{K}$ are in $\mathcal{K}_0$ and $\tilde{N}$, $\tilde{L}$ belong to $\mathcal{K}_\infty$.

The rest of the paper is organized as follows. In Section 2, we present some basic properties of the functions in the classes $\mathcal{K}_0$ and $\mathcal{K}_\infty$. In particular, we recall some sharp estimates on some suitable potential functions. In Section 3, we prove our main result given in Theorem 1.8.

2. Karamata classes $\mathcal{K}_0$ and $\mathcal{K}_\infty$

In what follows, we list some basic properties of the functions belonging to Karamata classes $\mathcal{K}_0$ and $\mathcal{K}_\infty$. Then, we give estimates on some potential functions.

Lemma 2.1 ([16-18])

(i) Let $p \in \mathbb{R}$ and $L_1, L_2 \in \mathcal{K}_0$ (resp. $\mathcal{K}_\infty$). Then the functions $L_1 + L_2$, L_1L_2 and L_1^p are in the class $\mathcal{K}_0$ (resp. $\mathcal{K}_\infty$).

(ii) Let $\varepsilon > 0$ and $L \in \mathcal{K}_0$ (resp. $\mathcal{K}_\infty$). Then we have

$$\lim_{t\to 0^+} t^{\varepsilon} L(t) = 0 \ \text{ (resp. } \lim_{t\to\infty} t^{-\varepsilon} L(t) = 0).$$

Lemma 2.2 (Karamata's Theorem [16, 18])

(I) Let $\gamma \in \mathbb{R}$ and $L \in \mathcal{K}_0$ defined on $(0, \eta]$, $\eta > 0$. Then we have the following assertions:

(i) If $\gamma > -1$, then $\int_0^{\eta} t^{\gamma} L(t) dt$ converges and $\int_0^t s^{\gamma} L(s) ds \underset{t\to 0^+}{\sim} \frac{t^{1+\gamma} L(t)}{1+\gamma}$.

(ii) If $\gamma < -1$, then $\int_0^{\eta} t^{\gamma} L(t) dt$ diverges and $\int_t^{\eta} s^{\gamma} L(s) ds \underset{t\to 0^+}{\sim} -\frac{t^{1+\gamma} L(t)}{1+\gamma}$.

(II) Let $L \in \mathcal{K}_\infty$ and $\gamma \in \mathbb{R}$. Then we have the following:

(i) If $\gamma < -1$, then $\int_1^{\infty} t^{\gamma} L(t) dt$ converges and $\int_t^{\infty} s^{\gamma} L(s) ds \underset{t\to\infty}{\sim} -\frac{t^{1+\gamma} L(t)}{1+\gamma}$.

(ii) If $\gamma > -1$, then $\int_1^{\infty} t^{1-\gamma} L(t) dt$ diverges and $\int_1^t s^{\gamma} L(s) ds \underset{t\to 0}{\sim} \frac{t^{1+\gamma} L(t)}{1+\gamma}$.

Lemma 2.3 [16, 19]

(i) Let $L \in \mathcal{K}_0$ defined on $(0, \eta]$, $\eta > 1$. Then $\lim_{t\to 0^+} \frac{L(t)}{\int_t^{\eta} \frac{L(s)}{s} ds} = 0$. Particularly, $t \mapsto \int_t^{\eta} \frac{L(s)}{s} ds \in \mathcal{K}_0$. If further, $\int_0^{\eta} \frac{L(s)}{s} ds$ converges, then

$$\lim_{t\to 0^+} \frac{L(t)}{\int_o^t \frac{L(s)}{s} ds} = 0 \text{ and } t \mapsto \int_0^t \frac{L(s)}{s} ds \in \mathcal{K}_0.$$

(ii) Let $L \in \mathcal{K}_\infty$ then $\lim_{t\to\infty} \frac{L(t)}{\int_1^t \frac{L(s)}{s} ds} = 0$. Particularly, $t \mapsto \int_1^{t+1} \frac{L(s)}{s} ds \in \mathcal{K}_\infty$. If further, $\int_1^{\infty} \frac{L(s)}{s} ds$ converges, then

$$\lim_{t\to\infty} \frac{L(t)}{\int_t^{\infty} \frac{L(s)}{s} ds} = 0 \text{ and } t \mapsto \int_t^{\infty} \frac{L(s)}{s} ds \in \mathcal{K}_\infty.$$

To simplify our statements, we introduce the following assumption.

(C) h is a function defined on D by:

$$h(x) = (\rho(x))^{-\nu} L_1(\rho(x)) |x|^{-\tau} L_2(|x|),$$

where $\nu \le 2 \le \tau$, $L_1 \in \mathcal{K}_0$ defined on $(0, \eta]$, $\eta > 1$ and $L_2 \in \mathcal{K}_\infty$ satisfy

$$\int_0^\eta t^{1-\nu} L_1(t)dt \text{ and } \int_1^\infty t^{1-\tau} L_2(t)dt < \infty. \tag{5}$$

Remark 2.4 We note that due to Lemma 2.2, we need to verify (5) only if $\nu = 2$ or $\tau = 2$.

Now, we recall the following key sharp estimates.

Proposition 2.5 [11] Let h satisfying (C). Then for $x \in D$, we have

$$Vh(x) \approx \frac{(\rho(x))^{\min(2-\nu,1)}}{|x|^{\min(\tau-2,n-2)}} \Phi_{L_1,\nu}(\rho(x)) \Psi_{L_2,\tau}(|x|),$$

where for $t \in (0, \eta)$,

$$\Phi_{L_1,\nu}(t) := \begin{cases} 1, & \text{if } \nu < 1, \\ \int_t^\eta \frac{L_1(\xi)}{\xi} d\xi, & \text{if } \nu = 1, \\ L_1(t), & \text{if } 1 < \nu < 2, \\ \int_0^t \frac{L_1(\xi)}{\xi} d\xi, & \text{if } \nu = 2, \end{cases}$$

and for $t \in [1, \infty)$,

$$\Psi_{L_2,\tau}(t) := \begin{cases} \int_t^\infty \frac{L_2(\xi)}{\xi} d\xi, & \text{if } \tau = 2, \\ L_2(t), & \text{if } 2 < \tau < n, \\ \int_t^{t+1} \frac{L_2(\xi)}{\xi} d\xi, & \text{if } \tau = n, \\ 1, & \text{if } \tau > n. \end{cases}$$

3. Proof of Theorem 1.8

In this section, we aim to prove Theorem 1.8.

Proposition 3.1 If h is a function satisfying (C), then $h \in K(D)$.

Proof. The proof results from Proposition 2.8 in [20] and Kelvin transformation.

The next result plays an important role in the proof of Theorem 1.8.

Proposition 3.2 Suppose that the condition (H) is satisfied. In addition, suppose that there exist two nonnegative functions θ and ψ satisfying:

(i) θ and ψ are in $C_0(D)$.

(ii) The functions $p\theta^\alpha\psi^a$ and $q\theta^b\psi^\beta$ are in $K(D)$ such that on D:

$$V(p\theta^\alpha\psi^a) \approx \theta \text{ and } V(q\theta^b\psi^\beta) \approx \psi. \tag{6}$$

Then system (3) has a positive solution $(u, v) \in C_0(D) \times C_0(D)$ such that for $x \in D$,

$$u(x) \approx \theta(x) \text{ and } v(x) \approx \psi(x).$$

Proof. Using (6), we deduce the existence of $c > 1$ such that

$$\frac{1}{c}\theta \le V(p\theta^{\alpha}\psi^{a}) \le c\theta \tag{7}$$

and

$$\frac{1}{c}\psi \le V(q\theta^{b}\psi^{\beta}) \le c\psi. \tag{8}$$

Such mentioned above, we have $\chi = (1 - |\alpha|)(1 - |\beta|) - |ab| > 0$. Set $c_1 = c^{\frac{1-|\beta|+|a|}{\chi}}$ and $c_2 = c^{\frac{1-|\alpha|+|b|}{\chi}}$. With the aim of using a fixed point argument, we define the non-empty convex closed set Λ by

$$\Lambda = \{(u,v) \in \left(C_0(D)\right)^2 : \frac{1}{c_1}\theta \le u \le c_1\theta;\ \frac{1}{c_2}\psi \le v \le c_2\psi\}.$$

We consider the operator T, defined on Λ by

$$T(u,v) = (V(pu^{\alpha}v^{a}), V(qu^{b}v^{\beta})).$$

We aim at proving that the operator T has a fixed point in $(C_0(D))^2$. First, we prove that $T\Lambda$ is relatively compact in $(C_0(D))^2$ endowed with the norm $\|.\|$ which is defined by $\|(u, v)\| = \|u\|_{\infty} + \|v\|_{\infty}$. Let $(u, v) \in \Lambda$, then we have

$$\frac{1}{c_1^{|\alpha|}c_2^{|a|}}p\theta^{\alpha}\psi^{a} \le pu^{\alpha}v^{a} \le c_1^{|\alpha|}c_2^{|a|}p\theta^{\alpha}\psi^{a} \tag{9}$$

and

$$\frac{1}{c_1^{|b|}c_2^{|\beta|}}b\theta^{b}\psi^{\beta} \le bu^{b}v^{\beta} \le c_1^{|b|}c_2^{|\beta|}b\theta^{b}\psi^{\beta}. \tag{10}$$

Using the fact that $p\theta^{\alpha}\psi^{a}$ and $b\theta^{b}\psi^{\beta}$ are in $K(D)$ and applying Proposition 1.3 (*ii*), we conclude that the sets of functions

$$\left\{x \mapsto V(pu^{\alpha}v^{a})(x),\ (u,v) \in \Lambda\right\}$$

and

$$\left\{x \mapsto V(bu^{b}v^{\beta})(x),\ (u,v) \in \Lambda\right\}$$

are relatively compact in $C_0(D)$. Which implies that $T\Lambda$ is relatively compact in $(C_0(D))^2$. Next, we prove that $T\Lambda \subset \Lambda$. Let $(u, v) \in \Lambda$. According to (7), (8), (9) and (10), we have

$$\frac{1}{cc_1^{|\alpha|}c_2^{|a|}}\theta \le V(p\theta^{\alpha}\psi^{a}) \le cc_1^{|\alpha|}c_2^{|a|}\theta$$

and

$$\frac{1}{cc_1^{|b|}c_2^{|\beta|}}\psi \le V(b\theta^{b}\psi^{\beta}) \le cc_1^{|b|}c_2^{|\beta|}\psi.$$

Since $cc_1^{|\alpha|}c_2^{|a|} = c_1$, $cc_1^{|b|}c_2^{|\beta|} = c_2$ and $T\Lambda \subset (C_0(D))^2$, we deduce that $T\Lambda \subset \Lambda$.

Finally, we shall show the continuity of the operator T in $(C_0(D))^2$ with respect to the norm $\| . \|$. For this end, we consider a sequence $((u_k, v_k))_k$ in Λ which converges to $(u, v) \in \Lambda$ with respect to the norm $\| . \|$.

For $k \in \mathbb{N}$, we have for each $x \in D$,

$$|V(pu_k^\alpha v_k^a)(x) - V(pu^\alpha v^a)(x)| \leq \int_D G_D(x, y)p(y)|(u_k^\alpha v_k^a)(y) - (u^\alpha v^a)(y)|\, dy.$$

Further from (9), we have

$$p|u_k^\alpha v_k^a - u^\alpha v^a| \leq 2c_1^{|\alpha|}c_2^{|a|} p\theta^\alpha \psi^a.$$

Since $V(p\theta^\alpha\psi^a) < \infty$, we conclude by the dominated convergence theorem that for each $x \in D$,

$$V(pu_k^\alpha v_k^a)(x) - V(pu^\alpha v^a)(x) \to 0 \text{ as } k \to \infty.$$

Similarly, we obtain that for each $x \in D$, $V(qu_k^b v_k^\beta)(x) - V(qu^b v^\beta)(x) \to 0$ as $k \to \infty$.

Moreover, since $T\Lambda$ is relatively compact in $(C_0(D))^2$, we deduce that the pointwise convergence implies the uniform convergence, namely,

$$\| T(u_k, v_k) - T(u, v)\| \text{ converges to } 0 \text{ as } k \to \infty.$$

This proves that T is a continuous mapping from Λ into itself. Hence, applying the Schäuder's fixed point theorem, we conclude the existence of $(u, v) \in \Lambda$ satisfies $T(u, v) = (u, v)$. Then we have $(u, v) \in C_0(D) \times C_0(D)$ such that

$$u = V(pu^\alpha v^a) \text{ and } v = V(qu^b v^\beta). \tag{11}$$

Using the fact that $(u, v) \in (C_0(D))^2$, hypothesis (H) and (11), we obtain that the functions $pu^\alpha v^a$ and $V(pu^\alpha v^a)$ are in $L^1_{loc}(D)$. This implies that, in the distributional sense,

$$-\Delta(V(pu^\alpha v^a)) = pu^\alpha v^a \text{ in } D.$$

Similarly, we have in the distributional sense,

$$-\Delta(V(qu^b v^\beta)) = qu^b v^\beta \text{ in } D.$$

Finally, since $(u, v) \in \Lambda$, we conclude that (u, v) is a positive continuous solution of (3) satisfying for each $x \in D$,

$$u(x) \approx \theta(x) \text{ and } v(x) \approx \psi(x).$$

This ends the proof.

Now, we are devoted to prove our main result.

Proof of Theorem 1.8

We shall distinguish several cases. In each case, we will give the explicit expressions of the functions θ and ψ which are of the form:

$$\theta(x) = \frac{(\rho(x))^{\tilde{\lambda}}}{|x|^{\tilde{\mu}}}\tilde{M}(\rho(x))\tilde{N}(|x|)$$

and

$$\psi(x) = \frac{(\rho(x))^{\tilde{\sigma}}}{|x|^{\tilde{\gamma}}} \tilde{K}(\rho(x))\tilde{L}(|x|),$$

where $\tilde{\lambda}$, $\tilde{\mu}$, $\tilde{\sigma}$, $\tilde{\gamma}$ are nonnegative real numbers, the functions $\tilde{M}$, $\tilde{K}$ are in $\mathcal{K}_0$ and $\tilde{N}$, $\tilde{L}$ belong to $\mathcal{K}_\infty$. We first verify that the functions θ and ψ are in $C_0(D)$. Then, we consider the functions $p\theta^\alpha\psi^a$ and $q\theta^b\psi^\beta$. From hypothesis (H), we obtain that for $x \in D$,

$$p\theta^\alpha\psi^a(x) \approx \frac{(\rho(x))^{-\lambda+\alpha\tilde{\lambda}+a\tilde{\sigma}}}{|x|^{\mu+\alpha\tilde{\mu}+a\tilde{\gamma}}}(M\tilde{M}^\alpha\tilde{K}^a)(\rho(x))(N\tilde{N}^\alpha\tilde{L}^a)(|x|)$$

and

$$q\theta^b\psi^\beta(x) \approx \frac{(\rho(x))^{-\sigma+b\tilde{\lambda}+\beta\tilde{\sigma}}}{|x|^{\gamma+b\tilde{\mu}+\beta\tilde{\gamma}}}(K\tilde{M}^b\tilde{K}^\beta)(\rho(x))(L\tilde{N}^b\tilde{L}^\beta)(|x|).$$

It is enough to prove that the functions $h : x \mapsto \frac{(\rho(x))^{-\lambda+\alpha\tilde{\lambda}+a\tilde{\sigma}}}{|x|^{\mu+\alpha\tilde{\mu}+a\tilde{\gamma}}}(M\tilde{M}^\alpha\tilde{K}^a)(\rho(x))(N\tilde{N}^\alpha\tilde{L}^a)(|x|)$ and $k : x \mapsto \frac{(\rho(x))^{-\sigma+b\tilde{\lambda}+\beta\tilde{\sigma}}}{|x|^{\gamma+b\tilde{\mu}+\beta\tilde{\gamma}}}(K\tilde{M}^b\tilde{K}^\beta)(\rho(x))(L\tilde{N}^b\tilde{L}^\beta)(|x|))$ satisfy respectively the condition (C). On the one hand, by the virtue of Proposition 3.1, we conclude that the functions $p\theta^\alpha\psi^a$ and $q\theta^b\psi^\beta$ are in $K(D)$. On the other hand, in view of Proposition 2.5, we estimate the potentials $V(p\theta^\alpha\psi^a)$ and $V(q\theta^b\psi^\beta)$ and by straightforward computations, we reach (6). This allows us to apply Proposition 3.2 which implies that system (3) has a positive solution $(u, v) \in C_0(D) \times C_0(D)$ satisfying on D,

$$u \approx \theta \text{ and } v \approx \psi.$$

Thus, Theorem 1.8 is proved.

Note that throughout the proof, we use Lemmas 2.1 and 2.3 to verify that some functions are in $\mathcal{K}_0$ or $\mathcal{K}_\infty$.

Case 1 Assume (H), (A_1) and (B_1).

We define the functions θ and ψ on D by

$$\theta(x) = \frac{\rho(x)}{|x|^{n-2}}$$

and

$$\psi(x) = \frac{(\rho(x))^{\delta_1}}{|x|^{\nu_1}} \widetilde{K}(\rho(x))\tilde{L}(|x|),$$

where

$$\widetilde{K}(\rho(x)) = \begin{cases} 1 & \text{if} \quad \sigma < 1+b+\beta, \\ \left(\int_{\rho(x)}^{\eta} \frac{K(t)}{t}dt\right)^{\frac{1}{1-\beta}} & \text{if} \quad \sigma = 1+b+\beta, \\ K^{\frac{1}{1-\beta}}(\rho(x)) & \text{if} \quad 1+b+\beta < \sigma < 2+b, \\ \left(\int_0^{\rho(x)} \frac{K(t)}{t}dt\right)^{\frac{1}{1-\beta}} & \text{if} \quad \sigma = 2+b, \end{cases} \tag{12}$$

and

$$\widetilde{L}(|x|)=\begin{cases}1 & \text{if} \quad \gamma > n-(\beta+b)(n-2),\\ \left(\int_1^{1+|x|}\frac{L(t)}{t}dt\right)^{\frac{1}{1-\beta}} & \text{if} \quad \gamma = n-(\beta+b)(n-2),\\ L^{\frac{1}{1-\beta}}(|x|) & \text{if} \quad 2-b(n-2)<\gamma<n-(\beta+b)(n-2),\\ \left(\int_{|x|}^{\infty}\frac{L(t)}{t}dt\right)^{\frac{1}{1-\beta}} & \text{if} \quad \gamma = 2-b(n-2).\end{cases} \tag{13}$$

Using Hypotheses (A_1), (B_1), Proposition 1.5, Lemmas 2.1 and 2.3, we get that the functions θ and ψ are in $C_0(D)$.
Using (H), we have for $x \in D$,

$$h(x)=\frac{(\rho(x))^{-\lambda+\alpha+a\delta_1}}{|x|^{\mu+\alpha(n-2)+a\nu_1}}(M\widetilde{K}^{a})(\rho(x))(N\widetilde{L}^{a})(|x|)$$

$$:=(\rho(x))^{-\nu}L_1(\rho(x))\,|x|^{-\tau}\ L_2(|x|).$$

Since $\nu = \lambda - \alpha - a\delta_1 < 1$, $\tau = \mu + \alpha(n-2) + a\nu_1 > n$, $L_1 \in \mathcal{K}_0$ and $L_2 \in \mathcal{K}_\infty$, we deduce by Lemma 2.2 that (5) is fulfilled. Hence, the function h satisfies the assumption (C).

On the other hand, by hypothesis (H) we obtain that

$$k(x)=\frac{(\rho(x))^{-\sigma+b+\beta\delta_1}}{|x|^{\gamma+b(n-2)+\beta\nu_1}}(K\widetilde{K}^{\beta})(\rho(x))(L\widetilde{L}^{\beta})(|x|)$$

$$:=(\rho(x))^{-\nu}L_1(\rho(x))\,|x|^{-\tau}\ L_2(|x|).$$

By simple calculus we obtain that for $x \in D$, $(\rho(x))^{-\nu}L_1(\rho(x)) = f(\rho(x))$, where

$$f(\rho(x)):=\begin{cases}(\rho(x))^{-\sigma+b+\beta}K(\rho(x)) & \text{if} \quad \sigma<1+b+\beta,\\ (\rho(x))^{-1}K(\rho(x))\left(\int_{\rho(x)}^{\eta}\frac{K(t)}{t}dt\right)^{\frac{\beta}{1-\beta}} & \text{if} \quad \sigma = 1+b+\beta,\\ (\rho(x))^{\frac{-\sigma+b+2\beta}{1-\beta}}(K(\rho(x)))^{\frac{1}{1-\beta}} & \text{if} \quad 1+b+\beta<\sigma<2+b,\\ (\rho(x))^{-2}K(\rho(x))\left(\int_0^{\rho(x)}\frac{K(t)}{t}dt\right)^{\frac{\beta}{1-\beta}} & \text{if} \quad \sigma = 2+b.\end{cases} \tag{14}$$

Besides, a simple computation shows that for $x \in D$, $|x|^{-\tau}L_2(|x|) = g(|x|)$, where

$$g(|x|) := \begin{cases} |x|^{-\gamma-(\beta+b)(n-2)}\ L(|x|) & \text{if} \quad \gamma > n-(\beta+b)(n-2), \\ |x|^{-n}\ L(|x|)\left(\int_1^{1+|x|}\frac{L(t)}{t}dt\right)^{\frac{\beta}{1-\beta}} & \text{if} \quad \gamma = n-(\beta+b)(n-2), \\ |x|^{\frac{-\gamma-b(n-2)+2\beta}{1-\beta}}\ (L(|x|))^{\frac{1}{1-\beta}} & \text{if} \quad 2-b(n-2) < \gamma < n-(\beta+b)(n-2), \\ |x|^{-2}\ L(|x|)\left(\int_{|x|}^{\infty}\frac{L(t)}{t}dt\right)^{\frac{\beta}{1-\beta}} & \text{if} \quad \gamma = 2-b(n-2). \end{cases} \tag{15}$$

So, from (14) and (15), one can see that $v \le 2 \le \tau$, $L_1 \in \mathcal{K}_0$, $L_2 \in \mathcal{K}_\infty$ satisfies (5). It follows that the function k satisfies the hypothesis (C).

Case 2 Assume (H), (A_1) and (B_2). We define the functions θ and ψ on D by

$$\theta(x) = \frac{\rho(x)}{|x|^{n-2}}\widetilde{N}(|x|),$$

where

$$\widetilde{N}(|x|) = \begin{cases} \left(\int_1^{1+|x|}\frac{N(t)}{t}dt\right)^{\frac{1}{1-\alpha}} & \text{if} \quad \gamma > n-(\beta+b)(n-2), \\ \left(\int_1^{1+|x|}\frac{(NL^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{\frac{1-\beta}{\omega}} & \text{if} \quad 2-b(n-2) < \gamma < n-(\beta+b)(n-2), \end{cases} \tag{16}$$

and

$$\psi(x) = \frac{(\rho(x))^{\delta_1}}{|x|^{\nu_1}}\widetilde{K}(\rho(x))\widetilde{L}(|x|),$$

where $\widetilde{K}$ is the function given by (12) and

$$\widetilde{L}(|x|) = \begin{cases} 1 & \text{if} \quad \gamma > n-(\beta+b)(n-2), \\ L^{\frac{1}{1-\beta}}(|x|)\left(\int_1^{1+|x|}\frac{(NL^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{\frac{b}{\omega}} & \text{if} \quad 2-b(n-2) < \gamma < n-(\beta+b)(n-2). \end{cases} \tag{17}$$

Due to Proposition 1.5, Lemmas 2.1 and 2.3, the functions θ and ψ are in $C_0(D)$.

Now, we consider two subcases.

Subcase 1 If $\gamma > n-(\beta+b)(n-2)$ then $\mu = n-(\alpha+a)(n-2)$.

From hypothesis (H), we have for $x \in D$,

$$h(x) = \frac{(\rho(x))^{-\lambda+\alpha+a\delta_1}}{|x|^n}(M\widetilde{K}^a)(\rho(x))N(|x|)\left(\int_1^{1+|x|}\frac{N(t)}{t}dt\right)^{\frac{\alpha}{1-\alpha}}$$

$$:= (\rho(x))^{-\nu}L_1((\rho(x)))|x|^{-\tau}\ L_2(|x|).$$

We note that $\nu = \lambda - \alpha - a\delta_1 < 1$, $\tau = n$ and the functions L_1 and L_2 are respectively in $\mathcal{K}_0$ and $\mathcal{K}_\infty$ satisfying (5). Hence the function h fulfills the condition (C).

Now, by (H) we have

$$k(x) = \frac{(\rho(x))^{-\sigma+b+\beta\delta_1}}{|x|^{\gamma+(\beta+b)(n-2)}}(K\widetilde{K}^{\beta})(\rho(x))L(|x|)\left(\int_1^{1+|x|}\frac{N(t)}{t}dt\right)^{\frac{b}{1-\alpha}}$$

$$:= (\rho(x))^{-\nu}L_1(\rho(x))|x|^{-\tau}L_2(|x|),$$

$$= f(\rho(x))|x|^{-\tau}L_2(|x|),$$

where f is the function defined by (14). So, we have $\nu \leq 2$ and $L_1 \in \mathcal{K}_0$ such that $\int_0^{\eta} t^{1-\nu}L_1(t)dt < \infty$. Moreover, since $\tau > n$ and $L_2 \in \mathcal{K}_\infty$, Lemma 2.2 gives that $\int_1^{\infty} t^{1-\nu}L_2(t)dt < \infty$. Thus, k satisfies (C).

Subcase 2 If $2 - b(n-2) < \gamma < n - (\beta + b)(n-2)$ then $\mu = n - \alpha(n-2) - a\,\frac{\gamma-2+b(n-2)}{1-\beta}$.

From hypothesis (H), we have

$$h(x) = \frac{(\rho(x))^{-\lambda+\alpha+a\delta_1}}{|x|^n}(M\widetilde{K}^{a})(\rho(x))(NL^{\frac{a}{1-\beta}})(|x|)\left(\int_1^{1+|x|}\frac{(NL^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{-1+\frac{1-\beta}{\omega}}$$

$$:= (\rho(x))^{-\nu}L_1(\rho(x))|x|^{-\tau}L_2(|x|).$$

We have $\nu = \lambda - \alpha - r\delta_1 < 1$, $\tau = n$ and $L_1 \in \mathcal{K}_0$, $L_2 \in \mathcal{K}_\infty$ satisfying (5).

Consequently, h satisfies the assumption (C).

On the other hand, by (H), we have

$$k(x) = \frac{(\rho(x))^{-\sigma+b+\beta\delta_1}}{|x|^{\frac{\gamma+b(n-2)-2\beta}{1-\beta}}}(K\widetilde{K}^{\beta})(\rho(x))\left(\int_1^{1+|x|}\frac{(NL^{\frac{r}{1-\beta}})(t)}{t}dt\right)^{\frac{b}{\omega}}L^{\frac{1}{1-\beta}}(|x|)$$

$$= (\rho(x))^{-\nu}L_1(\rho(x))|x|^{-\tau}L_2(|x|)$$

$$= f(\rho(x))|x|^{-\tau}L_2(|x|),$$

where f is the function defined by (14). Hence, we get that $\nu \leq 2$ and $L_1 \in \mathcal{K}_0$ such that $\int_0^{\eta} t^{1-\nu}L_1(t)dt < \infty$. Besides, since $\tau \in (2, n)$ and $L_2 \in \mathcal{K}_\infty$, we obtain by Lemma 2.2 that $\int_1^{\infty} t^{1-\tau}L_2(t)dt < \infty$. We deduce that the function k fulfills the condition (C).

Case 3 Assume (H), (A_1) and (B_3).

We define the functions θ and ψ on D by

$$\theta(x) = \frac{\rho(x)}{|x|^{\nu_2}}(N^{\frac{1-\beta}{\omega}}L^{\frac{a}{\omega}})(|x|)$$

and

$$\psi(x)=\frac{(\rho(x))^{\delta_1}}{|x|^{\nu_3}}\widetilde{K}(\rho(x))(N^{\frac{b}{\omega}}L^{\frac{1-\alpha}{\omega}})(|x|),$$

where $\widetilde{K}$ is the function given by (12).

From Proposition 1.5, Lemmas 2.1 and 2.3, we deduce that the functions θ and ψ are in $C_0(D)$.

Using (H), we have for $x \in D$,

$$h(x)=(\rho(x))^{-\lambda+\alpha+a\delta_1}M(\rho(x))|x|^{-\mu-\alpha\nu_2-a\nu_3}(N^{\frac{1-\beta}{\omega}}L^{\frac{a}{\omega}})(|x|).$$

Using (A_1) and (H), we have $\lambda-\alpha-a\delta_1<1$ and $M\in\mathcal{K}_0$. Due to Lemma 2.2, we have $\int_0^\eta t^{1-\lambda+\alpha+a\delta_1}M(t)dt<\infty$. From (B_3) and Remark 1.7, we get that $2<\mu+\alpha\nu_2+a\nu_3<n$ and since $N^{\frac{1-\beta}{\omega}}L^{\frac{a}{\omega}}\in\mathcal{K}_\infty$, Lemma 2.2 implies that $\int_1^\infty t^{1-\mu-\alpha\nu_2-a\nu_3}(N^{\frac{1-\beta}{\omega}}L^{\frac{a}{\omega}})(t)dt<\infty$.

Therefore, the function h satisfies (C).

Now, by hypothesis (H), we have for $x \in D$,

$$k(x)=(\rho(x))^{-\sigma+b+\beta\delta_1}(K\widetilde{K}^{\beta})(\rho(x))|x|^{-\gamma-b\nu_2-\beta\nu_3}(L^{\frac{1-\alpha}{\omega}}N^{\frac{b}{\omega}})(|x|).$$

Obviously, we have for $x \in D$,

$$(\rho(x))^{-\sigma+b+\beta\delta_1}(K\widetilde{K}^{\beta})(\rho(x))=(\rho(x))^{-\nu}L_1(\rho(x))=f(\rho(x)),$$

where f is the function defined by (14). So, we have $\nu\le 2$ and $L_1\in\mathcal{K}_0$ satisfying the condition of integrability $\int_0^\eta t^{1-\nu}L_1(t)dt<\infty$.

The function $L^{\frac{1-\alpha}{\omega}}N^{\frac{b}{\omega}}$ is in $\mathcal{K}_\infty$. Since by Remark 1.7 and hypothesis (B_3), we have $\gamma+b\nu_2+\beta\nu_3\in(2,n)$, then Lemma 2.2 implies that $\int_1^\infty t^{1-\gamma-b\nu_2-\beta\nu_3}(N^{\frac{b}{\omega}}L^{\frac{1-\alpha}{\omega}})(t)dt$ converges.

We deduce that k fulfills the hypothesis (C).

Case 4 Assume (H), (A_1) and (B_4).

We define the functions θ and ψ on D by

$$\theta(x)=\rho(x)\begin{cases}\left(\int_{|x|}^\infty\frac{N(t)}{t}dt\right)^{\frac{1}{1-\alpha}} & \text{if}\quad \gamma>n-\beta(n-2),\\ \left(\int_{|x|}^\infty\frac{(NL^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{\frac{1-\beta}{\omega}} & \text{if}\quad 2<\gamma<n-\beta(n-2),\end{cases}$$

and

$$\psi(x)=\frac{(\rho(x))^{\delta_1}}{|x|^{\nu_4}}\widetilde{K}(\rho(x))\begin{cases}1 & \text{if}\quad \gamma>n-\beta(n-2),\\ L^{\frac{1}{1-\beta}}(|x|)\left(\int_{|x|}^\infty\frac{(NL^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{\frac{b}{\omega}} & \text{if}\quad 2<\gamma<n-\beta(n-2),\end{cases}$$

where $\widetilde{K}$ is the function defined by (12).

By hypotheses (H), (B_4), Proposition 1.5 and Lemmas 2.1 and 2.3, we have θ and ψ are in $C_0(D)$.

Now, we consider two subcases.

Subcase 1 If $\gamma > n - \beta(n-2)$ then $\mu = 2 - a(n-2)$.

Using (H), we have for $x \in D$,

$$h(x) = (\rho(x))^{-\lambda+\alpha+a\delta_1}(M\widetilde{K}^a)(\rho(x))|x|^{-2}\ N(|x|)\left(\int_{|x|}^{\infty}\frac{N(t)}{t}dt\right)^{\frac{\alpha}{1-\alpha}}$$

$$:= (\rho(x))^{-\nu}L_1(\rho(x))|x|^{-\tau}\ L_2(|x|).$$

We remark that $\nu = \lambda - \alpha - a\delta_1 < 1$, $\tau = 2$. From Lemmas 2.1, 2.2 and 2.3 we have $L_1 \in \mathcal{K}_0$ such that $\int_0^{\eta} t^{1-\nu}L_1(t)dt < \infty$. Using (B_4) and applying Lemmas 2.1 and 2.3, the function $L_2 \in \mathcal{K}_\infty$ and we have

$$\int_1^{\infty} t^{1-\tau}L_2(t)dt = \int_1^{\infty}\frac{L_2(t)}{t}dt$$

$$= \int_1^{\infty}\frac{N(t)}{t}\left(\int_t^{\infty}\frac{N(\xi)}{\xi}d\xi\right)^{\frac{\alpha}{1-\alpha}}dt$$

$$= (1-\alpha)\left(\int_1^{\infty}\frac{N(\xi)}{\xi}d\xi\right)^{\frac{1}{1-\alpha}} < \infty.$$

Hence, the function h satisfies (C).

Now, by (H) we have

$$k(x) = (\rho(x))^{-\sigma+b+\beta\delta_1}(K\widetilde{K}^{\beta})(\rho(x))|x|^{-\gamma-\beta(n-2)}\left(\int_{1+|x|}^{\infty}\frac{N(t)}{t}dt\right)^{\frac{b}{1-\alpha}}L(|x|)$$

$$:= (\rho(x))^{-\nu}L_1(\rho(x))|x|^{-\tau}\ L_2(L(|x|)),$$

$$= f(\rho(x))|x|^{-\tau}\ L_2(L(|x|)),$$

where f is the function defined by (14). Then, we have $\nu \le 2$ and $L_1 \in \mathcal{K}_0$ such that $\int_0^{\eta} t^{1-\nu}L_1(t)dt < \infty$. Moreover, since $\tau = \gamma + \beta(n-2) > n > 2$ and $L_2 \in \mathcal{K}_\infty$, we deduce from Lemma 2.2 that $\int_1^{\infty} t^{1-\tau}L_2(t)dt < \infty$.

We deduce that k fulfills the condition (C).

Subcase 2 If $2 < \gamma < n - \beta(n-2)$, then $\mu = 2 - a\frac{\gamma-2}{1-\beta}$.

From hypothesis (H), we have

$$h(x) = (\rho(x))^{-\lambda+\alpha+a\delta_1}(M\widetilde{K}^a)(\rho(x))|x|^{-2}\ (NL^{\frac{a}{1-\beta}})(|x|)\left(\int_{|x|}^{\infty}\frac{(NL^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{-1+\frac{1-\beta}{\omega}}$$

$$:= (\rho(x))^{-\nu}L_1(\rho(x))|x|^{-\tau}\ L_2(|x|).$$

We have $\nu = \lambda - \alpha - a\delta_1 < 1$, $\tau = 2$. Using (B_4) and Lemmas 2.1, 2.2 and 2.3, we can easily see that L_1, L_2 are respectively in $\mathcal{K}_0$ and $\mathcal{K}_\infty$ such that (5) is satisfied.

So, h fulfills the hypothesis (C).

On the other hand, by (H), we have

$$k(x) = (\rho(x))^{-\sigma+b+\beta\delta_1}(K\widetilde{K}^{\beta})(\rho(x))\,|\,x\,|^{\frac{-\gamma+2\beta}{1-\beta}}\left(\int_{|x|}^{\infty}\frac{(NL^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{\frac{b}{\omega}}L^{\frac{1}{1-\beta}}(|\,x\,|)$$

$$:= (\rho(x))^{-\nu}L_1(\rho(x))\,|\,x\,|^{-\tau}\;L_2(|\,x\,|).$$

By simple calculus, we have

$$(\rho(x))^{-\nu}L_1(\rho(x)) = f(\rho(x)),$$

where f is the function defined by (14). So, we obtain that $\nu \le 2$ and $L_1 \in \mathcal{K}_0$ such that $\int_0^{\eta} t^{1-\nu}L_1(t)dt < \infty$. Besides, taking into account that $\tau = \frac{\gamma-2\beta}{1-\beta} \in (2, n)$ and $L_2 \in \mathcal{K}_\infty$, Lemma 2.2 implies that $\int_1^{\infty} t^{1-\tau}L_2(t)dt < \infty$. Hence, k satisfies the condition (C).

Case 5 Assume (H), (A_2) and (B_1).

We define the functions θ and ψ on D by

$$\theta(x) = \frac{\rho(x)}{|\,x\,|^{n-2}}\widetilde{M}(\rho(x)),$$

where

$$\widetilde{M}(\rho(x)) = \begin{cases} \left(\int_{\rho(x)}^{\eta}\frac{M(t)}{t}dt\right)^{\frac{1}{1-\alpha}} & \text{if} \quad \sigma < 1+\beta+b, \\ \left(\int_{\rho(x)}^{\eta}\frac{(MK^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{\frac{1-\beta}{\omega}} & \text{if} \quad 1+\beta+b < \sigma < 2+b, \end{cases} \tag{18}$$

and

$$\psi(x) = \frac{(\rho(x))^{\delta_1}}{|\,x\,|^{\nu_1}}\widetilde{K}(\rho(x))\widetilde{L}(|\,x\,|),$$

where $\widetilde{L}$ is the function given by (13) and

$$\widetilde{K}(\rho(x)) = \begin{cases} 1 & \text{if} \quad \sigma < 1+b+\beta, \\ K^{\frac{1}{1-\beta}}(\rho(x))\left(\int_{\rho(x)}^{\eta}\frac{(MK^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{\frac{b}{\omega}} & \text{if} \quad 1+b+\beta < \sigma < 2+b. \end{cases} \tag{19}$$

Using Proposition 1.5 and Lemmas 2.1 and 2.3, we get that the functions θ and ψ are in $C_0(D)$.

Now, we consider two subcases.

Subcase 1 If $\sigma < 1 + b + \beta$ then $\lambda = 1 + \alpha + a$.

Using (H), we have for $x \in D$,

$$h(x) = \frac{(\rho(x))^{-1}}{|x|^{\mu+\alpha(n-2)+av_1}} M(\rho(x)) \left(\int_{\rho(x)}^{\eta} \frac{M(t)}{t} dt \right)^{\frac{\alpha}{1-\alpha}} (N\widetilde{L}^a)(|x|)$$

$$:= (\rho(x))^{-\nu} L_1((\rho(x))) |x|^{-\tau} L_2(|x|).$$

It is clear that $\nu = 1$, $\tau = \mu + \alpha(n-2) + av_1 > n$, $L_1 \in \mathcal{K}_0$ and $L_2 \in \mathcal{K}_\infty$ such that $\int_0^\eta t^{1-\nu} L_1(t)dt < \infty$ and $\int_1^\infty t^{1-\tau} L_2(t)dt < \infty$. Hence, h satisfies (C).

Now, by (H) we have

$$k(x) = \frac{(\rho(x))^{-\sigma+b+\beta}}{|x|^{\gamma+b(n-2)+\beta v_1}} K(\rho(x)) \left(\int_{\rho(x)}^{\eta} \frac{M(t)}{t} dt \right)^{\frac{b}{1-\alpha}} (L\widetilde{L}^\beta)(|x|)$$

$$:= (\rho(x))^{-\nu} L_1(\rho(x)) |x|^{-\tau} L_2(|x|).$$

Since $\nu = \sigma - b - \beta < 1$ and $L_1 \in \mathcal{K}_0$, Lemma 2.2 implies that $\int_0^\eta t^{1-\nu} L_1(t)dt < \infty$.

On the other hand, by a simple computation, we get for $x \in D$,

$$|x|^{-\tau} L_2(|x|) = g(|x|),$$

where g is the function defined by (15). So, one have $\tau \geq 2$ and $L_2 \in \mathcal{K}_\infty$ such that $\int_1^\infty t^{1-\tau} L_2(t)dt < \infty$.

Then, we deduce that k satisfies the assumption (C).

Subcase 2 If $1 + b + \beta < \sigma < 2 + b$ then we have $\lambda = 1 + \alpha + a\frac{2-\sigma+b}{1-\beta}$.

From hypothesis (H), we have

$$h(x) = \frac{(\rho(x))^{-1}}{|x|^{\mu+\alpha(n-2)+av_1}} (MK^{\frac{a}{1-\beta}})(\rho(x)) \left(\int_{\rho(x)}^{\eta} \frac{(MK^{\frac{a}{1-\beta}})(t)}{t} dt \right)^{-1+\frac{1-\beta}{\omega}} (N\widetilde{L}^a)(|x|)$$

$$:= (\rho(x))^{-\nu} L_1(\rho(x)) |x|^{-\tau} L_2(|x|).$$

We have $\nu = 1$, $\tau = \mu + \alpha(n-2) + av_1 > n$ and $L_1 \in \mathcal{K}_0$, $L_2 \in \mathcal{K}_\infty$ satisfying $\int_0^\eta t^{1-\nu} L_1(t)dt < \infty$ and $\int_1^\infty t^{1-\tau} L_2(t)dt < \infty$. Hence, the function h fulfills (C).

On the other hand, by (H), we have

$$k(x) = \frac{(\rho(x))^{\frac{-\sigma+b+2\beta}{1-\beta}}}{|x|^{\gamma+b(n-2)+\beta v_1}} K^{\frac{1}{1-\beta}}(\rho(x)) \left(\int_{(\rho(x))}^{\eta} \frac{(MK^{\frac{a}{1-\beta}})(t)}{t} dt \right)^{\frac{b}{\omega}} (L\widetilde{L}^\beta)(|x|)$$

$$:= (\rho(x))^{-\nu} L_1(\rho(x)) |x|^{-\tau} L_2(|x|).$$

Since $L_1 \in \mathcal{K}_0$ and $\nu = \frac{\sigma-b-2\beta}{1-\beta} \in (1, 2)$, Lemma 2.2 gives that $\int_0^\eta t^{1-\nu} L_1(t)dt < \infty$.

Besides, by simple calculus, we have

$$|x|^{-\tau} L_2(|x|) = g(|x|),$$

where g is the function defined by (15). So, we have $\tau \geq 2$ and $L_2 \in \mathcal{K}_\infty$ such that $\int_1^\infty t^{1-\tau} L_2(t)dt < \infty$. We conclude that k verifies (C).

Case 6 Assume (H), (A_2) and (B_2).

We define the functions θ and ψ on D by

$$\theta(x) = \frac{\rho(x)}{|x|^{n-2}} \widetilde{M}(\rho(x))\widetilde{N}(|x|),$$

where $\widetilde{M}$ and $\widetilde{N}$ are the functions given respectively by (18) and (16) and

$$\psi(x) = \frac{(\rho(x))^{\delta_1}}{|x|^{\nu_1}} \widetilde{K}(\rho(x))\widetilde{L}(|x|),$$

where $\widetilde{K}$ and $\widetilde{L}$ are the functions given respectively by (19) and (17). Using Proposition 1.5, Lemmas 2.1 and 2.3, we get that the functions θ and ψ are in $C_0(D)$.

Now, we consider four subcases.

Subcase 1 If $\sigma < 1 + b + \beta$ and $\gamma > n - (\beta + b)(n - 2)$ then $\lambda = 1 + \alpha + a$ and $\mu = n - (\alpha + a)(n - 2)$.

From hypothesis (H), we have

$$\begin{aligned} h(x) &= \frac{(\rho(x))^{-1}}{|x|^n} M(\rho(x)) \left(\int_{\rho(x)}^{\eta} \frac{M(t)}{t} dt \right)^{\frac{\alpha}{1-\alpha}} N(|x|) \left(\int_1^{1+|x|} \frac{N(t)}{t} dt \right)^{\frac{\alpha}{1-\alpha}} \\ &:= (\rho(x))^{-\nu} L_1(\rho(x)) |x|^{-\tau} L_2(|x|). \end{aligned}$$

We note that $\nu = 1$ and $\tau = n$. Applying Lemmas 2.1, 2.3 and 2.2, we can easily see that $L_1 \in \mathcal{K}_0$, $L_2 \in \mathcal{K}_\infty$ such that (5) is fulfilled. So, the assumption (C) is well satisfied by the function h.

Now, by (H) we have

$$\begin{aligned} k(x) &= \frac{(\rho(x))^{-\sigma+b+\beta}}{|x|^{\gamma+(\beta+b)(n-2)}} K(\rho(x)) \left(\int_{\rho(x)}^{\eta} \frac{M(t)}{t} dt \right)^{\frac{b}{1-\alpha}} L(|x|) \left(\int_1^{1+|x|} \frac{N(t)}{t} dt \right)^{\frac{b}{1-\alpha}} \\ &:= (\rho(x))^{-\nu} L_1(\rho(x)) |x|^{-\tau} L_2(|x|). \end{aligned}$$

It is clear that $\nu = \sigma - b - \beta < 1$ and $\tau = \gamma + (\beta + b)(n - 2) > n$. Due to Lemmas 2.1, 2.3 and 2.2, we deduce that $L_1 \in \mathcal{K}_0$, $L_2 \in \mathcal{K}_\infty$ satisfying (5). Hence, k verifies the condition (C).

Subcase 2 If $\sigma < 1 + b + \beta$ and $2 - b(n - 2) < \gamma < n - (\beta + b)(n - 2)$ then $\lambda = 1 + \alpha + a$ and $\mu = n - \alpha(n - 2) - a\,\frac{\gamma-2+b(n-2)}{1-\beta}$.

From (H), we have

$$h(x) = \frac{(\rho(x))^{-1}}{|x|^n} M(\rho(x)) \left(\int_{\rho(x)}^{\eta} \frac{M(t)}{t} dt \right)^{\frac{\alpha}{1-\alpha}}$$

$$\times (NL^{\frac{a}{1-\beta}})(|x|) \left(\int_1^{1+|x|} \frac{(NL^{\frac{a}{1-\beta}})(t)}{t} dt \right)^{-1+\frac{1-\beta}{\omega}}$$

$$:= (\rho(x))^{-\nu} L_1(\rho(x))(|x|)^{-\tau} L_2(|x|).$$

Since $\nu = 1$, $\tau = n$ and the functions L_1 and L_2 are respectively in $\mathcal{K}_0$ and $\mathcal{K}_\infty$, then Lemma 2.2 implies that (5) is reached. So, we deduce that h satisfies (C).

Moreover, we have

$$k(x) = \frac{(\rho(x))^{-\sigma+b+\beta}}{|x|^{\frac{\gamma+b(n-2)-2\beta}{1-\beta}}} K(\rho(x)) \left(\int_{\rho(x)}^{\eta} \frac{M(t)}{t} dt \right)^{\frac{b}{1-\alpha}} (L^{\frac{1}{1-\beta}})(|x|)$$

$$\times \left(\int_1^{1+|x|} \frac{(LN^{\frac{b}{1-\alpha}})(t)}{t} dt \right)^{\frac{b}{\omega}}$$

$$:= (\rho(x))^{-\nu} L_1(\rho(x)) |x|^{-\tau} L_2(|x|).$$

It is obvious to see that the function k verifies the hypothesis (C) with $\nu = \sigma - b - \beta < 1$ and $\tau = \frac{\gamma+b(n-2)-2\beta}{1-\beta} \in (2, n)$.

Subcase 3 If $1 + b + \beta < \sigma < 2 + b$ and $\gamma > n - (\beta + b)(n - 2)$ then $\lambda = 1 + \alpha + a\frac{2-\sigma+b}{1-\beta}$ and $\mu = n - (\alpha + a)(n - 2)$.

Using (H), we get

$$h(x) = \frac{(\rho(x))^{-1}}{|x|^n} (MK^{\frac{a}{1-\beta}})(\rho(x)) \left(\int_{\rho(x)}^{\eta} \frac{(MK^{\frac{a}{1-\beta}})(t)}{t} dt \right)^{-1+\frac{1-\beta}{\omega}}$$

$$\times N(|x|) \left(\int_1^{1+|x|} \frac{N(t)}{t} dt \right)^{\frac{\alpha}{1-\alpha}}$$

$$:= (\rho(x))^{-\nu} L_1(\rho(x)) |x|^{-\tau} L_2(|x|).$$

Since $L_1 \in \mathcal{K}_0$ and $L_2 \in \mathcal{K}_\infty$, then one can see that h satisfies (C) with $\nu = 1$ and $\tau = n$.

Besides, we have

$$k(x) = \frac{(\rho(x))^{\frac{-\sigma+b+2\beta}{1-\beta}}}{|x|^{\gamma+(\beta+b)(n-2)}} K^{\frac{1}{1-\beta}}(\rho(x)) \left(\int_{\rho(x)}^{\eta} \frac{(MK^{\frac{a}{1-\beta}})(t)}{t} dt \right)^{\frac{b}{\omega}} L(|x|) \left(\int_1^{1+|x|} \frac{N(t)}{t} dt \right)^{\frac{b}{1-\alpha}}$$

$$:= (\rho(x))^{-\nu} L_1(\rho(x)) |x|^{-\tau} L_2(|x|).$$

We note that $\nu = \frac{\sigma-b-2\beta}{1-\beta} \in (1, 2)$, $\tau > n$ and $L_1 \in \mathcal{K}_0$, $L_2 \in \mathcal{K}_\infty$ satisfy the condition (5). Hence, k fulfills the hypothesis (C).

Subcase 4 If $1 + b + \beta < \sigma < 2 + b$ and $2 - b(n - 2) < \gamma < n - (\beta + b)(n - 2)$ then $\lambda = 1 + \alpha + a\frac{2-\sigma+b}{1-\beta}$ and $\mu = n - \alpha(n - 2) - a\frac{\gamma-2+b(n-2)}{1-\beta}$.

We have on D,

$$h(x)=\frac{(\rho(x))^{-1}}{|x|^{n}}(MK^{\frac{a}{1-\beta}})(\rho(x))\left(\int_{\rho(x)}^{\eta}\frac{(MK^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{-1+\frac{1-\beta}{\omega}}$$

$$\times(NL^{\frac{a}{1-\beta}})(|x|)\left(\int_{1}^{1+|x|}\frac{(NL^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{-1+\frac{1-\beta}{\omega}}$$

$$:=(\rho(x))^{-\nu}L_1(\rho(x))|x|^{-\tau}L_2(|x|).$$

One can see that the function h verifies the hypothesis (C) with $\nu = 1$ and $\tau = n$.

On the other hand, we have

$$k(x)=\frac{(\rho(x))^{\frac{-\sigma+b+2\beta}{1-\beta}}}{|x|^{\frac{\gamma+b(n-2)-2\beta}{1-\beta}}}K^{\frac{1}{1-\beta}}(\rho(x))\left(\int_{\rho(x)}^{\eta}\frac{(MK^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{\frac{b}{\omega}}L^{\frac{1}{1-\beta}}(|x|)$$

$$\times\left(\int_{1}^{1+|x|}\frac{(NL^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{\frac{b}{\omega}}$$

$$:=(\rho(x))^{-\nu}L_1(\rho(x))|x|^{-\tau}L_2(|x|).$$

It is clear that k fulfills (C) with $\nu \in (1, 2)$ and $\tau \in (2, n)$.

Case 7 Assume (H), (A_2) and (B_3).

We define the functions θ and ψ on D by

$$\theta(x)=\frac{\rho(x)}{|x|^{\nu_2}}\widetilde{M}(\rho(x))(N^{\frac{1-\beta}{\omega}}L^{\frac{a}{\omega}})(|x|),$$

where $\widetilde{M}$ is the function given by (18) and

$$\psi(x)=\frac{(\rho(x))^{\delta_1}}{|x|^{\nu_3}}\widetilde{K}(\rho(x))(N^{\frac{b}{\omega}}L^{\frac{1-\alpha}{\omega}})(|x|),$$

where $\widetilde{K}$ is the function given by (19).

It is obvious from Proposition 1.5, Lemmas 2.1 and 2.3, that the functions θ and ψ are in $C_0(D)$.

Using hypotheses (H) and (A_2), we obtain

$$h(x)=\frac{(\rho(x))^{-1}}{|x|^{2+\nu_2}}(N^{\frac{1-\beta}{\omega}}L^{\frac{a}{\omega}})(|x|)$$

$$\times \begin{cases} M(\rho(x))\left(\int_{\rho(x)}^{\eta} \frac{M(t)}{t}dt\right)^{\frac{\alpha}{1-\alpha}} & \text{if } \sigma < 1+b+\beta, \\ (MK^{\frac{a}{1-\beta}})(\rho(x))\left(\int_{\rho(x)}^{\eta} \frac{(MK^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{-1+\frac{1-\beta}{\omega}} & \text{if } 1+b+\beta < \sigma < 2+b, \end{cases}$$

$$:= (\rho(x))^{-\nu} L_1(\rho(x)) |x|^{-\tau} L_2(|x|).$$

We notice that $\nu = 1$ and by hypothesis (B_3), we have $\tau = 2 + \nu_2 \in (2, n)$. Hence, in view of Lemmas 2.1, 2.3 and 2.2, the functions L_1, L_2 belong respectively to $\mathcal{K}_0$ and $\mathcal{K}_\infty$ and satisfy (5). We conclude that h verifies (C).

Moreover, we have on D,

$$k(x) = \frac{(\rho(x))^{-\sigma+b+\beta\delta_1}}{|x|^{2+\nu_3}}(N^{\frac{b}{\omega}} L^{\frac{1-\alpha}{\omega}})(|x|)$$

$$\times \begin{cases} K(\rho(x))\left(\int_{\rho(x)}^{\eta} \frac{M(t)}{t}dt\right)^{\frac{b}{1-\alpha}} & \text{if } \sigma < 1+b+\beta, \\ K^{\frac{1}{1-\beta}}(\rho(x))\left(\int_{\rho(x)}^{\eta} \frac{(MK^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{\frac{b}{\omega}} & \text{if } 1+b+\beta < \sigma < 2+b, \end{cases}$$

$$:= (\rho(x))^{-\nu} L_1(\rho(x)) |x|^{-\tau} L_2(|x|).$$

Remark that (C) is satisfied with $\nu = \sigma - b - \beta\delta_1 < 2$ and $\tau = 2 + \nu_3 \in (2, n)$.

Case 8 Assume (H), (A_2) and (B_4).

We define the functions θ and ψ on D by

$$\theta(x) = \rho(x)\widetilde{M}(\rho(x)) \begin{cases} \left(\int_{|x|}^{\infty} \frac{N(t)}{t}dt\right)^{\frac{1}{1-\alpha}} & \text{if } \quad \gamma > n - \beta(n-2), \\ \left(\int_{|x|}^{\infty} \frac{(NL^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{\frac{1-\beta}{\omega}} & \text{if } \quad 2 < \gamma < n - \beta(n-2), \end{cases}$$

where $\widetilde{M}$ is the function given by (18) and

$$\psi(x) = \frac{\rho(x)^{\delta_1}}{|x|^{\nu_4}}\widetilde{K}(\rho(x)) \begin{cases} 1 & \text{if } \quad \gamma > n - \beta(n-2), \\ L^{\frac{1}{1-\beta}}(|x|)\left(\int_{|x|}^{\infty} \frac{(NL^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{\frac{b}{\omega}} & \text{if } \quad 2 < \gamma < n - \beta(n-2), \end{cases}$$

where $\widetilde{K}$ is the function given by (19).

Using (B_4) and Proposition 1.5, Lemmas 2.1 and 2.3, we get that the functions θ and ψ are in $C_0(D)$.

Now, we consider four subcases.
Subcase 1 If $\sigma < 1 + b + \beta$ and $\gamma > n - \beta(n-2)$ then $\lambda = 1 + \alpha + a$ and $\mu = 2 - a(n-2)$.
From hypothesis (H), we have

$$h(x) = \frac{(\rho(x))^{-1}}{|x|^2} M(\rho(x)) \left(\int_{\rho(x)}^{\eta} \frac{M(t)}{t} dt \right)^{\frac{\alpha}{1-\alpha}} N(|x|) \left(\int_{|x|}^{\infty} \frac{N(t)}{t} dt \right)^{\frac{\alpha}{1-\alpha}}$$

$$:= (\rho(x))^{-\nu} L_1(\rho(x)) |x|^{-\tau} L_2(|x|).$$

Using (B_4) and Lemmas 2.1, 2.3 and 2.2, we can easily see that the function h fulfills (C) with $\nu = 1$ and $\tau = 2$.
Now, by (H) we have

$$k(x) = \frac{(\rho(x))^{-\sigma+b+\beta}}{|x|^{\gamma+\beta(n-2)}} K(\rho(x)) \left(\int_{\rho(x)}^{\eta} \frac{M(t)}{t} dt \right)^{\frac{b}{1-\alpha}} L(|x|) \left(\int_{|x|}^{\infty} \frac{N(t)}{t} dt \right)^{\frac{b}{1-\alpha}}$$

$$:= (\rho(x))^{-\nu} L_1(\rho(x)) |x|^{-\tau} L_2(|x|).$$

It is clear that k satisfies (C) with $\nu = \sigma - b - \beta < 1$ and $\tau = \gamma + \beta(n-2) > n$.
Subcase 2 If $\sigma < 1 + b + \beta$ and $2 < \gamma < n - \beta(n-2)$ then $\lambda = 1 + \alpha + a$ and $\mu = 2 - a\frac{\gamma-2}{1-\beta}$.
From (H), we have

$$h(x) = \frac{(\rho(x))^{-1}}{|x|^2} M(\rho(x)) \left(\int_{\rho(x)}^{\eta} \frac{M(t)}{t} dt \right)^{\frac{\alpha}{1-\alpha}}$$

$$\times (NL^{\frac{a}{1-\beta}})(|x|) \left(\int_{|x|}^{\infty} \frac{(NL^{\frac{a}{1-\beta}})(t)}{t} dt \right)^{-1+\frac{1-\beta}{\omega}}$$

$$:= (\rho(x))^{-\nu} L_1(\rho(x)) |x|^{-\tau} L_2(|x|).$$

We notice that $\nu = 1$ and $\tau = 2$. From Lemmas 2.1, 2.3 and 2.2, we have $L_1 \in \mathcal{K}_0$ such that $\int_0^{\eta} t^{1-\nu} L_1(t) dt < \infty$. Using (B_4) and Lemmas 2.1 and 2.3, the function L_2 belonging to $\mathcal{K}_\infty$ and we have

$$\int_1^{\infty} t^{1-\tau} L_2(t) dt = \int_1^{\infty} \frac{L_2(t)}{t} dt$$

$$= \int_1^{\infty} \frac{(NL^{\frac{a}{1-\beta}})(t)}{t} \left(\int_1^{\infty} \frac{(NL^{\frac{a}{1-\beta}})(\xi)}{\xi} d\xi \right)^{-1+\frac{1-\beta}{\omega}} dt$$

$$= \frac{\omega}{1-\beta} \left(\int_1^{\infty} \frac{(NL^{\frac{a}{1-\beta}})(\xi)}{\xi} d\xi \right)^{\frac{1-\beta}{\omega}} < \infty.$$

It follows that h verifies (C).

Moreover, we have

$$k(x) = \frac{(\rho(x))^{-\sigma+b+\beta}}{|x|^{\frac{\gamma-2\beta}{1-\beta}}} K(\rho(x)) \left(\int_{\rho(x)}^{\eta} \frac{M(t)}{t} dt \right)^{\frac{b}{1-\alpha}} L^{\frac{1}{1-\beta}}(|x|)$$

$$\times \left(\int_{|x|}^{\infty} \frac{(NL^{\frac{a}{1-\beta}})(t)}{t} dt \right)^{\frac{b}{\omega}}$$

$$:= (\rho(x))^{-\nu} L_1(\rho(x)) |x|^{-\tau} L_2(|x|).$$

Using (B_4) and Lemmas 2.1, 2.3 and 2.2, we deduce that the function k satisfies (C) with $\nu = \sigma - b - \beta < 1$ and $\tau = \frac{\gamma-2\beta}{1-\beta} \in (2, n)$.

Subcase 3 If $1 + b + \beta < \sigma < 2 + b$ and $\gamma > n - \beta(n - 2)$ then $\lambda = 1 + \alpha + a\frac{2-\sigma+b}{1-\beta}$ and $\mu = 2 - a(n - 2)$.

Using (H), we get

$$h(x) = \frac{(\rho(x))^{-1}}{|x|^2} (MK^{\frac{a}{1-\beta}})(\rho(x)) \left(\int_{\rho(x)}^{\eta} \frac{(MK^{\frac{a}{1-\beta}})(t)}{t} dt \right)^{-1+\frac{1-\beta}{\omega}}$$

$$\times N(|x|) \left(\int_{|x|}^{\infty} \frac{N(t)}{t} dt \right)^{\frac{\alpha}{1-\alpha}}$$

$$:= (\rho(x))^{-\nu} L_1(\rho(x)) |x|^{-\tau} L_2(|x|).$$

So, one can see that $\nu = 1$ and $\tau = 2$. Furthermore, by (B_4) and Lemmas 2.1, 2.3 and 2.2, we deduce that $L_1 \in \mathcal{K}_0$, $L_2 \in \mathcal{K}_\infty$ satisfies the condition (5).

Consequently, h fulfills (C).

On the other hand, we have

$$k(x) = \frac{(\rho(x))^{\frac{-\sigma+b+2\beta}{1-\beta}}}{|x|^{\gamma+\beta(n-2)}} K^{\frac{1}{1-\beta}}(\rho(x)) \left(\int_{\rho(x)}^{\eta} \frac{(MK^{\frac{a}{1-\beta}})(t)}{t} dt \right)^{\frac{b}{\omega}}$$

$$\times L(|x|) \left(\int_{|x|}^{\infty} \frac{N(t)}{t} dt \right)^{\frac{b}{1-\alpha}}$$

$$:= (\rho(x))^{-\nu} L_1(\rho(x)) |x|^{-\tau} L_2(|x|).$$

We note that k satisfies (C) with $\nu = \frac{\sigma-b-2\beta}{1-\beta} \in (1, 2)$ and $\tau = \gamma + \beta(n - 2) > n$.

Subcase 4 If $1 + b + \beta < \sigma < 2 + b$ and $2 < \gamma < n - \beta(n - 2)$ then $\lambda = 1 + \alpha + a\frac{2-\sigma+b}{1-\beta}$ and $\mu = 2 - a\frac{\gamma-2}{1-\beta}$.

We have on D,

$$h(x)=\frac{(\rho(x))^{-1}}{|x|^2}(MK^{\frac{a}{1-\beta}})(\rho(x))\left(\int_{\rho(x)}^{\eta}\frac{(MK^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{-1+\frac{1-\beta}{\omega}}$$

$$\times(NL^{\frac{a}{1-\beta}})(|x|)\left(\int_{|x|}^{\infty}\frac{(NL^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{-1+\frac{1-\beta}{\omega}}$$

$$:=(\rho(x))^{-\nu}L_1(\rho(x))|x|^{-\tau}\ L_2(|x|).$$

Using (B_4) and Lemmas 2.1, 2.3 and 2.2, we can easily see that the function h verifies (C) with $\nu = 1$ and $\tau = 2$. On the other hand, we have

$$k(x)=\frac{(\rho(x))^{\frac{-\sigma+b+2\beta}{1-\beta}}}{|x|^{\frac{\gamma-2\beta}{1-\beta}}}K^{\frac{1}{1-\beta}}(\rho(x))\left(\int_{\rho(x)}^{\eta}\frac{(MK^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{\frac{b}{\omega}}$$

$$\times L^{\frac{1}{1-\beta}}(|x|)\left(\int_{|x|}^{\infty}\frac{(NL^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{\frac{b}{\omega}}$$

$$:=(\rho(x))^{-\nu}L_1(\rho(x))|x|^{-\tau}\ L_2(|x|).$$

Since $\nu \in (1, 2)$ and $\tau \in (2, n)$, then (B_4), Lemmas 2.1, 2.3 and 2.2 imply that k fulfills the hypothesis (C).

Case 9 Assume (H), (A_3) and (B_1).

We define the functions θ and ψ on D by

$$\theta(x)=\frac{(\rho(x))^{\delta_2}}{|x|^{n-2}}(M^{\frac{1-\beta}{\omega}}K^{\frac{a}{\omega}})(\rho(x))$$

and

$$\psi(x)=\frac{(\rho(x))^{\delta_3}}{|x|^{\nu_1}}(M^{\frac{b}{\omega}}K^{\frac{1-\alpha}{\omega}})(\rho(x))\tilde{L}(|x|),$$

where $\tilde{L}$ is the function defined by (13).

By using hypothesis (H), Proposition 1.5 and Lemmas 2.1 and 2.3, it is obvious to see that the functions θ and ψ belong to $C_0(D)$.

From hypothesis (H) and Remark 1.7, we have

$$h(x)=\frac{(\rho(x))^{\delta_2-2}}{|x|^{\mu+\alpha(n-2)+a\nu_1}}(M^{\frac{1-\beta}{\omega}}K^{\frac{a}{\omega}})(\rho(x))(N\tilde{L}^{a})(|x|).$$

Using (A_3) and (B_1), we deduce that h satisfies the assumption (C) with $\nu = 2 - \delta_2 \in (1, 2)$ and $\tau = \mu + \alpha(n - 2) + a\nu_1 > n$.

On the other hand, by hypothesis (H) and Remark 1.7, we obtain that

$$k(x)=\frac{(\rho(x))^{\delta_3-2}}{|x|^{\gamma+b(n-2)+\beta\nu_1}}(K^{\frac{1-\alpha}{\omega}}M^{\frac{b}{\omega}})(\rho(x))(L\widetilde{L}^{\beta})(|x|).$$

By hypothesis (A_3), we have $2-\delta_3\in(1,2)$. Therefore, Lemmas 2.1 and 2.2 imply that $K^{\frac{1-\alpha}{\omega}}M^{\frac{b}{\omega}}\in\mathcal{K}_0$ satisfying $\int_0^{\eta}t^{-1+\delta_3}K^{\frac{1-\alpha}{\omega}}M^{\frac{b}{\omega}}(t)dt<\infty$.

Now, by simple calculus, we obtain that for $x\in D$,

$$|x|^{-\gamma-b(n-2)-\beta\nu_1}(L\widetilde{L}^{\beta})(|x|)=|x|^{-\tau}L_2(|x|)=g(|x|),$$

where g is the function defined by (15). So, we have $\tau\geq 2$ and $L_2\in\mathcal{K}_\infty$ satisfying the condition of integrability $\int_1^{\infty}t^{1-\tau}L_2(t)dt<\infty$. We deduce that k verifies the condition (C).

Case 10 Assume (H), (A_3) and (B_2).

We define the functions θ and ψ on D by

$$\theta(x)=\frac{(\rho(x))^{\delta_2}}{|x|^{n-2}}(M^{\frac{1-\beta}{\omega}}K^{\frac{a}{\omega}})(\rho(x))\widetilde{N}(|x|),$$

where $\widetilde{N}$ is the function defined by (16) and

$$\psi(x)=\frac{(\rho(x))^{\delta_3}}{|x|^{\nu_1}}(M^{\frac{b}{\omega}}K^{\frac{1-\alpha}{\omega}})(\rho(x))\widetilde{L}(|x|),$$

where

$$\widetilde{L}(|x|)=\begin{cases}1 & \text{if}\quad \gamma>n-(\beta+b)(n-2),\\ L^{\frac{1}{1-\beta}}(|x|)\left(\int_1^{1+|x|}\frac{(NL^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{\frac{b}{\omega}} & \text{if}\quad 2-b(n-2)<\gamma<n-(\beta+b)(n-2).\end{cases}$$

It is obvious from Proposition 1.5 and Lemmas 2.1 and 2.3 that the functions θ and ψ are in $C_0(D)$.

Using hypotheses (H) and Remark 1.7, we obtain

$$h(x)=\frac{(\rho(x))^{\delta_2-2}}{|x|^{n}}(M^{\frac{1-\beta}{\omega}}K^{\frac{a}{\omega}})(\rho(x))$$

$$\times\begin{cases}N(|x|)\left(\int_1^{1+|x|}\frac{N(t)}{t}dt\right)^{\frac{\alpha}{1-\alpha}} & \text{if}\quad \gamma>n-(\beta+b)(n-2),\\ (NL^{\frac{a}{1-\beta}})(|x|)\left(\int_1^{1+|x|}\frac{(NL^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{-1+\frac{1-\beta}{\omega}} & \text{if}\quad 2-b(n-2)<\gamma<n-(\beta+b)(n-2),\end{cases}$$

$$:=(\rho(x))^{-\nu}L_1(\rho(x))|x|^{-\tau}L_2(|x|).$$

We notice that h satisfies (C) with $\nu = 2 - \delta_2 \in (1, 2)$ and $\tau = n$.

Moreover, we have on D

$$k(x) = \frac{(\rho(x))^{\delta_3 - 2}}{|x|^{\gamma + b(n-2) + \beta\nu_1}} (M^{\frac{b}{\omega}} K^{\frac{1-\alpha}{\omega}})(\rho(x))$$

$$\times \begin{cases} L(|x|)\left(\int_1^{1+|x|} \frac{N(t)}{t} dt\right)^{\frac{b}{1-\alpha}} & \text{if } \quad \gamma > n - (\beta + b)(n-2), \\ L^{\frac{1}{1-\beta}}(|x|)\left(\int_1^{1+|x|} \frac{(NL^{\frac{a}{1-\beta}})(t)}{t} dt\right)^{\frac{b}{\omega}} & \text{if } \quad 2 - b(n-2) < \gamma < n - (\beta + b)(n-2), \end{cases}$$

$$:= (\rho(x))^{-\nu} L_1(\rho(x)) |x|^{-\tau} L_2(|x|).$$

We can easily see that k fulfills the condition (C) with $\nu \in (1, 2)$ and $\tau > 2$.

Case 11 Assume (H), (A_3) and (B_3).

We define the functions θ and ψ on D by

$$\theta(x) = \frac{(\rho(x))^{\delta_2}}{|x|^{\nu_2}} (M^{\frac{1-\beta}{\omega}} K^{\frac{a}{\omega}})(\rho(x))(N^{\frac{1-\beta}{\omega}} L^{\frac{a}{\omega}})(|x|)$$

and

$$\psi(x) = \frac{(\rho(x))^{\delta_3}}{|x|^{\nu_3}} (M^{\frac{b}{\omega}} K^{\frac{1-\alpha}{\omega}})(\rho(x))(N^{\frac{b}{\omega}} L^{\frac{1-\alpha}{\omega}})(|x|).$$

Due to Proposition 1.5 and Lemma 2.1, we get that θ and ψ are in $C_0(D)$.

Using hypothesis (H), we obtain

$$h(x) = (\rho(x))^{-\lambda + \alpha\delta_2 + a\delta_3} (M^{\frac{1-\beta}{\omega}} K^{\frac{a}{\omega}})(\rho(x)) |x|^{-\mu - \alpha\nu_2 - a\nu_3} (N^{\frac{1-\beta}{\omega}} L^{\frac{a}{\omega}})(|x|)$$

$$:= (\rho(x))^{-\nu} L_1((\rho(x))) |x|^{-\tau} L_2(|x|).$$

From (A_3), (B_3) and Remark 1.7, we have $\nu = \lambda - \alpha\delta_2 - a\delta_3 \in (1, 2)$ and $\tau = \mu + \alpha\nu_2 + a\nu_3 \in (2, n)$. Applying Lemmas 2.1 and 2.2, we deduce that h satisfies (C).

Moreover, we have

$$k(x) = \frac{(\rho(x))^{-\sigma + b\delta_2 + \beta\delta_3}}{|x|^{\gamma + b\nu_2 + \beta\nu_3}} (M^{\frac{b}{\omega}} K^{\frac{1-\alpha}{\omega}})(\rho(x))(N^{\frac{b}{\omega}} L^{\frac{1-\alpha}{\omega}})(|x|)$$

$$:= (\rho(x))^{-\nu} L_1(\rho(x)) |x|^{-\tau} L_2(|x|).$$

We notify that k verifies (C) with $\nu = \sigma - b\delta_2 - \beta\delta_3 \in (1, 2)$ and $\tau = \gamma + b\nu_2 + \beta\nu_3 \in (2, n)$.

Case 12 Assume (H), (A_3) and (B_4).

We define the functions θ and ψ on D by

$$\theta(x)=(\rho(x))^{\delta_2}(M^{\frac{1-\beta}{\omega}}K^{\frac{a}{\omega}})(\rho(x))\begin{cases}\left(\int_{|x|}^{\infty}\frac{N(t)}{t}dt\right)^{\frac{1}{1-\alpha}} & \text{if} \quad \gamma>n-\beta(n-2),\\ \left(\int_{|x|}^{\infty}\frac{(NL^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{\frac{1-\beta}{\omega}} & \text{if} \quad 2<\gamma<n-\beta(n-2),\end{cases}$$

and

$$\psi(x)=\frac{(\rho(x))^{\delta_3}}{|x|^{\nu_4}}(M^{\frac{b}{\omega}}K^{\frac{1-\alpha}{\omega}})(\rho(x))\begin{cases}1 & \text{if} \quad \gamma>n-\beta(n-2),\\ L^{\frac{1}{1-\beta}}(|x|)\left(\int_{|x|}^{\infty}\frac{(NL^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{\frac{b}{\omega}} & \text{if} \quad 2<\gamma<n-\beta(n-2).\end{cases}$$

It is obvious from (B_4), Proposition 1.5 and Lemmas 2.1 and 2.3 that the functions θ and ψ belong to $C_0(D)$.
Using hypothesis (H) and Remark 1.7, we obtain

$$h(x)=\frac{(\rho(x))^{\delta_2-2}}{|x|^2}(M^{\frac{1-\beta}{\omega}}K^{\frac{a}{\omega}})(\rho(x))$$

$$\times\begin{cases}N(|x|)\left(\int_{|x|}^{\infty}\frac{N(t)}{t}dt\right)^{\frac{\alpha}{1-\alpha}} & \text{if} \quad \gamma>n-\beta(n-2),\\ (NL^{\frac{a}{1-\beta}})(|x|)\left(\int_{|x|}^{\infty}\frac{(NL^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{-1+\frac{1-\beta}{\omega}} & \text{if} \quad 2<\gamma<n-\beta(n-2),\end{cases}$$

$$:=(\rho(x))^{-\nu}L_1(\rho(x))|x|^{-\tau}\ L_2(|x|).$$

By hypothesis (B_3), we obtain that $\nu=2-\delta_2\in(1,2)$ and we have $\tau=2$. Hence in view of (A_4) and Lemmas 2.1, 2.3 and 2.2, the function h satisfies (C).
On the other hand, we have

$$k(x)=\frac{(\rho(x))^{\delta_3-2}}{|x|^{\gamma+\beta\nu_4}}(M^{\frac{b}{\omega}}K^{\frac{1-\alpha}{\omega}})(\rho(x))$$

$$\times\begin{cases}L(|x|)\left(\int_{|x|}^{\infty}\frac{N(t)}{t}dt\right)^{\frac{b}{1-\alpha}} & \text{if} \quad \gamma>n-\beta(n-2),\\ L^{\frac{a}{1-\beta}}(|x|)\left(\int_{|x|}^{\infty}\frac{(NL^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{\frac{b}{\omega}} & \text{if} \quad 2<\gamma<n-\beta(n-2),\end{cases}$$

$$:=(\rho(x))^{-\nu}L_1(\rho(x))|x|^{-\tau}\ L_2(|x|).$$

We remark that k verifies the condition (C) with $\nu \in (1, 2)$ and $\tau = \gamma + \beta \nu_4 > 2$.

Case 13 Assume (H), (A_4) and (B_1).

We define the functions θ and ψ on D by

$$\theta(x) = \frac{1}{|x|^{n-2}} \widetilde{M}(\rho(x)),$$

where

$$\widetilde{M}(\rho(x)) = \begin{cases} \left(\int_0^{\rho(x)} \frac{M(t)}{t} dt \right)^{\frac{1}{1-\alpha}} & \text{if} \quad \sigma < 1+\beta, \\ \left(\int_0^{\rho(x)} \frac{(MK^{\frac{a}{1-\beta}})(t)}{t} dt \right)^{\frac{1-\beta}{\omega}} & \text{if} \quad 1+\beta < \sigma < 2, \end{cases} \tag{20}$$

and

$$\psi(x) = \frac{(\rho(x))^{\delta_4}}{|x|^{\nu_1}} \widetilde{K}(\rho(x)) \widetilde{L}(|x|),$$

where $\widetilde{L}$ is the function given by (13) and

$$\widetilde{K}(\rho(x)) = \begin{cases} 1 & \text{if} \quad \sigma < 1+\beta, \\ K^{\frac{1}{1-\beta}}(\rho(x)) \left(\int_0^{\rho(x)} \frac{(MK^{\frac{a}{1-\beta}})(t)}{t} dt \right)^{\frac{b}{\omega}} & \text{if} \quad 1+\beta < \sigma < 2. \end{cases} \tag{21}$$

By hypotheses (H), (A_4), (B_1), Proposition 1.5 and Lemmas 2.1, 2.3 and 2.2 we have θ and ψ are in $C_0(D)$.

Now, we consider two subcases.

Subcase 1 If $\sigma < 1 + \beta$ then $\lambda = 2 + a$.

Using (H), we have for $x \in D$,

$$h(x) = \frac{(\rho(x))^{-2}}{|x|^{\mu+\alpha(n-2)+a\nu_1}} M(\rho(x)) \left(\int_0^{\rho(x)} \frac{M(t)}{t} dt \right)^{\frac{\alpha}{1-\alpha}} (N\widetilde{L}^a)(|x|)$$

$$:= (\rho(x))^{-\nu} L_1(\rho(x)) |x|^{-\tau} L_2(|x|).$$

We remark that $\nu = 2$, $\tau = \mu + \alpha(n-2) + a\nu_1 > n$. From (B_1), (A_4), Lemmas 2.1, 2.3 and 2.2, we have $L_1 \in \mathcal{K}_0$, $L_2 \in \mathcal{K}_\infty$ satisfying (5). It follows that h fulfills the hypothesis (C).

Now, by (H) we have

$$k(x) = \frac{(\rho(x))^{-\sigma+\beta}}{|x|^{\gamma+b(n-2)+\beta\nu_1}} K(\rho(x)) \left(\int_0^{\rho(x)} \frac{M(t)}{t} dt \right)^{\frac{b}{1-\alpha}} (L\widetilde{L}^\beta)(|x|)$$

$$:= (\rho(x))^{-\nu} L_1(\rho(x)) |x|^{-\tau} L_2(|x|).$$

One can easily see that $\nu = \sigma - \beta < 1$. Using Lemmas 2.1, 2.3 and 2.2, we obtain that $L_1 \in \mathcal{K}_0$ satisfies $\int_0^\eta t^{1-\nu} L_1(t)\, dt < \infty$.

By elementary calculus, we obtain that for $x \in D$,

$$|x|^{-\tau} L_2(|x|) = g(|x|),$$

where g is the function defined by (13). So, we have $\tau \geq 2$ and $L_2 \in \mathcal{K}_\infty$ such that $\int_1^\infty t^{1-\tau} L_2(t)dt < \infty$. We deduce that k satisfies (C).

Subcase 2 If $1 + \beta < \sigma < 2$ then $\lambda = 2 + a\frac{2-\sigma}{1-\beta}$.

From hypothesis (H), we have

$$h(x) = \frac{(\rho(x))^{-2}}{|x|^{\mu+\alpha(n-2)+a\nu_1}} (MK^{\frac{a}{1-\beta}})(\rho(x)) \left(\int_0^{\rho(x)} \frac{(MK^{\frac{a}{1-\beta}})(t)}{t} dt \right)^{-1+\frac{1-\beta}{\rho}} (N\tilde{L}^a)(|x|)$$

$$:= (\rho(x))^{-\nu} L_1(\rho(x)) |x|^{-\tau} L_2(|x|).$$

Using (A_4), (B_1) and Lemmas 2.1, 2.3 and 2.2, we can easily see that h fulfills (C) with $\nu = 2$ and $\tau = \mu + \alpha(n-2) + a\nu_1 > n$.

On the other hand, by (H), we have

$$k(x) = \frac{(\rho(x))^{\frac{-\sigma+2\beta}{1-\beta}}}{|x|^{\gamma+b(n-2)+\beta\nu_1}} K^{\frac{1}{1-\beta}}(\rho(x)) \left(\int_0^{\rho(x)} \frac{(MK^{\frac{a}{1-\beta}})(t)}{t} dt \right)^{\frac{b}{\omega}} (L\tilde{L}^\beta)(|x|)$$

$$:= (\rho(x))^{-\nu} L_1(\rho(x)) |x|^{-\tau} L_2(|x|).$$

Using (A_4) and Lemmas 2.1 and 2.3, we get that $L_1 \in \mathcal{K}$ and since $\nu = \frac{\sigma-2\beta}{1-\beta} \in (1, 2)$, Lemma 2.2 implies that $\int_0^\eta t^{1-\nu} L_1(t)dt < \infty$.

Now, by simple calculus, we have

$$|x|^{-\tau} L_2(|x|) = g(|x|),$$

where g is the function defined by (13). Then, we have $\tau \geq 2$ and $L_2 \in \mathcal{K}_\infty$ such that $\int_1^\infty t^{1-\tau} L_2(t)dt < \infty$.

Therefore, the function k verifies the assumption (C).

Case 14 Assume (H), (A_4) and (B_2).

We define the functions θ and ψ on D by

$$\theta(x) = \frac{1}{|x|^{n-2}} \widetilde{M}(\rho(x)) \widetilde{N}(|x|),$$

where $\widetilde{M}$ and $\widetilde{N}$ are respectively given by (20) and (16) and

$$\psi(x) = \frac{(\rho(x))^{\delta_4}}{|x|^{\nu_1}} \widetilde{K}(\rho(x))$$

$$\times \begin{cases} 1 & \text{if } \gamma > n-(\beta+b)(n-2), \\ L^{\frac{1}{1-\beta}}(|x|)\left(\int_1^{1+|x|} \frac{(NL^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{\frac{b}{\omega}} & \text{if } 2-b(n-2)\gamma < n-(\beta+b)(n-2), \end{cases}$$

where $\widetilde{K}$ is the function given by (21).

Using hypothesis (A_4), Proposition 1.5 and Lemmas 2.1 and 2.3, we get that the functions θ and ψ are in $C_0(D)$.

Now, we consider four subcases.

Subcase 1 If $\sigma < 1 + \beta$ and $\gamma > n - (\beta + b)(n - 2)$, then $\lambda = 2 + a$ and $\mu = n - (\alpha + a)(n - 2)$.

From hypothesis (H), we have

$$h(x) = \frac{(\rho(x))^{-2}}{|x|^n} M(\rho(x))\left(\int_0^{\rho(x)} \frac{M(t)}{t}dt\right)^{\frac{\alpha}{1-\alpha}} N(|x|)\left(\int_1^{1+|x|} \frac{N(t)}{t}dt\right)^{\frac{\alpha}{1-\alpha}}$$

$$:= (\rho(x))^{-\nu} L_1(\rho(x))\,|x|^{-\tau}\, L_2(|x|).$$

From (A_4), Lemmas 2.1, 2.3 and 2.2, we can easily see that h satisfies (C) with $\nu = 2$ and $\tau = n$.

Now, by (H), we have

$$k(x) = \frac{(\rho(x))^{-\sigma+\beta}}{|x|^{\gamma+(\beta+b)(n-2)}} K(\rho(x))\left(\int_0^{\rho(x)} \frac{M(t)}{t}dt\right)^{\frac{b}{1-\alpha}} L(|x|)\left(\int_1^{1+|x|} \frac{N(t)}{t}dt\right)^{\frac{b}{1-\alpha}}$$

$$:= (\rho(x))^{-\nu} L_1(\rho(x))\,|x|^{-\tau}\, L_2(|x|).$$

Using hypothesis (A_4) and Lemmas 2.1, 2.3 and 2.2, we deduce that k fulfills (C) with $\nu = \sigma - \beta < 1$ and $\tau = \gamma + (\beta + b)(n - 2) > n$.

Subcase 2 If $\sigma < 1 + \beta$ and $2 - b(n - 2) < \gamma < n - (\beta + b)(n - 2)$, then $\lambda = 2 + a$ and $\mu = n - \alpha(n - 2) - a\frac{\gamma-2+b(n-2)}{1-\beta}$.

From (H), we have

$$h(x) = \frac{(\rho(x))^{-2}}{|x|^n} M(\rho(x))\left(\int_0^{\rho(x)} \frac{M(t)}{t}dt\right)^{\frac{\alpha}{1-\alpha}}$$

$$\times (NL^{\frac{a}{1-\beta}})(|x|)\left(\int_1^{1+|x|} \frac{(NL^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{-1+\frac{1-\beta}{\rho}}$$

$$:= (\rho(x))^{-\nu} L_1(\rho(x))\,|x|^{-\tau}\, L_2(|x|).$$

From (A_4) and by Lemmas 2.1, 2.3 and 2.2, the function L_1 is in $\mathcal{K}_0$ satisfying $\int_0^{\eta} t^{1-\nu} L_1(t)dt < \infty$. Besides, since $L_2 \in \mathcal{K}_\infty$ and $\tau = n$, we obtain by Lemma 2.2 that $\int_1^{\infty} t^{1-\tau} L_2(t)dt < \infty$. So, the function h verifies the condition (C).

Moreover, we have

$$k(x)=\frac{(\rho(x))^{-\sigma+\beta}}{|x|^{\frac{\gamma-2\beta+b(n-2)}{1-\beta}}}K(\rho(x))\left(\int_0^{\rho(x)}\frac{M(t)}{t}dt\right)^{\frac{b}{1-\alpha}}L^{\frac{1}{1-\beta}}(|x|)\left(\int_1^{1+|x|}\frac{(NL^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{\frac{b}{\omega}}$$

$$:=(\rho(x))^{-\nu}L_1(\rho(x))|x|^{-\tau}L_2(|x|).$$

We note that k satisfies the hypothesis (C) with $\nu=\sigma-\beta<1$ and $\tau=\frac{\gamma-2\beta+b(n-2)}{1-\beta}\in(2,n)$.

Subcase 3 If $1+\beta<\sigma<2$ and $\gamma>n-(\beta+b)(n-2)$, then $\lambda=2+a\frac{2-\sigma}{1-\beta}$ and $\mu=n-(\alpha+a)(n-2)$. Using (H), we get

$$h(x)=\frac{(\rho(x))^{-2}}{|x|^n}(MK^{\frac{a}{1-\beta}})(\rho(x))\left(\int_0^{\rho(x)}\frac{(MK^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{-1+\frac{1-\beta}{\omega}}$$

$$\times N(|x|)\left(\int_1^{1+|x|}\frac{N(t)}{t}dt\right)^{\frac{\alpha}{1-\alpha}}$$

$$:=(\rho(x))^{-\nu}L_1(\rho(x))|x|^{-\tau}L_2(|x|).$$

So, one can see that h fulfills the hypothesis (C) with $\nu=2$ and $\tau=n$.

Besides, we have

$$k(x)=\frac{(\rho(x))^{\frac{-\sigma+2\beta}{1-\beta}}}{|x|^{\gamma+(\beta+b)(n-2)}}K^{\frac{1}{1-\beta}}(\rho(x))\left(\int_0^{\rho(x)}\frac{(MK^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{\frac{b}{\omega}}L(|x|)\left(\int_1^{1+|x|}\frac{N(t)}{t}dt\right)^{\frac{b}{1-\alpha}}$$

$$:=(\rho(x))^{-\nu}L_1(\rho(x))|x|^{-\tau}L_2(|x|).$$

We note that $\nu=\frac{\sigma-2\beta}{1-\beta}\in(1,2)$ and $\tau>n$. From (A_4), Lemmas 2.1, 2.3 and 2.2, we have $L_1\in\mathcal{K}_0$, $L_2\in\mathcal{K}_\infty$ satisfying (5). Hence, k satisfies (C).

Subcase 4 If $1+\beta<\sigma<2$ and $2-b(n-2)<\gamma<n-(\beta+b)(n-2)$, then $\lambda=2+a\frac{2-\sigma}{1-\beta}$ and $\mu=n-\alpha(n-2)-a\frac{\gamma-2+b(n-2)}{1-\beta}$.

We have on D,

$$h(x)=\frac{(\rho(x))^{-2}}{|x|^n}(MK^{\frac{a}{1-\beta}})(\rho(x))\left(\int_0^{\rho(x)}\frac{(MK^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{-1+\frac{1-\beta}{\omega}}$$

$$\times(NL^{\frac{a}{1-\beta}})(|x|)\left(\int_1^{1+|x|}\frac{(NL^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{-1+\frac{1-\beta}{\omega}}$$

$$:=(\rho(x))^{-\nu}L_1(\rho(x))|x|^{-\tau}L_2(|x|).$$

In view of (A_4), Lemmas 2.1, 2.3 and 2.2, the functions L_1, L_2 belong respectively to $\mathcal{K}_0$ and $\mathcal{K}_\infty$ and satisfy (5). Hence, h fulfills (C).

On the other hand, we have

$$k(x) = \frac{(\rho(x))^{\frac{-\sigma+2\beta}{1-\beta}}}{|x|^{\frac{\gamma-2\beta+b(n-2)}{1-\beta}}} K^{\frac{1}{1-\beta}}(\rho(x)) \left(\int_0^{\rho(x)} \frac{(MK^{\frac{a}{1-\beta}})(\xi)}{\xi} d\xi \right)^{\frac{b}{\omega}} L^{\frac{1}{1-\beta}}(|x|)$$

$$\times \left(\int_1^{1+|x|} \frac{(NL^{\frac{a}{1-\beta}})(t)}{t} dt \right)^{\frac{b}{\omega}}$$

$$:= (\rho(x))^{-\nu} L_1(\rho(x)) |x|^{-\tau} L_2(|x|).$$

Using (A_4), Lemmas 2.1, 2.3 and 2.2, k satisfies the assumption (C) with $\nu \in (1, 2)$ and $\tau \in (2, n)$.

Case 15 Assume (H), (A_4) and (B_3).

We define the functions θ and ψ on D by

$$\theta(x) = \frac{1}{|x|^{\nu_2}} \widetilde{M}(\rho(x))(N^{\frac{1-\beta}{\omega}} L^{\frac{a}{\omega}})(|x|),$$

where $\widetilde{M}$ is the function defined by (20) and

$$\psi(x) = \frac{(\rho(x))^{\delta_4}}{|x|^{\nu_3}} \widetilde{K}(\rho(x))(N^{\frac{b}{\omega}} L^{\frac{1-\alpha}{\omega}})(|x|),$$

where $\widetilde{K}$ is the function given by (21).

It is obvious from (A_4) and Lemmas 2.1 and 2.3 that the functions θ and ψ belong to $C_0(D)$.

Using hypothesis (H) and Remark 1.7, we obtain

$$h(x) = \frac{(\rho(x))^{-2}}{|x|^{2+\nu_2}} (N^{\frac{1-\beta}{\omega}} L^{\frac{a}{\omega}})(|x|)$$

$$\times \begin{cases} M(\rho(x)) \left(\int_0^{\rho(x)} \frac{M(t)}{t} dt \right)^{\frac{\alpha}{1-\alpha}} & \text{if } \sigma < 1+\beta, \\ (MK^{\frac{a}{1-\beta}})(\rho(x)) \left(\int_0^{\rho(x)} \frac{(MK^{\frac{a}{1-\beta}})(t)}{t} dt \right)^{-1+\frac{1-\beta}{\omega}} & \text{if } 1+\beta < \sigma < 2, \end{cases}$$

$$:= (\rho(x))^{-\nu} L_1(\rho(x)) |x|^{-\tau} L_2(|x|).$$

We notice that $\nu = 2$ and by assumption (B_3), we have $\tau = 2 + \nu_2 \in (2, n)$. Hence, by the virtue of (A_4), Lemmas 2.1, 2.3 and 2.2, the function h fulfills (C).

Moreover, we have on D

$$k(x)=\frac{(\rho(x))^{-\sigma+\beta\delta_4}}{|x|^{2+\nu_3}}(N^{\frac{b}{\omega}}L^{\frac{1-\alpha}{\omega}})(|x|)$$

$$\times\begin{cases} K(\rho(x))\left(\int_0^{\rho(x)}\frac{M(t)}{t}dt\right)^{\frac{b}{1-\alpha}} & \text{if } \sigma<1+\beta, \\ K^{\frac{1}{1-\beta}}(\rho(x))\left(\int_0^{\rho(x)}\frac{(MK^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{\frac{b}{\omega}} & \text{if } 1+\beta<\sigma<2, \end{cases}$$

$$:=(\rho(x))^{-\nu}L_1(\rho(x))|x|^{-\tau}\ L_2(|x|).$$

Remark that $\nu=\sigma-\beta\delta_4<2$ and $\tau=2+\nu_3\in(2,n)$. Using same arguments as above, we deduce that k satisfies the hypothesis (C).

Case 16 Assume (H), (A_4) and (B_4).

We define the functions θ and ψ on D by

$$\theta(x)=\widetilde{M}(\rho(x))\begin{cases} \left(\int_{|x|}^{\infty}\frac{N(t)}{t}dt\right)^{\frac{1}{1-\alpha}} & \text{if } \gamma>n-\beta(n-2), \\ \left(\int_{|x|}^{\infty}\frac{(NL^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{\frac{1-\beta}{\omega}} & \text{if } 2<\gamma<n-\beta(n-2), \end{cases}$$

where $\widetilde{M}$ is the function given by (20) and

$$\psi(x)=\frac{(\rho(x))^{\delta_4}}{|x|^{\nu_4}}\widetilde{K}(\rho(x))\begin{cases} 1 & \text{if } \gamma>n-\beta(n-2), \\ L^{\frac{1}{1-\beta}}(|x|)\left(\int_{|x|}^{\infty}\frac{(NL^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{\frac{b}{\omega}} & \text{if } 2<\gamma<n-\beta(n-2), \end{cases}$$

where $\widetilde{K}$ is the function given by (21).

Using (A_4), (B_4), Proposition 1.5 and Lemmas 2.1 and 2.3, we get that the functions θ and ψ belong to $C_0(D)$.

Now, we consider four subcases.

Subcase 1 If $\sigma<1+\beta$ and $\gamma>n-\beta(n-2)$ then $\lambda=2+a$ and $\mu=2-a(n-2)$.

From hypothesis (H), we have

$$h(x)=\frac{(\rho(x))^{-2}}{|x|^2}M(\rho(x))\left(\int_0^{\rho(x)}\frac{M(t)}{t}dt\right)^{\frac{\alpha}{1-\alpha}}N(|x|)\left(\int_{|x|}^{\infty}\frac{N(t)}{t}dt\right)^{\frac{\alpha}{1-\alpha}}$$

$$:=(\rho(x))^{-\nu}L_1(\rho(x))|x|^{-\tau}\ L_2(|x|).$$

Using (A_4), (B_4), Lemmas 2.1, 2.3 and 2.2, we can easily see that h verifies (C) with $\nu=2$ and $\tau=2$.

Now, by (H), we have

$$k(x)=\frac{(\rho(x))^{-\sigma+\beta}}{|x|^{\gamma+\beta(n-2)}}K(\rho(x))\left(\int_0^{\rho(x)}\frac{M(t)}{t}dt\right)^{\frac{b}{1-\alpha}}L(|x|)\left(\int_{|x|}^{\infty}\frac{N(t)}{t}dt\right)^{\frac{b}{1-\alpha}}$$

$$:=(\rho(x))^{-\nu}L_1(\rho(x))|x|^{-\tau}L_2(|x|).$$

From (A_4), (B_4), Lemmas 2.1, 2.3 and 2.2, we deduce that the function k satisfies (C) with $\nu=\sigma-\beta<1$ and $\tau=\gamma+\beta(n-2)>n$.

Subcase 2 If $\sigma<1+\beta$ and $2<\gamma<n-\beta(n-2)$ then $\lambda=2+a$ and $\mu=2-a\frac{\gamma-2}{1-\beta}$.

From (H), we have

$$h(x)=\frac{(\rho(x))^{-2}}{|x|^2}M(\rho(x))\left(\int_0^{\rho(x)}\frac{M(t)}{t}dt\right)^{\frac{\alpha}{1-\alpha}}$$

$$\times(NL^{\frac{a}{1-\beta}})(|x|)\left(\int_{|x|}^{\infty}\frac{(NL^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{-1+\frac{1-\beta}{\omega}}$$

$$:=(\rho(x))^{-\nu}L_1(\rho(x))|x|^{-\tau}L_2(|x|).$$

Using (A_4), (B_4), Lemmas 2.1, 2.3 and 2.2, we get that the functions L_1 and L_2 are respectively in $\mathcal{K}_0$ and $\mathcal{K}_\infty$ satisfying (5). So, h fulfills the assumption (C).

Moreover, we have

$$k(x)=\frac{(\rho(x))^{-\sigma+\beta}}{|x|^{\frac{\gamma-2\beta}{1-\beta}}}K(\rho(x))\left(\int_0^{\rho(x)}\frac{M(t)}{t}dt\right)^{\frac{b}{1-\alpha}}L^{\frac{1}{1-\beta}}(|x|)$$

$$\times\left(\int_{|x|}^{\infty}\frac{(NL^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{\frac{b}{\omega}}$$

$$:=(\rho(x))^{-\nu}L_1(\rho(x))|x|^{-\tau}L_2(|x|).$$

In view of (A_4), (B_4), Lemmas 2.1, 2.3 and 2.2, we deduce that k satisfies (C) with $\nu=\sigma-\beta<1$ and $\tau=\frac{\gamma-2\beta}{1-\beta}\in(2,n)$.

Subcase 3 If $1+\beta<\sigma<2$ and $\gamma>n-\beta(n-2)$ then $\lambda=2+a\frac{2-\sigma}{1-\beta}$ and $\mu=2-a(n-2)$.

Using (H), we get

$$h(x)=\frac{(\rho(x))^{-2}}{|x|^2}(MK^{\frac{a}{1-\beta}})(\rho(x))\left(\int_0^{\rho(x)}\frac{(MK^{\frac{a}{1-\beta}})(t)}{t}dt\right)^{-1+\frac{1-\beta}{\omega}}$$

$$\times N(|x|)\left(\int_{|x|}^{\infty}\frac{N(t)}{t}dt\right)^{\frac{\alpha}{1-\alpha}}$$

$$:= (\rho(x))^{-\nu} L_1(\rho(x)) | x |^{-\tau} L_2(| x |).$$

So, one can see that h satisfies the hypothesis (C) with $\nu = 2$ and $\tau = 2$.
Besides, we have

$$k(x) = \frac{(\rho(x))^{\frac{-\sigma+2\beta}{1-\beta}}}{| x |^{\gamma+\beta(n-2)}} K^{\frac{1}{1-\beta}}(\rho(x)) \left(\int_0^{\rho(x)} \frac{(MK^{\frac{a}{1-\beta}})(t)}{t} dt \right)^{\frac{b}{\omega}} L(| x |)$$

$$\times \left(\int_{|x|}^{\infty} \frac{N(t)}{t} dt \right)^{\frac{b}{1-\alpha}}$$

$$:= (\rho(x))^{-\nu} L_1(\rho(x)) | x |^{-\tau} L_2(| x |).$$

We note that k fulfills (C) with $\nu = \frac{\sigma-2\beta}{1-\beta} \in (1, 2)$ and $\tau = \gamma + \beta(n-2) > n$.
Subcase 4 If $1 + \beta < \sigma < 2$ and $2 < \gamma < n - \beta(n-2)$ then $\lambda = 2 + a\frac{2-\sigma}{1-\beta}$ and $\mu = 2 - a\frac{\gamma-2}{1-\beta}$.
We have on D,

$$h(x) = \frac{(\rho(x))^{-2}}{| x |^2} (MK^{\frac{a}{1-\beta}})(\rho(x)) \left(\int_0^{\rho(x)} \frac{(MK^{\frac{a}{1-\beta}})(t)}{t} dt \right)^{-1+\frac{1-\beta}{\omega}}$$

$$(NL^{\frac{a}{1-\beta}})(| x |) \left(\int_{|x|}^{\infty} \frac{(NL^{\frac{a}{1-\beta}})(t)}{t} dt \right)^{-1+\frac{1-\beta}{\omega}}$$

$$:= (\rho(x))^{-\nu} L_1(\rho(x)) | x |^{-\tau} L_2(| x |).$$

Remark that $\nu = \tau = 2$. In view of (A_4), (B_4), Lemmas 2.1, 2.3 and 2.2, h satisfies the hypothesis (C).
On the other hand, we have

$$k(x) = \frac{(\rho(x))^{\frac{-\sigma+2\beta}{1-\beta}}}{| x |^{\frac{\gamma-2\beta}{1-\beta}}} K^{\frac{1}{1-\beta}}(\rho(x)) \left(\int_0^{\rho(x)} \frac{(MK^{\frac{a}{1-\beta}})(t)}{t} dt \right)^{\frac{b}{\omega}} L^{\frac{1}{1-\beta}}(| x |)$$

$$\times \left(\int_{|x|}^{\infty} \frac{(NL^{\frac{a}{1-\beta}})(t)}{t} dt \right)^{\frac{b}{\omega}}$$

$$:= (\rho(x))^{-\nu} L_1(\rho(x)) | x |^{-\tau} L_2(| x |).$$

It is obvious that the function k fulfills (C) with $\nu \in (1, 2)$ and $\tau \in (2, n)$.
To expound our main result, we present the following example.
Example 3.3 Let $\alpha, \beta \in (-1, 1)$, $a, b \in \mathbb{R}$ such that

$$|a| < 1 - |\alpha| \text{ and } |b| < 1 - |\beta|.$$

This implies that $\chi = (1 - |\alpha|)(1 - |\beta|) - |rs| > 0$ and so $\omega = (1 - \alpha)(1 - \beta) - rs > 0$.

Let p and q be two positive continuous functions on D such that for $x \in D$

$$p(x) \approx (\rho(x))^{-\lambda} |x|^{-\mu} \text{ and } q(x) \approx |x|^{-2+b(n-2)} \left(\log(\frac{3}{|x|}) \right)^{-2},$$

where $\lambda, \mu \in \mathbb{R}$.

It is clear that hypothesis (H) is well satisfied. We note that

$$\delta_1 = 1, \quad \delta_2 = \frac{(1-\beta)(2-\lambda)+2a}{\omega}, \quad \delta_3 = \frac{2(1-\alpha)+b(2-\lambda)}{\omega}, \quad \delta_4 = 1,$$

$$\nu_1 = 0, \quad \nu_2 = \frac{(1-\beta)(\mu-2)-rs(n-2)}{\omega}, \quad \nu_3 = \frac{-b(1-\alpha)(n-2)+b(\mu-2)}{\omega}, \quad \nu_4 = \frac{-b(n-2)}{1-\beta}.$$

• If $\lambda < 1 + \alpha + a$, $\mu > n - \alpha(n - 2)$, then hypotheses (A_1) and (B_1) are satisfied and by Theorem 1.8, system (3) possesses a positive solution $(u, v) \in C_0(D) \times C_0(D)$ such that for each $x \in D$:

$$u(x) \approx \frac{\rho(x)}{|x|^{n-2}}$$

and

$$v(x) \approx \rho(x) \left(\log(\frac{3}{|x|}) \right)^{\frac{-1}{1-\beta}}.$$

• If $\lambda < 1 + \alpha + a$, $0 < \nu_2 < n - 2$ and $0 < \nu_3 < n - 2$ then hypotheses (A_1) and (B_3) are satisfied and by Theorem 1.8, system (3) possesses a positive solution $(u, v) \in C_0(D) \times C_0(D)$ such that for each $x \in D$:

$$u(x) \approx \frac{\rho(x)}{|x|^{\nu_2}} \left(\log(\frac{3}{|x|}) \right)^{\frac{-2a}{\omega}}$$

and

$$v(x) \approx \frac{\rho(x)}{|x|^{\nu_3}} \left(\log(\frac{3}{|x|}) \right)^{\frac{-2(1-\alpha)}{\omega}}.$$

• If $\lambda < 1 + \alpha + a$, $\mu = 2 + \frac{rs(n-2)}{1-\beta}$, $b < 0$ and $2a > 1 - \beta$, then hypotheses (A_1) and (B_4) are fulfilled and by Theorem 1.8, system (3) possesses a positive solution $(u, v) \in C_0(D) \times C_0(D)$ such that for each $x \in D$:

$$u(x) \approx \rho(x) \left(\log(\frac{3}{|x|}) \right)^{\frac{1-\beta-2a}{\omega}}$$

and

$$v(x) \approx \frac{\rho(x)}{|x|^{\frac{-b(n-2)}{1-\beta}}}\left(\log(\frac{3}{|x|})\right)^{\frac{b+2\alpha-2}{\omega}}.$$

• If $\lambda = 1 + \alpha + r$, $\mu > n - \alpha(n-2)$, then hypotheses (A_2) and (B_1) are fulfilled and by Theorem 1.8, system (3) possesses a positive solution $(u, v) \in C_0(D) \times C_0(D)$ such that for each $x \in D$:

$$u(x) \approx \frac{\rho(x)}{|x|^{n-2}}\left(\log(\frac{3}{\rho(x)})\right)^{\frac{1}{1-\alpha}}$$

and

$$v(x) \approx \rho(x)\left(\log(\frac{3}{|x|})\right)^{\frac{-1}{1-\beta}}.$$

• If $\lambda = 1 + \alpha + a$, $0 < v_2 < n - 2$ and $0 < v_3 < n - 2$ then hypotheses (A_2) and (B_3) are satisfied and by Theorem 1.8, system (3) possesses a positive solution $(u, v) \in C_0(D) \times C_0(D)$ such that for each $x \in D$:

$$u(x) \approx \frac{\rho(x)}{|x|^{v_2}}\left(\log(\frac{3}{\rho(x)})\right)^{\frac{1}{1-\alpha}}\left(\log(\frac{3}{|x|})\right)^{\frac{-2a}{\omega}}$$

and

$$v(x) \approx \frac{\rho(x)}{|x|^{v_3}}\left(\log(\frac{3}{|x|})\right)^{\frac{-2(1-\alpha)}{\omega}}.$$

• If $\lambda = 1 + \alpha + a$, $\mu = 2 + \frac{rs(n-2)}{1-\beta}$, $b < 0$ and $2a > 1 - \beta$, then hypotheses (A_2) and (B_4) are satisfied and by Theorem 1.8, system (3) possesses a positive solution $(u, v) \in C_0(D) \times C_0(D)$ such that for each $x \in D$:

$$u(x) \approx \rho(x)\left(\log(\frac{3}{\rho(x)})\right)^{\frac{1}{1-\alpha}}\left(\log(\frac{3}{|x|})\right)^{\frac{-2a+1-\beta}{\omega}}$$

and

$$v(x) \approx \frac{\rho(x)}{|x|^{\frac{-b(n-2)}{1-\beta}}}\left(\log(\frac{3}{|x|})\right)^{\frac{b+2\alpha-2}{\omega}}.$$

• If $0 < \delta_2 < 1$, $0 < \delta_3 < 1$ and $\mu > n - \alpha(n-2)$ then hypotheses (A_3) and (B_1) are fulfilled and by Theorem 1.8, system (3) possesses a positive solution $(u, v) \in C_0(D) \times C_0(D)$ such that for each $x \in D$:

$$u(x) \approx \frac{(\rho(x))^{\delta_2}}{|x|^{n-2}}$$

and

$$v(x) \approx (\rho(x))^{\delta_3} \left(\log(\frac{3}{|x|}) \right)^{\frac{-1}{1-\beta}}.$$

• If $0 < \delta_2 < 1$, $0 < \delta_3 < 1$, $0 < \nu_2 < n - 2$ and $0 < \nu_3 < n - 2$ then hypotheses (A_3) and (B_3) are fulfilled and by Theorem 1.8, system (3) possesses a positive solution $(u, v) \in C_0(D) \times C_0(D)$ such that for each $x \in D$:

$$u(x) \approx \frac{(\rho(x))^{\delta_2}}{|x|^{\nu_2}} \left(\log(\frac{3}{|x|}) \right)^{\frac{-2a}{\omega}}$$

and

$$v(x) \approx \frac{(\rho(x))^{\delta_3}}{|x|^{\nu_3}} \left(\log(\frac{3}{|x|}) \right)^{\frac{-2(1-\alpha)}{\omega}}.$$

• If $0 < \delta_2 < 1$, $0 < \delta_3 < 1$, $\mu = 2 + \frac{rs(n-2)}{1-\beta}$, $b < 0$ and $2a > 1 - \beta$ then hypotheses (A_3) and (B_4) are satisfied and by Theorem 1.8, system (3) possesses a positive solution $(u, v) \in C_0(D) \times C_0(D)$ such that for each $x \in D$:

$$u(x) \approx (\rho(x))^{\delta_2} \left(\log(\frac{3}{|x|}) \right)^{\frac{-2a+1-\beta}{\omega}}$$

and

$$v(x) \approx \frac{(\rho(x))^{\delta_3}}{|x|^{\frac{-b(n-2)}{1-\beta}}} \left(\log(\frac{3}{|x|}) \right)^{\frac{b+2\alpha-2}{\omega}}.$$

References

[1] Ghergu M. Lane-Emden systems with negative exponents. *Journal of Functional Analysis*. 2010; 528: 3259-3318.

[2] Zhang Z. Positive solutions of Lane-Emden systems with negative exponents: Existence, boundary behavior and uniqueness. *Nonlinear Analysis*. 2011; 74: 5544-5553.

[3] Kawano N, Kusano T. On positive entire solutions of a class of second order semilinear elliptic systems. *Mathematische Zeitschrift*. 1984; 186: 287-297.

[4] Noussair E, Swanson CA. Asymptotics for semilinear elliptic systems. *Canadian Mathematical Bulletin*. 1991; 34: 514-519.

[5] Dalmasso R. Existence and uniqueness of positive solutions of semilinear elliptic systems. *Nonlinear Analysis*. 2000; 39: 559-568.

[6] De Figueiredo DG, Felmer P. A Liouville-type theorem for elliptic systems. *The Annali della Scuola Normale Superiore di Pisa*. 1994; 21: 387-397.

[7] Lair AV. A necessary and sufficient condition for the existence of large solutions to sublinear elliptic systems. *Journal of Mathematical Analysis and Applications*. 2010; 365: 103-108.

[8] Lair AV, Wood AW. Existence of entire large solutions of semilinear elliptic systems. *Journal of Differential Equations*. 2000; 164: 380-394.

[9] Maniwa M. Uniqueness and existence of positive solutions for some semilinear elliptic systems. *Nonlinear Analysis*.

2004; 59: 993-999.
[10] Serrin J, Zou H. Existence of positive solutions of the Lane-Emdem system. *Atti del Seminario Matematico e Fisico dell'Universita di Modena e Reggio Emilia. Modena*. 1998; 46: 369-380.
[11] Mâagli H, Alzahrani AK, Zine El Abidine Z. Asymptotic behavior of positive solutions of a semilinear Dirichlet problem in exterior domains. *Electronic Journal of Differential Equations*. 2018; 137: 1-14.
[12] Noussair ES, Swanson CA. Positive solutions of quasilinear elliptic equations in exterior domains. *Journal of Mathematical Analysis and Applications*. 1980; 75: 121-133.
[13] Chung KL, Zhao Z. *From Brownian Motion to Schrödinger's Equation*. Academic Press; 1978.
[14] Bachar I, Mâagli H, Zeddini N. Estimates on the Green Function and existence of positive solutions of nonlinear singular elliptic equations. *Communications in Contemporary Mathematics*. 2003; 3: 401-434.
[15] Karamata J. Sur un mode de croissance rgulire des fonctions. *Mathematica(Cluj)*. 1930; 4: 38-53.
[16] Chemmam R, Dhifli A, Mâagli H. Asymptotic behavior of ground state solutions for sublinear and singular nonlinear Dirichlet problem. *Electronic Journal of Differential Equations*. 2011; 88: 1-12.
[17] Marić V. Regular variation and differential equations. *Lecture Notes in Mathematics*. 2000; 1726.
[18] Seneta R. Regular varying functions. *Lectures Notes in Math*. 508 Springer-Verlag, Berlin; 1976.
[19] Chemmam R, Mâagli H, Masmoudi S, Zribi M. Combined effects in nonlinear singular elliptic problems in a bounded domain. *Advances in Nonlinear Analysis*. 2012; 1: 301-318.
[20] Bachar I, Mâagli H, Rădulescu VD. Singular solutions of a nonlinear elliptic equation in a punctured domain. *Electronic Journal of Qualitative Theory of Differential Equations*. 2017; 3: 1-19.