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Research Article

Existence Results of Nonlinear Fractional Differential Equations with Nonlocal Conditions

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Abstract: This paper explores a new class of nonlinear fractional differential equations coupled with integral terms and nonlocal boundary conditions. We consider ${}^cD^{\phi}\psi(\rho) = \mathscr{H}(\rho,\, \psi(\rho),\, (\Upsilon\psi)(\rho))$, $\rho \in [0,\, \varpi]$, $1 < \phi \le 2$, subject to the following conditions $\psi(h) = -\psi(\varpi)$, $\psi^{'}(h) = -\psi^{'}(\varpi)$, $0 \le h < \varpi$. The focus is on establishing the existence and uniqueness of solutions using a combination of fixed point theorems and the contraction principals. Our results extend some previous work and contribute to understanding fractional differential equations with complex boundary behaviors, offering new insights into their applications.

Keywords: fractional derivatives, differential equations, fractional differential equations, nonlocal boundary conditions, existence

MSC: 74H10, 26A33, 34B10, 34B15

1. Introduction

In this paper we discuses the existence of a solution of the following problem

$$\begin{cases} {}^cD^{\phi}\psi(\rho) = \mathscr{H}(\rho,\, \psi(\rho),\, (\Upsilon\psi)(\rho)), \quad \rho \in [0,\, \varpi], \quad 1 < \phi \le 2, \\ \\ \psi(h) = -\psi(\varpi), \;\; \psi^{'}(h) = -\psi^{'}(\varpi), \quad 0 \le h < \varpi, \end{cases} \tag{1}$$

where ${}^cD^{\phi}$ is the Caputo derivative of order ϕ, $\mathscr{H} : [0,\, \varpi] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, and for $\beta : [0,\, \varpi] \times [0,\, \varpi] \longrightarrow [0,\, \infty)$, $(\Upsilon\psi)(\rho) = \int_0^{\rho} \beta(\rho,\, \tau)\psi(\tau)d\tau$ with $\beta_0 = \max\{\int_0^{\rho} \beta(\rho,\, \tau)d\tau : (\rho,\, \tau) \in [0,\, \varpi] \times [0,\, \varpi]\}$. The main novelty of this work lies in the investigation of the existence of solutions for a class of fractional differential equations involving the Caputo derivative of order $\phi \in (1,\, 2]$, subject to the boundary conditions of the form $\psi(h) = -\psi(\varpi)$ and $\psi'(h) = -\psi'(\varpi)$. The work extends classical existence theory by employing advanced fixed point techniques.

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Fractional Differential Equations (FDEs) expand the ordinary differential equations framework to encompass derivatives of non-integer orders. This extension is precious for modeling intricate systems that exhibit memory and hereditary characteristics commonly encountered in finance, physics, biology, and engineering [1, 2].

A notable characteristic of FDEs is their capacity to depict systems with nonlocal dynamics, where the future state is contingent on the entire history of the system rather than solely on its present state. This is particularly pertinent for phenomena like heat conduction in materials with fractal structures and the dynamic behavior of financial markets [3, 4].

Mathematically, fractional differential equations are explicitly formulated using fractional derivatives and integrals, which can be defined in various ways. The Caputo derivative, often preferred in applications, allows for initial conditions regarding integer-order derivatives. This feature makes it more intuitive for practitioners in engineering and science, as they can relate it to their existing knowledge of ordinary differential equations. The application of FDEs spans a broad spectrum of disciplines. FDEs are well-suited to model viscoelasticity because the fractional derivative can represent the material's memory-dependent response—its strain depends not just on the current stress but also on its past states. The boundary conditions in (1) can correspond to a scenario where the material is clamped or constrained at two points with the displacement and its rate of change. In control theory, fractional controllers can yield enhanced system performance and robustness. In biomedical engineering, FDEs are employed to model disease spread and the dynamics of biological tissues. Environmental scientists utilize fractional models to explore pollutant dispersion and the impacts of climate change [5–7].

Numerous studies have investigated the existence of solutions for fractional differential equations of order $\phi \in (1, 2]$. For instance, the author established existence results for nonlinear integrodifferential equations of fractional order by employing the contraction mapping principle and Krasnoselskii's fixed point theorem [8]. Ahmad et al. [9] extended this analysis by leveraging Leray-Schauder degree theory to demonstrate the existence of solutions for fractional differential equations under specific boundary conditions. Building on these methods, the work in [10] explored existence results for boundary value problems defined by $a_i\psi(h) = b_i\psi(\varpi)$, $i = 0, 1$, where, $a_i, b_i \in \mathbb{R}^+$, incorporate a point $0 \le h < \varpi$, using fixed point theorems. Finally, [11] derived existence criteria through the Schauder fixed point theorem and the nonlinear alternative of Leray-Schauder type, further broadening the scope of analytical tools applicable to such problems. For more interesting results see [12–15].

Numerical methods are a practical approach for finding solutions due to the complexity of the system (1), especially since analytical solutions may be intractable. The predictor-corrector method is a well-established choice for solving Caputo fractional differential equations and can be adapted to the problem (1). It is very easy to be implemented using a program on a computer [16].

The structure of this paper is divided into many key sections. Section 2 outlines the materials and methods used in the research, providing the foundation of the existence results. In Section 3, the results obtained from the study are presented and analyzed, offering insights into the findings. Section 4 follows with specific examples that illustrate and support the results. The last section concludes the paper, highlighting the key points and offering final thoughts on the study's implications. This structure ensures a clear and logical progression from methodology to results, followed by detailed examples and concluding reflections.

2. Preliminaries

Definition 1 [1] For a given function $\zeta \in AC^m(0, \varpi)$ we define the Caputo fractional derivative of order $\phi > 0$ as follows:

$$^{c}D^{\phi}\zeta(s) = \frac{1}{\Gamma(m-\phi)}\int_0^s (s-\rho)^{m-\phi-1}\zeta^{(m)}(\rho)d\rho,$$

where $m = [\phi]+1$, $[\phi]$ is the integer part of ϕ and Γ is the Gamma function.

Definition 2 [1] For a defined function $\zeta \in L^1(0, \varpi)$ The Riemann-Liouville Fractional Integral (RLFI) of order ϕ, I^{ϕ}

$$I^{\phi}\zeta(h) = \frac{1}{\Gamma(\phi)}\int_0^h (h-\rho)^{\phi-1}\zeta(\rho)d\rho, \quad \phi > 0.$$

Lemma 1 [1] The general solution of ${}^cD^{\phi}\psi(\rho) = 0$, where $\phi > 0$, is given by

$$\psi(\rho) = k_0 + k_1\rho + k_2\rho^2 + \cdots + k_{m-1}\rho^{m-1}, \tag{2}$$

where $k_i \in \mathbb{R}$, $i = 0, 1, 2, \cdots, m-1$, $m = [\phi]+1$.

Lemma 2 The unique solution of

$$\begin{cases} {}^cD^{\phi}\psi(\rho) = z(\rho), \quad \rho \in [0, \varpi], \quad 1 < \phi \leq 2, \\ \psi(h) = -\psi(\varpi), \ \psi'(h) = -\psi'(\varpi), \quad 0 \leq h < \varpi, \end{cases} \tag{3}$$

is given by:

$$\psi(\rho) = \int_0^{\rho}\frac{(\rho-\tau)^{\phi-1}}{\Gamma(\phi)}z(\tau)d\tau - \frac{1}{2}\left[\int_0^h \frac{(h-\tau)^{\phi-1}}{\Gamma(\phi)}z(\tau)d\tau + \int_0^{\varpi}\frac{(\varpi-\tau)^{\phi-1}}{\Gamma(\phi)}z(\tau)d\tau\right]$$

$$+\frac{(h+\varpi)-2\rho}{4}\left[\int_0^h \frac{(h-\tau)^{\phi-2}}{\Gamma(\phi-1)}z(\tau)d\tau + \int_0^{\varpi}\frac{(\varpi-\tau)^{\phi-2}}{\Gamma(\phi-1)}z(\tau)d\tau\right]. \tag{4}$$

Proof. In the view of Lemma 1, $\psi(\rho) = I^{\phi}z(\rho) - k_0 - k_1\rho$ is the general solution for (3). Using the boundary conditions we see that,

$$k_0 = \frac{1}{2}\left[\int_0^h \frac{(h-\tau)^{\phi-1}}{\Gamma(\phi)}z(\tau)d\tau + \int_0^{\varpi}\frac{(\varpi-\tau)^{\phi-1}}{\Gamma(\phi)}z(\tau)d\tau\right]$$

$$-\frac{(h+\varpi)}{4}\left[\int_0^h \frac{(h-\tau)^{\phi-2}}{\Gamma(\phi-1)}z(\tau)d\tau + \int_0^{\varpi}\frac{(\varpi-\tau)^{\phi-2}}{\Gamma(\phi-1)}z(\tau)d\tau\right].$$

$$k_1 = \frac{1}{2}\left[\int_0^h \frac{(h-\tau)^{\phi-2}}{\Gamma(\phi-1)}z(\tau)d\tau + \int_0^{\varpi}\frac{(\varpi-\tau)^{\phi-2}}{\Gamma(\phi-1)}z(\tau)d\tau\right].$$

Therefore,

$$\psi(\rho) = \int_0^{\rho}\frac{(\rho-\tau)^{\phi-1}}{\Gamma(\phi)}z(\tau)d\tau - \frac{1}{2}\left[\int_0^h \frac{(h-\tau)^{\phi-1}}{\Gamma(\phi)}z(\tau)d\tau + \int_0^{\varpi}\frac{(\varpi-\tau)^{\phi-1}}{\Gamma(\phi)}z(\tau)d\tau\right]$$

$$+\frac{(h+\varpi)-2\rho}{4}\left[\int_0^h \frac{(h-\tau)^{\phi-2}}{\Gamma(\phi-1)}z(\tau)d\tau+\int_0^{\varpi} \frac{(\varpi-\tau)^{\phi-2}}{\Gamma(\phi-1)}z(\tau)d\tau\right]. \qquad \square$$

Theorem 1 [17] Let X be a Banach space, and $M \subset X$ be nonempty closed convex set. Let χ_1, χ_2 be two operators satisfying $\chi_1\psi$, $\chi_2\hat{\psi} \in M$ for ψ, $\hat{\psi} \in M$; χ_1 is compact and continuous; and χ_2 is a contraction operator. Then $\exists \mu \in M$ such that $\mu = \chi_1\mu + \chi_2\mu$.

3. Results

Define a Banach space $(\mathbb{R}, ||.||)$ of all continuous functions $\mathscr{B} = C([0, \varpi], \mathbb{R})$ endowed with the norm $\|\psi\| = \sup\limits_{\rho\in[0, \varpi]} |\psi(\rho)|$.

Define $\mathscr{Y} : \mathscr{B} \longrightarrow \mathscr{B}$ by

$$\begin{aligned}(\mathscr{Y}\psi)(\rho) = &\int_0^{\rho} \frac{(\rho-\tau)^{\phi-1}}{\Gamma(\phi)}\mathscr{H}(\tau, \psi(\tau), (\Upsilon\psi)(\tau))d\tau \\ &-\frac{1}{2}\left[\int_0^h \frac{(h-\tau)^{\phi-1}}{\Gamma(\phi)}\mathscr{H}(\tau, \psi(\tau), (\Upsilon\psi)(\tau))d\tau+\int_0^{\varpi} \frac{(\varpi-\tau)^{\phi-1}}{\Gamma(\phi)}\mathscr{H}(\tau, \psi(\tau), (\Upsilon\psi)(\tau))d\tau\right] \\ &+\frac{(h+p)-2\rho}{4}\left[\int_0^h \frac{(h-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathscr{H}(\tau, \psi(\tau), (\Upsilon\psi)(\tau))d\tau\right. \\ &\left.+\int_0^{\varpi} \frac{(\varpi-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathscr{H}(\tau, \psi(\tau), (\Upsilon\psi)(\tau))d\tau\right]. \end{aligned} \tag{5}$$

Theorem 2 For a continuous function $\mathscr{H} : [0, \varpi] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfying $||\mathscr{H}(\rho, \psi(\rho), (\Upsilon\psi)(\rho)) - \mathscr{H}(\rho, \hat{\psi}(\rho), (\Upsilon\hat{\psi})(\rho))|| \leq K_1||\psi - \hat{\psi}|| + K_2||\Upsilon\psi - \Upsilon\hat{\psi}||$ for all $\rho \in [0, \varpi]$ with

$$\frac{K_1+\beta_0K_2}{\Gamma(\phi+1)}\left[3\varpi^{\phi}+h^{\phi}+\frac{\phi(h+\varpi)(h^{\phi-1}+\varpi^{\phi-1})}{2}\right] < 1.$$

Then, the FDE problem (1) has a unique solution.

Proof. Define $\mathscr{Y}$ as in (5), and set $\sup\limits_{\rho\in[0, \varpi]} \big|\big|\mathscr{H}(\rho, 0, 0)\big|\big| = N$. Consider the ball $B = \{\psi \in \mathscr{B} : ||\psi|| \leq r\}$ where r is a chosen radius such that

$$r \geq \frac{N}{\Gamma(\phi+1)}\left[3\varpi^{\phi}+h^{\phi}+\frac{\phi(h+\varpi)(h^{\phi-1}+\varpi^{\phi-1})}{2}\right].$$

Now for $\psi \in B$,

$$
\begin{aligned}
\|(\mathscr{Y}\psi)(\rho)\| \leq & \max_{\rho\in[0,\,\varpi]}\left[\int_0^{\rho}\frac{(\rho-\tau)^{\phi-1}}{\Gamma(\phi)}|\mathscr{H}(\tau,\,\psi(\tau),\,(\Upsilon\psi)(\tau))|\,d\tau\right.\\
&+\frac{1}{2}\left(\int_0^{h}\frac{(h-\tau)^{\phi-1}}{\Gamma(\phi)}|\mathscr{H}(\tau,\,\psi(\tau),\,(\Upsilon\psi)(\tau))|\,d\tau\right.\\
&\left.+\int_0^{\varpi}\frac{(\varpi-\tau)^{\phi-1}}{\Gamma(\phi)}|\mathscr{H}(\tau,\,\psi(\tau),\,(\Upsilon\psi)(\tau))|\,d\tau\right)\\
&+\frac{|(h+\varpi)-2\rho|}{4}\left(\int_0^{h}\frac{(h-\tau)^{\phi-2}}{\Gamma(\phi-1)}|\mathscr{H}(\tau,\,\psi(\tau),\,(\Upsilon\psi)(\tau))|\,d\tau\right.\\
&\left.\left.+\int_0^{\varpi}\frac{(\varpi-\tau)^{\phi-2}}{\Gamma(\phi-1)}|\mathscr{H}(\tau,\,\psi(\tau),\,(\Upsilon\psi)(\tau))|\,d\tau\right)\right]\\
\leq & \max_{\rho\in[0,\,\varpi]}\left[\int_0^{\rho}\frac{(\rho-\tau)^{\phi-1}}{\Gamma(\phi)}|\mathscr{H}(\tau,\,\psi,\,\Upsilon\psi)-\mathscr{H}(\tau,\,0,\,0)+\mathscr{H}(\tau,\,0,\,0)|\,d\tau\right.\\
&+\frac{1}{2}\left(\int_0^{h}\frac{(h-\tau)^{\phi-1}}{\Gamma(\phi)}|\mathscr{H}(\tau,\,\psi,\,\Upsilon\psi)-\mathscr{H}(\tau,\,0,\,0)+\mathscr{H}(\tau,\,0,\,0)|\,d\tau\right.\\
&\left.+\int_0^{\varpi}\frac{(\varpi-\tau)^{\phi-1}}{\Gamma(\phi)}|\mathscr{H}(\tau,\,\psi,\,\Upsilon\psi)-\mathscr{H}(\tau,\,0,\,0)+\mathscr{H}(\tau,\,0,\,0)|\,d\tau\right)\\
&+\frac{|(h+\varpi)-2\rho|}{4}\left(\int_0^{h}\frac{(h-\tau)^{\phi-2}}{\Gamma(\phi-1)}|\mathscr{H}(\tau,\,\psi,\,\Upsilon\psi)-\mathscr{H}(\tau,\,0,\,0)+\mathscr{H}(\tau,\,0,\,0)|\,d\tau\right.\\
&\left.\left.+\int_0^{\varpi}\frac{(\varpi-\tau)^{\phi-2}}{\Gamma(\phi-1)}|\mathscr{H}(\tau,\,\psi,\,\Upsilon\psi)-\mathscr{H}(\tau,\,0,\,0)+\mathscr{H}(\tau,\,0,\,0)|\,d\tau\right)\right]\\
\leq & \ [(K_1+\beta_0K_2)r+N]\left[\int_0^{\rho}\frac{(\rho-\tau)^{\phi-1}}{\Gamma(\phi)}d\tau+\frac{1}{2}\left(\int_0^{h}\frac{(h-\tau)^{\phi-1}}{\Gamma(\phi)}d\tau+\int_0^{\varpi}\frac{(\varpi-\tau)^{\phi-1}}{\Gamma(\phi)}d\tau\right)\right.\\
&\left.+\frac{|(h+\varpi)-2\rho|}{4}\left(\int_0^{h}\frac{(h-\tau)^{\phi-2}}{\Gamma(\phi-1)}d\tau+\int_0^{\varpi}\frac{(\varpi-\tau)^{\phi-2}}{\Gamma(\phi-1)}d\tau\right)\right]\\
\leq & \ [(K_1+\beta_0K_2)r+N]\left[\frac{1}{2\Gamma(\phi+1)}\left(3\varpi^{\phi}+h^{\phi}+\frac{\phi(h+\varpi)(h^{\phi-1}+\varpi^{\phi-1})}{2}\right)\right]\leq r.
\end{aligned}
$$

Next, for all $\rho\in[0,\,\varpi]$ and for $\psi,\,\hat{\psi}\in\mathscr{B}$, we have

$$
\begin{aligned}
&\|(\mathscr{Y}\psi)(\rho)-(\mathscr{Y}\hat{\psi})(\rho)\| \\
&\leq \max_{\rho\in[0,\,\varpi]}\left[\int_0^{\rho}\frac{(\rho-\tau)^{\phi-1}}{\Gamma(\phi)}\,|\mathscr{H}(\tau,\,\psi(\tau),\,(\Upsilon\psi)(\tau))-\mathscr{H}(\tau,\,\hat{\psi}(\tau),\,(\Upsilon\hat{\psi})(\tau))|\,d\tau\right. \\
&\quad+\frac{1}{2}\left(\int_0^{h}\frac{(h-\tau)^{\phi-1}}{\Gamma(\phi)}\,|\mathscr{H}(\tau,\,\psi(\tau),\,(\Upsilon\psi)(\tau))-\mathscr{H}(\tau,\,\hat{\psi}(\tau),\,(\Upsilon\hat{\psi})(\tau))|\,d\tau\right. \\
&\quad\left.+\int_0^{\varpi}\frac{(\varpi-\tau)^{\phi-1}}{\Gamma(\phi)}\,|\mathscr{H}(\tau,\,\psi(\tau),\,(\Upsilon\psi)(\tau))-\mathscr{H}(\tau,\,\hat{\psi}(\tau),\,(\Upsilon\hat{\psi})(\tau))|\,d\tau\right) \\
&\quad+\frac{|(h+\varpi)-2\rho|}{4}\left(\int_0^{h}\frac{(h-\tau)^{\phi-2}}{\Gamma(\phi-1)}\,|\mathscr{H}(\tau,\,\psi(\tau),\,(\Upsilon\psi)(\tau))-\mathscr{H}(\tau,\,\hat{\psi}(\tau),\,(\Upsilon\hat{\psi})(\tau))|\,d\tau\right. \\
&\quad\left.\left.+\int_0^{\varpi}\frac{(\varpi-\tau)^{\phi-2}}{\Gamma(\phi-1)}\,|\mathscr{H}(\tau,\,\psi(\tau),\,(\Upsilon\psi)(\tau))-\mathscr{H}(\tau,\,\hat{\psi}(\tau),\,(\Upsilon\hat{\psi})(\tau))|\,d\tau\right)\right] \\
&\leq \frac{K_1+\beta_0K_2}{2\Gamma(\phi+1)}\left(3\varpi^{\phi}+h^{\phi}+\frac{\phi(h+\varpi)(h^{\phi-1}+\varpi^{\phi-1})}{2}\right)\|\psi-\hat{\psi}\| \\
&\leq \delta_{K_1,\,K_2,\,\beta_0,\,\varpi,\,h,\,\phi}\,\|\psi-\hat{\psi}\|,
\end{aligned}
$$

where, $\delta_{K_1,\,K_2,\,\beta_0,\,\varpi,\,h,\,\phi}$ depends on the parameters studied in the problem. Whenever $\delta_{K_1,\,K_2,\,\beta_0,\,\varpi,\,h,\,\phi}<1$, the operator $\mathscr{Y}$ is a contraction. Thus, the existence of a unique solution is concluded by the Banach fixed point theorem. □

Theorem 3 For a continuous function $\mathscr{H}:[0,\,\varpi]\times\mathbb{R}\times\mathbb{R}$ satisfying $||\mathscr{H}(\rho,\,\psi(\rho),\,(\Upsilon\psi)(\rho))-\mathscr{H}(\rho,\,\hat{\psi}(\rho),\,(\Upsilon\hat{\psi})(\rho))||\leq\omega(\rho)||\psi-\hat{\psi}||+\hat{\omega}(\rho)||\Upsilon\psi-\Upsilon\hat{\psi}||$ for all $\rho\in[0,\,\varpi]$, $\psi,\,\hat{\psi}\in\mathbb{R}$, $\Upsilon\in\mathbb{R}$, and $\omega,\,\hat{\omega}\in L^1([0,\,\varpi],\,\mathbb{R}^+)$ with

$$
\frac{||\omega||_{L^1}+\beta_0||\hat{\omega}||_{L^1}}{2\Gamma(\phi+1)}\left[3\varpi^{\phi}+h^{\phi}+\frac{\phi(h+\varpi)(h^{\phi-1}+\varpi^{\phi-1})}{2}\right]<1.
$$

The FDE problem (1) has a unique solution.

Proof. For all $\rho\in[0,\,\varpi]$ and for $\psi,\,\hat{\psi}\in\mathscr{B}$, we have

$$
\begin{aligned}
&\|(\mathscr{Y}\psi)(\rho)-(\mathscr{Y}\hat{\psi})(\rho)\| \\
&\leq \max_{\rho\in[0,\,\varpi]}\left[\int_0^{\rho}\frac{(\rho-\tau)^{\phi-1}}{\Gamma(\phi)}\,|\mathscr{H}(\tau,\,\psi(\tau),\,(\Upsilon\psi)(\tau))-\mathscr{H}(\tau,\,\hat{\psi},\,(\Upsilon\hat{\psi})(\tau))|\,d\tau\right. \\
&\quad+\frac{1}{2}\left(\int_0^{h}\frac{(h-\tau)^{\phi-1}}{\Gamma(\phi)}\,|\mathscr{H}(\tau,\,\psi(\tau),\,(\Upsilon\psi)(\tau))-\mathscr{H}(\tau,\,\hat{\psi},\,(\Upsilon\hat{\psi})(\tau))|\,d\tau\right.
\end{aligned}
$$

$$+\int_0^{\varpi}\frac{(\varpi-\tau)^{\phi-1}}{\Gamma(\phi)}|\mathscr{H}(\tau,\, \psi(\tau),\, (\Upsilon\psi)(\tau))-\mathscr{H}(\tau,\, \hat{\psi},\, (\Upsilon\hat{\psi})(\tau))|d\tau\Bigg)$$

$$+\frac{|(h+\varpi)-2\rho|}{4}\Bigg(\int_0^{h}\frac{(h-\tau)^{\phi-2}}{\Gamma(\phi-1)}|\mathscr{H}(\tau,\, \psi(\tau),\, (\Upsilon\psi)(\tau))-\mathscr{H}(\tau,\, \hat{\psi},\, (\Upsilon\hat{\psi})(\tau))|d\tau$$

$$+\int_0^{\varpi}\frac{(\varpi-\tau)^{\phi-2}}{\Gamma(\phi-1)}|\mathscr{H}(\tau,\, \psi(\tau),\, (\Upsilon\psi)(\tau))-\mathscr{H}(\tau,\, \hat{\psi},\, (\Upsilon\hat{\psi})(\tau))|d\tau\Bigg)\Bigg]$$

$$\leq\frac{\|\omega\|_{L^1}+\beta_0\|\hat{\omega}\|_{L^1}}{2\Gamma(\phi+1)}\left(3p^{\phi}+h^{\phi}+\frac{\phi(h+p)(h^{\phi-1}+p^{\phi-1})}{2}\right)\|\psi-\hat{\psi}\|.$$

Therefore, $||(\mathscr{Y}\psi)(\rho)-(\mathscr{Y}\hat{\psi})(\rho)||\leq\frac{||\omega||_{L^1}+\beta_0||\hat{\omega}||_{L^1}}{2\Gamma(\phi+1)}\left[3\varpi^{\phi}+h^{\phi}+\frac{\phi(h+\varpi)(h^{\phi-1}+\varpi^{\phi-1})}{2}\right]||\psi-\hat{\psi}||$, $\mathscr{Y}$ is a contraction and $\frac{||\omega||_{L^1}+\beta_0||\hat{\omega}||_{L^1}}{2\Gamma(\phi+1)}\left[3\varpi^{\phi}+h^{\phi}+\frac{\phi(h+\varpi)(h^{\phi-1}+\varpi^{\phi-1})}{2}\right]<1$. Then the unique solution exists for 1.1 by the contraction principal. □

Theorem 4 For a continuous function $\mathscr{H}:[0,\,\varpi]\times\mathbb{R}\times\mathbb{R}$ satisfying $||\mathscr{H}(\rho,\,\psi(\rho),\,(\Upsilon\psi)(\rho))-\mathscr{H}(\rho,\,\hat{\psi}(\rho),\,(\Upsilon\hat{\psi})(\rho))||\leq K_1||\psi-\hat{\psi}||+K_2||\Upsilon\psi-\Upsilon\hat{\psi}||$ for all $\rho\in[0,\,\varpi]$ and $||\mathscr{H}(\rho,\,\psi(\rho),\,(\Upsilon\psi)(\rho))||\leq\omega(\rho)$ for all $(\rho,\,\psi,\,\Upsilon\psi)\in[0,\,\varpi]\times\mathbb{R}\times\mathbb{R}$ and $\omega\in L^1([0,\,\varpi],\,\mathbb{R}^+)$ with $\left[\frac{K_1+\beta_0K_2}{\Gamma(\phi+1)}\right]\left[\varpi^{\phi}+h^{\phi}+\frac{\phi(h+\varpi)(h^{\phi-1}+\varpi^{\phi-1})}{2}\right]<1$. Then, the FDE problem (1) has at least one solution.

Proof. Consider a ball $B=\{\psi\in\mathscr{B}:||\psi||\leq r\}$, where r is fixed such that

$$r\geq\frac{||\omega||_{L^1}}{2\Gamma(\phi+1)}\left[3\varpi^{\phi}+h^{\phi}+\frac{\phi(h+\varpi)(h^{\phi-1}+\varpi^{\phi-1})}{2}\right].$$

Define two operators χ_1, χ_2 as

$$(\chi_1\psi)(\rho)=\int_0^{\rho}\frac{(\rho-\tau)^{\phi-1}}{\Gamma(\phi)}\mathscr{H}(\tau,\,\psi(\tau),\,(\Upsilon\psi)(\tau))d\tau$$

$$(\chi_2\psi)(\rho)=-\frac{1}{2}\left[\int_0^{h}\frac{(h-\tau)^{\phi-1}}{\Gamma(\phi)}\mathscr{H}(\tau,\,\psi(\tau),\,(\Upsilon\psi)(\tau))d\tau+\int_0^{\varpi}\frac{(\varpi-\tau)^{\phi-1}}{\Gamma(\phi)}\mathscr{H}(\tau,\,\psi(\tau),\,(\Upsilon\psi)(\tau))d\tau\right]$$

$$+\frac{|(h+\varpi)-2\rho|}{4}\left[\int_0^{h}\frac{(h-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathscr{H}(\tau,\,\psi(\tau),\,(\Upsilon\psi)(\tau))d\tau\right.$$

$$\left.+\int_0^{\varpi}\frac{(\varpi-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathscr{H}(\tau,\,\psi(\tau),\,(\Upsilon\psi)(\tau))d\tau\right]$$

Now for any ψ, $\hat{\psi}\in B$ we get

$$||\chi_1\psi+\chi_2\hat{\psi}|| \leq \max_{\rho\in[0,\,\varpi]}\Bigg[\int_0^{\rho}\frac{(\rho-\tau)^{\phi-1}}{\Gamma(\phi)}\big|\mathscr{H}(\tau,\,\psi(\tau),\,(\Upsilon\psi)(\tau))\big|d\tau$$

$$+\frac{1}{2}\Bigg[\int_0^{h}\frac{(h-\tau)^{\phi-1}}{\Gamma(\phi)}|\mathscr{H}(\tau,\,\psi(\tau),\,(\Upsilon\psi)(\tau))|d\tau+\int_0^{\varpi}\frac{(\varpi-\tau)^{\phi-1}}{\Gamma(\phi)}\mathscr{H}(\tau,\,\psi(\tau),\,(\Upsilon\psi)(\tau))d\tau\Bigg]$$

$$+\frac{|(h+\varpi)-2\rho|}{4}\Bigg[\int_0^{h}\frac{(h-\tau)^{\phi-2}}{\Gamma(\phi-1)}\Big|\mathscr{H}(\tau,\,\psi(\tau),\,(\Upsilon\psi)(\tau))\Big|d\tau$$

$$+\int_0^{\varpi}\frac{(\varpi-\tau)^{\phi-2}}{\Gamma(\phi-1)}\Big|\mathscr{H}(\tau,\,\psi(\tau),\,(\Upsilon\psi)(\tau))\Big|d\tau\Bigg]\Bigg].$$

We get that,

$$||\chi_1\psi+\chi_2\hat{\psi}|| \leq \frac{||\omega||_{L_1}}{2\Gamma(\phi+1)}\left[3\varpi^{\phi}+h^{\phi}+\frac{\phi(h+\varpi)(h^{\phi-1}+\varpi^{\phi-1})}{2}\right]\leq r.$$

Therefore, $\chi_1\psi+\chi_2\hat{\psi}\in B$.

By assumption χ_2 is a contraction. Moreover, χ_1 is continuous due to the continuity of $\mathscr{H}$. Additionally, since $||\chi_1\psi|| \leq \dfrac{||\omega||\varpi^{\phi}}{\Gamma(\phi+1)}$, it follows that χ_1 is uniformly bounded on B.

Next we prove the compactness of χ_1. Define $\Theta=[0,\,\varpi]\times B\times B$ and $\sup_{(\rho,\,\psi,\,\Upsilon\psi)\in\Theta}\big|\big|\mathscr{H}(\rho,\,\psi,\,\Upsilon\psi)\big|\big|=\mathscr{H}_{max}$, then,

$$\big|\big|(\chi_1\psi)(\rho_1)-(\chi_1\psi)(\rho_2)\big|\big|=\Bigg|\Bigg|\frac{1}{\Gamma(\phi)}\int_0^{\rho_1}\big[(\rho_1-\tau)^{\phi-1}-(\rho_2-\tau)^{\phi-1}\big]\mathscr{H}(\tau,\,\psi(\tau),\,(\Upsilon\psi)(\tau))d\tau$$

$$+\frac{1}{\Gamma(\phi)}\int_{\rho_2}^{\rho_1}(\rho_2-\tau)^{\phi-1}\mathscr{H}(\tau,\,\psi(\tau),\,(\Upsilon\psi)(\tau))d\tau\Bigg|\Bigg|.$$

Therefore,

$$\big|\big|(\chi_1\psi)(\rho_1)-(\chi_1\psi)(\rho_2)\big|\big|\leq\frac{\mathscr{H}_{max}}{\Gamma(\phi+1)}\big|2(\rho_2-\rho_1)^{\phi}+\rho_1^{\phi}-\rho_2^{\phi}\big|,$$

as $\rho_2\longrightarrow\rho_1$ the norm tends to zero and independent of ϕ. Hence ϕ is relatively compact on B and therefore compact on B by Arzela-Ascoli Theorem [18]. By Theorem 1 we conclude the FDE problem (1) has at least one solution. □

Remark 1 The results presented in Theorem 2 generalize those of Theorem 3.1 in [8], while Theorem 4 extends the results of Theorem 3.2 in [8]. Moreover, Theorem 3 in this paper is entirely new and has no direct counterpart in [8], thereby contributing original findings to the field.

4. Example

Example 1 Consider the following problem:

$$\begin{cases} {}^cD^{\frac{1}{2}}\psi(\rho) = \dfrac{1}{(\rho+2)^4} + \dfrac{||\psi||}{1+||\psi||} + \int_0^\rho \dfrac{e^{-(\tau-\rho)}}{16}\psi(\tau)d\tau, \quad \rho \in [0, 1], \\ \\ \psi(0) = -\psi(1), \;\; \psi'(0) = -\psi'(1). \end{cases} \tag{6}$$

Clearly, $h = 0$, $\varpi = 1$, $\mathscr{H}(\rho, \psi) = \dfrac{1}{(\rho+3)^6} + \dfrac{||\psi||}{1+||\psi||}$, $\beta(\rho, \tau) = \dfrac{e^{-(\tau-\rho)}}{16}$, $\beta_0 = \max\{\int_0^\rho \beta(\rho, \tau)d\tau : (\rho, \tau) \in [0, 1]\times[0, 1]\} \approx 0.107$.

Also, $||\mathscr{H}(\rho, \psi(\rho), (\Upsilon\psi)(\rho)) - \mathscr{H}(\rho, \hat{\psi}(\rho), (\Upsilon\hat{\psi})(\rho))|| \leq \dfrac{1}{16}(||\psi-\hat{\psi}|| + ||\Upsilon\psi - \Upsilon\hat{\psi}||)$, and $K_1 = K_2 = \dfrac{1}{16}$,

$$\frac{K_1+\beta_0K_2}{\Gamma(\phi+1)}\left[3\varpi^\phi + h^\phi + \frac{\phi(h+\varpi)(h^{\phi-1}+\varpi^{\phi-1})}{2}\right] = \frac{1+\beta_0}{16\Gamma\left(\dfrac{3}{2}\right)}\left[3+\frac{1}{4}\right] \approx 0.254 < 1.$$

Therefore, by Theorem 2 we see that problem (6) has a unique solution.

5. Conclusions

The paper addresses a class of fractional integrodifferential equations subject to nonlocal boundary conditions, advancing the theoretical framework for such nonlocal problems. By leveraging fixed-point theorems—precisely the contraction principle and Krasnoselskii's fixed-point theorem—we establish rigorous criteria for the existence and uniqueness of solutions. With some additional terms to the solution, we see that Theorems 2 and 3 guarantee a unique solution under continuity conditions on the nonlinear term. In contrast, Theorem 4 ensures the existence of at least one solution under compactness assumptions, thereby broadening the applicability of the results. Moreover, Theorems 2 and 4 presented in this paper extend and generalize previous findings: Theorem 2 generalizes the results of Theorem 3.1 in [8], while Theorem 4 extends those of Theorem 3.2 in [8]. In addition, Theorem 3 is entirely new.

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Data availability

The article contains all the necessary data that was utilized to back up the findings of this work.

Conflict of interest

The authors does not have any conflict of interest.

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