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Research Article

Linear Stability Analysis of the Couette Flow for the Two Dimensional Non-Isentropic Euler-Poisson System

Xueke Pu*, Xulong Wu, Lian Yang

School of mathematics and information science, Guangzhou University, Guangzhou, 510006, People's Republic of China
E-mail: puxueke@gmail.com

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Abstract: In the present paper, we study the linear stability of perturbations around the Couette flow for a two dimensional compressible non-isentropic Euler-Poisson system in the domain $\mathbb{T}\times\mathbb{R}$. A Lyapunov type instability result for the density and the temperature is obtained, and in particular, the inviscid damping for the solenoidal component of the velocity field is proved.

***Keywords*:** non-isentropic Euler-Poisson equations, Couette flow, linear stability, inviscid damping

MSC: 35M30, 35Q35, 76E05

1. Introduction

We consider the following non-isentropic Euler-Poisson system for ions in a two dimensional channel $\mathbb{T}\times\mathbb{R}$,

$$\begin{cases} \partial_t\widetilde{\eta}+\nabla\cdot(\widetilde{\eta}\widetilde{v})=0, \\ m\partial_t\widetilde{v}+m(\widetilde{v}\cdot\nabla)\widetilde{v}+R\widetilde{\theta}\nabla(\log\widetilde{\eta})+R\nabla\widetilde{\theta}=\nabla\widetilde{\phi}, \\ \partial_t\widetilde{\theta}+\widetilde{v}\cdot\nabla\widetilde{\theta}+(\gamma-1)\widetilde{\theta}\nabla\cdot\widetilde{v}=0, \\ \Delta\widetilde{\phi}=\widetilde{\eta}-e^{-\widetilde{\phi}}, \end{cases} \tag{1}$$

where $(x, y)\in\mathbb{T}\times\mathbb{R}$ and $\mathbb{T}=\mathbb{R}/\mathbb{Z}$. The unknowns $\left(\widetilde{\eta}, \widetilde{v}, \widetilde{\phi}, \widetilde{\theta}\right)$ are functions of (t, x, y) representing the density, the velocity, the electric potential and the temperature respectively and the coefficient m is the mass, R is the resistance and $\gamma>1$ is the ratio of specific heats. For convenience, we assume that the constants $m=R=1$.

The Euler-Poisson system is a simplified model from the two fluid Euler-Maxwell system, which forms the foundation of the two-fluid theory in a plasma, where two compressible electron and ion fluids interact with their own

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self-consistent electromagnetic field [1]. By letting the light speed tend to infinity, one obtains the two-fluid Euler-Poisson system for both electrons and ions and can be viewed as a simplified model for the motion of a plasma. Then, if one further fix the ion background (i.e., with constant ion density and zero ion velocity), one obtains the famous Euler-Poisson system for the electron fluid dynamics [2]. On the other hand, if one regards the electron-ion mass ratio to be zero, one obtains the famous Boltzmann law for the electron mass density (i.e., $n_e \sim e^{-\phi}$), then the Poisson equation reduces to the last equation in (1) and the Euler-Poisson for the heavier ion dynamics is obtained [3]. These Euler-Poisson systems are origins of many important nonlinear dispersive equations, such as the Korteweg-de Vries (KdV) and the Nonlinear Schrödinger (NLS) equations [4, 5]. In the past three decades, stability and global existence of smooth solutions of the Euler-Poisson system near constant equilibrium were studied, since the seminal work of Guo [2]. For the three dimensional electron Euler-Poisson system, the global smooth irrotational solutions was firstly constructed with small velocity, based on the dispersive Klein-Gordon effect. Later on, the global smooth irrotational solutions for the 2D and even 1D electronic Euler-Poisson system were constructed in [6, 7]. The global smooth irrotational solutions was constructed by Guo and Pausader [3] for the three dimensional Euler-Poisson system for ions, where the dispersive relation is drastically different from that of the electron Euler-Poisson system. These are important nonlinear stability of the Euler-Poisson system near constant equilibrium. However, one also wants to know whether nonlinear stability or linear stability results hold for the Euler-Poisson system near non-constant equilibriums. As a first step, we want to study the stability of the Couette flow, an important shear flow, which is non-constant and has a long history in the study of the fluid dynamics.

The Couette flow is an important shear flow in fluid dynamics, since the classical results of Rayleigh [8] and Kelvin [9] for the incompressible fluid and up to date there are extensive studies in both mathematics and physics. Here we cannot list all of these results, but to list only a few that are closely related our study in the Euler-Poisson system. In recent years, many fruitful results are focused on the study of the nonlinear stability of the Couette flow, in particular, its close relationship between the enhanced dissipation, transition thresholds, inviscid damping and lift up effects in fluid dynamics in two or three dimensional cases and in various boundary conditions. One may refer to [10–14], to list only a very few, and the references cited therein for more detailed information.

Compared to incompressible fluids, there are significantly fewer literature on the study of the stability of the Couette flow in compressible fluids. In Mathematics, Kagei [15] proved that the plane Couette flow in an infinite layer is asymptotically stable when the Reynolds and Mach numbers are sufficiently small. Subsequently, Li and Zhang [16] lifted the restriction of the Reynolds and Mach numbers with Navier-slip boundary condition at the bottom boundary, investigated that the plane Couette flow is asymptotically stable for small perturbation around the 3D compressible Navier-Stokes equations. In the case of high Reynolds number, Zeng et al. [17] investigated the linear stability result of the Couette flow in the isentropic 3D compressible Navier-Stokes equations on the domain $\mathbb{T}\times\mathbb{R}\times\mathbb{T}$. Antonelli et al. [18, 19] studied the inviscid damping and enhanced dissipation phenomena for the homogeneous Couette flow or shear flow near Couette flow in a 2D isentropic compressible fluid, and obtained a generic Lyapunov type instability for the density and the irrotational component of the velocity field in the inviscid case and that the perturbations have a transient growth after which it decays exponentially fast in the viscid case. Later on, following the spirit of [18, 19], Zhai [20] investigated an inviscid damping result on the non-isentropic compressible Euler equations on the domain $\mathbb{T}\times\mathbb{R}$ and Pu et al. [21] obtained the inviscid damping result on the isentropic Euler-Poisson system on the domain $\mathbb{T}\times\mathbb{R}$. For the nonlinear stability threshold problem for the compressible Couette flow at high Reynolds number, as far as we know, Huang et al. [22] recently have made progress and obtained the enhanced dissipation and stability threshold for the compressible Couette flow in the two dimensional Navier-Stokes equation in $\mathbb{T}\times\mathbb{R}$. In particular, there is no stability analysis of the Couette flow for the non-isentropic Euler-Poisson system studied in these papers.

Let us now return to the stability of the Couette flow in the compressible non-isentropic Euler-Poisson system (1) in this paper. Till this work, as far as know, there is no stability analysis of the Couette flow of the Euler-Poisson system except the analysis made in the isentropic case by the Pu et al. [21]. It is the task of this paper to study the linear stability of the Couette flow for the non-isentropic Euler-Poisson system in this paper, as a first step towards the nonlinear stability.

It is obvious that the Couette flow,

$$\widetilde{\eta}_s = 1, \quad \widetilde{v}_s = (y, 0)^T, \quad \widetilde{\phi}_s = 0, \quad \widetilde{\theta}_s = 1, \tag{2}$$

is a special solution of (1), independent of time. To study the linear stability, we perturb the system in the following way

$$\widetilde{\eta} = \eta + \widetilde{\eta}_s, \quad \widetilde{v} = u + \widetilde{v}_s, \quad \widetilde{\phi} = \phi + \widetilde{\phi}_s, \quad \widetilde{\theta} = \theta + \widetilde{\theta}_s,$$

and linearize the Euler-Poisson system (1) around the Couette flow to obtain the following system

$$\partial_t \eta + y\partial_x \eta + \nabla \cdot u = 0, \tag{3}$$

$$\partial_t u + y\partial_x u + \begin{pmatrix} u^y \\ 0 \end{pmatrix} + \nabla \eta + \nabla \theta = \nabla \phi, \tag{4}$$

$$\partial_t \theta + y\partial_x \theta + (\gamma - 1)\nabla \cdot u = 0, \tag{5}$$

$$\Delta \phi = \eta + \phi. \tag{6}$$

To study the above system, we introduce $\psi = \nabla \cdot u$ and $\omega = \nabla^{\perp} \cdot u$ with $\nabla^{\perp} = (-\partial_y, \partial_x)^{\top}$ and we get the Helmholtz projection operator as follows

$$u = (u^x, u^y)^{\top} = \nabla^{\perp}\Delta^{-1}\omega + \nabla\Delta^{-1}\psi := \mathbb{P}[u] + \mathbb{Q}[u]. \tag{7}$$

The first part $\mathbb{P}[u]$ is the incompressible part, and the $\mathbb{Q}[u]$ is compressible part. It's easy to verify that

$$u^y = \partial_y(\Delta)^{-1}\psi + \partial_x(\Delta)^{-1}\omega. \tag{8}$$

Therefore, we can rewrite the system (3)-(6) in terms of $(\eta, \psi, \omega, \theta)$ that

$$\partial_t \eta + y\partial_x \eta + \psi = 0, \tag{9}$$

$$\partial_t \psi + y\partial_x \psi + 2\partial_x u^y + \Delta\eta + \Delta\theta = -\Delta(-\Delta + 1)^{-1}\eta, \tag{10}$$

$$\partial_t \omega + y\partial_x \omega - \psi = 0, \tag{11}$$

$$\partial_t \theta + y\partial_x \theta + (\gamma - 1)\psi = 0. \tag{12}$$

If we denote the average of a function in the x direction by

$$f_0(y) = \frac{1}{2\pi} \int_{\mathbb{T}} f(x, y) dx,$$

and then take the x-average of equations (9)-(12), we obtain the following system for the zero mode

$$\partial_t \eta_0 + \psi_0 = 0, \tag{13}$$

$$\partial_t \psi_0 + \partial_{yy} \eta_0 + \partial_{yy} \theta_0 = -\Delta (-\Delta + 1)^{-1} \eta_0, \tag{14}$$

$$\partial_t \omega_0 - \psi_0 = 0, \tag{15}$$

$$\partial_t \theta_0 + (\gamma - 1) \psi_0 = 0. \tag{16}$$

Adding (13) to (15) yields $\partial_t (\eta_0 + \omega_0) = 0$, then we have

$$\eta_0 + \omega_0 = \eta_0^{in} + \omega_0^{in}, \tag{17}$$

where $(\eta_0^{in}, \omega_0^{in})$ represents the initial data. According to (13) and (14), we can get

$$\partial_{tt} \eta_0 - \partial_{yy} \eta_0 - \partial_{yy} \theta_0 - \Delta (-\Delta + 1)^{-1} \eta_0 = 0 \text{ in } \mathbb{T} \times \mathbb{R}. \tag{18}$$

Combinbing (13) with (16), we can obtain

$$\partial_t \eta_0 = \frac{1}{(\gamma - 1)} \partial_t \theta_0. \tag{19}$$

If we supply system (13)-(16) with the initial condition

$$\eta_0^{in} = \psi_0^{in} = \omega_0^{in} = \theta_0^{in} = 0, \tag{20}$$

then $\eta_0(t) = \psi_0(t) = \omega_0(t) = \theta_0(t) = 0$ is the unique solution to the system (13)-(16) with initial condition (20). Actually, by virtue of (19), taking the $L^2(\mathbb{R})$ inner product of (18) with $\partial_t \eta_0$, then it follows from integration by parts that

$$\frac{1}{2} \frac{d}{dt} \left(\|\partial_t \eta_0\|_{L^2}^2 + \|\partial_y \eta_0\|_{L^2}^2 + \frac{1}{\gamma - 1} \|\partial_y \theta_0\|_{L^2}^2 + \left\| \nabla (-\Delta + 1)^{-\frac{1}{2}} \eta_0 \right\|_{L^2}^2 \right) = 0,$$

which implies that $\eta_0(t) = \theta_0(t) = 0$ is the unique solution to the equation (18) with initial condition $\eta_0^{in} = \theta_0^{in} = 0$. Thanks to (17) and (13), we also obtain that $\psi_0(t) = \omega_0(t) = 0$. Based on this fact, for simplicity of notation, we need only to study the dynamics of $(\eta,\ \psi,\ \omega,\ \theta)$ with $\eta_0^{in} = \psi_0^{in} = \omega_0^{in} = \theta_0^{in} = 0$, instead of studying the dynamics for $(\eta - \eta_0,\ \psi - \psi_0,\ \omega - \omega_0,\ \theta - \theta_0)$. That is to say, we can decouple the system into the zero mode and nonzero modes.

The main results are stated as follows.

Theorem 1 Suppose $(\eta^{in},\ \omega^{in},\ \theta^{in}) \in H_x^1 H_y^2$ and $\psi^{in} \in H_x^{-\frac{1}{2}} L_y^2$ with $\left(\eta_0^{in},\ \psi_0^{in},\ \omega_0^{in},\ \theta_0^{in}\right) = (0,\ 0,\ 0,\ 0)$. Let $(\eta,\ u,\ \theta,\ \phi)$ be a smooth solution for the system (3)-(6). Then the following estimates hold

$$
\begin{aligned}
&\|\mathbb{P}[u]^x(t)\|_{L^2} + \|\phi(t)\|_{L^2} \\
&\lesssim \frac{1}{\sqrt{\gamma}\langle t\rangle^{\frac{1}{2}}} \left(\left\| \frac{\eta^{in} + \theta^{in}}{\sqrt{\gamma}} \right\|_{H_x^{-\frac{1}{2}} L_y^2} + \left\| \psi^{in} \right\|_{H_x^{-\frac{1}{2}} H_y^{-1}} + \left\| \frac{\eta^{in} + \theta^{in} + \gamma\omega^{in}}{\gamma} \right\|_{H_x^{-\frac{1}{2}} H_y^{\frac{1}{2}}} \right) \\
&\quad + \frac{1}{\langle t\rangle} \left\| \frac{\eta^{in} + \theta^{in} + \gamma\omega^{in}}{\gamma} \right\|_{H_x^{-1} H_y^1},
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
&\|\mathbb{P}[u]^y(t)\|_{L^2} \\
&\lesssim \frac{1}{\sqrt{\gamma}\langle t\rangle^{\frac{3}{2}}} \left(\left\| \frac{\eta^{in} + \theta^{in}}{\sqrt{\gamma}} \right\|_{H_x^{-\frac{1}{2}} H_y^1} + \left\| \psi^{in} \right\|_{H_x^{-\frac{1}{2}} L_y^2} + \left\| \frac{\eta^{in} + \theta^{in} + \gamma\omega^{in}}{\gamma} \right\|_{H_x^{-\frac{1}{2}} H_y^{\frac{3}{2}}} \right) \\
&\quad + \frac{1}{\langle t\rangle^2} \left\| \frac{\eta^{in} + \theta^{in} + \gamma\omega^{in}}{\gamma} \right\|_{H_x^{-1} H_y^2},
\end{aligned} \tag{22}
$$

and

$$
\begin{aligned}
&\|\mathbb{Q}[u](t)\|_{L^2} + \gamma^{\frac{3}{2}} \|\eta(t)\|_{L^2} + \gamma^{\frac{3}{2}} \|\theta(t)\|_{L^2} \\
&\lesssim \langle t\rangle^{\frac{1}{2}} \left\| \sqrt{\gamma}\left((\gamma - 1)\eta^{in} - \theta^{in}\right) \right\|_{L^2} \\
&\quad + \langle t\rangle^{\frac{1}{2}} \left((\gamma+1)^2 \left(\left\| \frac{\eta^{in} + \theta^{in}}{\sqrt{\gamma}} \right\|_{L^2} + \left\| \psi^{in} \right\|_{H^{-1}} + \left\| \frac{\eta^{in} + \theta^{in} + \gamma\omega^{in}}{\gamma} \right\|_{H^{\frac{1}{2}}} \right) \right).
\end{aligned} \tag{23}
$$

Theorem 2 Assume that $(\eta^{in},\ \omega^{in},\ \theta^{in}) \in L_x^2 H_y^{-\frac{1}{2}}$ and that $\psi^{in} \in H_x^{-\frac{3}{2}} H_y^{-2}$ with $\left(\eta_0^{in},\ \psi_0^{in},\ \omega_0^{in},\ \theta_0^{in}\right) = (0, 0, 0, 0)$. Let $(\eta,\ u,\ \theta,\ \phi)$ be a smooth solution for the system (3)-(6). Then it holds that

$$
\|\mathbb{Q}[u](t)\|_{L^2} + \sqrt{\gamma}\|\eta(t) + \theta(t)\|_{L^2} \gtrsim \mathscr{C}^{in}\left(\eta^{in},\ \psi^{in},\ \omega^{in},\ \theta^{in}\right)\langle t\rangle^{\frac{1}{2}},
$$

where the constant $\mathscr{C}^{in}\left(\eta^{in}, \psi^{in}, \omega^{in}, \theta^{in}\right)$ is a suitable combination of $\left\|\eta^{in}\right\|_{L_x^2 H_y^{-\frac{1}{2}}}$, $\left\|\psi^{in}\right\|_{H_x^{-\frac{3}{2}} H_y^{-2}}$, $\left\|\omega^{in}\right\|_{L_x^2 H_y^{-\frac{1}{2}}}$ and $\left\|\theta^{in}\right\|_{L_x^2 H_y^{-\frac{1}{2}}}$.

The difficulty in this article is that the new transport term appear after the introduction of temperature, which brings more complexity. Firstly, we use the Helmholtz projection operators to divide it into irrotational and divergence free parts. Then we use coordinate transformation to find the transportation terms of the Couette flow. In this way, we can rewrite the system as (26)-(29) in the moving frame. Finally, we use Fourier transform and Grönwall inequality to estimate the solutions and the following sections are devoted to the proof of the two theorems.

The outline of this paper is as follows. We will give some notations and introduce a set of coordinate transformations in Section 2. In Section 3, we present the proof of linear stability of the non-isentropic Euler-Poisson system near the Couette flow.

2. Notations

The Fourier transform of function $f(x, y)$, henceforth denoted by $\widehat{f}(k, \xi)$, is defined as

$$\widehat{f}(k, \xi) = \frac{1}{2\pi} \iint_{\mathbb{T}\times\mathbb{R}} e^{-i(kx+\xi y)} f(x, y)dxdy.$$

The corresponding inverse Fourier transform is

$$f(x, y) = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} e^{i(kx+\xi y)} \widehat{f}(k, \xi)d\xi.$$

Define the anisotropic Sobolev space

$$H_x^{s_1} H_y^{s_2}(\mathbb{T}\times\mathbb{R}) = \left\{ f\colon \|f\|^2_{H_x^{s_1} H_y^{s_2}(\mathbb{T}\times\mathbb{R})} = \sum_k \int \langle k\rangle^{2s_1} \langle \xi\rangle^{2s_2} \left|\widehat{f}\right|^2 (k, \xi)d\xi < +\infty \right\},$$

where $\langle a\rangle = \sqrt{(1+a^2)}$ for $a \in \mathbb{R}$.

Define the usual Sobolev space

$$H^s(\mathbb{T}\times\mathbb{R}) = \left\{ f\colon \|f\|^2_{H^s(\mathbb{T}\times\mathbb{R})} = \sum_k \int \langle k, \xi\rangle^{2s} \left|\widehat{f}\right|^2 (k, \xi)d\xi < +\infty \right\}.$$

Here $\langle a, b\rangle = \sqrt{(1+a^2+b^2)}$ for $a, b \in \mathbb{R}$.

A coordinate transformation strategy [18] is introduced to handle the transport terms in (9)-(12), thereby facilitating the decoupling of spatial propagation modes

$$X = x - yt, \; Y = y. \tag{24}$$

In the new coordinate system, the following differential operators are obtained,

$$\partial_x = \partial_X, \ \partial_y = \partial_Y - t\partial_X, \ \Delta =: \Delta_L = \partial_{XX} + (\partial_Y - t\partial_X)^2.$$

We denote the symbol for $-\Delta_L$ by

$$\alpha(t, k, \xi) = k^2 + (\xi - kt)^2.$$

And the symbol of the operator $2\partial_X(\partial_Y - t\partial_X)$ is

$$\partial_t \alpha(t, k, \xi) = -2k(\xi - kt).$$

3. The proof of the main theorem

The subsequent analysis focuses on the dynamical behavior of the system governed by (9)-(12). By using the coordinate transformations (24), we proceed to define the functions

$$\Pi(t, X, Y) = \eta(t, X + tY, Y), \quad \Psi(t, X, Y) = \psi(t, X + tY, Y),$$

$$\Gamma(t, X, Y) = \omega(t, X + tY, Y), \quad \Theta(t, X, Y) = \theta(t, X + tY, Y).$$

Owing to (8), we obtain

$$U^y = (\partial_Y - t\partial_X)\Delta_L^{-1}\Psi + \partial_X \Delta_L^{-1}\Gamma. \tag{25}$$

Then, the linear system (9)-(12) reduces to the following system in the new coordinates,

$$\partial_t \Pi + \Psi = 0, \tag{26}$$

$$\partial_t \Psi + 2\partial_X(\partial_Y - t\partial_X)\Delta_L^{-1}\Psi + 2\partial_{XX}\Delta_L^{-1}\Gamma + \Delta_L(\Pi + \Theta) + \Delta_L(-\Delta_L + 1)^{-1}\Pi = 0, \tag{27}$$

$$\partial_t \Gamma - \Psi = 0, \tag{28}$$

$$\partial_t \Theta + (\gamma - 1)\Psi = 0. \tag{29}$$

In the view of (9)-(12), there holds

$$
\begin{aligned}
&(\partial_t + y\partial_x)(\eta + \omega) = 0, \\
&(\partial_t + y\partial_x)(\theta + (\gamma - 1)\omega) = 0.
\end{aligned} \tag{30}
$$

In particular, (30) implies that $\eta + \omega$ and $\theta + (\gamma - 1)\omega$ are transported by Couette flow. Hence, if we further define

$$
\begin{aligned}
F(t, X, Y) &\overset{def}{=} \Pi(t, X, Y) + \Gamma(t, X, Y), \\
G(t, X, Y) &\overset{def}{=} \Theta(t, X, Y) + (\gamma - 1)\Gamma(t, X, Y),
\end{aligned} \tag{31}
$$

then we have

$$
\partial_t F = 0, \quad \partial_t G = 0,
$$

which implies that

$$
F = F^{in} = \eta^{in} + \omega^{in}, \quad G = G^{in} = \theta^{in} + (\gamma - 1)\omega^{in}. \tag{32}
$$

Moreover, one can infer from (31) and (32) that

$$
\Gamma(t, X, Y) = \frac{F^{in}(X, Y) + G^{in}(X, Y)}{\gamma} - \frac{\Pi(t, X, Y) + \Theta(t, X, Y)}{\gamma}. \tag{33}
$$

Next, we continue to introduce the following notation

$$
\Omega = \frac{\Pi(t, X, Y) + \Theta(t, X, Y)}{\gamma}. \tag{34}
$$

In view of (25), (33) and (34), we get

$$
\begin{aligned}
U^y &= (\partial_Y - t\partial_X)\Delta_L^{-1}\Psi + \partial_X \Delta_L^{-1}\Gamma \\
&= (\partial_Y - t\partial_X)\Delta_L^{-1}\Psi + \frac{1}{\gamma}\partial_X \Delta_L^{-1}(F^{in} + G^{in}) - \frac{1}{\gamma}\partial_X \Delta_L^{-1}(\Pi + \Theta) \\
&= (\partial_Y - t\partial_X)\Delta_L^{-1}\Psi + \frac{1}{\gamma}\partial_X \Delta_L^{-1}(F^{in} + G^{in}) - \partial_X \Delta_L^{-1}\Omega.
\end{aligned}
$$

As a result, (26)-(29) reduces to the following system

$$
\begin{aligned}
&\partial_t \Omega = -\Psi,\\
&\partial_t \Psi = -2\partial_X\left(\partial_Y - t\partial_X\right)\Delta_L^{-1}\Psi + (2\partial_{XX}\Delta_L^{-1} - \gamma\Delta_L - \Delta_L(-\Delta_L+1)^{-1})\Omega\\
&\quad -(\frac{2}{\gamma}\partial_{XX}\Delta^{-1_L} + \frac{\gamma-1}{\gamma}\Delta_L(-\Delta_L+1)^{-1})F^{in}\\
&\quad -(\frac{2}{\gamma}\partial_{XX}\Delta_L^{-1} - \frac{1}{\gamma}\Delta_L(-\Delta_L+1)^{-1})G^{in}.
\end{aligned}
\tag{35}
$$

Compared to (26)-(29), the above system (35) is a closed 2×2 system only involving Ω and Ψ. Taking the Fourier transform on system (35), we get the following system

$$
\begin{aligned}
&\partial_t \widehat{\Omega} = -\widehat{\Psi},\\
&\partial_t \widehat{\Psi} = \frac{\partial_t \alpha}{\alpha}\widehat{\Psi} + (\frac{2k^2}{\alpha} + \gamma\alpha + \frac{\alpha}{1+\alpha})\widehat{\Omega}\\
&\quad -(\frac{2k^2}{\gamma\alpha} - \frac{(\gamma-1)\alpha}{\gamma(1+\alpha)})\widehat{F}^{in} - (\frac{2k^2}{\gamma\alpha} + \frac{\alpha}{\gamma(1+\alpha)})\widehat{G}^{in}.
\end{aligned}
\tag{36}
$$

To deal with the system (36), we need to find an appropriate symmetrization for this system. Define

$$
A(t) = (A_1(t), A_2(t))^\top = \left(\frac{\sqrt{\gamma}\,\widehat{\Omega}(t)}{\alpha^{1/4}}, \frac{\widehat{\Psi}(t)}{\alpha^{3/4}}\right)^\top.
$$

By a straightforward calculation, $A(t)$ satisfies a non-autonomous two dimensional dynamical system

$$
\begin{cases}
\frac{d}{dt}A(t) = L(t)A(t) + M(t)\widehat{F}^{in} + N(t)\widehat{G}^{in},\\
A(0) = A^{in},
\end{cases}
\tag{37}
$$

where

$$
L(t) = \begin{bmatrix} -\frac{1}{4}\alpha^{-1}\partial_t\alpha & -\sqrt{\gamma}\alpha^{1/2} \\ \sqrt{\gamma}\alpha^{1/2} + \frac{2k^2}{\sqrt{\gamma}\alpha^{3/2}} + \frac{\alpha^{1/2}}{\sqrt{\gamma}(1+\alpha)} & \frac{1}{4}\alpha^{-1}\partial_t\alpha \end{bmatrix},
$$

$$M(t)=(0,-\frac{2k^2}{\gamma\alpha^{7/4}}+\frac{(\gamma-1)\alpha^{1/4}}{\gamma(1+\alpha)})^\top,\quad N(t)=(0,-\frac{2k^2}{\gamma\alpha^{7/4}}+\frac{\alpha^{1/4}}{\gamma(1+\alpha)})^\top,$$

and

$$A^{in}=\left(\frac{\sqrt{\gamma}\widehat{\Omega}^{in}}{(k^2+\xi^2)^{1/4}},\frac{\widehat{\Psi}^{in}}{(k^2+\xi^2)^{3/4}}\right)^\top. \tag{38}$$

Hence, it suffices to study the homogeneous problem of system (37). For this purpose, we denote by $A(t)$ a solution to system (37) with $\widehat{F}^{in}=\widehat{G}^{in}=0$.

Let

$$\mathscr{E}(t)=\left(\sqrt{\frac{d}{b}}|A_1|^2\right)(t)+\left(\sqrt{\frac{b}{d}}|A_2|^2\right)(t)+2\left(\frac{a}{\sqrt{bd}}\operatorname{Re}\left(A_1\bar{A_2}\right)\right)(t),$$

where

$$a(t)=\frac{1}{4}\alpha^{-1}\partial_t\alpha,\ b(t)=\sqrt{\gamma}\alpha^{1/2}, \tag{39}$$

$$d(t)=\sqrt{\gamma}\alpha^{1/2}+\frac{2k^2}{\sqrt{\gamma}\alpha^{3/2}}+\frac{\alpha^{1/2}}{\sqrt{\gamma}(1+\alpha)}. \tag{40}$$

Then we can get the upper and lower bounds for $\mathscr{E}(t)$ and the solution $A(t)$.

Lemma 1 There exist positive constants c_1, c_2, C_1, C_2 that do not depend on k and ξ such that

$$c_1\mathscr{E}(0)\le\mathscr{E}(t)\le C_1\mathscr{E}(0), \tag{41}$$

and

$$c_2\left|A^{in}\right|\le|A(t)|\le C_2\left|A^{in}\right|. \tag{42}$$

Proof. Define

$$\zeta=\sqrt{\frac{d}{b}}=\left(1+\frac{1}{\gamma(1+\alpha)}+\frac{2k^2}{\gamma\alpha^2}\right)^{1/2}, \tag{43}$$

$$\beta=\sqrt{bd}=(\gamma\alpha+\frac{2k^2}{\alpha}+\frac{\alpha}{1+\alpha})^{1/2}. \tag{44}$$

From the definition of ζ we get that

$$1 \le \zeta^2 \le 1 + \frac{2k^2}{\gamma\alpha^2} + \frac{1}{\gamma(1+\alpha)} \le 1 + \frac{3}{\gamma}. \tag{45}$$

Moreover, we have

$$\begin{aligned}\frac{|a|}{\beta} &= \frac{|\partial_t \alpha|}{4\alpha}(\gamma\alpha + \frac{2k^2}{\alpha} + \frac{\alpha}{1+\alpha})^{-1/2} \\ &= \frac{|\partial_t \alpha|}{4\alpha}(\gamma\alpha)^{-1/2}(1 + \frac{2k^2}{\gamma\alpha^2} + \frac{1}{\gamma(1+\alpha)})^{-1/2} \\ &\le \frac{|k|}{2\sqrt{\alpha}}(\gamma\alpha)^{-1/2}(\frac{2k^2}{\gamma\alpha^2})^{-1/2} = \frac{1}{2\sqrt{2}} \le \frac{\sqrt{2}}{2},\end{aligned} \tag{46}$$

where we have used the fact that $|\partial_t \alpha| \le 2|k|\sqrt{\alpha}$.

Setting

$$\widetilde{\mathscr{E}}(t) = \left(\zeta |A_1|^2\right)(t) + \left(\frac{1}{\zeta}|A_2|^2\right)(t).$$

Using the Young inequality and combining with (45) and (46), we can obtain

$$-\frac{\sqrt{2}}{2}\widetilde{\mathscr{E}}(t) \le 2\left(\frac{|a|}{\beta}\operatorname{Re}\left(A_1\bar{A}_2\right)\right)(t) \le \frac{\sqrt{2}}{2}\widetilde{\mathscr{E}}(t),$$

from which we can get

$$\left(1 - \frac{\sqrt{2}}{2}\right)\widetilde{\mathscr{E}}(t) \le \mathscr{E}(t) \le \left(1 + \frac{\sqrt{2}}{2}\right)\widetilde{\mathscr{E}}(t). \tag{47}$$

The coerciveness of $\mathscr{E}(t)$ is ensured by the above inequality and (45).

According to (39), (40), (43) and (44), system (37) becomes

$$\zeta \frac{d}{dt}A_1 = -a\zeta A_1 - \beta A_2, \tag{48}$$

$$\frac{1}{\zeta}\frac{d}{dt}A_2 = \beta A_1 + \frac{a}{\zeta}A_2. \tag{49}$$

Multiplying equations (48) by $\bar{A}_1$ and (49) by $\bar{A}_2$, and we add the two equations to get

$$\frac{\zeta}{2}\frac{d}{dt}|A_1|^2+\frac{1}{2\zeta}\frac{d}{dt}|A_2|^2=-a(\zeta|A_1|^2-\frac{1}{\zeta}|A_2|^2). \tag{50}$$

Note the fact that

$$\frac{a}{\beta}\frac{d}{dt}\mathrm{Re}(A_1\bar{A}_2)=a(\zeta|A_1|^2-\frac{1}{\zeta}|A_2|^2). \tag{51}$$

Adding (50) to (51), we obtain

$$\frac{\zeta}{2}\frac{d}{dt}|A_1|^2+\frac{1}{2\zeta}\frac{d}{dt}|A_2|^2+\frac{a}{\beta}\frac{d}{dt}\mathrm{Re}(A_1\bar{A}_2)=0,$$

which yields

$$\frac{d\mathscr{E}}{dt}=\frac{d}{dt}(\log\zeta)\,\zeta|A_1|^2+\frac{d}{dt}\left(\log(\frac{1}{\zeta})\right)\frac{1}{\zeta}|A_2|^2+2\frac{d}{dt}\left(\frac{a}{\beta}\right)\mathrm{Re}(A_1\bar{A}_2).$$

By using Young's inequality, we can deduce from (47) that

$$\begin{aligned}\frac{d\mathscr{E}}{dt}&\leq\left(\left|\frac{d}{dt}\left(\frac{a}{\beta}\right)\right|+\left|\frac{d}{dt}(\log\zeta)\right|\right)\widetilde{\mathscr{E}}\\&\leq\left(2+\sqrt{2}\right)\left(\left|\frac{d}{dt}\left(\frac{a}{\beta}\right)\right|+\left|\frac{d}{dt}(\log\zeta)\right|\right)\mathscr{E}.\end{aligned} \tag{52}$$

An argument similar to that of (52) gives

$$\frac{d\mathscr{E}}{dt}\geq-\left(2+\sqrt{2}\right)\left(\left|\frac{d}{dt}\left(\frac{a}{\beta}\right)\right|+\left|\frac{d}{dt}(\log\zeta)\right|\right)\mathscr{E}. \tag{53}$$

For the purpose of applying Grönwall's inequality, we need to estimate the integrals

$$\int_0^{+\infty}\left|\frac{d}{dt}\left(\frac{a}{\beta}\right)\right|dt\quad\text{and}\quad\int_0^{+\infty}\left|\frac{d}{dt}(\log\zeta)\right|dt.$$

For the first one, by a direct calculation, we have

$$\frac{d}{dt}\left(\frac{a}{\beta}\right) = \frac{1}{2}\frac{k^2}{\alpha\beta} - \frac{1}{4}\frac{(\partial_t\alpha)^2}{\alpha^2\beta} - \frac{1}{8}\frac{\gamma(\partial_t\alpha)^2 - \frac{2k^2(\partial_t\alpha)^2}{\alpha^2} + \frac{(\partial_t\alpha)^2}{(1+\alpha)^2}}{\alpha\beta^3}$$

$$= \frac{\mathscr{G}}{8\alpha^3\beta^3(1+\alpha)^2},$$

where

$$\mathscr{G} = 4k^2\alpha^2(1+\alpha)^2\beta^2 - 2(\partial_t\alpha)^2\alpha(1+\alpha)^2\beta^2$$

$$-\gamma(\partial_t\alpha)^2\alpha^2(1+\alpha)^2 - 2k^2(\partial_t\alpha)^2(1+\alpha)^2 + (\partial_t\alpha)^2\alpha^2,$$

is a polynomial in time of order 10. Denote by $t_i(i = 1, 2, ..., 10)$ the possible positive roots for $\mathscr{G}$. We set $i_0 = 0$ provided that $\mathscr{G}(0) \leq 0$ and take $i_0 = 1$ for the other cases. Set $t_0 = 0$ and that $t_{11} = +\infty$. By virtue of (46), we get

$$\int_0^{+\infty}\left|\frac{d}{dt}\left(\frac{a}{\beta}\right)\right| d\vartheta = \sum_{i=1}^{11}(-1)^{i+i_0}\left(\frac{a}{\beta}(t_i) - \frac{a}{\beta}(t_{i-1})\right)$$
$$\leq 2\sum_{i=0}^{11}\frac{|a|}{\beta}(t_i) \leq 12\sqrt{2}. \tag{54}$$

For the second integral term, we deduce from (45) that

$$\int_0^{+\infty}\left|\frac{d}{dt}(\log\zeta)\right| d\vartheta = \frac{1}{2}\int_0^{\frac{\xi}{k}}\left|\frac{d}{dt}\left(\log(\zeta)^2\right)\right| d\vartheta + \frac{1}{2}\int_{\frac{\xi}{k}}^{+\infty}\left|\frac{d}{dt}\left(\log(\zeta)^2\right)\right| d\vartheta$$
$$\leq \frac{1}{2}\left[\left(\log(\zeta)^2\right)\left(\frac{\xi}{k}\right) - \left(\log(\zeta)^2\right)(0)\right] + \frac{1}{2}\left[\left(\log(\zeta)^2\right)\left(\frac{\xi}{k}\right) - \left(\log(\zeta)^2\right)(+\infty)\right] \tag{55}$$
$$\leq \log\left(1 + \frac{3}{\gamma}\right),$$

where we have used the fact that

$$\partial_t(\zeta^2) = -\left(\frac{4k^2}{\gamma\alpha^3} + \frac{1}{(1+\alpha)^2}\right)\partial_t\alpha,$$

changes sign only at $t = \xi/k$ (if $\xi/k \geq 0$). Applying Grönwall's inequality to (52) and (53), respectively, from (54) and (55) we obtain (41).

Next, it remains to prove (42). Owing to (41) and (47), it can be seen that

$$c_1'\widetilde{\mathscr{E}}(0) \leq \widetilde{\mathscr{E}}(t) \leq C_1'\widetilde{\mathscr{E}}(0). \tag{56}$$

From (45), we get

$$\left(1+\frac{3}{\gamma}\right)^{-1}\widetilde{\mathscr{E}}(t) \leq |A(t)|^2 \leq \left(1+\frac{3}{\gamma}\right)\widetilde{\mathscr{E}}(t). \tag{57}$$

Combining (56) with (57) yields (42). Hence, we have proved the lemma. □

According to the Duhamel's formula, the solution $A(t)$ to the system (37) is given by

$$A(t) = S_L(t, 0)A^{in} + \int_0^t S_L(t, s)(M(s)\widehat{F}^{in} + N(s)\widehat{G}^{in})ds. \tag{58}$$

Here, S_L is the solution operator related to the equation

$$\frac{d}{dt}A(t) = L(t)A(t),$$

and satisfies the group property

$$S_L(t, 0)S_L(0, s) = S_L(t, s),$$

for any $t, s > 0$.

Based on Lemma 1, we give the proof of Theorem 1.

Proof. [Proof of Theorem 1] Due to Lemma 1, we have

$$\begin{aligned}
\int_0^\infty S_L(t, s)M(s)ds &\lesssim \int_0^\infty |M(s)|ds \\
&\lesssim \int_0^\infty \left|\frac{2k^2}{\gamma\alpha^{7/4}} + \frac{(\gamma-1)\alpha^{1/4}}{\gamma(1+\alpha)}\right| ds \\
&\lesssim \frac{1}{\gamma|k|^{\frac{3}{2}}}\int_0^\infty \frac{ds}{(1+(\xi/k-s)^2)^{7/4}} + \frac{\gamma-1}{\gamma|k|^{\frac{3}{2}}}\int_0^\infty \frac{ds}{(1+(\xi/k-s)^2)^{3/4}} \\
&\lesssim \frac{1}{\gamma|k|^{\frac{3}{2}}},
\end{aligned} \tag{59}$$

and

$$\int_0^\infty S_L(t, s)N(s)ds \lesssim \frac{1}{\gamma|k|^{\frac{3}{2}}}, \tag{60}$$

for any $t \geq 0$. Using Lemma 1 once again, we infer from (58), (59) and (60) that

$$|A(t, k, \xi)| \lesssim |A^{in}(k, \xi)| + \left|\frac{\widehat{F}^{in} + \widehat{G}^{in}}{\gamma}\right|(k, \xi). \tag{61}$$

Next, combing (33) and (34), we get

$$\Gamma(t, X, Y) = \frac{F^{in}(X, Y) + G^{in}(X, Y)}{\gamma} - \Omega(t, X, Y),$$

which implies

$$|\widehat{\Gamma}(t, k, \xi)| \leq \left|\frac{\widehat{F}^{in} + \widehat{G}^{in}}{\gamma}\right|(k, \xi) + \left|\widehat{\Omega}(t, k, \xi)\right|. \tag{62}$$

Then, from (6) and (7), we obtain

$$\begin{cases} \|\phi(t)\|_{L^2}^2 = \left\|(-\Delta+1)^{-1}\eta(t)\right\|_{L^2}^2 = \sum_k \int (\alpha+1)^{-2}\left|\widehat{\Pi}(t)\right|^2 d\xi, \\ \|\mathbb{P}[u]^x(t)\|_{L^2}^2 = \left\|\left(\partial_y\Delta^{-1}\omega\right)(t)\right\|_{L^2}^2 = \sum_k \int \frac{(\xi-kt)^2}{\alpha^2}\left|\widehat{\Gamma}(t)\right|^2 d\xi. \end{cases} \tag{63}$$

Plugging (62) into the (63) and using the definition of $A(t)$, we have

$$\begin{aligned} &\|\mathbb{P}[u]^x(t)\|_{L^2}^2 + \|\phi(t)\|_{L^2}^2 \\ \lesssim &\sum_k \int \left(\frac{(\xi-kt)^2}{\gamma\alpha^{3/2}}\left|\frac{\sqrt{\gamma}\widehat{\Omega}}{\alpha^{1/4}}\right|^2 + \frac{(\xi-kt)^2}{\alpha^2}\left|\frac{\widehat{F}^{in}+\widehat{G}^{in}}{\gamma}\right|^2 + \frac{1}{\gamma\alpha^{3/2}}\left|\frac{\sqrt{\gamma}\widehat{\Omega}}{\alpha^{1/4}}\right|^2\right) d\xi \\ \lesssim &\sum_k \int \left(\frac{(\xi-kt)^2}{\gamma\alpha^{3/2}}|A|^2 + \frac{(\xi-kt)^2}{\alpha^2}\left|\frac{\widehat{F}^{in}+\widehat{G}^{in}}{\gamma}\right|^2 + \frac{1}{\gamma\alpha^{3/2}}|A|^2\right) d\xi. \end{aligned}$$

Now, since $\langle t\rangle \lesssim \langle \xi/k - t\rangle \langle \xi/k\rangle$, we observe that

$$\frac{1}{\sqrt{\alpha}} = \frac{1}{|k|\langle \xi/k - t\rangle}\frac{\langle \xi/k\rangle}{\langle \xi/k\rangle} \lesssim \frac{1}{\langle t\rangle}\frac{\langle \xi\rangle}{\langle k\rangle}.$$

Hence, in the view of (61), (32) and (38), we further get

$$\begin{aligned}
&\|\mathbb{P}[u]^x(t)\|_{L^2}^2 + \|\phi(t)\|_{L^2}^2 \\
&\lesssim \sum_k \int \left(\frac{1}{\gamma\alpha^{1/2}} \left(\left|A^{in}\right| + \left|\frac{\widehat{F}^{in} + \widehat{G}^{in}}{\gamma}\right|^2 \right) + \frac{1}{\alpha}\left|\frac{\widehat{F}^{in} + \widehat{G}^{in}}{\gamma}\right|^2 \right) d\xi \\
&\lesssim \frac{1}{\gamma\langle t\rangle} \left(\left\|\frac{\eta^{in} + \theta^{in}}{\sqrt{\gamma}}\right\|_{H_x^{-\frac{1}{2}} L_y^2}^2 + \left\|\psi^{in}\right\|_{H_x^{-\frac{1}{2}} H_y^{-1}}^2 + \left\|\frac{\eta^{in} + \theta^{in} + \gamma\omega^{in}}{\gamma}\right\|_{H_x^{-\frac{1}{2}} H_y^{\frac{1}{2}}}^2 \right) \\
&\quad + \frac{1}{\langle t\rangle^2} \left\|\frac{\eta^{in} + \theta^{in} + \gamma\omega^{in}}{\gamma}\right\|_{H_x^{-1} H_y^1}^2 .
\end{aligned}$$

Similarly for $\mathbb{P}[u]^x$, we have

$$\begin{aligned}
p\|\mathbb{P}[u]^y(t)\|_{L^2}^2 \lesssim \frac{1}{\gamma\langle t\rangle^3} &\left(\left\|\frac{\eta^{in} + \theta^{in}}{\sqrt{\gamma}}\right\|_{H_x^{-\frac{1}{2}} H_y^1}^2 + \left\|\psi^{in}\right\|_{H_x^{-\frac{1}{2}} L_y^2}^2 + \left\|\frac{\eta^{in} + \theta^{in} + \gamma\omega^{in}}{\gamma}\right\|_{H_x^{-\frac{1}{2}} H_y^{\frac{3}{2}}}^2 \right) \\
&+ \frac{1}{\langle t\rangle^4} \left\|\frac{\eta^{in} + \theta^{in} + \gamma\omega^{in}}{\gamma}\right\|_{H_x^{-1} H_y^2}^2 .
\end{aligned}$$

Finally, we have to estimate the compressible part of the velocity, the density and the temperature. Using the Helmholtz decomposition and the Plancherel's theorem, we obtain

$$\begin{aligned}
\|\mathbb{Q}[u](t)\|_{L^2}^2 + \gamma\|\eta(t) + \theta(t)\|_{L^2}^2 &= \left\|\left(\partial_x\Delta^{-1}\psi\right)(t)\right\|_{L^2}^2 + \left\|\left(\partial_y\Delta^{-1}\psi\right)(t)\right\|_{L^2}^2 + \gamma\|\eta(t) + \theta(t)\|_{L^2}^2 \\
&= \sum_k \int \left(\frac{\left|\widehat{\Psi}(t)\right|^2}{\alpha}(t, k, \xi) + \gamma\left|\widehat{\Pi}(t) + \widehat{\Theta}(t)\right|^2 (t, k, \xi) \right) d\xi \\
&= \sum_k \int \left(\frac{\left|\widehat{\Psi}(t)\right|^2}{\alpha}(t, k, \xi) + \gamma^3\left|\widehat{\Omega}(t)\right|^2 (t, k, \xi) \right) d\xi .
\end{aligned} \tag{64}$$

Note the fact that $\alpha \lesssim \langle t\rangle^2 \langle k, \xi\rangle^2$. From (61) and (64), it follows that

$$
\begin{aligned}
\|\mathbb{Q}[u](t)\|_{L^2}^2+\gamma\|\eta(t)+\theta(t)\|_{L^2}^2 &= \sum_k \int \sqrt{\alpha}\left(\left|\frac{\widehat{\Psi}(t)}{\alpha^{3/4}}\right|^2+\left|\frac{\gamma\sqrt{\gamma}\widehat{\Omega}(t)}{\alpha^{1/4}}\right|^2\right) d\xi \\
&= \sum_k \int \sqrt{\alpha}\left(|A(t)|^2+\gamma^2|A(t)|^2\right) d\xi \\
&\lesssim (1+\gamma^2)\langle t\rangle \left(\|A^{in}\|_{H^{\frac{1}{2}}}^2+\left\|\frac{\eta^{in}+\theta^{in}+\gamma\omega^{in}}{\gamma}\right\|_{H^{\frac{1}{2}}}^2\right) \\
&\lesssim (\gamma+1)^2\langle t\rangle \left(\left\|\frac{\eta^{in}+\theta^{in}}{\sqrt{\gamma}}\right\|_{L^2}+\|\psi^{in}\|_{H^{-1}}+\left\|\frac{\eta^{in}+\theta^{in}+\gamma\omega^{in}}{\gamma}\right\|_{H^{\frac{1}{2}}}\right).
\end{aligned}
\tag{65}
$$

On the other hand, from (9) and (12), we have

$$(\partial_t+y\partial_x)((\gamma-1)\eta-\theta)=0,$$

which implies that

$$(\gamma-1)\eta-\theta=(\gamma-1)\eta^{in}-\theta^{in}.$$

Then

$$\|\sqrt{\gamma}((\gamma-1)\eta-\theta)\|_{L^2}^2=\|\sqrt{\gamma}((\gamma-1)\eta^{in}-\theta^{in})\|_{L^2}^2.$$

It's obvious that

$$
\begin{cases}
\gamma^{3/2}\eta=\sqrt{\gamma}((\gamma-1)\eta-\theta)+\sqrt{\gamma}(\eta+\theta), \\
\gamma^{3/2}\theta=-\sqrt{\gamma}((\gamma-1)\eta-\theta)+\sqrt{\gamma}(\gamma-1)(\eta+\theta).
\end{cases}
$$

Hence, we have

$$
\begin{aligned}
\gamma^3\|\eta(t)\|_{L^2}^2 &= \left(\|\sqrt{\gamma}((\gamma-1)\eta-\theta)\|_{L^2}^2+\|\sqrt{\gamma}(\eta+\theta)\|_{L^2}^2\right) \\
&\lesssim \|\sqrt{\gamma}((\gamma-1)\eta^{in}-\theta^{in})\|_{L^2}^2+(\gamma+1)^2\langle t\rangle\left(\left\|\frac{\eta^{in}+\theta^{in}}{\sqrt{\gamma}}\right\|_{L^2}+\|\psi^{in}\|_{H^{-1}}+\left\|\frac{\eta^{in}+\theta^{in}+\gamma\omega^{in}}{\gamma}\right\|_{H^{\frac{1}{2}}}\right),
\end{aligned}
$$

and

$$\gamma^3 \|\theta(t)\|_{L^2}^2 = \left(\|\sqrt{\gamma}((\gamma-1)\eta - \theta)\|_{L^2}^2 + \left\|(\gamma-1)^2\sqrt{\gamma}(\eta+\theta^{)}\right\|_{L^2}^2 \right)$$

$$\lesssim \|\sqrt{\gamma}((\gamma-1)\eta^{in} - \theta^{in})\|_{L^2}^2$$

$$+ (\gamma+1)^2(\gamma-1)^2 \langle t \rangle \left(\left\| \frac{\eta^{in}+\theta^{in}}{\sqrt{\gamma}} \right\|_{L^2} + \|\psi^{in}\|_{H^{-1}} + \left\| \frac{\eta^{in}+\theta^{in}+\gamma\omega^{in}}{\gamma} \right\|_{H^{\frac{1}{2}}} \right),$$

which gives the estimate (23). Consequently, we finish the proof of Theorem 1. □

Now we prove Theorem 2.

Proof. [Proof of Theorem 2] Set

$$\mathscr{R}\left(t, A^{in}, F^{in}, G^{in}\right) = A^{in} + \int_0^t S_L(0, s)\left(M(s)\widehat{F}^{in} + N(s)\widehat{G}^{in}\right) ds.$$

According to (58), we have

$$A(t) = S_L(t, 0)A^{in} + \int_0^t S_L(t, s)(M(s)\widehat{F}^{in} + N(s)\widehat{G}^{in})ds = S_L(t, 0)\mathscr{R}\left(t, A^{in}, F^{in}, G^{in}\right).$$

By virtue of Lemma 3.1, we obtain

$$|A(t)| \geq C|\mathscr{R}\left(t, A^{in}, F^{in}, G^{in}\right)|.$$

Using the identity in (65), we get from the above inequality that

$$\|\mathbb{Q}[u](t)\|_{L^2}^2 + \gamma\|\eta(t) + \theta(t)\|_{L^2}^2 = \sum_k \int \sqrt{\alpha}\left(|A(t)|^2 + \gamma^2|A(t)|^2\right) d\xi$$

$$\gtrsim \sum_k \int (1+\gamma^2)\sqrt{\alpha}|\mathscr{R}\left(t, A^{in}, F^{in}, G^{in}\right)|^2 d\xi$$

$$\gtrsim \sum_k \int (1+\gamma^2)\langle \xi - kt \rangle |\mathscr{R}\left(t, A^{in}, F^{in}, G^{in}\right)|^2 d\xi$$

$$\gtrsim \sum_k \int (1+\gamma^2)\frac{\langle kt \rangle}{\langle \xi \rangle}|\mathscr{R}\left(t, A^{in}, F^{in}, G^{in}\right)|^2 d\xi,$$

where we have used the facts that $\sqrt{\alpha} \gtrsim \langle \xi - kt \rangle$ and $\langle \xi - kt \rangle \langle \xi \rangle \gtrsim \langle kt \rangle$. The definition of the anisotropic Sobolev space gives

$$\|\mathbb{Q}[u](t)\|_{L^2}^2+\gamma\|\eta(t)\|_{L^2}^2 \gtrsim \langle t\rangle \left\|\mathscr{R}\left(t, A^{in}, F^{in}, G^{in}\right)\right\|_{L_x^2 H_y^{-\frac{1}{2}}}^2 .$$

This completes the proof. □

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Declaration

Dedicated to Professor Boling Guo on the Occasion of His 90th Birthday.

Conflict of interest

The authors declare no competing financial interest.

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