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Research Article

Gradient Estimate for Solutions of the Equation $\Delta_p u = a|\nabla u|^q + be^{cu}$ on a Complete Riemannian Manifold

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Abstract: In this paper, a universal gradient estimate for a quasilinear elliptic equation $\Delta_p u = a|\nabla u|^q + be^{cu}$ on a Riemannian manifold is presented. As applications, a Liouville theorem and Harnack inequalities for positive solutions are established. These results cover gradient estimates for many equations, including the quasi-linear Hamilton-Jacobi equation, the Lane-Emden equation, and others.

***Keywords*:** non-linear elliptic equation, gradient estimate, p-Laplace

MSC: 53C20

1. Introduction

One trend in Riemannian geometry since the 1950s has been the study of how curvature affects global properties of partial differential equations and global quantities like the eigenvalues of the Laplacian. It is well-known that gradient estimates are fundamental and powerful techniques in the analysis of partial differential equations on Riemannian manifolds.

One of the most classical results about gradient estimates should be traced back to Cheng-Yau's gradient estimate for positive harmonic functions (see [1, 2]).

Theorem Let M be an n-dimensional complete Riemannian manifold with $Ric \geq -(n-1)\kappa$, where $\kappa \geq 0$ is a constant. Suppose that u is a positive harmonic function on a geodesic ball $B(o, R)$. Then

$$\sup_{B(o,\, R/2)} \frac{|\nabla u|}{u} \leq C_n \frac{1 + R\sqrt{\kappa}}{R},$$

where C_n is a constant depending only on n.

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An important feature of Cheng-Yau's estimate is that the right-hand side of the estimate depends only on n, κ and R, it does not depend on the injective radius or other global properties. Cheng-Yau's estimate turned out to be very useful. Harnack inequality can be derived immediately from Cheng-Yau's estimate. Liouville theorem for global positive harmonic functions on noncompact manifolds with nonnegative Ricci curvature is also a direct consequence of Cheng-Yau's estimate. Cheng-Yau's approach can also be used to derive the estimates of the spectrum of manifolds and investigate the geometry of manifolds (see [1, 3]).

A natural question is if similar gradient estimates hold for more general equations on manifold. First, we recall Cheng-Yau's original proof. They mainly used Bochner's formula on manifolds

$$\Delta\left(\frac{1}{2}|\nabla u|^2\right) = |\mathrm{Hess}\, u|^2 + \langle \nabla u,\ \nabla \Delta u\rangle + \mathrm{Ric}(\nabla u,\ \nabla u),$$

Laplacian comparison theorem and cut-off function technique. However, when we deal with nonlinear equations, Hessian of cut-off function may appear and Laplacian comparison theorem could not control the second derivative terms of cut-off function. In 2009, Kotschwar and Ni [4] generalized Cheng-Yau's estimate to positive p-harmonic functions by a similar technique. However, in order to control the Hessian of cut-off function, they strengthened the Ricci curvature conditions to sectional curvature conditions and use the Hessian comparison theorem.

Theorem (Kotschwar-Ni) Let $(M,\ g)$ be a complete non-compact Riemannian manifold with $\mathrm{Sec}_g \geq -\kappa$ with $\kappa \geq 0$, then for any p-harmonic function u, we have

$$\sup_{B(o,\ R/2)} \frac{|\nabla u|}{u} \leq C_{n,\ p} \frac{1 + R\sqrt{\kappa}}{R}.$$

A natural question is whether or not the sectional curvature condition in the above theorem can be weakened to Ricci curvature. In 2011, this problem was answered affirmatively by Wang and Zhang in [5]. Different from the maximum principle used in Cheng-Yau estimates, in [5] the authors used Nash-Moser iteration technique to avoid involving the second derivatives of cut-off function. The theorem is stated as follows:

Theorem Let $(M^n,\ g)$ be an n-dimensional complete Riemannian manifold with the Ricci curvature fulfilling $Ric \geq -(n-1)\kappa$, where κ is a nonnegative constant. Assume that u is a positive p-harmonic function on the ball $B(o,\ R) \subset M$. Then, there holds

$$\frac{|\nabla u|}{u} \leq C_{n,\ p} \frac{1 + \sqrt{\kappa} R}{R}.$$

Recently, the development of gradient estimates is along two main routines. One is to generalize the gradient estimate of harmonic function to a more general space. For example, Zhang-Zhu (see [6]) generalized Cheng-Yau's estimate to harmonic functions on Alexandrov space, and Xia (see [7]) generalized it to harmonic functions on Finsler manifolds. The other is to study the gradient estimate for solutions to more complicated and general nonlinear non-homogeneous elliptic equations. For example, see [8–15] for details.

In this paper, we are concerned with the equation

$$\begin{cases} \Delta_p u = a|\nabla u|^q + be^{cu}; \\ \\ p > 1, \quad q > p-1; \\ \\ a \neq 0, \quad ab \geq 0 \end{cases} \tag{1}$$

on a Riemannian manifold. The equation (1) could be viewed as generalizations of many classical equations. If $a = b = 0$, then equation (1) is just the p-Laplace equation; if $b = 0$ and $a = 1$, then equation (1) becomes the quasi-linear Hamilton-Jacobi equation, see [8, 16, 17] for more studies; if $a = 1$, $q = p$ and $b = (p-1)^{p-1}$, then equation (1) is the logarithm transformation of the Lane-Emden equation which has already been the subject of numerous publications (see [18–26]).

Inspired by the previous work [8], we will combine the point-wise estimate and the Nash-Moser iteration technique to establish an universal gradient estimate for positive solutions to equation (1) on a complete Riemannian manifold with Ricci curvature bounded from below. Our main theorem is stated as follows.

Theorem 1 Let $(M,\ g)$ be an n-dim $(n > 2)$ complete manifold with Ricci curvature $Ric_g \geq -(n-1)\kappa g$, where κ is a non-negative constant. Assume u is a solution to equation (1) on a geodesic ball $B(o,\ R) \subset M$. In addition, suppose that $bc \geq 0$ or $|a| > \frac{n-1}{4}|c|$. Then there holds true

$$\sup_{B(o,\ R/2)} |\nabla u| \leq C_{n,\ p,\ q,\ a,\ c} \left(\frac{1+\sqrt{\kappa}R}{R} \right)^{\frac{1}{1-p+q}}.$$

Remark 2 In the case $b = 0$, $a = 1$ and $q = p$, (1) reduces to

$$\Delta_p u - |\nabla u|^p = 0,$$

which is the logarithm transformation of the p-Laplace equation $\Delta_p v = 0$(Precisely, taking $u = -(p-1)\ln v$). Theorem 1 covers Wang-Zhang's gradient estimate in [5].

In the case $b = 0$ and $a = 1$, equation (1) reduces to quasi-linear Hamilton-Jacobi equation

$$\Delta_p u - |\nabla u|^q = 0.$$

Theorem 1 recovers the part results in [8].

Remark 3 Another related equations is the Lane-Emden equation

$$\Delta_p v + av^q = 0, \quad a > 0.$$

By a logarithmic transformation $u = -(p-1)\log v$, the above equation can be rewritten as

$$\Delta_p u - |\nabla u|^p - be^{cu} = 0,$$

where

$$b = a(p-1)^{p-1} \quad \text{and} \quad c = \frac{p-q-1}{p-1}.$$

According to Theorem 1, if $c > 0$ or $|\frac{q}{p-1} - 1| < \frac{4}{n-1}$, i.e.,

$$\frac{q}{p-1} < \frac{n+3}{n-1},$$

then

$$\frac{|\nabla v|}{v} \leq C_{n,\ p,\ q,\ a} \frac{1+\sqrt{\kappa}R}{R}.$$

Thus, as a corollary, Theorem 1 can recover the main theorem in [9].

By the gradient estimate above, we can get a Liouville-type result for positive solutions to (1) on noncompact complete Riemannian manifolds with nonnegative Ricci curvature.

Corollary 4 Let $(M^n,\ g)$ be an n-dimensional complete noncompact Riemannian manifold with nonnegative Ricci curvature. Assume the conditions in Theorem 1 are fulfilled. If $b = 0$, then any global solution to equation (1) must be a constant. If $b \neq 0$, then equation (1) admits no global solution.

By Theorem 1, it is not hard to get the Harnack inequality which is a natural corollary of the gradient estimate.

Theorem 5 Assume that M satisfies the same assumptions as in Theorem 1. If $u \in C^1(M)$ is a global solution to the equation (1) on M, then for any fixed $a \in M$ and any $x \in M$, we have

$$u(a) - C_{n,\ p,\ q,\ a,\ c}\kappa^{\frac{1}{2(1-p+q)}} d(x,\ a) \leq u(x) \leq u(a) + C_{n,\ p,\ q,\ a,\ c}\kappa^{\frac{1}{2(1-p+q)}} d(x,\ a), \tag{2}$$

where $d(x,\ a)$ denotes the geodesic distance from x to a.

We will follow the main ideas of [8] to approach the gradient estimate of equation (1) in the present paper, but we need to overcome some new technique difficulties. The structure of the paper is organized as follows. In Section 2, we give a meticulous estimate of $\mathscr{L}\left(|\nabla u|^{2\alpha}\right)$ where $\mathscr{L}$ is the linearized operator of the p-Laplacian operator Δ_p at u (see (3) for the explicit definition of the operator $\mathscr{L}$). We also recall Saloff-Coste's Sobolev embedding theorem in Section 2. In Section 3, we establish a universal integral estimate on $|\nabla u|^{2\alpha}$ and then use delicately the Nash-Moser iteration to prove the main results of this paper. The proofs of Corollary 4, Theorem 5 are also provided in Section 3.

2. Preliminaries

Throughout this paper, we denote an n-dim Riemannian manifold by $(M,\ g)$, and the corresponding Levi-Civita connection by ∇. For any function $\varphi \in C^1(M)$, we denote $\nabla\varphi \in \Gamma(T^*M)$ by

$$\nabla\varphi(X) = \nabla_X \varphi.$$

We denote the volume form of $(M,\ g)$ by $= \sqrt{\det(g_{ij})}dx_1 \wedge \ \ldots \ \wedge dx_n$, where $(x_1,\ \ldots,\ x_n)$ is a local coordinates on $(M,\ g)$, and for simplicity we may omit the volume form of integral over M.

The p-Laplace operator is defined by

$$\Delta_p u = \left(|\nabla u|^{p-2}\nabla u\right).$$

The solution of p-Laplace equation $\Delta_p u = 0$ is the critical point of the energy functional

$$E(u) = \int_M |\nabla u|^p.$$

Definition 6 A function v is said to be a (weak) solution of equation (1) on a region $\Omega \subset M$, if $v \in C^1(\Omega) \cap W^{1,\ p}_{loc}(\Omega)$ and for all $\psi \in W^{1,\ p}_0(\Omega)$ there holds

$$\int_\Omega |\nabla v|^{p-2} \langle \nabla v,\ \nabla \psi \rangle + \int_\Omega a|\nabla v|^q \psi + \int_\Omega be^{cu}\psi = 0.$$

It is worth mentioning that any solution v of equation (1) satisfies

$$v \in W^{2,\ 2}_{loc}(\Omega \setminus \Omega_{cr}) \quad \text{and} \quad v \in C^{1,\ \gamma}(\Omega)$$

for some $\gamma \in (0,\ 1)$ (for example, see [27–29]). Here $\Omega_{cr} = \{x : \nabla u(x) = 0\}$. By a very recent result [30, Corollary 1.6], one has known that the measure of critical point set of u, i.e., Ω_{cr} is zero.

Next, we recall the Saloff-Coste's Sobolev inequalities (see [31, Theorem 3.1]) which shall play a key role in our proof of the main theorem.

Lemma 7 [31] Let $(M,\ g)$ be a complete manifold with $Ric \geq -(n-1)\kappa$. For $n > 2$, there exists a positive constant C_n depending only on n, such that for all $B \subset M$ of radius R and volume V we have for $f \in C^\infty_0(B)$

$$\|f\|^2_{L^{\frac{2n}{n-2}}} \leq e^{C_n(1+\sqrt{\kappa}R)} V^{-\frac{2}{n}} R^2 \left(\int |\nabla f|^2 + R^{-2} f^2 \right).$$

For $n = 2$, the above inequality holds with n replaced by any fixed $n' > 2$.

Now we consider the linearized operator $\mathscr{L}$ of p-Laplacian:

$$\mathscr{L}(\psi) = \left(f^{p/2-1} A(\nabla \psi) \right), \tag{3}$$

where

$$A(\nabla \psi) = \nabla \psi + (p-2) f^{-1} \langle \nabla \psi,\ \nabla u \rangle \nabla u. \tag{4}$$

The following expression of $\mathscr{L}(f^\alpha)$ for any $\alpha > 0$ is useful in our proof.

Lemma 8 [9] For any $\alpha > 0$, the equality

$$\begin{aligned}\mathscr{L}(f^\alpha) = \alpha\left(\alpha + \frac{p}{2} - 2\right) f^{\alpha+\frac{p}{2}-3}|\nabla f|^2 + 2\alpha f^{\alpha+\frac{p}{2}-2}\left(|\nabla\nabla u|^2 + (\nabla u,\ \nabla u)\right) \\ + \alpha(p-2)(\alpha-1) f^{\alpha+\frac{p}{2}-4}\langle\nabla f,\ \nabla u\rangle^2 + 2\alpha f^{\alpha-1}\langle\nabla\Delta_p u,\ \nabla u\rangle\end{aligned} \tag{5}$$

holds point-wisely in $\{x : f(x) > 0\}$.

3. Proof of main theorem

We divide the proof of Theorem 1 into three parts. In the first part, we derive a fundamental integral inequality on $f = |\nabla u|^2$, which will be used in the second and third parts. In the second part, we give a L^{α_1}-estimate of f on a geodesic ball with radius $3R/4$, where L^{α_1} norm of f determines the initial state of the Nash-Moser iteration. Finally, we give a complete proof of our main theorem by an intensive use of the Nash-Moser iteration method.

3.1 *Estimates for the linearized operator of p-Laplace operator*

We first need to prove a pointwise estimate for $\mathscr{L}(f^\alpha)$.

Lemma 9 Let u be a solution of equation (1) on $(M,\ g)$ with $\mathrm{Ric} \geq -(n-1)\kappa$. Denote $f = |\nabla u|^2$ and $a_1 = \left|p - \dfrac{2(p-1)}{n-1}\right|$. Then there hold:

1. If $bc \geq 0$, then for any $\alpha \geq 1$, we have the point-wise estimate

$$\mathscr{L}(f^\alpha) \geq \frac{2\alpha a^2 f^{\alpha+q-\frac{p}{2}}}{n-1} - 2(n-1)\alpha\kappa f^{\alpha+\frac{p}{2}-1} - \alpha a_1|a| f^{\alpha+\frac{q}{2}-\frac{3}{2}}|\nabla f|$$

for any $x \in \{x : f(x) > 0\}$.

2. If

$$|a| > \frac{n-1}{4}|c|,$$

then there exists some $\alpha_0 = \alpha_0(n,\ p,\ q,\ a,\ c) > 0$ such that for any $\alpha \geq \alpha_0$ we obtain the following inequality

$$\mathscr{L}(f^\alpha) \geq 2\alpha\beta_{n,\ p,\ a,\ c,\ \alpha} f^{\alpha+\frac{p}{2}} - 2\alpha(n-1)\kappa f^{\alpha+\frac{p}{2}-1} - \alpha a_1|a| f^{\alpha+\frac{q}{2}-\frac{3}{2}}|\nabla f|,$$

for any $x \in \{x : f(x) > 0\}$, here

$$\beta_{n,\ p,\ a,\ c,\ \alpha} = \frac{a^2}{n-1} - \frac{(2\alpha-1)(n-1)+p-1}{4(2\alpha-1)}c^2 > 0.$$

Proof. Choosing an orthonormal frame $\{e_1,\ e_2,\ \ldots,\ e_n\}$ for the tangent bundle of M on a domain with $f \neq 0$ such that $e_1 = \dfrac{\nabla u}{|\nabla u|}$, straightforward calculations lead to $u_1 = f^{1/2}$ and

$$u_{11} = \frac{1}{2} f^{-1/2} f_1 = \frac{1}{2} f^{-1} \langle \nabla u,\ \nabla f \rangle . \tag{6}$$

In this case, $\Delta_p u$ can be written as follows (see [4, 5]),

$$\Delta_p u = f^{\frac{p}{2}-1} \left((p-1) u_{11} + \sum_{i=2}^{n} u_{ii} \right).$$

Substituting the above equality into equation (1) yields

$$(p-1) u_{11} + \sum_{i=2}^{n} u_{ii} = a f^{1+\frac{q-p}{2}} + b e^{cu} f^{1-\frac{p}{2}}. \tag{7}$$

Using the fact $u_1 = f^{1/2}$ again yields

$$|\nabla f|^2 / f = 4 \sum_{i=1}^{n} u_{1i}^2. \tag{8}$$

Applying Cauchy inequality, it is found that

$$|\nabla \nabla u|^2 \geq \sum_{i=1}^{n} u_{1i}^2 + \sum_{i=2} u_{ii}^2 \geq \frac{|\nabla f|^2}{4f} + \frac{1}{n-1} \left(\sum_{i=2} u_{ii} \right)^2 . \tag{9}$$

It follows from (1) that

$$\langle \nabla \Delta_p u,\ \nabla u \rangle = a q f^{\frac{q}{2}} u_{11} + b c e^{cu} f. \tag{10}$$

Combining (6), (8), (9), (10) and (5), one gets

$$\begin{aligned} \frac{f^{2-\alpha-\frac{p}{2}}}{2\alpha} \mathscr{L}(f^{\alpha}) \geq & \frac{1}{2} \left(\alpha + \frac{p-3}{2} \right) \frac{|\nabla f|^2}{f} + \frac{1}{n-1} \left(\sum_{i=2} u_{ii} \right)^2 + (\nabla u,\ \nabla u) \\ & + 2(p-2)(\alpha-1) u_{11}^2 + f^{1-\frac{p}{2}} \left(a q f^{\frac{q}{2}} u_{11} + b c e^{cu} f \right). \end{aligned} \tag{11}$$

Furthermore, noting that

$$|\nabla f|^2/f \geq 4u_{11}^2,$$

which follows from (8), and the fact $\alpha + \frac{p-3}{2} > 0$ since $\alpha \geq 1$ and $p > 1$, we can infer from (11) that

$$\begin{aligned} \frac{f^{2-\alpha-\frac{p}{2}}}{2\alpha} \mathscr{L}(f^{\alpha}) \geq & 2\left(\alpha + \frac{p-3}{2}\right) u_{11}^2 + \frac{1}{n-1}\left(\sum_{i=2} u_{ii}\right)^2 + (\nabla u, \nabla u) \\ & + 2(p-2)(\alpha-1)u_{11}^2 + f^{1-\frac{p}{2}}\left(aqf^{\frac{q}{2}}u_{11} + bce^{cu}f\right). \end{aligned} \tag{12}$$

By (7), we have

$$\begin{aligned} \left(\sum_{i=2} u_{ii}\right)^2 &= \left(af^{1+\frac{q-p}{2}} + be^{cu}f^{1-\frac{p}{2}} - (p-1)u_{11}\right)^2 \\ &= a^2 f^{2-p+q} + \left(be^{cu}f^{1-\frac{p}{2}} - (p-1)u_{11}\right)^2 + 2abe^{cu}f^{2-p+\frac{q}{2}} - 2f^{1+\frac{q-p}{2}}(p-1)u_{11}. \end{aligned}$$

The above inequality is substituted into (12), yielding the following inequality

$$\begin{aligned} \frac{f^{2-\alpha-\frac{p}{2}}}{2\alpha} \mathscr{L}(f^{\alpha}) \geq & (p-1)(2\alpha-1)u_{11}^2 - (n-1)\kappa f + a\left(q - \frac{2(p-1)}{n-1}\right) f^{1+\frac{q-p}{2}} u_{11} \\ & + \frac{a^2 f^{2-p+q}}{n-1} + bce^{cu}f^{2-\frac{p}{2}} + \frac{2ab}{n-1}e^{cu}f^{2-p+\frac{q}{2}} + \frac{1}{n-1}\left(be^{cu}f^{1-\frac{p}{2}} - (p-1)u_{11}\right)^2. \end{aligned} \tag{13}$$

Denote $a_1 = \left|q - \frac{2(p-1)}{n-1}\right|$, from (6), we observe that

$$2\left(q - \frac{2(p-1)}{n-1}\right) f u_{11} \geq -a_1 f^{\frac{1}{2}} |\nabla f|.$$

Inserting a_1 into (13), it is obvious that

$$\begin{aligned} \frac{f^{2-\alpha-\frac{p}{2}}}{2\alpha} \mathscr{L}(f^{\alpha}) \geq & (p-1)(2\alpha-1)u_{11}^2 - (n-1)\kappa f - \frac{|a|a_1}{2} f^{\frac{1+q-p}{2}} |\nabla f| + \frac{a^2 f^{2-p+q}}{n-1} \\ & + bce^{cu}f^{2-\frac{p}{2}} + \frac{2ab}{n-1}e^{cu}f^{2-p+\frac{q}{2}} + \frac{1}{n-1}\left(be^{cu}f^{1-\frac{p}{2}} - (p-1)u_{11}\right)^2. \end{aligned} \tag{14}$$

Case I: $a,\ b,\ c,\ p$ and q satisfy

$$ab \geq 0, \quad bc \geq 0.$$

In this case, the last 3 terms on right-hand side of (14) are all non-negative. We omit the non-negative terms to obtain

$$\mathscr{L}(f^{\alpha}) \geq 2\alpha f^{\alpha+\frac{p}{2}-2}\left(\frac{a^2 f^{2-p+q}}{n-1} - (n-1)\kappa f - -\frac{|a|a_1}{2} f^{\frac{1+q-p}{2}}|\nabla f|\right),$$

which is just the inequality in the first case of Lemma 9.

Case II: a, b, c, p and q satisfy

$$ab \geq 0 \quad \text{and} \quad |a| > \frac{n-1}{4}|c|.$$

In this case, the second condition reduces to

$$\frac{a^2}{n-1} - \frac{n-1}{4}c^2 > 0.$$

This implies

$$\lim_{\alpha\to\infty} \frac{a^2}{n-1} - c^2\frac{(2\alpha-1)(n-1)+p-1}{4(2\alpha-1)} > 0.$$

Thus, we can choose $\alpha_0 = \alpha_0(n,\ p,\ q,\ a,\ c)$ large enough such that, for any $\alpha \geq \alpha_0$ there holds true

$$\beta_{n,\ p,\ q,\ a,\ c,\ \alpha} = \frac{a^2}{n-1} - c^2\frac{(2\alpha-1)(n-1)+p-1}{4(2\alpha-1)} > 0. \tag{15}$$

By expanding the last term of the right-hand side of (14), we obtain

$$\begin{aligned}\frac{f^{2-\alpha-\frac{p}{2}}}{2\alpha}\mathscr{L}(f^{\alpha}) \geq{}& (p-1)\left(2\alpha-1+\frac{p-1}{n-1}\right)u_{11}^2 - (n-1)\kappa f - \frac{|a|a_1}{2}f^{\frac{1+q-p}{2}}|\nabla f| + \frac{a^2 f^{2-p+q}}{n-1}\\ &+ bce^{cu}f^{2-\frac{p}{2}} + \frac{2ab}{n-1}e^{cu}f^{2-p+\frac{q}{2}} + \frac{1}{n-1}\left(b^2e^{2cu}f^{2-p} - 2(p-1)be^{cu}f^{1-\frac{p}{2}}u_{11}\right).\end{aligned} \tag{16}$$

On the other hand, by using the inequality $a^2 - 2ab \geq -b^2$ we have

$$(p-1)\left(2\alpha-1+\frac{p-1}{n-1}\right)u_{11}^2-2\frac{(p-1)}{n-1}be^{cu}f^{1-\frac{p}{2}}u_{11}$$

$$\geq-\frac{(p-1)b^2e^{2cu}f^{2-p}}{((2\alpha-1)(n-1)+p-1)(n-1)}. \tag{17}$$

Combining (16) and (17) yields

$$\frac{f^{2-\alpha-\frac{p}{2}}}{2\alpha}\mathscr{L}(f^\alpha)\geq\frac{(2\alpha-1)b^2e^{2cu}f^{2-p}}{(2\alpha-1)(n-1)+p-1}-\kappa(n-1)f-\frac{|a|a_1}{2}f^{\frac{1+q-p}{2}}|\nabla f|$$

$$+bce^{cu}f^{2-\frac{p}{2}}+\frac{2ab}{n-1}e^{cu}f^{2-p+\frac{q}{2}}+\frac{a^2f^{2-p+q}}{n-1}. \tag{18}$$

Applying the inequality $a^2+2ab\geq-b^2$ again, we have

$$\frac{(2\alpha-1)b^2e^{2cu}f^{2-p}}{(2\alpha-1)(n-1)+p-1}+bce^{cu}f^{2-\frac{p}{2}}\geq-\frac{(2\alpha-1)(n-1)+p-1}{4(2\alpha-1)}c^2f^{2-p+q}. \tag{19}$$

Inserting (19) into (18), the following inequality is given

$$\frac{f^{2-\alpha-\frac{p}{2}}}{2\alpha}\mathscr{L}(f^\alpha)\geq\left(\frac{a^2}{n-1}-\frac{(2\alpha-1)(n-1)+p-1}{4(2\alpha-1)}c^2\right)f^{2-p+q}+\frac{2ab}{n-1}e^{cu}f^{2-p+\frac{q}{2}}$$

$$-(n-1)\kappa f-\frac{|a|a_1}{2}f^{\frac{1+q-p}{2}}|\nabla f|.$$

Hence,

$$\mathscr{L}(f^\alpha)\geq2\beta_{n,\ p,\ q,\ a,\ c,\ \alpha}\alpha f^{\alpha+q-\frac{p}{2}}-2\alpha(n-1)\kappa f^{\alpha+\frac{p}{2}-1}-a_1|a|\alpha f^{\alpha+\frac{q}{2}-\frac{3}{2}}|\nabla f|,$$

where $\beta_{n,\ p,\ q,\ a,\ c,\ \alpha}>0$ is defined in (15). Thus, we complete the proof of this lemma. □

From now on, we fix

$$\alpha=\alpha_0(n,\ p,\ q,\ a,\ c)$$

and use $a_1, a_2, \cdots$ to denote constants depending only on n, p, q, a, b and c. Denote $\beta=\beta_{n,\ p,\ q,\ a,\ c,\ \alpha_0}$. Moreover, from the definition of $\beta_{n,\ p,\ q,\ a,\ c,\ \alpha}$ (see (15)) we can see easily that

$$\beta < \frac{a^2}{n-1}.$$

So, we actually have proved that there holds

$$\mathscr{L}(f^{\alpha_0}) \geq 2\alpha_0 f^{\alpha_0+\frac{p}{2}-2}\left(\beta f^{2+q-p} - 2(n-1)\kappa f - \frac{a_1}{2} f^{\frac{1}{2}}|\nabla f|\right), \tag{20}$$

if one of the conditions (1) and (2) in Lemma 9 is satisfied.

3.2 *Deducing the main integral inequality*

Now, we need to establish a key integral inequality of $f = |\nabla u|^2$.

Lemma 10 Let $\Omega = B_R(o) \subset M$ be a geodesic ball. Under the same assumptions as in Lemma 9, we have the following integral inequality

$$\beta \int_\Omega f^{\alpha_0+\frac{p}{2}+t}\eta^2 + \frac{a_3}{t} e^{-t_0} V^{\frac{2}{n}} R^{-2} \left\| f^{\frac{\alpha_0+t-1}{2}+\frac{p}{4}}\eta \right\|^2_{L^{\frac{2n}{n-2}}}$$

$$\leq a_5 t_0^2 R^{-2} \int_\Omega f^{\alpha_0+\frac{p}{2}+t-1}\eta^2 + \frac{a_4}{t}\int_\Omega f^{\alpha_0+\frac{p}{2}+t-1}|\nabla\eta|^2,$$

where $\beta = \beta_{n,\, p,\, q,\, a,\, c,\, \alpha_0}$ is given in Lemma 9, a_3, a_4 and a_5 depend only on n, p, q, a and c.

Proof. By choosing $\psi = f_\varepsilon^t \eta^2$ as the test function of (20), where $\eta \in C_0^\infty(\Omega, \mathbb{R})$ is non-negative, $f_\varepsilon = (f-\varepsilon)^+$ with $\varepsilon > 0$, and $t > 1$ is to be determined later, we can deduce from (20) that

$$-\int_\Omega \left\langle f^{p/2-1}\nabla f^{\alpha_0} + (p-2) f^{p/2-2}\langle \nabla f^{\alpha_0},\, \nabla u\rangle \nabla u,\, \nabla\psi \right\rangle$$

$$\geq 2\beta\alpha_0 \int_\Omega f^{\alpha_0+q-\frac{p}{2}} f_\varepsilon^t \eta^2 - 2(n-1)\alpha_0\kappa \int_\Omega f^{\alpha_0+\frac{p}{2}-1} f_\varepsilon^t\eta^2 - a_1|a|\alpha_0 \int_\Omega f^{\alpha_0+\frac{q-3}{2}} f_\varepsilon^t |\nabla f|\eta^2.$$

Hence,

$$-\int_\Omega \alpha_0 t f^{\alpha_0+\frac{p}{2}-2} f_\varepsilon^{t-1}|\nabla f|^2\eta^2 + t\alpha_0(p-2) f^{\alpha_0+\frac{p}{2}-3} f_\varepsilon^{t-1}\left\langle \nabla f,\, \nabla u^2\right\rangle \eta^2$$

$$-\int_\Omega 2\eta\alpha_0 f^{\alpha_0+\frac{p}{2}-2} f_\varepsilon^t \langle \nabla f,\, \nabla\eta\rangle + 2\alpha_0\eta(p-2) f^{\alpha_0+\frac{p}{2}-3} f_\varepsilon^t \langle\nabla f,\, \nabla u\rangle\langle\nabla u,\, \nabla\eta\rangle \tag{21}$$

$$\geq 2\beta\alpha_0 \int_\Omega f^{\alpha_0+q-\frac{p}{2}} f_\varepsilon^t \eta^2 - 2(n-1)\alpha_0\kappa \int_\Omega f^{\alpha_0+\frac{p}{2}-1} f_\varepsilon^t\eta^2 - a_1|a|\alpha_0 \int_\Omega f^{\alpha_0+\frac{q-3}{2}} f_\varepsilon^t |\nabla f|\eta^2.$$

Next, we need to use the following two inequalities

$$f_\varepsilon^{t-1}|\nabla f|^2+(p-2)f_\varepsilon^{t-1}f^{-1}\langle\nabla f,\ \nabla u\rangle^2\geq a_2f_\varepsilon^{t-1}|\nabla f|^2, \tag{22}$$

where $a_2=\min\{1,\ p-1\}$, and

$$f_\varepsilon^{t}\langle\nabla f,\ \nabla\eta\rangle+(p-2)f_\varepsilon^{t}f^{-1}\langle\nabla f,\ \nabla u\rangle\langle\nabla u,\ \nabla\eta\rangle\geq-(p+1)f_\varepsilon^{t}|\nabla f||\nabla\eta|. \tag{23}$$

Now, substituting (22) and (23) into (21), dividing the both sides of the inequality obtained by α_0 and letting $\varepsilon\to0$, we can obtain

$$\begin{aligned}&2\beta\int_\Omega f^{\alpha_0+q-\frac{p}{2}+t}\eta^2+a_2t\int_\Omega f^{\alpha_0+\frac{p}{2}+t-3}|\nabla f|^2\eta^2\\&\leq2(n-1)\kappa\int_\Omega f^{\alpha_0+q-\frac{p}{2}+t-1}\eta^2+a_1|a|\int_\Omega f^{\alpha_0+\frac{q-3}{2}+t}|\nabla f|\eta^2+2(p+1)\int_\Omega f^{\alpha_0+\frac{p}{2}+t-2}|\nabla f||\nabla\eta|\eta.\end{aligned} \tag{24}$$

Since $u\in W_{loc}^{2,\ 2}(\Omega\setminus\Omega_{cr})\cap C^{1,\ \gamma}(\Omega)$, we have $f\in C^\gamma(\Omega)$ and $|\nabla f|\in L_{loc}^2$, and hence the integrals in the above make sense.

By Cauchy-inequality, we have

$$a_1|a|f^{\alpha_0+\frac{q-3}{2}+t}|\nabla f|\eta^2\leq\frac{a_2t}{4}f^{\alpha_0+\frac{p}{2}+t-3}|\nabla f|^2\eta^2+\frac{a_1^2a^2}{a_2t}f^{\alpha_0+q-\frac{p}{2}+t}\eta^2, \tag{25}$$

and

$$2(p+1)f^{\alpha_0+\frac{p}{2}+t-2}|\nabla f||\nabla\eta|\eta\leq\frac{a_2t}{4}f^{\alpha_0+\frac{p}{2}+t-3}|\nabla f|^2\eta^2+\frac{4(p+1)^2}{a_2t}f^{\alpha_0+\frac{p}{2}+t-1}|\nabla\eta|^2. \tag{26}$$

Now we choose t large enough such that

$$\frac{a_1^2a^2}{a_2t}\leq\beta. \tag{27}$$

Then, it follows from (24), (25), (26) and (27) that

$$\begin{aligned}&\beta\int_\Omega f^{\alpha_0+q-\frac{p}{2}+t}\eta^2+\frac{a_2t}{2}\int_\Omega f^{\alpha_0+\frac{p}{2}+t-3}|\nabla f|^2\eta^2\\&\leq2(n-1)\kappa\int_\Omega f^{\alpha_0+\frac{p}{2}+t-1}\eta^2+\frac{4(p+1)^2}{a_2t}\int_\Omega f^{\alpha_0+\frac{p}{2}+t-1}|\nabla\eta|^2.\end{aligned} \tag{28}$$

On the other hand, we have

$$\frac{1}{2}\left|\nabla\left(f^{\frac{\alpha_0+t-1}{2}+\frac{p}{4}}\eta\right)\right|^2 \le \left|\nabla f^{\frac{\alpha_0+t-1}{2}+\frac{p}{4}}\right|^2\eta^2 + f^{\alpha_0+t-1+\frac{p}{2}}|\nabla\eta|^2$$
$$= \frac{(2\alpha_0+2t+p-2)^2}{16} f^{\alpha_0+t+\frac{p}{2}-3}|\nabla f|^2\eta^2 + f^{\alpha_0+t-1+\frac{p}{2}}|\nabla\eta|^2. \tag{29}$$

Substituting (29) into (28) gives

$$\beta\int_\Omega f^{\alpha_0+q-\frac{p}{2}+t}\eta^2 + \frac{4a_2t}{(2\alpha_0+2t+p-2)^2}\int_\Omega\left|\nabla\left(f^{\frac{\alpha_0+t-1}{2}+\frac{p}{4}}\eta\right)\right|^2$$
$$\le 2(n-1)\kappa\int_\Omega f^{\alpha_0+t+\frac{p}{2}-1}\eta^2 + \frac{4(p+1)^2}{a_2t}\int_\Omega f^{\alpha_0+\frac{p}{2}+t-1}|\nabla\eta|^2 + \frac{8a_2t}{(2\alpha_0+2t+p-2)^2}\int_\Omega f^{\alpha_0+t+\frac{p}{2}-1}|\nabla\eta|^2.$$

We choose then a_3 and a_4 depending on n, p and q such that

$$\frac{a_3}{t} \le \frac{4a_2t}{(2\alpha_0+2t+p-2)^2} \quad \text{and} \quad \frac{8a_2t}{(2\alpha_0+2t+p-2)^2} + \frac{4(p+1)^2}{a_2t} \le \frac{a_4}{t}.$$

Hence

$$\beta\int_\Omega f^{\alpha_0+q-\frac{p}{2}+t}\eta^2 + \frac{a_3}{t}\int_\Omega\left|\nabla\left(f^{\frac{\alpha_0+t-1}{2}+\frac{p}{4}}\eta\right)\right|^2 \le 2(n-1)\kappa\int_\Omega f^{\alpha_0+t+\frac{p}{2}-1}\eta^2 + \frac{a_4}{t}\int_\Omega f^{\alpha_0+\frac{p}{2}+t-1}|\nabla\eta|^2. \tag{30}$$

Moreover, Saloff-Coste's Sobolev inequality tells us

$$e^{-C_n(1+\sqrt{\kappa}R)}V^{\frac{2}{n}}R^{-2}\left\|f^{\frac{\alpha_0+t-1}{2}+\frac{p}{4}}\eta\right\|^2_{L^{\frac{2n}{n-2}}(\Omega)} \le \int_\Omega\left|\nabla\left(f^{\frac{\alpha_0+t-1}{2}+\frac{p}{4}}\eta\right)\right|^2 + R^{-2}\int_\Omega f^{\alpha_0+t+\frac{p}{2}-1}\eta^2.$$

Substituting the above into (30) yields

$$\beta\int_\Omega f^{\alpha_0+q-\frac{p}{2}+t}\eta^2 + \frac{a_3}{t}e^{-C_n(1+\sqrt{\kappa}R)}V^{\frac{2}{n}}R^{-2}\left\|f^{\frac{\alpha_0+t-1}{2}+\frac{p}{4}}\eta\right\|^2_{L^{\frac{2n}{n-2}}}$$
$$\le 2(n-1)\kappa\int_\Omega f^{\alpha_0+t+\frac{p}{2}-1}\eta^2 + \frac{a_4}{t}\int_\Omega f^{\alpha_0+t+\frac{p}{2}-1}|\nabla\eta|^2 + \frac{a_3}{t}\int_\Omega R^{-2}f^{\alpha_0+\frac{p}{2}+t-1}\eta^2. \tag{31}$$

Now we set $t_0 = c_1(n,\ p,\ q,\ a,\ c)(1+\sqrt{\kappa}R)$ where

$$c_1(n,\ p,\ q,\ a,\ c) = \max\left\{C_n,\ \frac{a_1^2}{a_2\beta},\ 2\right\},$$

and choose t such that $t \geq t_0$. Since

$$2(n-1)\kappa R^2 \leq \frac{2(n-1)}{c_1^2(n,\ p,\ q)} t_0^2 \quad \text{and} \quad \frac{a_3}{t} \leq \frac{a_3}{c_1(n,\ p,\ q)},$$

there exists $a_5 = a_5(n,\ p,\ q,\ a,\ c) > 0$ such that

$$2(n-1)\kappa R^2 + \frac{a_3}{t} \leq a_5 t_0^2 = a_5 c_1^2(n,\ p,\ q)\left(1+\sqrt{\kappa}R\right)^2. \tag{32}$$

It follows from (31) and (32) that

$$\beta \int_\Omega f^{\alpha_0+q-\frac{p}{2}+t}\eta^2 + \frac{a_3}{t} e^{-t_0} V^{\frac{2}{n}} R^{-2} \left\| f^{\frac{\alpha_0+t-1}{2}+\frac{p}{4}}\eta \right\|^2_{L^{\frac{2n}{n-2}}} \leq a_5 t_0^2 R^{-2} \int_\Omega f^{\alpha_0+\frac{p}{2}+t-1}\eta^2 + \frac{a_4}{t}\int_\Omega f^{\alpha_0+\frac{p}{2}+t-1}|\nabla\eta|^2. \tag{33}$$

This is the required inequality and we finish the proof of this lemma. □

3.3 L^{α_1} *bound of gradient in a ball with radius* $3R/4$

Next, we turn to giving the following L^{α_1} upper bound of the gradient of positive solutions to equation (1).

Lemma 11 Let $(M,\ g)$ be a complete manifold with $Ric \geq -(n-1)\kappa$ and

$$\alpha_1 = \left(\alpha_0 + t_0 + \frac{p}{2} - 1\right)\frac{n}{n-2}.$$

Assume u is a positive solution to equation (1) on the geodesic ball $B(o,\ R) \subset M$ and $f = |\nabla u|^2$. Then there exists $a_8 = a_8(n,\ p,\ q,\ a,\ c) > 0$ such that

$$\|f\|_{L^{\alpha_1}(B_{3R/4}(o))} \leq a_8 V^{\frac{1}{\beta}} \left(\frac{t_0^2}{R^2}\right)^{\frac{1}{q-p+1}}, \tag{34}$$

where V is the volume of geodesic ball $B_R(o)$.

Proof. If

$$f \geq \left(\frac{2a_5 t_0^2}{\beta R^2}\right)^{\frac{1}{q-p+1}},$$

then we can obtain from (33)

$$a_5 t_0^2 R^{-2} \int_\Omega f^{\alpha_0+\frac{p}{2}+t-1}\eta^2 \leq \frac{\beta}{2}\int_\Omega f^{\alpha_0+q-\frac{p}{2}+t}\eta^2.$$

We denote

$$\Omega_1 = \left\{ f \geq \frac{2a_5 t_0^2}{\beta R^2} \right\}$$

and $\Omega_2 = \Omega \backslash \Omega_1$. Then, it is not difficult to see

$$\begin{aligned} a_5 t_0^2 R^{-2} \int_{\Omega} f^{\alpha_0 + \frac{p}{2} + t - 1} \eta^2 &= a_5 t_0^2 R^{-2} \int_{\Omega_1} f^{\alpha_0 + \frac{p}{2} + t - 1} \eta^2 + a_5 t_0^2 R^{-2} \int_{\Omega_2} f^{\alpha_0 + \frac{p}{2} + t - 1} \eta^2 \\ &\leq \frac{\beta}{2} \int_{\Omega} f^{\alpha_0 + \frac{p}{2} + t} \eta^2 + \frac{2a_5 t_0^2}{R^2} \left(\frac{2a_5 t_0^2}{\beta R^2} \right)^{\frac{\alpha_0 + \frac{p}{2} + t - 1}{q - p + 1}} V, \end{aligned} \tag{35}$$

where V is the volume of $B_R(o)$. By choosing $t = t_0$ we can obtain from (33) and (35)

$$\begin{aligned} &\frac{\beta}{2} \int_{\Omega} f^{\alpha_0 + q - \frac{p}{2} + t_0} \eta^2 + \frac{a_3}{t_0} e^{-t_0} V^{\frac{2}{n}} R^{-2} \left\| f^{\frac{\alpha_0 + t_0 - 1}{2} + \frac{p}{4}} \eta \right\|^2_{L^{\frac{2n}{n-2}}} \\ \leq &\frac{2a_5 t_0^2}{R^2} \left(\frac{2a_5 t_0^2}{\beta R^2} \right)^{\frac{\alpha_0 + \frac{p}{2} + t - 1}{q - p + 1}} V + \frac{a_4}{t_0} \int_{\Omega} f^{\alpha_0 + \frac{p}{2} + t_0 - 1} |\nabla \eta|^2. \end{aligned} \tag{36}$$

Now, we choose $\eta_1 \in C_0^\infty(B_R(o))$ satisfying

$$\begin{cases} 0 \leq \eta_1 \leq 1, \quad \eta_1 \equiv 1 \text{in} B_{\frac{3R}{4}}(o); \\ |\nabla \eta_1| \leq \frac{C(n)}{R}, \end{cases}$$

and in (36) let

$$\eta = \eta_1^{\frac{\alpha_0 + q - \frac{p}{2} + t_0}{q - p + 1}}.$$

We take a direct calculation to see that

$$a_4 R^2 |\nabla \eta|^2 \leq a_4 C^2 \left(\frac{\alpha_0 + q - \frac{p}{2} + t_0}{q - p + 1} \right)^2 \eta^{\frac{2\alpha_0 + p + 2t_0 - 2}{\alpha_0 + q - \frac{p}{2} + t_0}} \leq a_6 \alpha_0^2 \eta^{\frac{2\alpha_0 + p + 2t_0 - 2}{\alpha_0 + q - \frac{p}{2} + t_0}}. \tag{37}$$

By Hölder inequality and Young inequality, we have

$$\frac{a_4}{t_0}\int_\Omega f^{\frac{p}{2}+\alpha_0+t_0-1}|\nabla\eta|^2 \le \frac{a_6 t_0}{R^2}\int_\Omega f^{\frac{p}{2}+\alpha_0+t_0-1}\eta^{\frac{2\alpha_0+p+2t_0-2}{\alpha_0+q-p/2+t_0}}$$

$$\le \frac{a_6 t_0}{R^2}\left(\int_\Omega f^{\alpha_0+t_0+q-\frac{p}{2}}\eta^2\right)^{\frac{\alpha_0+p/2+t_0-1}{\alpha_0+q-p/2+t_0}} V^{\frac{q-p+1}{\alpha_0+t_0+q-p/2}} \tag{38}$$

$$\le \frac{\beta}{2}\left[\int_\Omega f^{\alpha_0+t_0+q-\frac{p}{2}}\eta^2 + \left(\frac{2a_6t_0}{\beta R^2}\right)^{\frac{\alpha_0+t_0+q-p/2}{q-p+1}} V\right].$$

Combining (36) and (38) we obtain

$$\left(\int_\Omega f^{\frac{n(p/2+\alpha_0+t_0-1)}{n-2}}\eta^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}$$

$$\le \frac{t_0}{a_3}e^{t_0}V^{1-\frac{2}{n}}R^2\left[\frac{2a_5t_0^2}{R^2}\left(\frac{2a_5t_0^2}{\beta R^2}\right)^{\frac{\alpha_0+\frac{p}{2}+t_0-1}{q-p+1}} + \frac{a_6t_0}{R^2}\left(\frac{2a_6t_0}{\beta R^2}\right)^{\frac{\alpha_0+\frac{p}{2}+t_0-1}{q-p+1}}\right] \tag{39}$$

$$\le a_7^{\alpha_0+t_0+\frac{p}{2}-1}e^{t_0}V^{1-\frac{2}{n}}t_0^3\left(\frac{t_0^2}{R^2}\right)^{\frac{\alpha_0+\frac{p}{2}+t_0-1}{q-p+1}},$$

where a_7 depending only on n, p, q, a and c satisfies

$$a_7^{\alpha_0+\frac{p}{2}+t_0-1} \ge \frac{2a_5}{a_3}\left(\frac{2a_5}{\beta}\right)^{\frac{\alpha_0+\frac{p}{2}+t_0-1}{q-p+1}} + \frac{a_6}{a_3t_0}\left(\frac{2a_6}{\beta t_0}\right)^{\frac{\alpha_0+\frac{p}{2}+t_0-1}{q-p+1}}.$$

Thus

$$\left\|f\eta^{\frac{2}{\alpha_0+t_0+p/2-1}}\right\|_{L^{\alpha_1}(\Omega)} \le a_7 e^{\frac{t_0}{\alpha_0+t_0+\frac{p}{2}-1}}V^{1/\alpha_1}t_0^{\frac{3}{\alpha_0+t_0+p/2-1}}\left(\frac{t_0^2}{R^2}\right)^{\frac{1}{q-p+1}} \le a_8 V^{\frac{1}{\alpha_1}}\left(\frac{t_0^2}{R^2}\right)^{\frac{1}{q-p+1}},$$

where a_8 depending only on n, p,q, a and c satisfies

$$a_8 = a_7\sup_{t_0\ge 1} t_0^{\frac{3}{\alpha_0+t_0+p/2-1}} e^{\frac{t_0}{\alpha_0+t_0+\frac{p}{2}-1}}.$$

Since $\eta \equiv 1$ in $B_{3R/4}$, we obtain that

$$\|f\|_{L^{\alpha_1}(B_{3R/4}(o))} \le a_8 V^{\frac{1}{\alpha_1}} \left(\frac{t_0^2}{R^2}\right)^{\frac{1}{q-p+1}}.$$

Thus, we complete the proof of this lemma. □

3.4 *Moser iteration*

Lemma 12 Let $(M,\ g)$ be a complete manifold with $Ric \ge -(n-1)\kappa$. Assume u is a solution to equation (1) on the geodesic ball $B(o,\ R) \subset M$ and $f = |\nabla u|^2$. Then there exists $a_{11} = a_{11}(n,\ p,\ q) > 0$ such that

$$\|f\|_{L^\infty(B_{R/2}(o))} \le a_{11} \frac{(1+\sqrt{\kappa}R)^2}{R^2}.$$

Proof. We discard the first term of the left hand side of (33) to obtain

$$\frac{a_3}{t} e^{-t_0} V^{\frac{2}{n}} R^{-2} \left\| f^{\frac{\alpha_0+t-1}{2}+\frac{p}{4}} \eta \right\|^2_{L^{\frac{2n}{n-2}}} \le a_5 t_0^2 R^{-2} \int_\Omega f^{\alpha_0+\frac{p}{2}+t-1} \eta^2 + \frac{a_4}{t} \int_\Omega f^{\alpha_0+\frac{p}{2}+t-1} |\nabla \eta|^2. \tag{40}$$

Let $r_k = \dfrac{R}{2} + \dfrac{R}{4^k}$ and $\Omega_k = B_{r_k}(o)$. It is easy to see that there exist cut-off functions $\eta_k \in C^\infty(\Omega_k)$ satisfying

$$\begin{cases} 0 \le \eta_k \le 1, \quad |\nabla \eta_k| \le \dfrac{C4^k}{R}; \\ \eta_k \equiv 1 \text{ in } B_{r_{k+1}}(o), \end{cases}$$

where $k = 1,\ 2,\ 3,\ \cdots$. Substituting η_k into (40) instead of η, we arrive at

$$\begin{aligned} a_3 e^{-t_0} V^{\frac{2}{n}} \left\| f^{\frac{\alpha_0+t-1}{2}+\frac{p}{4}} \eta_k \right\|^2_{L^{\frac{2n}{n-2}}(\Omega_k)} &\le a_5 t_0^2 t \int_{\Omega_k} f^{\alpha_0+\frac{p}{2}+t-1} \eta_k^2 + a_4 R^2 \int_{\Omega_k} f^{\alpha_0+\frac{p}{2}+t-1} |\nabla \eta_k|^2 \\ &\le \left(a_5 t_0^2 t + C^2 16^k\right) \int_{\Omega_k} f^{\alpha_0+\frac{p}{2}+t-1}. \end{aligned} \tag{41}$$

By picking $\alpha_{k+1} = \dfrac{n\alpha_k}{n-2}$, and letting $t = t_k$ such that

$$t_k + \frac{p}{2} + \alpha_0 - 1 = \alpha_k,$$

we can deduce from (41) that

$$a_3\left(\int_{\Omega_k} f^{\alpha_{k+1}}\eta_k^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \le e^{t_0}V^{-\frac{2}{n}}\left(a_5t_0^2\left(t_0+\frac{p}{2}+\alpha_0-1\right)\left(\frac{n}{n-2}\right)^k+C^2 16^k\right)\int_{\Omega_k} f^{\alpha_k},$$

where $k = 1,\ 2,\ 3\ \cdots$.

On the other hand, we can choose $a_9 = a_9(n,\ p,\ q,\ a,\ c)$ which satisfies

$$a_9t_0^3 \ge \max\left\{a_5t_0^2\left(\alpha_0+t_0+\frac{p}{2}-1\right),\ C^2\right\},$$

since $\frac{n}{n-2} < 16$. Then we have

$$a_3\left(\int_{\Omega_k} f^{\alpha_{k+1}}\eta_k^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \le 2a_9t_0^3e^{t_0}V^{-\frac{2}{n}}16^k\int_{\Omega_k} f^{\alpha_k}. \tag{42}$$

Taking $\frac{1}{\beta_k}$ power of the both sides of (42), we obtain

$$\|f\|_{L^{\alpha_{k+1}}(\Omega_{k+1})} \le \left(2a_9t_0^3e^{t_0}V^{-\frac{2}{n}}\right)^{\frac{1}{\alpha_k}} 16^{\frac{k}{\alpha_k}}\|f\|_{L^{\alpha_k}(\Omega_k)}.$$

Noting

$$\sum_{k=1}^{\infty}\frac{1}{\alpha_k} = \frac{\frac{1}{\alpha_1}}{1-\frac{n-2}{n}} = \frac{n}{2\alpha_1} \quad \text{and} \quad \sum_{k=1}^{\infty}\frac{k}{\alpha_k} < \infty,$$

we can derive

$$\|f\|_{L^\infty(B_{R/2}(o))} \le a_{10}V^{-\frac{1}{\alpha_1}}\|f\|_{L^{\alpha_1}(B_{3R/4}(o))},$$

where a_{10} depending only on $n,\ p,\ q,\ a,\ c$ satisfies

$$a_{10} \ge \left(2a_9t_0^3e^{t_0}\right)^{\frac{n}{2\alpha_1}} 16^{\sum_{k=1}^{\infty}\frac{k}{\alpha_k}}.$$

In view of (34), we obtain

$$\|f\|_{L^\infty(B_{R/2}(o))} \le a_{11}\frac{(1+\sqrt{\kappa}R)^2}{R^2},$$

where $a_{11} = a_{10}a_8c_1(n,\ p,\ q,\ a,\ c)$. □

Proof of Corollary 4 Choosing $\kappa = 0$ in Theorem 1 implies

$$\sup_{B_{R/2}(o)} |\nabla u| \leq C_{n,\ p,\ q,\ r}\frac{1}{R}. \tag{43}$$

By letting $R \to \infty$ in (43), we obtain

$$\nabla u = 0.$$

Hence v is a constant and $\Delta_p v = 0,\ |\nabla u| = 0$. This contradicts to equation (1) if $b \neq 0$. We complete the proof. □

Next, we provide the proof of Theorem 5.

Proof of Theorem 5 Letting $R \to \infty$ in the estimate in (1), we have

$$|\nabla u| \leq C_{n,\ p,\ q,\ a,\ c}\kappa^{\frac{1}{2(1-p+q)}}. \tag{44}$$

Let $d = d(x,\ a)$ be the distance between a and x. For any arc minimizing geodesic segment $\gamma(t) : [0,\ d] \to M$ which connects a and x, we have

$$u(x) = u(a) + \int_0^d \frac{d}{dt}(u \circ \gamma(t))\, dt. \tag{45}$$

We infer from (44) that

$$\left|\frac{d}{dt}(u \circ \gamma(t))\right| = \left|\langle \nabla u(\gamma(t)),\ \gamma'(t)\rangle\right| \leq |\nabla u(\gamma(t))| \leq C_{n,\ p,\ q,\ a,\ c}\kappa^{\frac{1}{2(1-p+q)}}. \tag{46}$$

It follows from (45) and (46) that

$$|u(x) - u(a)| \leq C_{n,\ p,\ q,\ a,\ c}\kappa^{\frac{1}{2(1-p+q)}} d(x,\ a),$$

which implies (2). □

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Conflict of interest

The authors declare no competing financial interest.

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