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Research Article

Existence and Asymptotic Behavior of Solutions for Schrödinger-Born-Infeld System in $\mathbb{R}^3$ with a General Nonlinearity

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Abstract: In this paper, we study the related properties of the solutions of a class of generalized Schrödinger-Born-Infeld system. In contemporary theoretical physics, the Schrödinger-Born-Infeld system also plays an active and important role. The coupling problem between the Schrödinger equation and the logarithmic type Born-Infeld equation gives rise to insights and new ideas about how space-time geometry and matter interactions can be coupled. Our work is to describe the interaction between matter and electromagnetic field from a dualistic viewpoint. We will synthetically apply variational methods, truncation techniques and other analytical tools to establish the existence and asymptotic behavior of solutions for the coupled system under certain conditions. Our studies will unveil a broad spectrum of systems of elliptic equations with logarithmic nonlinearity and rich properties and structures, which present new challenges.

Keywords: Schrödinger-Born-Infeld system, logarithmic electrodynamics, variational method

MSC: 35A15, 35A21, 35J10, 35B40

1. Introduction

In this paper, we investigate the following system

$$\begin{cases} -\Delta u + u + \lambda \phi u = f(u), & \mathrm{x} \in \mathbb{R}^3, \\ -\nabla \cdot \left(\frac{\nabla \phi}{1 - \frac{1}{2}|\nabla \phi|^2} \right) = \lambda u^2, & \mathrm{x} \in \mathbb{R}^3, \\ u(\mathrm{x}) \to 0,\ \phi(\mathrm{x}) \to 0, & \text{as } |\mathrm{x}| \to \infty, \end{cases} \tag{1}$$

where physical parameter $\lambda > 0$, u is a scalar function, ϕ is a potential function, $f \in \mathscr{C}^1(\mathbb{R}, \mathbb{R})$ is a general nonlinearity that satisfies the famous Berestycki and Lions conditions in [1]. System (1) is generated by the coupling of the

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Schrödinger equation and the logarithmic Born-Infeld equation [2, 3], which describes the interaction between matter and electromagnetic from a dualistic point of view. System (1) is a variation of the famous Schrödinger-Maxwell system introduced in [4]. In fact, the logarithmic Born-Infeld Lagrangian replaces the general Maxwell Lagrangian in system (1) and we will consider the electrostatic solution.

In recent years, research on Born-Infeld nonlinear electrodynamics has seen significant activity. Originally proposed by Born and Infeld [5], the theory was introduced to address the energy divergence problem associated with point charges in Maxwell electrodynamics, offering a framework to model the electron as a finite-energy point charge. A distinctive feature of Born-Infeld electrodynamics is its inherent electromagnetic asymmetry: while it supports finite-energy electric point charges, it excludes the possibility of finite-energy magnetic monopoles. This asymmetry provides a unique opportunity to modify Maxwell theory, preserving finite-energy electric charges while eliminating their magnetic counterparts. In modern theoretical physics, the Born-Infeld theory has found important applications beyond its original context. For instance, it naturally arises in string theory and brane dynamics [6, 7], where it describes the nonlinear behavior of gauge fields on D-branes. Additionally, the conceptual framework of Born-Infeld theory has inspired attempts to modify the Einstein-Hilbert action in gravity, aiming to regularize singularities and address fundamental issues in gravitational theory [8].

We work on the Minkowski spacetime, characterized by its temporal and spatial coordinates $x^0 = t,\ (x^i) = \mathbf{x},\ i = 1, 2, 3$. The metric $(\eta_{\mu\nu}) = \mathrm{diag}(1, -1, -1, -1)$. The Maxwell electromagnetic action density is

$$\mathscr{L}_M = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2), \quad \mu, \nu = 0, 1, 2, 3,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, ∂_μ denotes the partial derivative with respect to x_μ, and A_μ is the gauge field. Additionally, $\mathbf{E},\ \mathbf{B}$ are the magnetic field and electric field respectively.

Take the charge density as ρ, the associated Maxwell equations are

$$\nabla \cdot \mathbf{B} = 0,\ \frac{\partial \mathbf{B}}{\partial t} - \nabla \times \mathbf{E} = \mathbf{0}, \tag{2}$$

$$\nabla \cdot \mathbf{D} = \rho,\ \frac{\partial \mathbf{D}}{\partial t} - \nabla \times \mathbf{H} = \textbf{-j}, \tag{3}$$

among then, $\mathbf{D}$ and $\mathbf{H}$ represent the electric displacement and magnetic intensity fields, respectively, and $\mathbf{j}$ is current density. Moreover, we have the equation relating the pairs $\mathbf{D}$, $\mathbf{H}$, $\mathbf{E}$, $\mathbf{B}$

$$\mathbf{D} = \varepsilon(\mathbf{E}, \mathbf{B})\mathbf{E}, \quad \mathbf{B} = \mu(\mathbf{E}, \mathbf{B})\mathbf{H},$$

where $\varepsilon(\mathbf{E}, \mathbf{B}) = l'(\mathscr{L}_M),\ \mu(\mathbf{E}, \mathbf{B}) = \frac{1}{l'(\mathscr{L}_M)}$ are the field-dependent dielectric and permeability coefficients, respectively. Significantly, the function l satisfies the normalization condition [9]

$$l(0) = 0, \quad l'(0) = 1. \tag{4}$$

In this paper, we consider the electrostatic case, according to (2), we have

$$\mathbf{E} = -\nabla\phi, \quad \mathscr{L}_{\mathrm{M}} = \frac{1}{2}\mathbf{E}^2. \tag{5}$$

And

$$l(\mathscr{L}_M) = -b^2 \ln(1 - \frac{\mathscr{L}_M}{b^2}), \tag{6}$$

where $b > 0$ is a Born parameter. It is called logarithmic model [3, 5, 9, 10] in Born-Infeld theory and belongs to a well-known Born-Infeld-like nonlinear electrodynamics [11]. This model was proposed to study exact solutions of spherically symmetric static black hole. And logarithmic electrodynamics are characterized by having a finite self-energy solution for a point-like charge [12], they predict a birefringence effect in the presence of an electromagnetic background field. In particular, the model can be viewed as a slight variation of the classical Born-Infeld theory. However, from a mathematical perspective, the Born-Infeld theory is highly nonlinear in structure, making it one of the most challenging problems to analyze.

Obviously, (6) satisfies the normalization condition (4), then from the relation (3), we get

$$\mathbf{D} = \frac{\mathbf{E}}{1 - \frac{1}{2b^2}|\mathbf{E}|^2}.$$

By calculation, without loss of generality, taking the Born parameter $b = 1$ yields

$$-\nabla \cdot \left(\frac{\nabla\phi}{1 - \frac{1}{2}|\nabla\phi|^2} \right) = \rho. \tag{7}$$

It is evident that the logarithmic Born-Infeld equation (7), as the second equation of the problem (1) we are studying, exhibits singularity.

In order to find the solution of the problem (1), we will choose a variations method with constraints to calculate. Under the appropriate assumption of f, the action functional of system (1) is as follows

$$\mathscr{I}_\lambda(u, \phi) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2\right) \mathrm{dx} + \frac{\lambda}{2}\int_{\mathbb{R}^3} \phi u^2 \mathrm{dx} + \frac{1}{2}\int_{\mathbb{R}^3} \ln(1 - \frac{1}{2}|\nabla\phi|^2)\mathrm{dx} - \int_{\mathbb{R}^3} F(u)\mathrm{dx}, \tag{8}$$

where $F(u) = \int_0^u f(t)\mathrm{d}t$. The functional space X is

$$X = D^{1,2}(\mathbb{R}^3) \cap \left\{\phi \in \mathscr{C}^{0,1}(\mathbb{R}^3) \middle| |\nabla\phi|_\infty \leq \alpha\right\}, \tag{9}$$

where $0 < \alpha < \sqrt{2}$ is a constant, $D^{1,2}(\mathbb{R}^3)$ is the completion of $\mathscr{C}_c^\infty(\mathbb{R}^3)$ with respect to the norm $|\nabla \cdot|_2$, the norm in $D^{1,2}(\mathbb{R}^3)$ is defined as follows

$$\|\phi\|_{D^{1,2}(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} |\nabla\phi|^2 \mathrm{dx} \right)^{\frac{1}{2}}. \tag{10}$$

The weak solution of system (1) is a pair $(u_\lambda, \phi_\lambda) \in H^1(\mathbb{R}^3) \times X$, where $\lambda > 0$, such that

$$\begin{cases} \int_{\mathbb{R}^3} (\nabla u_\lambda \cdot \nabla v + u_\lambda v + \lambda \phi_\lambda u_\lambda v)\,\mathrm{dx} = \int_{\mathbb{R}^3} f(u_\lambda) v \mathrm{dx}, & v \in \mathscr{C}_0^\infty(\mathbb{R}^3), \\ \int_{\mathbb{R}^3} \frac{\nabla\phi_\lambda \nabla\varphi}{1-\frac{1}{2}|\nabla\phi_\lambda|^2} \mathrm{dx} = \lambda \int_{\mathbb{R}^3} u_\lambda^2 \varphi \mathrm{dx}, & \varphi \in \mathscr{C}_0^\infty(\mathbb{R}^3). \end{cases} \tag{11}$$

Noting that the functional $\mathscr{I}_\lambda$ (8) has the strongly indefinite nature. We employ space

$$H_r^1(\mathbb{R}^3) = \left\{ u \in H^1(\mathbb{R}^3) |\ u \text{ is radially symmetric} \right\},$$

and the standard inner product and norm on $H_r^1(\mathbb{R}^3)$ are defined as follows

$$(u, v) = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + uv)\,\mathrm{dx}, \quad \|u\| = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + |u|^2) \mathrm{dx} \right)^{\frac{1}{2}}. \tag{12}$$

Moreover, for any $u \in H_r^1(\mathbb{R}^3)$, the solution for the second equation in system (1) is also radial. Therefore, define the following set

$$X_r(\mathbb{R}^3) = \{ \phi \in X |\ \phi \text{ is radially symmetric} \}.$$

In [13], Yu considered the coupling of the Klein-Gordon with the Born-Infeld lagrangian and studied the electrostatic case expressed by the following system

$$\begin{cases} -\Delta u + (m^2 - (\omega+\phi)^2) u = |u|^{p-1} u, & \mathrm{x} \in \mathbb{R}^3, \\ -\mathrm{div}\left(\frac{\nabla\phi}{\sqrt{1-|\nabla\phi|^2}} \right) = u^2(\omega+\phi), & \mathrm{x} \in \mathbb{R}^3, \\ u(\mathrm{x}) \to 0,\ \phi(\mathrm{x}) \to 0, & \text{as } \mathrm{x} \to \infty. \end{cases}$$

According to the form of the second equation, the variational method can not be applied in the usual function space. This limitation arises because the term $1/\sqrt{1-|\nabla\phi(\mathrm{x})|^2}$ is well-defined only for $\mathrm{x} \in \mathbb{R}^3$ satisfying $|\nabla\phi(\mathrm{x})| < 1$. As a result, this inequality must be treated as an essential constraint within the functional framework. In both bounded smooth domain case and $\mathbb{R}^3$ case, he obtained the existence of the least-action solitary waves via variational approach and perturbation method.

In [14], Li et al. considered the following Schrödinger-Born-Infeld system

$$\begin{cases} -\Delta u + u + \lambda \phi u = f(u), & \mathrm{x} \in \mathbb{R}^3, \\ -\mathrm{div}\left(\frac{\nabla \phi}{\sqrt{1-|\nabla \phi|^2}}\right) = \lambda u^2, & \mathrm{x} \in \mathbb{R}^3, \\ u(\mathrm{x}) \to 0,\ \phi(\mathrm{x}) \to 0, & \text{as } |\mathrm{x}| \to \infty, \end{cases} \tag{13}$$

where $\lambda > 0$. Using variational methods, they established the existence of nontrivial solutions for small values of λ and analyzed the asymptotic behavior of these solutions as λ approaches zero.

In [15], Azzollini et al. studied the problem of $\lambda = 1$ and $f(t) = |t|^{p-2}t$ with $p \in (\frac{7}{2}, 6)$ in system (13). They used a monotonicity trick [16, 17] and variational approach to obtain a radial ground state solution.

In [18], Liu and Siciliano focused on a special form of system (13), namely the system of $\lambda = 1$ and the nonlinear term $f(u) + \mu|u|^4u$, where $\mu > 0$. By employing a novel perturbation approach, they established the existence and multiplicity of nontrivial solutions in both subcritical and critical cases. Regarding similar issues, we can also refer to [19, 20] and their relate literatures.

In [21], Dai and Zhang discussed the existence of solutions to a nonlinear problem involving an exponential model of the Born-Infeld nonlinear electromagnetism.

In [2], Ye studied the logarithmic equations in Born-Infeld nonlinear electromagnetism, and proved the existence and related properties of solutions in the entire three-dimensional space using techniques such as variational methods and convex function analysis. This is the early results of our research team.

In [22], Yousif et al. applied the logarithmic non-polynomial spline numerical method to solve the nonlinear non-homogeneous time fractional order Burgers-Huxley equation.

To our knowledge, system (1) is not commonly seen. The primary challenge in this work lies in the fact that, despite the variational structure of system (1), standard variational methods cannot be directly applied. Firstly, this is primarily due to the nonlinear nature of the electrostatic potential equation in the system. Compared with the general Poisson equation, the solution $\phi_\lambda(u)$ of the second equation in system (1) lacks both homogeneity properties and an explicit formula. As a result, the scaling transformation method cannot be applied. Secondly, using a perturbation approach, when $f(t)$ satisfied a generalized Ambrosetti-Rabinowitz condition, the existence of bounded Palais-Smale sequences obtained in [18]. However, under our assumptions, for general nonlinear term, the range of p belongs to $(2, 6)$ and the asymptotically linear growth, the method is not applicable, we need to find other ways to prove the boundedness of Palais-Smale sequence. We will synthetically apply variational methods, truncation techniques [23–26], the monotonicity trick of Jeanjean [16] and other analytical tools. To ensure the existence of bounded Palais-Smale sequence of the functional associated to system (1), so we need to constraint parameter λ. In this way, our work is unusual.

In this paper, we will prove that when λ is small enough, system (1) has at least one nontrivial solution, and when λ tended to zero, these solutions will converge to a nontrivial solution for a class of Schrödinger equation associated with it.

We assume that the nonlinear term f satisfies the following conditions

(f_1) $f \in \mathscr{C}^1(\mathbb{R}, \mathbb{R})$ and $f(t)t \geq 0$ for $t \in \mathbb{R}$;

(f_2) $\lim_{t\to 0} \frac{f(t)}{t} = \lim_{|t|\to\infty} \frac{f(t)}{t^5} = 0$;

(f_3) There exists a constant $t_0 > 0$ such that $F(t_0) > \frac{t_0^2}{\nu_1}$, where $0 < \nu_1 < 2$, $F(t) = \int_0^t f(\xi)\mathrm{d}\xi$;

(f_4) There exists a constant $\nu_2 > 3$ such that $0 \leq \nu_2 F(t) \leq f(t)t$, for $t \in \mathbb{R}$.

Now, we state our main results.

Theorem 1 Assume that f satisfies $(f_1) - (f_3)$, then there exists $\lambda_0 > 0$ such that for every $\lambda \in (0, \lambda_0)$, (1) has at least one nontrivial solution $(u_\lambda, \phi_\lambda) \in H_r^1(\mathbb{R}^3) \times X_r$.

Theorem 2 Assume that f satisfies $(f_1)-(f_4)$, then for every $\lambda\in(0,\lambda_0)$, (1) has at least one radial ground state solution.

Theorem 3 Assume that f satisfies $(f_1)-(f_3)$, then for every $\lambda\in(0,\lambda_0)$ the solution (u_λ,ϕ_λ) obtained in Theorem 1.1 corresponding to $\lambda\to 0^+$ satisfies

$$u_\lambda\to\bar{u}\ in\ H_r^1(\mathbb{R}^3),\quad \phi_\lambda\to 0\ in\ D^{1,2}(\mathbb{R}^3),$$

where $\bar{u}\in H_r^1(\mathbb{R}^3)$ is a nontrivial solution of the following Schrödinger equation

$$-\Delta u+u=f(u),\ \mathrm{x}\in\mathbb{R}^3. \tag{14}$$

Remark 1 Under the conditions of Theorems 1-2, the solutions (u_λ,ϕ_λ) are classical. That is to say, both of u_λ and ϕ_λ belong to $\mathscr{C}^2(\mathbb{R}^3)$.

This paper is organized as follows. In Section 2, we will present some related lemmas and the variation setting. We will use techniques such as monotonicity and truncation to study the minimizer (ϕ_λ,u_λ) of system (1). In Section 3, we will prove Theorem 1.1, Theorem 1.2 and Theorem 1.3, respectively.

Throughout this paper, we use C_q, $q\in[2,6]$ to denote the constant of Sobolev's imbedding from $H_r^1(\mathbb{R}^3)$ to $L^q(\mathbb{R}^3)$. For any $q\in[1,+\infty]$, the standard $L^q(\mathbb{R}^3)$ norm is denoted by $|\cdot|_q$. The constant of Sobolev's imbedding $W=\inf_{\varphi\in D^{1,2}(\mathbb{R}^3)\setminus\{0\}}\frac{|\nabla\varphi|_2^2}{|\varphi|_6^2}$.

2. Preliminary

In this section, we develop a variational formulation to analyze system (1). Since the functional $\mathscr{I}_\lambda$ is strongly indefinite, apply the method for the classical Schrödinger-Poisson system (see [4, 23, 27–30] for more details) to system (1).

For every $u\in H_r^1(\mathbb{R}^3)$ and $\lambda>0$, we define the following functional

$$E_{\lambda,u}(\phi)=\int_{\mathbb{R}^3}-\ln\left(1-\frac{1}{2}|\nabla\phi|^2\right)\mathrm{dx}-\lambda\int_{\mathbb{R}^3}\phi u^2\mathrm{dx}. \tag{15}$$

According to Theorem 1.3.1, Theorem 1.3.2 and Lemma 3.4.1 in [2], we can know that the functional $E_{\lambda,u}$ has a unique nonnegative minimizer $\phi_\lambda(u)$ on X, and $E_{\lambda,u}\le 0$, $\phi_\lambda(u)\in X_r$. At the same time, $\phi_\lambda(u)$ is also the unique weak solution $\phi_\lambda(u)\in X_r$ of the following equation

$$-\nabla\cdot\left(\frac{\nabla\phi}{1-\frac{1}{2}|\nabla\phi|^2}\right)=\lambda u^2,\quad \mathrm{x}\in\mathbb{R}^3,\ u\in H_r^1(\mathbb{R}^3).$$

Hence, system (1) can be simplified into the following equation

$$-\Delta u+u+\lambda\phi_\lambda(u)u=f(u),\quad \mathrm{x}\in\mathbb{R}^3.$$

Furthermore, by Hölder inequality, $\frac{1}{2}s \leq -\ln(1-\frac{1}{2}s) \leq \frac{s}{2-s}$, $\forall s \in [0, 2)$ and Sobolev's embedding inequality, we have

$$\frac{1}{2}|\nabla\phi_\lambda(u)|_2^2 \leq \lambda \int_{\mathbb{R}^3} \phi_\lambda(u)u^2\mathrm{dx} \leq \lambda W^{-\frac{1}{2}} C_{\frac{12}{5}}^2 |\nabla\phi_\lambda(u)|_2 \|u\|^2, \quad \forall u \in H_r^1(\mathbb{R}^3), \tag{16}$$

so we can get

$$|\nabla\phi_\lambda(u)|_2 \leq 2\lambda W^{-\frac{1}{2}} C_{\frac{12}{5}}^2 \|u\|^2, \quad \forall u \in H_r^1(\mathbb{R}^3). \tag{17}$$

Lemma 1 If $u_n \rightharpoonup u$ in $H_r^1(\mathbb{R}^3)$, then for every $\lambda > 0$,

(i) $\phi_\lambda(u_n) \to \phi_\lambda(u)$ in $L^\infty(\mathbb{R}^3)$;

(ii) $\int_{\mathbb{R}^3} \phi_\lambda(u_n)u_n^2\mathrm{dx} \to \int_{\mathbb{R}^3} \phi_\lambda(u)u^2\mathrm{dx}$, $\int_{\mathbb{R}^3} \phi_\lambda(u_n)u_n v\mathrm{dx} \to \int_{\mathbb{R}^3} \phi_\lambda(u)uv\mathrm{dx}$, $v \in H_r^1(\mathbb{R}^3)$.

Proof. (i) Through Lemma 3.4.3, Lemma 3.4.4 and Lemma 3.4.5 in [2], we can obtain that if $(u_n)_n \subset L^p(\mathbb{R}^3)$ with $p \in [1, +\infty)$, and $u_n \to u$ in $L^p(\mathbb{R}^3)$, then $\phi(u_n)$ converges to weakly in X and uniformly in $\mathbb{R}^3$. Therefore, (i) can be directly obtained. For (ii), observe that uniformly in $v \in H_r^1(\mathbb{R}^3)$, by Hölder inequality, we have

$$\int_{\mathbb{R}^3} |\phi_\lambda(u_n)u_n - \phi_\lambda(u)u||v|\mathrm{dx} \leq \int_{\mathbb{R}^3} |\phi_\lambda(u_n)||u_n - u||v|\mathrm{dx} + \int_{\mathbb{R}^3} |\phi_\lambda(u_n) - \phi_\lambda(u)||u||v|\mathrm{dx} = o_n(1),$$

therefore, we obtain $\int_{\mathbb{R}^3} \phi_\lambda(u_n)u_n v\mathrm{dx} \to \int_{\mathbb{R}^3} \phi_\lambda(u)uv\mathrm{dx}$. Similarly, we have the first conclusion of (ii) in Lemma 2.1.

Similar to the discussion on Proposition 2.4 in [15], we have the following lemma.

Lemma 2 The functional

$$I_\lambda(u) = \mathscr{I}_\lambda(u, \phi_\lambda(u)) = \frac{1}{2}\|u\|^2 + \frac{\lambda}{2}\int_{\mathbb{R}^3} \phi_\lambda(u)u^2\mathrm{dx} + \frac{1}{2}\int_{\mathbb{R}^3} \ln(1 - \frac{1}{2}|\nabla\phi_\lambda|^2)\mathrm{dx} - \int_{\mathbb{R}^3} F(u)\mathrm{dx} \tag{18}$$

is of class $\mathscr{C}^1$ and its Fréchet derivative at $u \in H_r^1(\mathbb{R}^3)$ is given by

$$\langle I_\lambda'(u), v\rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + uv + \lambda\phi_\lambda(u)uv)\mathrm{dx} - \int_{\mathbb{R}^3} f(u)v\mathrm{dx}, \quad \forall v \in H_r^1(\mathbb{R}^3). \tag{19}$$

Lemma 3 Let $\lambda > 0$ be fixed,

(i) if $(u_\lambda, \phi_\lambda) \in H_r^1(\mathbb{R}^3) \times X_r$ is a weak solution of system (1), then $\phi_\lambda = \phi_\lambda(u_\lambda)$ and u_λ is a critical point of I_λ;

(ii) if $u_\lambda \in H_r^1(\mathbb{R}^3) \setminus \{0\}$ is a critical point of I_λ, then $(u_\lambda, \phi_\lambda(u_\lambda))$ is a weak nontrivial solution of system (1).

Proof. (i) This result can be obtained from Lemma 3.2.4 in [2] and Lemma 2.2.

(ii) Finding the weak solution of system (1) is equivalent to finding the critical point of functional I_λ on $H_r^1(\mathbb{R}^3)$. To verify the mountain pass geometric structure of the functional I_λ, we will use truncation method to handle the nonlocal term $\phi_\lambda(u)$. Now, we define the truncation function $\chi \in \mathscr{C}^\infty([0, +\infty], [0, 1])$ to satisfy

$$
\begin{cases}
\chi(s) = 1, & s \in [0, 1], \\
0 \le \chi(s) \le 1, & s \in (1, 2), \\
\chi(s) = 0, & s \in [2, +\infty), \\
-2 \le \chi'(s) \le 0, & s \in [0, +\infty).
\end{cases} \tag{20}
$$

For each constant $T > 0$ and $u \in H_r^1(\mathbb{R}^3)$, we define the function

$$
g_T(u) = \chi\left(\frac{\|u\|^2}{T^2}\right)
$$

and the truncated functional renders

$$
I_\lambda^T(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2}g_T(u)G_\lambda(u) - \int_{\mathbb{R}^3} F(u)\mathrm{dx}, \tag{21}
$$

where the functional $G_\lambda(u) = E_{\lambda,\,u}(\phi_\lambda(u))$. According to (f_1) and (f_2), we can see that the functional $I_\lambda^T \in \mathscr{C}^1(H_r^1(\mathbb{R}^3), \mathbb{R})$, and Fréchet derivative at $u \in H_r^1(\mathbb{R}^3)$ as follows

$$
\begin{aligned}
\left\langle (I_\lambda^T)'(u),\, v\right\rangle = & \left(1 - \frac{1}{T^2}\chi'(\frac{\|u\|^2}{T^2})G_\lambda(u)\right)\int_{\mathbb{R}^3}(\nabla u \cdot \nabla v + uv)\mathrm{dx} \\
& + \lambda g_T(u)\int_{\mathbb{R}^3}\phi_\lambda(u)uv\mathrm{dx} - \int_{\mathbb{R}^3} f(u)v\mathrm{dx}, \quad \forall v \in H_r^1(\mathbb{R}^3).
\end{aligned} \tag{22}
$$

Then $u_\lambda \in H_r^1(\mathbb{R}^3)$ is a critical point of I_λ^T, which is equivalent to the pair $(u_\lambda,\, \phi_\lambda(u_\lambda)) \in H_r^1(\mathbb{R}^3) \times X_r$ is a weak solution of

$$
\begin{cases}
(-\Delta u + u)\left(1 - \frac{1}{T^2}\chi'(\frac{\|u\|^2}{T^2})E_{\lambda,\,u}(\phi)\right) + \lambda g_T(u)\phi u = f(u), & \mathrm{x} \in \mathbb{R}^3, \\
-\nabla \cdot \left(\frac{\nabla\phi}{1 - \frac{1}{2}|\nabla\phi|^2}\right) = \lambda u^2, & \mathrm{x} \in \mathbb{R}^3, \\
u(\mathrm{x}) \to 0, \quad \phi(\mathrm{x}) \to 0, & \text{as } |\mathrm{x}| \to \infty.
\end{cases} \tag{23}
$$

In light of the definition of χ, we can see that

$$
I_\lambda(u) = I_\lambda^T(u) \ \text{ and } \ I_\lambda'(u) = (I_\lambda^T)'(u), \quad \text{if } \|u\| \le T.
$$

The proof is completed.

Under the assumptions of $(f_1)-(f_4)$, we will establish the boundedness of Palais-Smale (PS) sequences for functional I_λ^T through Lemma 2.4 [16, 17, 31].

Lemma 4 Let B be Banach space and let $\Sigma \subset \mathbb{R}^+$ is an interval. Consider the family of $\mathscr{C}^1$-functionals on B

$$I_\mu(u) = J(u) - \mu K(u), \quad \forall\, \mu \in \Sigma,$$

where $K(u) \geq 0$ and either $J(u) \to +\infty$ or $K(u) \to +\infty$ as $\|u\| \to +\infty$, and such that $I_\mu(0) = 0$. For any $\mu \in \Sigma$, we set

$$\Gamma_\mu = \left\{\gamma \in \mathscr{C}([0,\, 1],\, B) | \gamma(0) = 0,\, I_\mu(\gamma(1)) < 0\right\}. \tag{24}$$

If for every $\mu \in \Sigma$ the set Γ_μ is nonempty and

$$c_\mu = \inf_{\gamma \in \Gamma_\mu} \max_{t \in [0,\, 1]} I_\mu(\gamma(t)) > 0, \tag{25}$$

then there is a sequence $\{w_n\} \subset B$ such that

(i) $\{w_n\}$ is bounded in B;

(ii) $I_\mu(w_n) \to c_\mu$;

(iii) $I'_\mu(w_n) \to 0$ in B^{-1}, where B^{-1} is the dual of B.

In our case, let $B = H_r^1(\mathbb{R}^3)$, we define the following perturbed functional

$$I_\mu(u) = I_{\lambda,\, \mu}^T(u) = J(u) - \mu K(u), \tag{26}$$

where

$$J(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2} g_T(u) G_\lambda(u), \quad K(u) = \int_{\mathbb{R}^3} F(u)\mathrm{dx}.$$

By (f_1), we can deduce that $K(u) \geq 0$. In view of the definitions of g_T and G_λ, we get $J(u) \to +\infty$ as $\|u\| \to \infty$. By (26), it is easy to obtain that $I_\mu(0) = 0$.

To apply Lemma 2.4, it is essential to determine an interval Σ that meets the required criteria for our analysis. That is, for any $\mu \in \Sigma$, (24) and (25) hold. As a consequence of (f_3), there exists a function $\hat{u} \in H_r^1(\mathbb{R}^3)$ such that $\int_{\mathbb{R}^3} F(\hat{u})\mathrm{dx} > \frac{1}{\nu_1}|\hat{u}|_2^2$. Then there exists $\eta \in (0,\, 1)$ such that

$$\eta \int_{\mathbb{R}^3} F(\hat{u})\mathrm{dx} > \frac{|\hat{u}|_2^2}{\nu_1}. \tag{27}$$

Consequently, we obtain the interval $\Sigma = [\eta,\, 1]$, which will be shown to satisfy the required conditions.

Lemma 5 For any $\mu \in \Sigma = [\eta,\, 1]$, (24) and (25) are valid.

Proof. Firstly, let's verify that (24) holds. For every $\mu \in [\eta, 1]$, set $\tilde{u}_0 = u_0(\frac{\cdot}{\rho})$ with $\rho > 0$. Define γ_0: $[0, 1] \to H_r^1(\mathbb{R}^3)$ and

$$\gamma_0(t) = \begin{cases} 0, & t = 0, \\ \tilde{u}_0(\frac{\cdot}{t}), & t \in (0, 1]. \end{cases} \tag{28}$$

It is easy to see through calculation that γ_0 is a continuous path from 0 to $\tilde{u}_0$. Due to the definition of functional G_λ and (2.3), we have

$$-2\lambda^2 W^{-1} C_{\frac{12}{5}}^4 \|v\|^4 \leq -\lambda \int_{\mathbb{R}^3} \phi_\lambda(v) v^2 \mathrm{dx} \leq G_\lambda(v) \leq 0, \quad \forall v \in H_r^1(\mathbb{R}^3). \tag{29}$$

According to the definition of g_T, (28) and (29), one has

$$\begin{aligned} I_{\lambda,\mu}^T(\gamma_0(1)) &= \frac{1}{2}\|\gamma_0(1)\|^2 - \frac{1}{2} g_T(\gamma_0(1)) G_\lambda(\gamma_0(1)) - \mu \int_{\mathbb{R}^3} F(\gamma_0(1)) \mathrm{dx} \\ &\leq \frac{1}{2}(\rho|\nabla u_0|_2^2 + \rho^3 |u_0|_2^2) - \eta\rho^3 \int_{\mathbb{R}^3} F(u_0) \mathrm{dx} \\ &\quad + \chi\left(\frac{\rho|\nabla u_0|_2^2 + \rho^3|u_0|_2^2}{T^2}\right) \lambda^2 W^{-1} C_{\frac{12}{5}}^4 (\rho|\nabla u_0|_2^2 + \rho^3|u_0|_2^2)^2. \end{aligned} \tag{30}$$

Let $\rho > 0$ large enough, we get $\chi\left(\frac{\rho|\nabla u_0|_2^2+\rho^3|u_0|_2^2}{T^2}\right) = 0$, from (27), we can see that $I_{\lambda,\mu}^T(\gamma_0(1)) < 0$. Obviously, $\gamma_0 \in \Gamma_\mu$, for each $\mu \in [\eta, 1]$.

Next, let's prove that (25) is valid. By (f_1) and (f_2), it can be inferred that there exists $a_\varepsilon > 0$, for every $\varepsilon \in (0, 1)$ such that

$$0 \leq F(t) \leq \frac{\varepsilon}{2} t^2 + a_\varepsilon t^6, \quad t \in \mathbb{R}.$$

Thus, we have

$$\begin{aligned} I_{\lambda,\mu}^T(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{2} g_T(u) G_\lambda(u) - \mu \int_{\mathbb{R}^3} F(u) \mathrm{dx} \\ &\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^3} F(u) \mathrm{dx} \\ &\geq \frac{1-\varepsilon}{2}\|u\|^2 - \frac{a_\varepsilon}{W^3}\|u\|^6. \end{aligned} \tag{31}$$

From this, there exists a sufficiently small $d > 0$ such that for every $u \in H_r^1(\mathbb{R}^3)$ with $0 \leq \|u\| \leq d$ and $\mu \in [\eta, 1]$, giving rise to $I_{\lambda,\mu}^T(u) > 0$. Moreover, for any $\|u\| = d$, one has

$$I^T_{\lambda,\,\mu}(u) \geq \frac{1-\varepsilon}{2}d^2 - \frac{a_\varepsilon}{W^3}d^6 > 0.$$

Now fix $\gamma \in \Gamma_\mu$, it is obviously that $\|\gamma(1)\| > d$, since $\gamma(0) = 0$ and $I^T_{\lambda,\,\mu}(\gamma(1)) < 0$. Because γ is a continuous function, we know that there exists $t_\gamma \in (0,\, 1)$ such that $\|\gamma(t_\gamma)\| = d$. Then

$$c_\mu \geq \frac{1-\varepsilon}{2}d^2 - \frac{a_\varepsilon}{W^3}d^6 > 0. \tag{32}$$

Thus, (25) holds.

According to Lemmas 2.4-2.5, it can be seen that for each $\mu \in [\eta,\, 1]$, there exists a bounded $(PS)_{c_\mu}$ sequence $\{u_n\} \subset H^1_r(\mathbb{R}^3)$ for the functional $I^T_{\lambda,\,\mu}$.

Lemma 6 There exists $\lambda_T > 0$ such that for any $\mu \in [\eta,\, 1]$ and $\lambda \in (0,\, \lambda_T)$, each bounded (PS) sequence of $I^T_{\lambda,\,\mu}$ admits a convergent subsequence.

Proof. Let $\{u_n\}$ be a bounded (PS) sequence of $I^T_{\lambda,\,\mu}$, we know from Lemma 2.4 that

$$I^T_{\lambda,\,\mu}(u_n) \text{ is bounded}, \quad (I^T_{\lambda,\,\mu})'(u_n) \to 0 \ in\ (H^1_r(\mathbb{R}^3))^*, \tag{33}$$

among then, the Fréchet derivative of $I^T_{\lambda,\,\mu}$ at u_n is as follows

$$\left\langle (I^T_{\lambda,\,\mu})'(u_n),\, v \right\rangle = \left(1 - \frac{1}{T^2}\chi'\left(\frac{\|u_n\|^2}{T^2}\right)G_\lambda(u_n)\right)\int_{\mathbb{R}^3}(\nabla u_n \cdot \nabla v + u_n v)\mathrm{dx}$$

$$+ \lambda g_T(u_n)\int_{\mathbb{R}^3}\phi_\lambda(u_n)u_n v\mathrm{dx} - \mu\int_{\mathbb{R}^3}f(u_n)v\mathrm{dx}, \quad \forall v \in H^1_r(\mathbb{R}^3). \tag{34}$$

Up to a subsequence, we assume that there exists $u \in H^1_r(\mathbb{R}^3)$ such that

$$u_n \rightharpoonup u \ in\ H^1_r(\mathbb{R}^3); \quad u_n \to u \ in\ L^q(\mathbb{R}^3),\ q \in (2,\, 6); \quad u_n(\mathrm{x}) \to u(\mathrm{x})\ a.e.\ \mathrm{x} \in \mathbb{R}^3.$$

From (12) and (34), one has

$$0 \leftarrow \left\langle (I^T_{\lambda,\,\mu})'(u_n) - (I^T_{\lambda,\,\mu})'(u),\, u_n - u \right\rangle$$

$$= \left(1 - \frac{1}{T^2}\chi'\left(\frac{\|u_n\|^2}{T^2}\right)G_\lambda(u_n)\right)\|u_n - u\|^2$$

$$+ \left(\frac{1}{T^2}\chi'\left(\frac{\|u\|^2}{T^2}\right)G_\lambda(u) - \frac{1}{T^2}\chi'\left(\frac{\|u_n\|^2}{T^2}\right)G_\lambda(u_n)\right)(u,\, u_n - u)$$

$$+\lambda g_T(u_n)\int_{\mathbb{R}^3}\phi_\lambda(u_n)u_n(u_n-u)\mathrm{dx}-\lambda g_T(u)\int_{\mathbb{R}^3}\phi_\lambda(u)u(u_n-u)\mathrm{dx}$$

$$+\mu\int_{\mathbb{R}^3}(f(u)-f(u_n))(u_n-u)\mathrm{dx},\quad \text{as } n\to\infty. \tag{35}$$

We estimate (35) below. By (20) and (29), we have

$$0\le\frac{1}{T^2}\left|\chi'(\frac{\|v\|^2}{T^2})G_\lambda(v)\right|\le 4\lambda^2W^{-1}C^4_{\frac{12}{5}}T^2\to 0,\quad \forall v\in H^1_r(\mathbb{R}^3),\quad \text{as } \lambda\to 0.$$

Therefore, we can conclude that $\left\{\frac{1}{T^2}\chi'(\frac{\|u_n\|^2}{T^2})G_\lambda(u_n)\right\}$ is bounded in $\mathbb{R}$, for any $\lambda>0$. Specifically, we set $\lambda_T=\frac{\sqrt{W}}{2C^2_{\frac{12}{5}}T}$, for every $\lambda\in(0,\lambda_T)$, we get

$$0<1-4\lambda^2W^{-1}C^4_{\frac{12}{5}}T^2\le 1-\frac{1}{T^2}\chi'(\frac{\|u_n\|^2}{T^2})G_\lambda(u_n)\le 1. \tag{36}$$

Then according to $u_n\rightharpoonup u$ in $H^1_r(\mathbb{R}^3)$ and $\lambda>0$, one has

$$\left(\frac{1}{T^2}\chi'(\frac{\|u\|^2}{T^2})G_\lambda(u)-\frac{1}{T^2}\chi'(\frac{\|u_n\|^2}{T^2})G_\lambda(u_n)\right)(u,\,u_n-u)\to 0,\ \text{as } n\to\infty. \tag{37}$$

In view of (ii) of Lemma 2.1, it is easy to obtain that

$$\int_{\mathbb{R}^3}\phi_\lambda(u_n)u_n^2\mathrm{dx}\to\int_{\mathbb{R}^3}\phi_\lambda(u)u^2\mathrm{dx},\quad \int_{\mathbb{R}^3}\phi_\lambda(u_n)u_nu\mathrm{dx}\to\int_{\mathbb{R}^3}\phi_\lambda(u)u^2\mathrm{dx},\quad \text{as } n\to\infty.$$

Together with

$$\int_{\mathbb{R}^3}\phi_\lambda(u)u(u_n-u)\mathrm{dx}\to 0,\quad \text{as } n\to\infty.$$

For every $\lambda>0$, we can conclude that

$$\lambda g_T(u_n)\int_{\mathbb{R}^3}\phi_\lambda(u_n)u_n(u_n-u)\mathrm{dx}-\lambda g_T(u)\int_{\mathbb{R}^3}\phi_\lambda(u)u(u_n-u)\mathrm{dx}\to 0,\quad \text{as } n\to\infty. \tag{38}$$

In view of (f_1) and (f_2), we can obtain that for every $\varepsilon>0$ there exists $b_\varepsilon>0$ such that

$$0\le|f(t)|\le\varepsilon|t|+b_\varepsilon|t|^{p-1}+\varepsilon|t|^5,\quad p\in(2,\,6),\quad t\in\mathbb{R}.$$

Then we have

$$\left|\int_{\mathbb{R}^3} f(u_n)(u_n-u)\mathrm{dx}\right| \leq \int_{\mathbb{R}^3} (\varepsilon|u_n|+b_\varepsilon|u_n|^{p-1}+\varepsilon|u_n|^5)|u_n-u|\mathrm{dx},$$

From the arbitrariness of ε and Hölder inequality, it is easy to obtain that

$$\int_{\mathbb{R}^3} f(u_n)(u_n-u)\mathrm{dx} \to 0, \quad \text{as } n\to\infty.$$

From the same discussion as above, we can get that

$$\int_{\mathbb{R}^3} f(u)(u_n-u)\mathrm{dx} \to 0, \quad \text{as } n\to\infty.$$

So we can conclude that

$$\mu\int_{\mathbb{R}^3} (f(u)-f(u_n))(u_n-u)\mathrm{dx} \to 0, \quad \text{as } n\to\infty. \tag{39}$$

Hence, by (36)-(39) and (35) imply that for every $\lambda\in(0,\lambda_T)$, we have

$$\|u_n-u\|\to 0, \quad \text{as } n\to\infty.$$

The proof of Lemma 2.6 is completed.

From Lemmas 2.4-2.6, we can infer that there exists $u_\mu\in H_r^1(\mathbb{R}^3)$ such that

$$I^T_{\lambda,\mu}(u_\mu)=c_\mu;\quad (I^T_{\lambda,\mu})'(u_\mu)=0,\ \forall\,\lambda\in(0,\lambda_T),\ \mu\in[\eta,1].$$

Hence, there exists $\{\mu_n\}\subset[\eta,1]$ such that

$$\mu_n\to 1,\quad n\to\infty;\quad I^T_{\lambda,\mu_n}(u_{\mu_n})=c_{\mu_n};\quad (I^T_{\lambda,\mu})'(u_{\mu_n})=0. \tag{40}$$

The following regularity result and Pohožaev type identity are important for studying the properties of the sequence $\{u_{\mu_n}\}$.

Lemma 7 For every $\lambda\in(0,\lambda_T)$, the solution $(u,\phi)\in H_r^1(\mathbb{R}^3)\times X_r$ to system (23) satisfies $u,\phi\in\mathscr{C}^2(\mathbb{R}^3)$.

Proof. A discussion similar to Lemma 2.6 in [14] led to $u\in\mathscr{C}^2(\mathbb{R}^3)$. Since $u\in H_r^1(\mathbb{R}^3)$, according to Lemma 3.5.1 in [2], it can be seen that $\phi\in\mathscr{C}^1(\mathbb{R}^3)$. Now we prove that $\phi\in\mathscr{C}^2(\mathbb{R}^3)$.

We define ψ: $[0,+\infty]\to\mathbb{R}$ such that $\psi(r)=\phi(|\mathrm{x}|)$, $\forall r\geq 0$, where $\mathrm{x}\in\mathbb{R}^3$, $|\mathrm{x}|=r$. Since ϕ_λ satisfies the second equation in a weak sense and is radial, we can get

$$D\left(\frac{\psi' r^2}{1-\frac{1}{2}|\psi'|^2}\right) = -\lambda u^2 r^2, \quad \text{in } (0, +\infty)$$

where the symbol D denotes the derivative in the sense of distributions. Because the right hand side of the above equation is a continuous function, it is the derivative in the classical sense. Since $\psi'(0)=0$, and integrating in $(0, r)$, we have

$$\frac{\psi'(r)}{1-\frac{1}{2}|\psi'(r)|^2} = -\frac{\lambda}{r^2}\int_0^r u^2(s)s^2 \mathrm{d}s = h(r) \in \mathscr{C}^1(0, +\infty). \tag{41}$$

From (41), we derive the following two results. Firstly, for $r>0$, we have

$$h'(r) = \frac{2\lambda}{r^3}\int_0^r u^2(s)s^2\mathrm{d}s - \lambda u^2(r)$$

and the $\lim_{r\to 0} h'(r) = -\frac{1}{3}\lambda u^2(0)$. Secondly, we get

$$\lim_{r\to 0}\frac{h(r)}{r} = \lim_{r\to 0} -\frac{\lambda}{r^3}\int_0^r u^2(s)s^2\mathrm{d}s = -\frac{1}{3}\lambda u^2(0).$$

Obviously, there exists $h'(0)$ and $\lim_{r\to 0} h'(r) = h'(0)$. Then $h \in \mathscr{C}^1([0, +\infty))$.

Through (41), we get

$$\psi'(r) = \frac{-1+\sqrt{1+2h^2(r)}}{h(r)}.$$

Therefore, we can easily see that $\phi \in \mathscr{C}^2(\mathbb{R}^3)$.

The following lemma gives us the Pohožaev type identity.

Lemma 8 Assume $(u, \phi) \in H_r^1(\mathbb{R}^3)\times X_r$ is a solution for system (23) of class $\mathscr{C}^2$, then the following Pohožaev type identity holds

$$\begin{aligned} 3\int_{\mathbb{R}^3} F(u)\mathrm{dx} =& \left(\frac{1}{2}|\nabla u|_2^2 + \frac{3}{2}|u|_2^2\right)\left(1-\frac{1}{T^2}\chi'\left(\frac{\|u\|^2}{T^2}\right)E_{\lambda, u}(\phi)\right) \\ &+ g_T(u)\left(2\int_{\mathbb{R}^3}\frac{|\nabla\phi|^2}{1-\frac{1}{2}|\nabla\phi|^2}\mathrm{dx} + \frac{3}{2}\int_{\mathbb{R}^3}\ln\left(1-\frac{1}{2}|\nabla\phi|^2\right)\mathrm{dx}\right). \end{aligned} \tag{42}$$

Proof. The first equation of system (23) is multiplied by $\mathrm{x}\cdot\nabla u$ and for every $R>0$, integrating on B_R, where B_R denotes the ball centered at the origin with radius R in $\mathbb{R}^3$, we get

$$\int_{B_R} -\Delta u(\mathrm{x}\cdot\nabla u)\mathrm{dx} = -\int_{\partial B_R}\left(\frac{|\mathrm{x}\cdot\nabla u|^2}{R}-\frac{R}{2}|\nabla u|^2\right)\mathrm{dS}-\frac{1}{2}\int_{B_R}|\nabla u|^2\mathrm{dx}, \tag{43}$$

$$\int_{B_R} u(\mathrm{x}\cdot\nabla u)\mathrm{dx} = \frac{R}{2}\int_{\partial B_R} u^2\mathrm{dS}-\frac{3}{2}\int_{B_R} u^2\mathrm{dx}, \tag{44}$$

$$\begin{aligned}\int_{B_R} \lambda g_T(u)\phi u(\mathrm{x}\cdot\nabla u)\mathrm{dx} = \frac{\lambda R}{2}\int_{\partial B_R} g_T(u)\phi u^2\mathrm{dS}-\frac{3\lambda}{2}\int_{B_R} g_T(u)\phi u^2\mathrm{dx}\\ -\frac{\lambda}{2}\int_{B_R} g_T(u)u^2(\mathrm{x}\cdot\nabla\phi)\mathrm{dx},\end{aligned} \tag{45}$$

$$-\int_{B_R} f(u)(\mathrm{x}\cdot\nabla u)\mathrm{dx} = -R\int_{\partial B_R} F(u)\mathrm{dS}+3\int_{B_R} F(u)\mathrm{dx}, \tag{46}$$

Moreover, for any $i,\ j = 1,\ 2,\ 3$, we have

$$\begin{aligned}\int_{B_R}\partial_i\left(\frac{\partial_i\phi}{1-\frac{1}{2}|\nabla\phi|^2}\right)x_j\partial_j\phi\mathrm{dx} \quad = \quad & -\int_{B_R}\frac{\partial_i\phi\partial_j\phi}{1-\frac{1}{2}|\nabla\phi|^2}\delta_{ij}\mathrm{dx}-\int_{B_R}\frac{\partial_i\phi\partial_{i,\,j}^2\phi}{1-\frac{1}{2}|\nabla\phi|^2}x_j\mathrm{dx}\\ & +\int_{\partial B_R}\frac{\partial_i\phi\partial_j\phi}{1-\frac{1}{2}|\nabla\phi|^2}\frac{x_ix_j}{|\mathrm{x}|}\mathrm{dS}.\end{aligned}$$

Through the second equation of system (23), we have

$$\begin{aligned}\int_{B_R}\lambda u^2(\mathrm{x}\cdot\nabla u)\mathrm{dx} = & \int_{B_R} -\nabla\cdot\left(\frac{\nabla\phi}{1-\frac{1}{2}|\nabla\phi|^2}\right)(\mathrm{x}\cdot\nabla u)\mathrm{dx}\\ & = -\sum_{i,\,j=1}^{3}\int_{B_R}\partial_i\left(\frac{\partial_i\phi}{1-\frac{1}{2}|\nabla\phi|^2}\right)x_j\partial_j\phi\mathrm{dx}\\ & = R\int_{\partial B_R} -\ln(1-\frac{1}{2}|\nabla\phi|^2)\mathrm{dS}-\sum_{i,\,j=1}^{3}\int_{\partial B_R}\frac{\partial_i\phi\partial_j\phi}{1-\frac{1}{2}|\nabla\phi|^2}\frac{x_ix_j}{|\mathrm{x}|}\mathrm{dS}\\ & \quad +\int_{B_R}\frac{|\nabla\phi|^2}{1-\frac{1}{2}|\nabla\phi|^2}\mathrm{dx}+3\int_{B_R}\ln(1-\frac{1}{2}|\nabla\phi|^2)\mathrm{dx},\end{aligned} \tag{47}$$

Hence, combining with (43)-(47), one has

$$\frac{1}{2}\int_{B_R}|\nabla u|^2\mathrm{dx}+\frac{3}{2}\int_{B_R}u^2\mathrm{dx}-3\int_{B_R}F(u)\mathrm{dx}+\frac{3\lambda}{2}\int_{B_R}g_T(u)\phi u^2\mathrm{dx}$$

$$+\frac{1}{2}\int_{B_R}\frac{|\nabla\phi|^2}{1-\frac{1}{2}|\nabla\phi|^2}g_T(u)\mathrm{dx}+\frac{3}{2}\int_{B_R}\ln(1-\frac{1}{2}|\nabla\phi|^2)g_T(u)\mathrm{dx}$$

$$= -\int_{\partial B_R}\left(\frac{|\mathrm{x}\cdot\nabla u|^2}{R}-\frac{R}{2}|\nabla u|^2\right)\mathrm{dS}+\frac{R}{2}\int_{\partial B_R}u^2\mathrm{dS}-R\int_{\partial B_R}F(u)\mathrm{dS}+\frac{\lambda R}{2}\int_{\partial B_R}g_T(u)\phi u^2\mathrm{dS}$$

$$+\frac{R}{2}\int_{\partial B_R}\ln(1-\frac{1}{2}|\nabla\phi|^2)g_T(u)\mathrm{dS}+\frac{1}{2}\sum_{i,\,j=1}^{3}\int_{\partial B_R}\frac{\partial_i\phi\partial_j\phi}{1-\frac{1}{2}|\nabla\phi|^2}\frac{x_ix_j}{|\mathrm{x}|}g_T(u)\mathrm{dS}. \tag{48}$$

When $R\to\infty$, all the boundary integrals go to zero, by (11) we get (42).

Lemma 9 Choose $T>0$ sufficiently large, there exists $\lambda_T'\le\lambda_T$ such that

$$\|u_{\mu_n}\|\le T,\quad \forall\,\lambda\in(0,\,\lambda_T')$$

holds, where $\{u_{\mu_n}\}$ is the critical point sequence of $I^T_{\lambda,\,\mu_n}$ mentioned in (40).

Proof. Firstly, let us estimate that $|\nabla u_{\mu_n}|_2$ is bounded. Since $(I^T_{\lambda,\,\mu_n})'(u_{\mu_n})=0$, we have

$$3\mu_n\int_{\mathbb{R}^3}F(u_{\mu_n})\mathrm{dx} = (\frac{1}{2}|\nabla u_{\mu_n}|_2^2+\frac{3}{2}|u_{\mu_n}|_2^2)\left(1-\frac{1}{T^2}\chi'(\frac{\|u_{\mu_n}\|^2}{T^2})G_\lambda(u_{\mu_n})\right)$$

$$+g_T(u_{\mu_n})\left(2\int_{\mathbb{R}^3}\frac{|\nabla\phi_\lambda(u_{\mu_n})|^2}{1-\frac{1}{2}|\nabla\phi_\lambda(u_{\mu_n})|^2}\mathrm{dx}+\frac{3}{2}\int_{\mathbb{R}^3}\ln(1-\frac{1}{2}|\nabla\phi_\lambda(u_{\mu_n})|^2)\mathrm{dx}\right).$$

According to $I^T_{\lambda,\,\mu_n}(u_{\mu_n})=c_{\mu_n}$ and $\int_{\mathbb{R}^3}\frac{|\nabla\phi_\lambda(u_{\mu_n})|^2}{1-\frac{1}{2}|\nabla\phi_\lambda(u_{\mu_n})|^2}\mathrm{dx}=\lambda\int_{\mathbb{R}^3}\phi_\lambda(u_{\mu_n})u^2_{\mu_n}\mathrm{dx}$, we have

$$|\nabla u_{\mu_n}|_2^2$$

$$=3c_{\mu_n}-\left(\frac{1}{2}|\nabla u_{\mu_n}|_2^2+\frac{3}{2}|u_{\mu_n}|_2^2\right)\frac{1}{T^2}\chi'(\frac{\|u_{\mu_n}\|^2}{T^2})G_\lambda(u_{\mu_n})+\frac{\lambda}{2}g_T(u_{\mu_n})\int_{\mathbb{R}^3}\phi_\lambda(u_{\mu_n})u^2_{\mu_n}\mathrm{dx}$$

$$\le 3c_{\mu_n}+\lambda^2W^{-1}C^4_{\frac{12}{5}}T^4. \tag{49}$$

For γ_0 obtained in Lemma 2.5, by (27) and (29), one has

$$
\begin{aligned}
c_{\mu_n} &\leq \max_{t\in[0,\,1]} I^T_{\lambda,\,\mu_n}(\gamma_0(t)) \\
&\leq \max_{t\in[0,\,1]} \left(\frac{1}{2}\|\gamma_0(t)\|^2 - \mu_n\int_{\mathbb{R}^3} F(\gamma_0(t))\mathrm{dx}\right) + \max_{t\in[0,\,1]} \frac{1}{2} g_T(\gamma_0(t))(-G(\gamma_0(t))) \\
&\leq \max_{\rho\in[0,\,\infty]} \left(\frac{1}{2}(\rho|\nabla u_0|_2^2 + \rho^3|u_0|_2^2) - \delta\rho^3\int_{\mathbb{R}^3} F(u_0)\mathrm{dx}\right) + \lambda^2 W^{-1} C^4_{\frac{12}{5}} T^4 \\
&= c^* + \lambda^2 W^{-1} C^4_{\frac{12}{5}} T^4.
\end{aligned} \tag{50}
$$

According to (49) and (50), we can get

$$
|\nabla u_{\mu_n}|_2^2 \leq 3c^* + 4\lambda^2 W^{-1} C^4_{\frac{12}{5}} T^4. \tag{51}
$$

Next, let us prove that $|u_{\mu_n}|_2$ is bounded. $\left\langle (I^T_{\lambda,\,\mu_n})'(u_{\mu_n}),\, u_{\mu_n}\right\rangle = 0$ implies that

$$
\begin{aligned}
&\mu_n \int_{\mathbb{R}^3} f(u_{\mu_n}) u_{\mu_n} \mathrm{dx} \\
=&\|u_{\mu_n}\|^2 \left(1 - \frac{1}{T^2}\chi'(\frac{\|u_{\mu_n}\|^2}{T^2}) G_\lambda(u_{\mu_n})\right) + \lambda g_T(u_{\mu_n}) \int_{\mathbb{R}^3} \phi_\lambda(u_{\mu_n}) u^2_{\mu_n} \mathrm{dx}.
\end{aligned} \tag{52}
$$

If $\|u_{\mu_n}\| \geq \sqrt{2}T$, (52) can be simplified as

$$
\|u_{\mu_n}\|^2 = \mu_n \int_{\mathbb{R}^3} f(u_{\mu_n}) u_{\mu_n} \mathrm{dx}.
$$

From (f_1) and (f_2), it can be seen that there exists $C > 0$ such that

$$
0 \leq f(t)t \leq \frac{1}{2}t^2 + Ct^6, \quad t \in \mathbb{R}.
$$

Combining with $\mu_n \in [0,\, 1]$, we have

$$
|u_{\mu_n}|_2^2 \leq 2C|u_{\mu_n}|_6^6 \leq 2CS^{-3}|\nabla u_{\mu_n}|_2^6.
$$

By (51), we obtain

$$|u_{\mu_n}|_2^2 \leq 2CW^{-3}|\nabla u_{\mu_n}|_2^6 \leq 2CW^{-3}(3c^* + 4\lambda^2 W^{-1} C_{\frac{12}{5}}^4 T^4)^3. \tag{53}$$

If $\|u_{\mu_n}\| < \sqrt{2}T$, since $\phi_\lambda(u_{\mu_n})$ is nonnegative, we rewrite (52) as

$$|u_{\mu_n}|_2^2 \leq \frac{\|u_{\mu_n}\|^2}{T^2}\chi'(\frac{\|u_{\mu_n}\|^2}{T^2})G_\lambda(u_{\mu_n}) + \mu_n \int_{\mathbb{R}^3} f(u_{\mu_n})u_{\mu_n}\mathrm{dx}.$$

Similar to the discussion above, by (29) and (51), for the above C in (53)

$$\begin{aligned}|u_{\mu_n}|_2^2 &\leq 2CW^{-3}|\nabla u_{\mu_n}|_2^6 + 2C\frac{\|u_{\mu_n}\|^2}{T^2}\chi'(\frac{\|u_{\mu_n}\|^2}{T^2})G_\lambda(u_{\mu_n}) \\ &\leq 2CW^{-3}|\nabla u_{\mu_n}|_2^6 + 64C\lambda^2 W^{-1} C_{\frac{12}{5}}^4 T^4 \\ &\leq 2CW^{-3}(3c^* + 4\lambda^2 W^{-1} C_{\frac{12}{5}}^4 T^4)^3 + 64C\lambda^2 W^{-1} C_{\frac{12}{5}}^4 T^4. \end{aligned} \tag{54}$$

It follows from (51), (53) and (54) that

$$\|u_{\mu_n}\|^2 \leq 3c^* + 68C\lambda^2 W^{-1} C_{\frac{12}{5}}^4 T^4 + 2CW^{-3}(3c^* + 4\lambda^2 W^{-1} C_{\frac{12}{5}}^4 T^4)^3. \tag{55}$$

Finally, we prove through the method of contradiction that $\|u_{\mu_n}\| \leq T$. We assume that for a certain n_0, there is $\|u_{\mu_n}\| > T,\ \forall\, n > n_0$. Generally, we assume that it is true for any n. Now, we derive from (55) that

$$T^2 < \|u_{\mu_n}\|^2 \leq 3c^* + 68C\lambda^2 W^{-1} C_{\frac{12}{5}}^4 T^4 + 2CW^{-3}(3c^* + 4\lambda^2 W^{-1} C_{\frac{12}{5}}^4 T^4)^3,$$

which is invalid for $\lambda > 0$ small enough and $T > 0$ large enough. Therefore, we can find $T_0 > 0$ such that

$$T_0^2 > 3c^* + 54CW^{-3}c^{*3} + 1,$$

and $\lambda'_{T_0} \leq \lambda_{T_0}$ such that

$$68C\lambda^2 W^{-1} C_{\frac{12}{5}}^4 T_0^4 + \cdots + 2 \times 4^3 C\lambda^6 W^{-6} C_{\frac{12}{5}}^{12} T_0^{12} < 1, \quad \forall \lambda \in (0, \lambda'_{T_0}).$$

The proof of lemma is completed.

3. Proof of Theorems

In this section, we will give the proof of Theorem 1.1, Theorem 1.2 and Theorem 1.3.

Proof of Theorem 1.1 Let $\lambda_0 = \lambda'_{T_0}$, where λ'_{T_0} and T_0 obtained in Lemma 2.9. For each $\lambda \in (0, \lambda_0)$, let u_{μ_n} be a critical point for $I^{T_0}_{\lambda,\mu_n}$ at level c_{μ_n}. By (20) and the definitions of $I^{T_0}_{\lambda,\mu_n}$, we can infer that $\{u_{\mu_n}\}$ is a bounded Palais-Smale sequence for I_λ, and $\|u_{\mu_n}\| \le T_0$. In fact, since $\mu_n \to 1$, according to (18) and (26), we can get

$$\begin{aligned} c_{\mu_n} &= I^{T_0}_{\lambda,\mu_n}(u_{\mu_n}) \\ &= \frac{1}{2}\|u_{\mu_n}\|^2 - \frac{1}{2} g_{T_0}(u_{\mu_n}) G_\lambda(u_{\mu_n}) - \mu_n \int_{\mathbb{R}^3} F(u_{\mu_n}) \mathrm{dx} \\ &= \frac{1}{2}\|u_{\mu_n}\|^2 - \frac{1}{2} G_\lambda(u_{\mu_n}) - \mu_n \int_{\mathbb{R}^3} F(u_{\mu_n}) \mathrm{dx} \\ &= I_\lambda(u_{\mu_n}) + o_n(1). \end{aligned}$$

Consequently, we have

$$\lim_{n\to\infty} I_\lambda(u_{\mu_n}) = \lim_{n\to\infty}\left(I^{T_0}_{\lambda,\mu_n}(u_{\mu_n}) + (\mu_n - 1)\int_{\mathbb{R}^3} F(u_{\mu_n}) \mathrm{dx}\right) = \lim_{n\to\infty} c_{\mu_n} = c_1.$$

Moreover, through (20) and (34), we get

$$\begin{aligned} 0 &= \left\langle (I^{T_0}_{\lambda,\mu_n})'(u_{\mu_n}), v \right\rangle \\ &= \left(1 - \frac{1}{T_0^2}\chi'\Big(\frac{\|u_{\mu_n}\|^2}{T_0^2}\Big) G_\lambda(u_{\mu_n})\right) \int_{\mathbb{R}^3} (\nabla u_{\mu_n} \cdot \nabla v + u_{\mu_n} v) \mathrm{dx} \\ &\quad + \lambda g_{T_0}(u_{\mu_n}) \int_{\mathbb{R}^3} \phi_\lambda(u_{\mu_n}) u_{\mu_n} v \mathrm{dx} - \mu_n \int_{\mathbb{R}^3} f(u_{\mu_n}) v \mathrm{dx} \\ &= \int_{\mathbb{R}^3} (\nabla u_{\mu_n} \cdot \nabla v + u_{\mu_n} v) \mathrm{dx} + \lambda \int_{\mathbb{R}^3} \phi_\lambda(u_{\mu_n}) u_{\mu_n} v \mathrm{dx} - \mu_n \int_{\mathbb{R}^3} f(u_{\mu_n}) v \mathrm{dx} \\ &= \langle I'_\lambda(u_{\mu_n}), v \rangle + o_n(1)\|v\|. \end{aligned}$$

Therefore, $\{u_{\mu_n}\}$ is a bounded $(PS)_{c_1}$ sequence of I_λ in $H^1_r(\mathbb{R}^3)$. Similar to Lemma 2.6, there exists $u_\lambda \in H^1_r(\mathbb{R}^3)$ such that

$$u_{\mu_n} \to u_\lambda \ \ in\ H^1_r(\mathbb{R}^3), \quad \text{as } n \to \infty. \tag{56}$$

By (32) and (56), we can get that $u_\lambda \neq 0$. The proof of Theorem 1.1 is completed.

Proof of Theorem 1.2 Define

$$k = \inf\{I_\lambda(u) \colon u \in \mathscr{M}\},$$

where the set $\mathscr{M} = \left\{u \in H_r^1(\mathbb{R}^3) \colon u \neq 0,\, I'_\lambda(u) = 0\right\}$. We can know from Theorem 1.1 that $\mathscr{M} \neq \emptyset$, so $k < +\infty$.

Since $\nu_2 > 3$, there exists $\delta_1,\, \delta_2 > 0$ such that

$$I_\lambda(u) \geq \delta_1 \|u\|^2 + \delta_2 \int_{\mathbb{R}^3} \frac{|\nabla\phi_\lambda(u)|^2}{1 - \frac{1}{2}|\nabla\phi_\lambda(u)|^2} \mathrm{dx}, \quad \forall u \in \mathscr{M}. \tag{57}$$

Obviously, $(u,\, \phi_\lambda(u))$ satisfies the following Pohožaev type identity

$$\begin{aligned} &3\int_{\mathbb{R}^3} F(u)\mathrm{dx} \\ =&\frac{1}{2}|\nabla u|_2^2 + \frac{3}{2}|u|_2^2 + 2\int_{\mathbb{R}^3} \frac{|\nabla\phi_\lambda(u)|^2}{1 - \frac{1}{2}|\nabla\phi_\lambda(u)|^2}\mathrm{dx} + \frac{3}{2}\int_{\mathbb{R}^3} \ln(1 - \frac{1}{2}|\nabla\phi_\lambda(u)|^2)\mathrm{dx}. \end{aligned} \tag{58}$$

And because $I'_\lambda(u) = 0$, we can get

$$\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2 + \lambda\phi_\lambda(u)u^2)\mathrm{dx} = \int_{\mathbb{R}^3} f(u)u\mathrm{dx} \tag{59}$$

By calculating (58)-(59), we combine with (f_4) and obtain

$$\begin{aligned} \int_{\mathbb{R}^3} F(u)\mathrm{dx} \leq& (\frac{2}{\nu_2} - \frac{1}{6})|\nabla u|_2^2 + (\frac{2}{\nu_2} - \frac{1}{2})|u|_2^2 + (\frac{2}{\nu_2} - \frac{2}{3})\int_{\mathbb{R}^3} \frac{|\nabla\phi_\lambda(u)|^2}{1 - \frac{1}{2}|\nabla\phi_\lambda(u)|^2}\mathrm{dx} \\ &- \frac{1}{2}\int_{\mathbb{R}^3} \ln(1 - \frac{1}{2}|\nabla\phi_\lambda(u)|^2)\mathrm{dx}. \end{aligned} \tag{60}$$

Substituting (60) into (18), and combining inequality $-\ln(1 - \frac{1}{2}s) \leq \frac{s}{2-s},\ s \in [0,\, 2)$,we get

$$\begin{aligned} I_\lambda(u) \geq& \left(\frac{2}{3} - \frac{2}{\nu_2}\right)|\nabla u|_2^2 + \left(1 - \frac{2}{\nu_2}\right)|u|_2^2 + \left(\frac{7}{6} - \frac{2}{\nu_2}\right)\int_{\mathbb{R}^3} \frac{|\nabla\phi_\lambda(u)|^2}{1 - \frac{1}{2}|\nabla\phi_\lambda(u)|^2}\mathrm{dx} \\ &+ \int_{\mathbb{R}^3} \ln(1 - \frac{1}{2}|\nabla\phi_\lambda(u)|^2)\mathrm{dx} \end{aligned}$$

$$\geq \left(\frac{2}{3}-\frac{2}{\nu_2}\right)|\nabla u|_2^2+\left(1-\frac{2}{\nu_2}\right)|u|_2^2+\left(\frac{2}{3}-\frac{2}{\nu_2}\right)\int_{\mathbb{R}^3}\frac{|\nabla\phi_\lambda(u)|^2}{1-\frac{1}{2}|\nabla\phi_\lambda(u)|^2}\mathrm{d}x. \tag{61}$$

Since $\nu_2 > 3$, every coefficient in inequality (61) is positive. Thus, (57) is valid.

For $u \in \mathscr{M}$, by $\langle I_\lambda'(u), u\rangle = 0$ and (f_2), one has

$$\|u\|^2 \leq \|u\|^2+\int_{\mathbb{R}^3}\lambda\phi_\lambda(u)u^2\mathrm{d}x = \int_{\mathbb{R}^3}f(u)u\mathrm{d}x \leq \frac{1}{2}\|u\|^2+C\|u\|^6, \tag{62}$$

where C is a positive constant. Therefore, $\|u\| \geq \delta_3 = \sqrt[4]{\frac{1}{2C}}$.

Combining (57) with (62), we can conclude that $k > 0$. Now we only need to prove that k can be achieved in $H_r^1(\mathbb{R}^3)$. Let $\{u_n\}$ be a sequence of nontrivial critical point of I_λ satisfying $I_\lambda(u_n) \to k$. By (57), we can know that $\{I_\lambda(u_n)\}$ is bounded, so we can infer that $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^3)$. In particular, $\{u_n\}$ is a bounded (PS) sequence of I_λ. The same argument as Lemma 2.6, there exists $\bar{u}_\lambda \in H_r^1(\mathbb{R}^3)$ such that

$$I_\lambda(\bar{u}_\lambda) = k, \quad I_\lambda'(\bar{u}_\lambda) = 0.$$

The proof of Theorem 1.2 is completed.

Proof of Theorem 1.3 For any $\lambda \in (0, \lambda_0)$, let $(u_\lambda, \phi_\lambda(u_\lambda))$ be the solution of system (1) obtained in Theorem 1.1. We can know from $\|u_{\mu_n}\| \leq T_0$ and (56) that $\|u_\lambda\| \leq T_0$. So $\{u_\lambda\}$ is bounded in $H_r^1(\mathbb{R}^3)$ with respect to λ. Therefore, there exists $\bar{u} \in H_r^1(\mathbb{R}^3)$ such that

$$\text{as } \lambda \to 0^+, \quad u_\lambda \rightharpoonup \bar{u} \ \text{ in } H_r^1(\mathbb{R}^3); \quad u_\lambda \to \bar{u} \ \text{ in } L^p(\mathbb{R}^3), \ \text{ for } p \in (2, 6). \tag{63}$$

Since $\langle I_\lambda'(u_\lambda), v\rangle = 0$, we have

$$\int_{\mathbb{R}^3}(\nabla u_\lambda\cdot\nabla v+u_\lambda v)\mathrm{d}x+\lambda\int_{\mathbb{R}^3}\phi_\lambda(u_\lambda)u_\lambda v\mathrm{d}x = \int_{\mathbb{R}^3}f(u_\lambda)v\mathrm{d}x, \quad \forall\, v \in H_r^1(\mathbb{R}^3). \tag{64}$$

Let $\lambda \to 0^+$ in (64), by (17), we have

$$0 < \lambda\int_{\mathbb{R}^3}\phi_\lambda(u_\lambda)u_\lambda^2\mathrm{d}x \leq 2\lambda^2W^{-1}C_{\frac{12}{5}}^4\|u_\lambda\|^4 \to 0; \quad \|\phi_\lambda(u_\lambda)-0\|_{D^{1,2}(\mathbb{R}^3)} = |\nabla\phi_\lambda(u_\lambda)|_2 \to 0.$$

For this, it can be concluded that

$$\phi_\lambda(u_\lambda) \to 0, \quad \text{in } D^{1,2}(\mathbb{R}^3).$$

Meanwhile, combining with (63), when $\lambda \to 0^+$ in (64), it can be simplified into the following form

$$\int_{\mathbb{R}^3} (\nabla \bar{u} \cdot \nabla v + \bar{u} v) \mathrm{dx} = \int_{\mathbb{R}^3} f(\bar{u}) v \mathrm{dx}.$$

Therefore, $\bar{u}$ is a weak solution of equation (14).

Now, let's prove that $u_\lambda \to \bar{u}$ in $H_r^1(\mathbb{R}^3)$, as $\lambda \to 0^+$ and $\bar{u} \neq 0$. Because u_λ is a nontrivial critical point of I_λ, so we can get that $\langle I'_\lambda(u_\lambda), u_\lambda \rangle = 0$. Through (f_1) and (f_2), we have

$$\int_{\mathbb{R}^3} (|\nabla u_\lambda|^2 + |u_\lambda|^2) \mathrm{dx} \leq \int_{\mathbb{R}^3} f(u_\lambda) v \mathrm{dx} \leq \frac{1}{2} \|u_\lambda\|^2 + C |u_\lambda|_6^6,$$

combining with the definition of constant W, it can be inferred that

$$\|u_\lambda\|^2 \leq C |u_\lambda|_6^6 \leq C W^{-3} \|u_\lambda\|^6. \tag{65}$$

From (65), we can obtain $\|u_\lambda\| \geq C^{-\frac{1}{4}} S^{\frac{3}{4}}, \ \forall \lambda \in (0, \lambda_0)$. Since

$$\begin{aligned} 0 &= \langle I'_\lambda(u_\lambda) - I'_\lambda(\bar{u}), u_\lambda - \bar{u} \rangle \\ &= \|u_\lambda - \bar{u}\|^2 + \lambda \int_{\mathbb{R}^3} \phi_\lambda(u_\lambda) u_\lambda (u_\lambda - \bar{u}) \mathrm{dx} + \int_{\mathbb{R}^3} (f(\bar{u}) - f(u_\lambda))(u_\lambda - \bar{u}) \mathrm{dx}, \end{aligned}$$

similar to the discussion in Lemma 2.6, from the boundedness of $\{u_\lambda\}$ and $\lambda \to 0^+$, we can obtain

$$\|u_\lambda - \bar{u}\| \to 0.$$

Hence, we can infer that $\|\bar{u}\| \geq C^{-\frac{1}{4}} W^{\frac{3}{4}}$, so $\bar{u} \neq 0$. The proof of Theorem 1.3 is completed.

4. Conclusions

In this paper, we investigate a coupled system of the Schrödinger equation and the logarithmic Born-Infeld equation in three-dimensional space. In contemporary theoretical physics, the Schrödinger-Born-Infeld system also plays an active and important role. The coupling problem between the Schrödinger equation and the logarithmic type Born-Infeld equation gives rise to insights and new ideas about how space-time geometry and matter interactions can be coupled. Compared to the Schrödinger-Born-Infeld coupled systems in references [14, 15], the second equation of system (1) is a logarithmic Born-Infeld equation, which differs from the classical Born-Infeld equations in [14, 15]. Not only does this equation (see (7)) exhibit singularities, but the corresponding functional (see (8)) also possesses singularities. Meanwhile, the nonlinear term f in system (1) is more general (see (f_1)-(f_4)). Therefore, we have synthetically applied variational methods, truncation techniques and other analytical tools to study system (1), established the existence of solutions for the coupled Schrödinger-Born-Infeld system (1) (see Theorem 1.1 and Theorem 1.2) as well as the asymptotic behavior of the solutions with respect to parameter variations (see Theorem 1.3). Our approach provides valuable insights for addressing similar Born-Infeld coupled systems.

In addition, there are many open questions regarding system (1) we should pay attention to. For instance, in this paper, we have verified the existence and asymptotic behavior of solutions to system (1) under the constraint condition $|\phi|_\infty \leq \alpha$ with $0 < \alpha < \sqrt{2}$. However, does a solution still exist when $\alpha = \sqrt{2}$? This problem is worth thinking deeply and studying.

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Author contributions

Ruifeng Zhang: investigation, writing-original draft, writing-review and editing, methodology, formal analysis. Ruixin Zhang: methodology, writing-review and editing, writing-original draft, investigation, formal analysis. Xiangyi Zhang: writing-review and editing, writing-original draft, investigation.

Data availability statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflict of interest

The author declares no competing financial interest.

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