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Research Article

Prioritization of Geothermal Energy Systems for Industrial Applications by Using Hesitant Bipolar Fuzzy Multi-Criteria Decision-Making Technique Based on Dombi Operators

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Abstract: The proposed research fills a significant gap in the decision-making technique for evaluating geothermal energy systems in industrial processes by introducing a new approach involving Hesitant Bipolar Fuzzy (HBF) Sets (HBFSs) with Dombi operators. The existing literature has mostly focused on uncertainty only, overlooking the aspect that decisions tend to be imprecise, bipolar, and hesitant in reality. To overcome this gap, we first introduce Dombi operators in the context of HBFSs, thereby improving the parametric flexibility in handling more complex uncertain information. Based on these operators, we establish an HBF Multi-Criteria Decision-Making (MCDM) method for the ranking of geothermal energy systems. The applicability of our proposed methodology for prioritizing different types of geothermal energy systems for industrial applications is illustrated in a detailed case study that supports the theoretical framework. The benefit of the suggested method is also supplemented by the comparison of the proposed method with the previous methods and evidence of the capability to handle uncertainty and make more precise and confident decisions. This study offers an important theoretical as well as practical contribution to decision-making practices and the choice of sustainable energy systems for geothermal energy options under uncertainty, offering decision-makers a robust framework of analysis. Moreover, we have the following key findings or outcomes of proposed research. • Development of HBF Dombi Weighted Averaging (HBFDWA) operators. • Development of HBF Dombi Ordered Weighted Averaging (HBFDOWA) operators. • Development of HBF Dombi Weighted Geometric (HBFDWG) operators. • Development of HBF Dombi Ordered Weighted Geometric (HBFDOWG) operators. • A case study is performed based on the developed operators to rank geothermal energy systems. • A comparative analysis is performed to show the superiority of the proposed approach. • A sensitivity analysis is discussed to show the influences of the parameter.

***Keywords*:** geothermal energy systems, industry, MCDM technique, hesitant bipolar fuzzy set

1. Introduction

Geothermal systems utilize heat that comes from deep within the Earth, heat from the molten core of the Earth and the breakdown of radioactive materials deep within the planet. Geothermal is a compound Greek word; geo means earth and thermal means heat. One clear attribute of this source of energy is renewability and low impacts on the natural environment;

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the latter qualifies the source of energy as a central component of transitioning to renewable energy sources. Geothermal power is tapped directly from the earth's heat it should be pointed out however that the earth's core temperatures reach up to 5,000 °C. It radiates outward in the heat and heats surrounding rocks in the mantle. Water that enters by way of cracks and fissures in the earth's crust is heated to produce such phenomena as hot springs and geysers. In some cases, it is hot water that becomes sequestered in the ground in so-called geothermal reservoirs for energy production. Geothermal energy is harnessed in various ways; direct use, where geothermal hot water heats buildings or is utilized for industrial applications, and geothermal power plants, where geothermal reservoir steam or hot water is utilized to drive turbines connected to generators. The advantages of geothermal energy include its nature as a renewable form of energy that does not get depleted, low emission rates compared to power plants that utilize fossil fuels, and an increased potential for energy self-sufficiency with the utilization of locally available resources. Being among the cleaner forms of power, geothermal energy systems cannot be avoided in the battle against carbon emissions and the push towards energy independence. Figure 1 and Figure 2 show the usage distribution of geothermal systems and energy output by geothermal systems respectively.

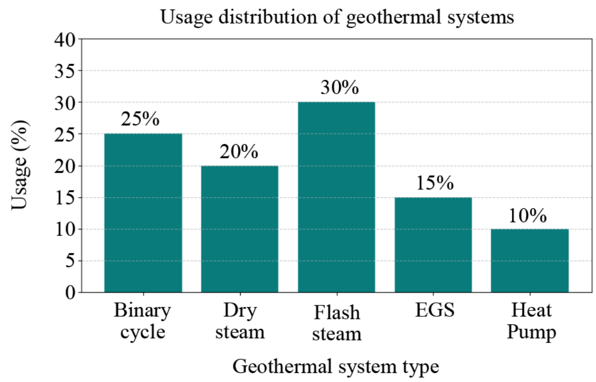

Figure 1. Usage distribution of geothermal systems

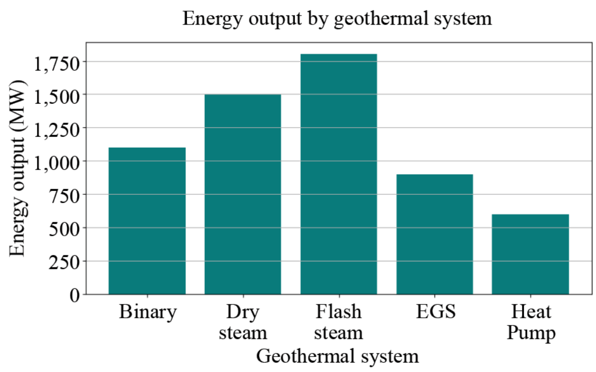

Figure 2. Energy output by geothermal system

Q: What is the need for geothermal energy systems in industrial applications?

Industrial uses of geothermal energy systems are extremely significant as they offer numerous advantages that address the environmental and operation needs of the modern industrial world. The most compelling reason is to provide a reliable and continuous heat source for many processes to reduce the consumption of fossil fuels and to facilitate the sustainable development of industries. Geothermal energy can offer the consistently high temperatures required in cooking, textile processing, and chemical processing industries at lower costs and greenhouse gas emissions in manufacturing industries. The capacity of the system to control temperatures makes it ideal for applications in industries that need specific temperatures like agriculture and aquaculture. Geothermal can be utilized in district cooling and heating systems within industrial parks wherein numerous facilities can be provided with cold and heat from a single source. This centralization is not just efficient in energy consumption but also assists in keeping general infrtimesructure costs low for specific companies. In heavy industries, it can be employed in processes involving high heat input like drying, sterilization, and ptimeseurization, and would be a far greener method of heating. This makes the technology to be a well-suited baseload energy source as it can produce power for 24 hours with minimal variation. Besides, geothermal systems have a significant contribution to industrial decarburization, which is crucial since industries are compelled to reduce their carbon footprint. Geothermal energy can be embraced by industries to achieve their emission reduction objectives, and the embracement also has the potential to allow industries to benefit from environmental incentives and carbon credits. The technology also benefits from having a small footprint since geothermal installations have fewer space requirements compared to most other types of renewable energy and are, therefore, best suited for industries with limited space. Moreover, geothermal systems also have a long lifespan of between 20-30 years, hence providing industries with long-term sources of energy that can go a long way in helping to reduce their vulnerabilities to spiking energy market prices. Such systems are a worthwhile investment for industries located in places with suitable geothermal resources to be harnessed. Access to local geothermal resources reduces foreign energy supply dependency, enhances energy security, and can positively contribute to regional economic growth through the creation of jobs and technological advancement. In addition, the direct utilization of geothermal energy where the heat is utilized at decreasing and decreasing gradients makes the industries optimize the use of energy hence making their operations more sustainable and less expensive. The following Figure 3 illustrates the world geothermal market growth.

Figure 3. Global geothermal market growth

Q: How the prioritization of geothermal energy systems for industrial applications is an MCDM problem?

Prioritization of geothermal systems for industrial use is an MCDM problem since it requires analysis of several, sometimes contrtimesing factors. Technical factors such as availability of resources, heat gradient, and system efficiency are weighed against economic factors including installation cost, operational costs, and return on investment. Environmental impacts such as land use, water consumption, and emissions are also crucial. Moreover, social aspects of public acceptance, the creation of employment, and local development also come into play. As these vary in relative importance based on the region and priorities of the stakeholders, MCDM offers a systematic method for determining trade-offs. It allows decision-makers to compare options clearly and make well-informed decisions that take into account both short-term and long-term effects.

1.1 *MCDM and HBFS contributions*

HBFSs play a significant role in solving MCDM problems by effectively handling uncertainty, hesitation, and the dual nature of criteria (positive and negative). Their strength lies in modeling situations where decision-makers face ambiguity or conflicting opinions, such as evaluating both the benefits and drawbacks of geothermal projects. Unlike traditional fuzzy sets, HBFSs preserve hesitation and bipolarity, making them ideal for integrating diverse expert judgments and uncertain information. Their mathematical structure allows for combining different assessments while maintaining the nuanced nature of the data, which is essential in group decision-making with varying confidence levels and conflicting criteria.

1.2 *Study framework*

Section 1 introduces the study, while section 2 outlines the research problem and key contributions. Section 3 reviews the relevant literature, and section 4 revises the basic notions of the proposed work. In section 5, we develop novel Dombi operators within the HBFS framework and discuss their results. Section 6 presents an HBF MCDM method to tackle real-world problems involving vagueness and hesitancy, supported by a case study on geothermal energy system prioritization. The proposed theory is compared with existing models in section 7. Section 8 concludes the whole study.

2. Research problem and contributions

Why we need this framework, and the existing research gaps are discussed step by step below.

• The selection of geothermal energy systems for industrial use is inherently an MCDM problem, which necessitates the application of appropriate MCDM approaches. However, the existing literature primarily addresses this problem under the assumption that all input data and criteria are fully known and clearly defined, which is often not the case in real-world scenarios.

• A key problem occurs when a number of the criteria used in the evaluation of geothermal energy systems are imprecise, bipolar (having both positive and negative facets), and characterized by uncertainty. To our knowledge, the literature has not as yet addressed this issue from this point of view, nor has it recognized the added complexity due to these features.

• This reveals that there exists an urgent requirement for a decision-making method that can deal with vagueness, bipolarity, and hesitation at the same time. Traditional decision-making models often operate on clear and well-framed information, which restricts their usability in more real-world, complex situations involving doubt and double polarity.

• In real-world applications, especially in geothermal energy decision-making, experts often face incomplete, imprecise, or conflicting information. These conditions create a significant gap in current decision-making methodologies, particularly when the process is influenced by various layers of uncertainty that conventional models are not equipped to handle.

• Dombi aggregation operators, which are grounded in fuzzy set theory, are recognized for their parameterized flexibility and ability to model diverse levels of interaction between criteria. These operators are especially effective in handling nonlinear relationships and offer consistent, adaptable aggregation behavior, which makes them highly suitable for complex decision-making tasks.

• Despite their advantages, Dombi aggregation operators have not yet been constructed or explored within the context of HBFSs. This reveals another significant gap in the literature the lack of operators that can leverage the strengths of both Dombi methods and HBF information to enhance decision-making capabilities.

• The integration of Dombi operators into the HBFS framework presents a valuable opportunity to develop more advanced decision-making tools. By combining the parametric flexibility of Dombi operators with the uncertainty-handling capacity of HBFSs, this research aims to create a powerful methodology capable of managing real-life decision problems characterized by hesitancy and bipolarity.

2.1 *Contribution*

To handle the mentioned problems, we need an improved decision-making framework that fits these details and still accurately collects various pieces of information. Therefore, we are providing the following important contributions.

• To address this gap, we propose a novel family of Dombi aggregation operators in the HBF environment, including HBFDWA, HBFDOWA, HBFDWG, and HBFDOWG operators. These operators offer parameterized control for modeling various degrees of interaction among decision criteria under uncertainty.

• These operators are specifically designed to aggregate information characterized by hesitancy, bipolarity, and fuzziness, thereby enhancing the decision-making process in complex and uncertain environments.

• Based on these operators, we develop a hesitant bipolar fuzzy MCDM method that can handle vagueness, positive-negative polarity, and hesitation simultaneously-key challenges in evaluating geothermal energy systems.

• The proposed method is applied to a real-world case study for the prioritization of geothermal energy systems, showcasing its practical value and ability to produce reliable results in uncertain decision scenarios.

• We use existing methods in comparison to illustrate that our suggested model is more effective, more readily combined, and less biased than other methods.

• Overall, the approach in this study addresses important difficulties in the theory and offers a sturdy framework for managing uncertainty when managing geothermal projects in real practice.

3. Literature review

Geothermal energy systems have attracted much interest in the recent ptimes because of their effectiveness in the provision of renewable energy. Soltani et al. [1] provided a detailed discussion on the environmental, economic, and social benefits and challenges of geothermal energy systems to show that the decision to implement these systems is not one-dimensional. This research also shows that the assessment of geothermal technologies cannot be done in a vacuum and is based on technical merits alone. The authors Kumar et al. [2] have also reviewed the current state of technology in geothermal energy systems and the related issues in the development of the technology. Their work highlights the importance of the development of new strategies to address current constraints in geothermal energy applications. Von Jouanne and Brekken [3] analyzed the technological synergy and viability of ocean and geothermal energy systems in the broader context. Their research gives an understanding of the systematic approach towards the use of geothermal energy. A critical systems analysis of geothermal energy systems for power and polygeneration was done by Lee et al. [4]. They focused on the current developments in the field and future trends, pointing out that the evaluation methods were still lacking. Soltani et al. [5] studied the use of nanotechnology in geothermal energy systems and proposed new ways of enhancing the systems' efficiency and performance. This research shows how technology can be used to improve geothermal energy systems. Huenges et al. [6] discussed the prospects of geothermal systems for local power supply, which offered initial insights into the feasibility of geothermal use in homes. Aljundi et al. [7] investigated the applications of geothermal energy systems and expanded the discussion beyond simple system efficiency to include sustainability aspects. This approach is in line with the current knowledge in the evaluation of geothermal energy. Ahmed et al. [8] provided a critical analysis of SHS for heating and cooling and showed the versatility of geothermal technologies in various settings. Johnston et al. [9] have reviewed the advanced technologies in geothermal energy, which gave an understanding of the current development in geothermal energy systems. The use of MCDM techniques in the assessment of geothermal energy projects has been the focus of growing interest. Raos et al. [10] have proposed an MCDM approach to evaluate investments in EGS projects. This research seeks to fill the existing gap of lack of properly structured decision-making frameworks for selecting geothermal energy projects. In another study conducted by Raos et al. [11], the authors built upon the methodology for MCDM in EGS, showing that the analysis methods are becoming more complex. Their work offers a better way of evaluating geothermal energy projects than the current approaches. Raos et al. [12] had previously initiated the discussion on MCDM in the context of EG technologies and the prospects of developing more sophisticated methods of evaluation.

3.1 *MCDM technique*

MCDM emerged as a core method for addressing multifaceted, or multi-criteria, complex problems with many, usually contradictory, criteria. Massam's [13] early work provided an essential foundation for the discussion of how

MCDM might be brought into urban and regional planning. Toloie and Homayonfar [14] surveyed MCDM literature between 1999 and 2009, reflecting the trend from conventional techniques such as AHP and TOPSIS towards hybrid approaches that involve fuzzy logic, rough sets, and evolutionary algorithms. Bonissone et al. [15] added to this progress with an integrated approach blending MCDM with computational intelligence methods to facilitate a more flexible and robust handling of uncertainty and subjectivity in engineering and industrial issues. Kaya et al. [16] concentrated on the case of energy policy-making, where MCDM approaches were illustrated to facilitate the assessment of different energy sources and technologies in terms of environmental, economic, and technical criteria. Likewise, Emovon and Oghenenyerovwho [17] illustrated the use of MCDM to ensure optimal material choice, emphasizing its capability to aid sustainable engineering design by reconciling multiple attributes such as cost, performance, and environment. Kumar et al. [18] presented a detailed review of MCDM techniques in renewable energy systems, highlighting their significance in determining the most sustainable and efficient sources of energy through systematic assessment processes. Besides general frameworks, different domain-specific uses of MCDM have also been put forth. Tsaur et al. [19] used fuzzy MCDM to assess airline service quality since customer perceptions and expectations are generally imprecise and subjective. Dursun and Karsak [20] used fuzzy MCDM for personnel selection issues so that organizations could make rational hiring decisions under uncertain conditions. Chu and Lin [21] extended fuzzy MCDM by adjusting the preference aggregation, thus enhancing model accuracy in handling human judgment inconsistencies. Environmental planning has also been enhanced through MCDM, evidenced by Chen et al. [22], who employed fuzzy MCDM to choose the most suitable watershed management plans. Van de Kaa et al. [23] applied a fuzzy MCDM approach to assess and rank photovoltaic technologies in aid of strategic decision-making in sustainable energy investments. On the theory front, various next-generation fuzzy models have augmented MCDM. Khan et al. [24] proposed the notion of N-Fuzzy Bi-Topological Spaces, which improves uncertainty modeling and can be a theoretical foundation for upcoming fuzzy MCDM models. Alreshidi et al. [25] came up with a trapezoidal type-2 Pythagorean fuzzy TODIM method that includes unknown and hesitancy weights, which makes it suitable for decision-making environments with incomplete or vague information. Kumar and Pamucar [26] presented a comprehensive and systematic overview of MCDM methodologies spanning two decades and classified them based on methodological progression, theoretical contributions, and practical applicability.

More recently, new fuzzy and hybrid models have further extended the frontier of MCDM. Petchimuthu et al. [27] introduced a strong MCDM method based on complex q-rung picture fuzzy generalized power prioritized Yager operators, particularly suitable for energy transformation and infrtimesructure analysis. Gul [28] applied the VIKOR approach with a bipolar fuzzy preference model that incorporated δ-covering-based bipolar fuzzy rough sets, thereby allowing for more sophisticated decision-making in conflicting and uncertain situations. Seikh and Chatterjee [29] formulated a hybrid SWARA-ARAS model within the framework of a confidence-level-based interval-valued Fermatean fuzzy setup to prioritize renewable energy sources in India, showing a versatile and systematic model for national energy planning. Seikh and Mandal [30] presented Dombi aggregation operators in intuitionistic fuzzy environments for MADM to add richness to the decision-making process by incorporating membership, non-membership, and hesitancy degrees into evaluations. Ur Rehman [31] presented a new approach to choosing Database Management Systems with probability complex fuzzy aggregation operators to enhance the decision-making process by incorporating probabilistic and complex fuzzy logic to provide more realistic and precise evaluations. Mahmood et al. [32] created generalized similarity measures using cubic complex hesitant fuzzy sets, allowing for more subtle decision-making in situations with both hesitation and higher-order uncertainty. Ahmmad et al. [33] utilized complex fuzzy rough Dombi aggregation operators in market segmentation, demonstrating the applicability of rough and complex fuzzy models for strategic business choices.

In the energy sector, Hussain et al. [34] evaluated solar panels with an MCDM methodology based on intuitionistic fuzzy Aczel-Alsina Heronian mean operator that offered a stable framework for both membership and non-membership degrees integration in solar technology assessment. In the same manner, Jabeen et al. [35] applied interval-valued picture fuzzy Aczel–Alsina aggregation operators to find a solution to a multi-attribute decision-making problem that augmented further the applicability of fuzzy MCDM methods. Khan et al. [36] contributed by proposing the intuitionistic fuzzy rough Aczel-Alsina prioritized aggregation operator, which combined rough set theory and prioritized fuzzy reasoning efficiently. In robotics, Imran et al. [37] introduced an interval-valued intuitionistic fuzzy and Aczel-Alsina Bonferroni means-based group decision-making model, an integrated framework approach to multiple stakeholders' decision environments with conflicting criteria. All these studies collectively demonstrate the flexibility, flexibility, and increasing variety of MCDM approaches. From classical models to fuzzy, hesitant, bipolar, and intuitionistic environments, MCDM has evolved to accommodate the growing complexity of modern-day decision problems in fields such as planning, engineering design, energy, environment, and human resources.

4. Fundamentals

In this section of the manuscript, we discuss the basic notions of HBFSs and their related operations and properties.

Definition 1 [38] The HBFS $\widehat{\mathrm{H}}$ over the fixed set $\widehat{\mathrm{L}}$ is identified by:

$$\widehat{\mathrm{H}} = \left\{ \langle c,\, \bar{X}_{\widehat{\mathrm{H}}}(c) \rangle \mid c \in \widehat{\mathrm{L}} \right\} = \left\{ \langle c,\, \left(\bar{X}^{+}_{\widehat{\mathrm{H}}}(c),\, \bar{X}^{-}_{\widehat{\mathrm{H}}}(c) \right) \rangle \mid c \in \widehat{\mathrm{L}} \right\} \tag{1}$$

where $\bar{X}^{+}_{\widehat{\mathrm{H}}}(c) = \left\{ \bar{X}^{+}_{\widehat{\mathrm{H}}_j}(c) \mid j = 1,\, 2,\, \ldots,\, m \right\} \in [0,\ 1]$ is the set of finite values that shows the positive part of the membership grade, and $\bar{X}^{-}_{\widehat{\mathrm{H}}}(c) = \left\{ \bar{X}^{-}_{\widehat{\mathrm{H}}_k}(c) \mid k = 1,\, 2,\, \ldots,\, n \right\} \in [-1,\ 0]$ is the set of finite values that shows the negative part of the membership grade of each $c \in \widehat{\mathrm{L}}$. For easiness, an HBFN is denoted by $\bar{X} = (\bar{X}^{+},\, \bar{X}^{-})$.

Definition 2 [38] Let $\bar{X} = (\bar{X}^{+},\, \bar{X}^{-})$, $\bar{X}_1 = (\bar{X}_1^{+},\, \bar{X}_1^{-})$, and $\bar{X}_2 = (\bar{X}_2^{+},\, \bar{X}_2^{-})$ be three HBFNs. Then:

(1)

$$\bar{X}^{c} = \left(\bigcup_{\tilde{h}^{+} \in \bar{X}^{+}} \{(1 - \tilde{h}^{+})\},\ \bigcup_{\tilde{h}^{-} \in \bar{X}^{-}} \{(1 - \tilde{h}^{-})\} \right)$$

(2)

$$\bar{X}_1 \cup \bar{X}_2 = \left(\bigcup_{\tilde{h}_1^{+} \in \bar{X}_1^{+},\ \tilde{h}_2^{+} \in \bar{X}_2^{+}} \{\max(\tilde{h}_1^{+},\, \tilde{h}_2^{+})\},\ \bigcup_{\tilde{h}_1^{-} \in \bar{X}_1^{-},\ \tilde{h}_2^{-} \in \bar{X}_2^{-}} \{\min(\tilde{h}_1^{-},\, \tilde{h}_2^{-})\} \right)$$

(3)

$$\bar{X}_1 \cap \bar{X}_2 = \left(\bigcup_{\tilde{h}_1^{+} \in \bar{X}_1^{+},\ \tilde{h}_2^{+} \in \bar{X}_2^{+}} \{\min(\tilde{h}_1^{+},\, \tilde{h}_2^{+})\},\ \bigcup_{\tilde{h}_1^{-} \in \bar{X}_1^{-},\ \tilde{h}_2^{-} \in \bar{X}_2^{-}} \{\max(\tilde{h}_1^{-},\, \tilde{h}_2^{-})\} \right).$$

Definition 3 [38] Let $\bar{X}_1 = (\bar{X}_1^{+},\, \bar{X}_1^{-})$ and $\bar{X}_2 = (\bar{X}_2^{+},\, \bar{X}_2^{-})$ be two HBFNs. Then:

(1)

$$\bar{X}_1 \oplus \bar{X}_2 = \left(\bigcup_{\tilde{h}_1^{+} \in \bar{X}_1^{+},\ \tilde{h}_2^{+} \in \bar{X}_2^{+}} \{\tilde{h}_1^{+} + \tilde{h}_2^{+} - \tilde{h}_1^{+}\tilde{h}_2^{+}\},\ \bigcup_{\tilde{h}_1^{-} \in \bar{X}_1^{-},\ \tilde{h}_2^{-} \in \bar{X}_2^{-}} \{\tilde{h}_1^{-}\tilde{h}_2^{-}\} \right)$$

(2)

$$\bar{X}_1 \otimes \bar{X}_2 = \left(\bigcup_{\tilde{h}_1^+ \in \bar{X}_1^+,\ \tilde{h}_2^+ \in \bar{X}_2^+} \{\tilde{h}_1^+ \tilde{h}_2^+\},\ \bigcup_{\tilde{h}_1^- \in \bar{X}_1^-,\ \tilde{h}_2^- \in \bar{X}_2^-} \{\tilde{h}_1^- + \tilde{h}_2^- + \tilde{h}_1^- \tilde{h}_2^-\} \right).$$

Definition 4 [38] Let $\bar{X} = (\bar{X}^+,\ \bar{X}^-)$ be a HBFN and let $\lambda > 0$. Then:
(1)

$$\bar{X}^\lambda = \left(\bigcup_{\tilde{h}^+ \in \bar{X}^+} \{(\tilde{h}^+)^\lambda\},\ \bigcup_{\tilde{h}^- \in \bar{X}^-} \left\{-1 + (1 + \tilde{h}^-)^\lambda\right\} \right)$$

(2)

$$\lambda \bar{X} = \left(\bigcup_{\tilde{h}^+ \in \bar{X}^+} \left\{1 - (1 - \tilde{h}^+)^\lambda\right\},\ \bigcup_{\tilde{h}^- \in \bar{X}^-} \left\{(\tilde{h}^-)^\lambda\right\} \right).$$

Definition 5 [38] Let $\bar{X} = (\bar{X}^+,\ \bar{X}^-)$ be an HBFN. Then:
(1)

$$\mathrm{scor}(\bar{X}) = \frac{1}{2} \left(\frac{1}{l_{\tilde{h}^+}} \sum_{\tilde{h}^+ \in \bar{X}^+} \tilde{h}^+ - \frac{1}{l_{\tilde{h}^+}} \sum_{\tilde{h}^- \in \bar{X}^-} \tilde{h}^- \right) \in [0,\ 1]$$

(2)

$$\mathrm{accu}(\bar{X}) = \frac{1}{2} \left(\frac{1}{l_{\tilde{h}^+}} \sum_{\tilde{h}^+ \in \bar{X}^+} \tilde{h}^+ + \frac{1}{l_{\tilde{h}^+}} \sum_{\tilde{h}^- \in \bar{X}^-} \tilde{h}^- \right) \in [0,\ 1].$$

Theorem 1. Let us assume that $\bar{X}$, $\bar{X}_1$, and $\bar{X}_2$ are three HBFNs, and let $\lambda > 0$. Then we have:
1. $(\bar{X}^c)^\lambda = \lambda(\bar{X}^c)$.
2. $\bar{X}_1^c \cup \bar{X}_2^c = (\bar{X}_1 \cap \bar{X}_2)^c$.
3. $\bar{X}_1^c \cap \bar{X}_2^c = (\bar{X}_1 \cup \bar{X}_2)^c$.
4. $\bar{X}_1^c \oplus \bar{X}_2^c = (\bar{X}_1 \otimes \bar{X}_2)^c$.
5. $\bar{X}_1^c \otimes \bar{X}_2^c = (\bar{X}_1 \oplus \bar{X}_2)^c$.

Definition 6 [39] Let $\mathring{T}_1$ and $\mathring{T}_2$ be two real numbers. Then the Dombi t-norm and Dombi t-conorm are defined as:

$$\mathrm{Dom}_{(t\text{-norm})}(\mathring{T}_1,\ \mathring{T}_2) = \frac{1}{1 + \left(\left(\frac{1 - \mathring{T}_1}{\mathring{T}_1} \right)^\xi + \left(\frac{1 - \mathring{T}_2}{\mathring{T}_2} \right)^\xi \right)^{\frac{1}{\xi}}}$$

$$\mathrm{Dom}_{(t\text{-conorm})}(\mathring{T}_1,\ \mathring{T}_2) = 1 - \frac{1}{1 + \left(\left(\frac{\mathring{T}_1}{1-\mathring{T}_1}\right)^{\xi} + \left(\frac{\mathring{T}_2}{1-\mathring{T}_2}\right)^{\xi}\right)^{-\frac{1}{\xi}}}$$

where $\xi \geq 1$ and $\mathring{T}_1,\ \mathring{T}_2 \in (0,\ 1) \times (0,\ 1)$.

Definition 7 Let $\bar{X}$, $\bar{X}_1$, and $\bar{X}_2$ be three HBFNs, and let $\lambda > 0$. Then the Dombi operational laws for HBFNs are defined as.

(1)

$$\bar{X}_1 \oplus \bar{X}_2 = \left(\begin{array}{c} \left(\bigcup_{\tilde{h}_1^+ \in \bar{X}_1^+,\ \tilde{h}_2^+ \in \bar{X}_2^+} \left\{ 1 - \frac{1}{1+\left(\left(\frac{\tilde{h}_1^+}{1-\tilde{h}_1^+}\right)^{\xi} + \left(\frac{\tilde{h}_2^+}{1-\tilde{h}_2^+}\right)^{\xi}\right)^{\frac{1}{\xi}}} \right\}, \right. \\ \left. \bigcup_{\tilde{h}_1^- \in \bar{X}_1^-,\ \tilde{h}_2^- \in \bar{X}_2^-} \left\{ \frac{1}{1+\left(\left(\frac{1+\tilde{h}_1^-}{\left|-\tilde{h}_1^-\right|}\right)^{\xi} + \left(\frac{1+\tilde{h}_2^-}{\left|-\tilde{h}_2^-\right|}\right)^{\xi}\right)^{\frac{1}{\xi}}} \right\} \right) \end{array} \right)$$

(2)

$$\bar{X}_1 \otimes \bar{X}_2 = \left(\begin{array}{c} \left(\bigcup_{\tilde{h}_1^+ \in \bar{X}_1^+,\ \tilde{h}_2^+ \in \bar{X}_2^+} \left\{ \frac{1}{1+\left(\left(\frac{1-\tilde{h}_1^+}{\tilde{h}_1^+}\right)^{\xi} + \left(\frac{1-\tilde{h}_2^+}{\tilde{h}_2^+}\right)^{\xi}\right)^{\frac{1}{\xi}}} \right\}, \right. \\ \left. \bigcup_{\tilde{h}_1^- \in \bar{X}_1^-,\ \tilde{h}_2^- \in \bar{X}_2^-} \left\{ 1 - \frac{1}{1+\left(\left(\frac{\left|-\tilde{h}_1^-\right|}{1+\tilde{h}_1^-}\right)^{\xi} + \left(\frac{\left|-\tilde{h}_2^-\right|}{1+\tilde{h}_2^-}\right)^{\xi}\right)^{\frac{1}{\xi}}} \right\} \right) \end{array} \right)$$

(3)

$$\lambda \bar{X} = \left(\bigcup_{\tilde{h}^+ \in \bar{X}^+} \left\{ 1 - \frac{1}{1 + \left(\lambda \left(\frac{\tilde{h}^+}{1-\tilde{h}^+}\right)\right)^{\frac{1}{\xi}}} \right\},\ \bigcup_{\tilde{h}^- \in \bar{X}^-} \left\{ \frac{1}{1 + \left(\lambda \left(\frac{\left|-\tilde{h}^-\right|}{1+\tilde{h}^-}\right)\right)^{\frac{1}{\xi}}} \right\} \right)$$

(4)

$$\bar{X}^{\lambda}=\left(\bigcup_{\tilde{h}^{+}\in\bar{X}^{+}}\left\{\frac{1}{1+\left(\lambda\left(\frac{1-\tilde{h}^{+}}{\tilde{h}^{+}}\right)\right)^{\frac{1}{\xi}}}\right\},\ \bigcup_{\tilde{h}^{-}\in\bar{X}^{-}}\left\{1-\frac{1}{1+\left(\lambda\left(\frac{1+\tilde{h}^{-}}{|-\tilde{h}^{-}|}\right)\right)^{\frac{1}{\xi}}}\right\}\right)$$

5. HBF Dombi aggregation operators

In this section, we introduce Dombi arithmetic and Dombi geometric aggregation operators based on HBFNs.

5.1 *HBF Dombi arithmetic aggregation operators*

In this subsection of the manuscript, we introduce HBFDWA and HBFDOWA operators.

Definition 8 Let $\bar{X}_{\varphi}=\left(\bar{X}_{\varphi+},\ \bar{X}_{\varphi-}\right)(\varphi=1,\ 2,\ 3,\ \ldots,\ \Upsilon)$ be a collection of HBFNs, then the HBFDWA operator is defined as:

$$\mathrm{HBFDWA}(\bar{X}_1,\ \bar{X}_2,\ \ldots,\ \bar{X}_{\Upsilon})=\bigoplus_{\varphi=1}^{\Upsilon}\left(\tilde{w}_{\varphi}\bar{X}_{\varphi}\right) \tag{2}$$

where $\tilde{w}=(\tilde{w}_1,\ \tilde{w}_2,\ \ldots,\ \tilde{w}_{\Upsilon})^{\top}$ be the weight vector of $\bar{X}_{\varphi}=\left(\bar{X}_{\varphi+},\ \bar{X}_{\varphi-}\right)(\varphi=1,\ 2,\ 3,\ \ldots,\ \Upsilon)$ with $\tilde{w}_{\varphi}\in(0,\ 1)$ and $\sum_{\varphi=1}^{\Upsilon}\tilde{w}_{\varphi}=1$.

Theorem 2 Let $\bar{X}_{\varphi}=\left(\bar{X}_{\varphi+},\ \bar{X}_{\varphi-}\right)(\varphi=1,\ 2,\ 3,\ \ldots,\ \Upsilon)$ be a collection of HBFNs, then by (1) we have:

$$\mathrm{HBFDWA}(\bar{X}_1,\ \bar{X}_2,\ \ldots,\ \bar{X}_{\Upsilon})=\bigoplus_{\varphi=1}^{\Upsilon}\tilde{w}_{\varphi}\bar{X}_{\varphi}$$

$$=\left(\begin{array}{c}\bigcup_{\tilde{h}_1^{+}\in\bar{X}_1^{+},\ \ldots,\ \tilde{h}_{\Upsilon}^{+}\in\bar{X}_{\Upsilon}^{+}}\left\{1-\frac{1}{1+\left(\sum_{\varphi=1}^{\Upsilon}\tilde{w}_{\varphi}\left(\frac{\tilde{h}_{\varphi}^{+}}{1-\tilde{h}_{\varphi}^{+}}\right)^{\xi}\right)^{\frac{1}{\xi}}}\right\}, \\ \bigcup_{\tilde{h}_1^{-}\in\bar{X}_1^{-},\ \ldots,\ \tilde{h}_{\Upsilon}^{-}\in\bar{X}_{\Upsilon}^{-}}\left\{\frac{-1}{1+\left(\sum_{\varphi=1}^{\Upsilon}\tilde{w}_{\varphi}\left(\frac{1+\tilde{h}_{\varphi}^{-}}{|\tilde{h}_{\varphi}^{-}|}\right)^{\xi}\right)^{\frac{1}{\xi}}}\right\}\end{array}\right) \tag{3}$$

Where $\tilde{w}=(\tilde{w}_1,\ \tilde{w}_2,\ \ldots,\ \tilde{w}_{\Upsilon})^{\top}$ is the weight vector of $\bar{X}^{(\varphi)}=(\bar{X}^{(\varphi)+},\ \bar{X}^{(\varphi)-})$ for $\varphi=1,\ 2,\ \ldots,\ \Upsilon$ with $\tilde{w}_{\varphi}\in(0,\ 1)$ and $\sum_{\varphi=1}^{\Upsilon}\tilde{w}_{\varphi}=1$.

Proof. By mathematical induction.

Let $\Upsilon = 2$, then (3) becomes:

$$\text{HBFDWA}(\bar{X}_1,\ \bar{X}_2) = \bar{X}_1 \oplus \bar{X}_2 = \left(\begin{array}{c} \left(\bigcup_{\tilde{h}_1^+ \in \bar{X}_1^+,\ \tilde{h}_2^+ \in \bar{X}_2^+} \left\{ 1 - \frac{1}{1+\left(\tilde{w}_1 \left(\frac{\tilde{h}_1^+}{1-\tilde{h}_1^+} \right)^{\xi} + \tilde{w}_2 \left(\frac{\tilde{h}_2^+}{1-\tilde{h}_2^+} \right)^{\xi} \right)^{\frac{1}{\xi}}} \right\}, \right. \\ \left. \bigcup_{\tilde{h}_1^- \in \bar{X}_1^-,\ \tilde{h}_2^- \in \bar{X}_2^-} \left\{ \frac{1}{1+\left(\tilde{w}_1 \left(\frac{1+\tilde{h}_1^-}{|\tilde{h}_1^-|} \right)^{\xi} + \tilde{w}_2 \left(\frac{1+\tilde{h}_2^-}{|\tilde{h}_2^-|} \right)^{\xi} \right)^{\frac{1}{\xi}}} \right\} \right) \end{array} \right)$$

$$= \left(\bigcup_{\tilde{h}_\varphi^+ \in \bar{X}_\varphi^+} \left\{ 1 - \frac{1}{1+\left(\sum_{\varphi=1}^{2} \tilde{w}_\varphi \left(\frac{\tilde{h}_\varphi^+}{1-\tilde{h}_\varphi^+} \right)^{\xi} \right)^{\frac{1}{\xi}}} \right\},\ \bigcup_{\tilde{h}_\varphi^- \in \bar{X}_\varphi^-} \left\{ \frac{1}{1+\left(\sum_{\varphi=1}^{2} \tilde{w}_\varphi \left(\frac{1+\tilde{h}_\varphi^-}{|\tilde{h}_\varphi^-|} \right)^{\xi} \right)^{\frac{1}{\xi}}} \right\} \right)$$

This shows that (3) holds for $\Upsilon = 2$.
Now assume that (3) holds for $\Upsilon = \kappa$:

$$\text{HBFDWA}(\bar{X}_1,\ \bar{X}_2,\ \ldots,\ \bar{X}_\kappa) = \bigoplus_{\varphi=1}^{\kappa} \tilde{w}_\varphi \bar{X}_\varphi$$

$$= \left(\begin{array}{c} \left(\bigcup_{\tilde{h}_1^+ \in \bar{X}_1^+,\ \ldots,\ \tilde{h}_\Upsilon^+ \in \bar{X}_\Upsilon^+} \left\{ 1 - \frac{1}{1+\left(\sum_{\varphi=1}^{\kappa} \tilde{w}_\varphi \left(\frac{\tilde{h}_\varphi^+}{1-\tilde{h}_\varphi^+} \right)^{\xi} \right)^{\frac{1}{\xi}}} \right\}, \right. \\ \left. \bigcup_{\tilde{h}_1^- \in \bar{X}_1^-,\ \ldots,\ \tilde{h}_\Upsilon^- \in \bar{X}_\Upsilon^-} \left\{ \frac{1}{1+\left(\sum_{\varphi=1}^{\kappa} \tilde{w}_\varphi \left(\frac{1+\tilde{h}_\varphi^-}{|\tilde{h}_\varphi^-|} \right)^{\xi} \right)^{\frac{1}{\xi}}} \right\} \right) \end{array} \right)$$

Next, we show that (3) holds for $\Upsilon = \kappa + 1$:

$$\mathrm{HBFDWA}(\bar{X}_1,\, \bar{X}_2,\, \ldots,\, \bar{X}_\kappa,\, \bar{X}_{\kappa+1}) = \bigoplus_{\varphi=1}^{\kappa} \left(\tilde{w}_\varphi \bar{X}_\varphi\right) \oplus \left(\tilde{w}_{\kappa+1}\bar{X}_{\kappa+1}\right)$$

$$= \left(\left(\begin{array}{l} \bigcup_{\tilde{h}_1^+ \in \bar{X}_1^+,\, \ldots,\, \tilde{h}_\Upsilon^+ \in \bar{X}_\Upsilon^+} \left\{ 1 - \dfrac{1}{1 + \left(\sum_{\varphi=1}^{\kappa} \tilde{w}_\varphi \left(\dfrac{\tilde{h}_\varphi^+}{1-\tilde{h}_\varphi^+} \right)^{\xi} \right)^{\frac{1}{\xi}}} \right\}, \\ \bigcup_{\tilde{h}_1^- \in \bar{X}_1^-,\, \ldots,\, \tilde{h}_\Upsilon^- \in \bar{X}_\Upsilon^-} \left\{ \dfrac{1}{1 + \left(\sum_{\varphi=1}^{\kappa} \tilde{w}_\varphi \left(\dfrac{1+\tilde{h}_\varphi^-}{|\tilde{h}_\varphi^-|} \right)^{\xi} \right)^{\frac{1}{\xi}}} \right\} \end{array} \right) \right)$$

$$\oplus \left(\begin{array}{l} \bigcup_{\tilde{h}_1^+ \in \bar{X}_1^+,\, \ldots,\, \tilde{h}_\Upsilon^+ \in \bar{X}_\Upsilon^+, \tilde{h}_{\kappa+1}^+ \in \bar{X}_{\kappa+1}^+} \left\{ 1 - \dfrac{1}{1 + \left(\tilde{w}_{\kappa+1} \left(\dfrac{\tilde{h}_{\kappa+1}^+}{1-\tilde{h}_{\kappa+1}^+} \right)^{\xi} \right)^{\frac{1}{\xi}}} \right\}, \\ \bigcup_{\tilde{h}_1^- \in \bar{X}_1^-,\, \ldots,\, \tilde{h}_\Upsilon^- \in \bar{X}_\Upsilon^-, \tilde{h}_{\kappa+1}^- \in \bar{X}_{\kappa+1}^-} \left\{ \dfrac{1}{1 + \left(\tilde{w}_{\kappa+1} \left(\dfrac{1+\tilde{h}_{\kappa+1}^-}{-\tilde{h}_{\kappa+1}^-} \right)^{\xi} \right)^{\frac{1}{\xi}}} \right\} \end{array} \right)$$

$$= \left(\begin{array}{l} \bigcup_{\tilde{h}_1^+ \in \bar{X}_1^+,\, \ldots,\, \tilde{h}_\Upsilon^+ \in \bar{X}_\Upsilon^+, \tilde{h}_{\kappa+1}^+ \in \bar{X}_{\kappa+1}^+} \left\{ 1 - \dfrac{1}{1 + \left(\sum_{\varphi=1}^{\kappa+1} \tilde{w}_\varphi \left(\dfrac{\tilde{h}_\varphi^+}{1-\tilde{h}_\varphi^+} \right)^{\xi} \right)^{\frac{1}{\xi}}} \right\}, \\ \bigcup_{\tilde{h}_1^- \in \bar{X}_1^-,\, \ldots,\, \tilde{h}_\Upsilon^- \in \bar{X}_\Upsilon^-, \tilde{h}_{\kappa+1}^- \in \bar{X}_{\kappa+1}^-} \left\{ \dfrac{1}{1 + \left(\sum_{\varphi=1}^{\kappa+1} \tilde{w}_\varphi \left(\dfrac{1+\tilde{h}_\varphi^-}{-\tilde{h}_\varphi^-} \right)^{\xi} \right)^{\frac{1}{\xi}}} \right\} \end{array} \right)$$

Thus, equation (3) holds for $\Upsilon = \kappa + 1$.This implies that (3) holds for every Υ.

The aforementioned operator satisfies the boundedness, monotonicity, and idempotency requirements.

Definition 9 Let $\bar{X}_\varphi = (\bar{X}_{\varphi+},\ \bar{X}_{\varphi-})$ for $\varphi = 1,\ 2,\ 3,\ \ldots,\ \Upsilon$ be a collection of HBFNs. Then, the HBFDOWA operator is defined as:

$$\mathrm{HBFDOWA}(\bar{X}_1,\ \bar{X}_2,\ \ldots,\ \bar{X}_\Upsilon) = \bigoplus_{\varphi=1}^{\Upsilon} \tilde{w}_\varphi \bar{X}_{o(\varphi)} \tag{4}$$

where $\tilde{w} = (\tilde{w}_1,\ \tilde{w}_2,\ \ldots,\ \tilde{w}_\Upsilon)^\top$ be the weight vector of $\bar{X}_\varphi = \left(\bar{X}_{\varphi+},\ \bar{X}_{\varphi-}\right)(\varphi = 1,\ 2,\ 3,\ \ldots,\ \Upsilon)$ with $\tilde{w}_\varphi \in (0,\ 1)$ and $\sum_{\varphi=1}^{\Upsilon} \tilde{w}_\varphi = 1$, and $o(1),\ o(2),\ \ldots,\ o(\Upsilon)$ are the permutation of $o(\Upsilon)\,(\varphi = 1,\ 2,\ 3,\ \ldots,\ \Upsilon)$ such that $\bar{X}_{o(\Upsilon-1)} \geq \bar{X}_{o(\Upsilon)} \forall \Upsilon$.

Theorem 3 Let $\bar{X}_\varphi = (\bar{X}_\varphi^+,\ \bar{X}_\varphi^-)\,(\varphi = 1,\ 2,\ 3,\ \ldots,\ \Upsilon)$ be a collection of HBFNs. Then, by (4), we have:

$$\mathrm{HBFDOWA}(\bar{X}_1,\ \bar{X}_2,\ \ldots,\ \bar{X}_\Upsilon) = \left(\begin{array}{c} \bigcup_{\tilde{h}_1^+ \in \bar{X}_1^+,\ \ldots,\ \tilde{h}_\Upsilon^+ \in \bar{X}_\Upsilon^+} \left\{ 1 - \dfrac{1}{1 + \left(\sum_{\varphi=1}^{\Upsilon} \tilde{w}_{o(\varphi)} \left(\dfrac{\tilde{h}_{o(\varphi)}^+}{1 - \tilde{h}_{o(\varphi)}^+} \right)^{\xi} \right)^{\frac{1}{\xi}}} \right\}, \\ \bigcup_{\tilde{h}_1^- \in \bar{X}_1^-,\ \ldots,\ \tilde{h}_\Upsilon^- \in \bar{X}_\Upsilon^-} \left\{ \dfrac{1}{1 + \left(\sum_{\varphi=1}^{\Upsilon} \tilde{w}_{o(\varphi)} \left(\dfrac{1 + \tilde{h}_{o(\varphi)}^-}{-\tilde{h}_{o(\varphi)}^-} \right)^{\xi} \right)^{\frac{1}{\xi}}} \right\} \end{array} \right) \tag{5}$$

where $\tilde{w} = (\tilde{w}_1,\ \tilde{w}_2,\ \ldots,\ \tilde{w}_\Upsilon)^\top$ be the weight vector of $\bar{X}_\varphi = \left(\bar{X}_{\varphi+},\ \bar{X}_{\varphi-}\right)(\varphi = 1,\ 2,\ 3,\ \ldots,\ \Upsilon)$ with $\tilde{w}_\varphi \in (0,\ 1)$ and $\sum_{\varphi=1}^{\Upsilon} \tilde{w}_\varphi = 1$, and $o(1),\ o(2),\ \ldots,\ o(\Upsilon)$ are the permutation of $o(\Upsilon)\,(\varphi = 1,\ 2,\ 3,\ \ldots,\ \Upsilon)$ such that $\bar{X}_{o(\Upsilon-1)} \geq \bar{X}_{o(\Upsilon)} \forall \Upsilon$.

5.2 *HBF Dombi geometric aggregation operators*

In this subsection, we introduce HBFDWG and HBFDOWG operators.

Definition 10 Let $\bar{X}_\varphi = \left(\bar{X}_\varphi^+,\ \bar{X}_\varphi^-\right)(\varphi = 1,\ 2,\ 3,\ \ldots,\ \Upsilon)$ be a collection of HBFNs, then the HBFDWG operator is defined as:

$$\mathrm{HBFDWG}(\bar{X}_1,\ \bar{X}_2,\ \ldots,\ \bar{X}_\Upsilon) = \bigotimes_{\varphi=1}^{\Upsilon} \left(\bar{X}_\varphi\right)^{\tilde{w}_\varphi} \tag{6}$$

where $\tilde{w} = (\tilde{w}_1,\ \tilde{w}_2,\ \ldots,\ \tilde{w}_\Upsilon)^\top$ be the weight vector of $\bar{X}_\varphi = \left(\bar{X}_{\varphi+},\ \bar{X}_{\varphi-}\right)(\varphi = 1,\ 2,\ 3,\ \ldots,\ \Upsilon)$ with $\tilde{w}_\varphi \in (0,\ 1)$ and $\sum_{\varphi=1}^{\Upsilon} \tilde{w}_\varphi = 1$.

Theorem 4 Let $\bar{X}_\varphi = \left(\bar{X}_{\varphi+},\ \bar{X}_{\varphi-}\right)(\varphi = 1,\ 2,\ 3,\ \ldots,\ \Upsilon)$ be a collection of HBFNs, then by (6) we have:

$$\text{HBFDWG}(\bar{X}_1,\ \bar{X}_2,\ \ldots,\ \bar{X}_\Upsilon) = \left(\begin{array}{c} \bigcup_{\tilde{h}_1^+ \in \bar{X}_1^+,\ \ldots,\ \tilde{h}_\Upsilon^+ \in \bar{X}_\Upsilon^+} \left\{ \frac{1}{1+\left(\sum_{\varphi=1}^{\Upsilon} \tilde{w}_\varphi \left(\frac{1-\tilde{h}_\varphi^+}{\tilde{h}_\varphi^+} \right)^\xi \right)^{\frac{1}{\xi}}} \right\}, \\ \bigcup_{\tilde{h}_1^- \in \bar{X}_1^-,\ \ldots,\ \tilde{h}_\Upsilon^- \in \bar{X}_\Upsilon^-} \left\{ 1 - \frac{1}{1+\left(\sum_{\varphi=1}^{\Upsilon} \tilde{w}_\varphi \left(\frac{\left|\tilde{h}_\varphi^-\right|}{1+\tilde{h}_\varphi^-} \right)^\xi \right)^{\frac{1}{\xi}}} \right\} \end{array} \right) \tag{7}$$

Where $\tilde{w} = (\tilde{w}_1,\ \tilde{w}_2,\ \ldots,\ \tilde{w}_\Upsilon)^\top$ be the WV of $\bar{X}^{(\varphi)} = (\bar{X}^{(\varphi)+},\ \bar{X}^{(\varphi)-})\,(\varphi = 1,\ 2,\ \ldots,\ \Upsilon)$ with $\tilde{w}_\varphi \in (0,\ 1)$ and $\sum_{\varphi=1}^{\Upsilon} \tilde{w}_\varphi = 1$.

Definition 11 Let $\bar{X}_\varphi = (\bar{X}_{\varphi+},\ \bar{X}_{\varphi-})$ for $\varphi = 1,\ 2,\ 3,\ \ldots,\ \Upsilon$ be a collection of HBFNs. Then, the HBFDOWG operator is defined as:

$$\text{HBFDOWA}(\bar{X}_1,\ \bar{X}_2,\ \ldots,\ \bar{X}_\Upsilon) = \bigotimes_{\varphi=1}^{\Upsilon} \left(\bar{X}_{o(\varphi)}\right)^{\tilde{w}_\varphi} \tag{8}$$

where $\tilde{w} = (\tilde{w}_1,\ \tilde{w}_2,\ \ldots,\ \tilde{w}_\Upsilon)^\top$ be WV of $\bar{X}_\varphi = \left(\bar{X}_{\varphi+},\ \bar{X}_{\varphi-}\right)(\varphi = 1,\ 2,\ 3,\ \ldots,\ \Upsilon)$ with $\tilde{w}_\varphi \in (0,\ 1)$ and $\sum_{\varphi=1}^{\Upsilon} \tilde{w}_\varphi = 1$, and $o(1),\ o(2),\ \ldots,\ o(\Upsilon)$ are the permutation of $o(\Upsilon)\,(\varphi = 1,\ 2,\ 3,\ \ldots,\ \Upsilon)$ such that $\bar{X}_{o(\Upsilon-1)} \geq \bar{X}_{o(\Upsilon)} \forall \Upsilon$.

Theorem 5 Let $\bar{X}_\varphi = (\bar{X}_\varphi^+,\ \bar{X}_\varphi^-)\,(\varphi = 1,\ 2,\ 3,\ \ldots,\ \Upsilon)$ be a collection of HBFNs. Then, by (9), we have:

$$\text{HBFDOWG}(\bar{X}_1,\ \bar{X}_2,\ \ldots,\ \bar{X}_\Upsilon) = \left(\begin{array}{c} \bigcup_{\tilde{h}_1^+ \in \bar{X}_1^+,\ \ldots,\ \tilde{h}_\Upsilon^+ \in \bar{X}_\Upsilon^+} \left\{ \frac{1}{1+\left(\sum_{\varphi=1}^{\Upsilon} \tilde{w}_{o(\varphi)} \left(\frac{1-\tilde{h}_{o(\varphi)}^+}{\tilde{h}_{o(\varphi)}^+} \right)^\xi \right)^{\frac{1}{\xi}}} \right\}, \\ \bigcup_{\tilde{h}_1^- \in \bar{X}_1^-,\ \ldots,\ \tilde{h}_\Upsilon^- \in \bar{X}_\Upsilon^-} \left\{ -1 + \frac{1}{1+\left(\sum_{\varphi=1}^{\Upsilon} \tilde{w}_{o(\varphi)} \left(\frac{\left|\tilde{h}_{o(\varphi)}^-\right|}{1+\tilde{h}_{o(\varphi)}^-} \right)^\xi \right)^{\frac{1}{\xi}}} \right\} \end{array} \right) \tag{9}$$

where $\tilde{w} = (\tilde{w}_1,\ \tilde{w}_2,\ \ldots,\ \tilde{w}_\Upsilon)^\top$ be the WV of $\bar{X}_\varphi = \left(\bar{X}_{\varphi+},\ \bar{X}_{\varphi-}\right)(\varphi = 1,\ 2,\ 3,\ \ldots,\ \Upsilon)$ with $\tilde{w}_\varphi \in (0,\ 1)$ and $\sum_{\varphi=1}^{\Upsilon} \tilde{w}_\varphi = 1$, and $o(1),\ o(2),\ \ldots,\ o(\Upsilon)$ are the permutation of $o(\Upsilon)\,(\varphi = 1,\ 2,\ 3,\ \ldots,\ \Upsilon)$ such that $\bar{X}_{o(\Upsilon-1)} \geq \bar{X}_{o(\Upsilon)} \forall \Upsilon$.

6. HBF-MCDM approach

In this part, we suggest a MADM strategy in the context of HBFNs, based on the suggested operators. Let us assume that there are $\bar{\Upsilon}$ alternatives. $\mathscr{B}_{\triangle}\ (\triangle = 1,\ 2,\ \ldots,\ \bar{\Upsilon})$ and $\wp$ criteria $\acute{g}_z\ (z = 1,\ 2,\ \ldots,\ \wp)$ along with criteria weights $\acute{w}$ and $\acute{w}_{\wp} \in [0,\ 1] \forall \wp$ and $\sum_{z=1}^{\wp} \tilde{w}_z = 1$. Now we assume that the HBF decision matrix is $\mathscr{M} = (\beta_{\triangle z})_{\bar{\Upsilon} \times \wp} = \left(\bar{X}^{+}_{\triangle z},\ \bar{X}^{-}_{\triangle z}\right)_{\mathfrak{I} \times \zeta}$, where $\bar{X}^{+}_{\triangle z} \in [0,\ 1]$ and $\bar{X}^{-}_{\triangle z} \in [0,\ 1]$. Next, we have the following algorithm steps shown in Figure 4.

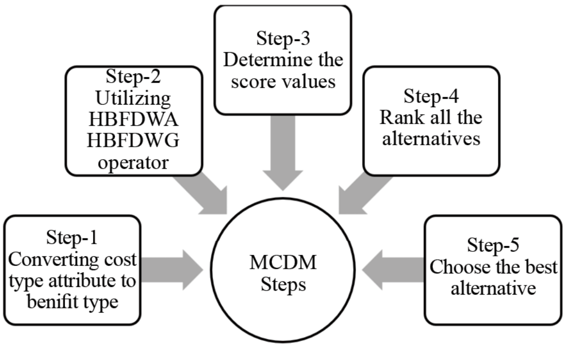

Figure 4. MCDM algorithm steps

Step 1: A benefit type and a cost type are the two possible attribute types in any MADM process. If an attribute is provided in the form of a cost type, then we convert it to a benefit type using the following transformation:

$$\beta_{\Delta z} = \begin{cases} \left(\bar{X}^{+}_{\Delta z},\ \bar{X}^{-}_{\Delta z}\right) & \text{for benefit-type attributes} \\ \left(\bar{X}^{-}_{\Delta z},\ \bar{X}^{+}_{\Delta z}\right)^{C} & \text{for cost-type attributes} \end{cases}$$

Step 2: Using the HBFDWA operator on the decision matrix $\mathscr{M}$, derive all the values β_{Δ}, $(\Delta = 1,\ 2,\ \ldots,\ \bar{\Upsilon})$ of the attribute $\acute{g}_{\Delta}$.

$$\text{HBFDWA}(\beta_{\Delta 1},\ \beta_{\Delta 2},\ \ldots,\ \beta_{\Delta \Upsilon}) = \left(\begin{array}{c} \bigcup_{\tilde{h}^{+}_{z} \in \bar{X}^{+}_{\Delta z}} \left\{ 1 - \dfrac{1}{1 + \left(\sum_{z=1}^{\Upsilon} \tilde{w}_z \left(\dfrac{\tilde{h}^{+}_{\Delta z}}{1 - \tilde{h}^{+}_{\Delta z}} \right)^{\xi} \right)^{\frac{1}{\xi}}} \right\}, \\ \bigcup_{\tilde{h}^{-}_{z} \in \bar{X}^{-}_{\Delta z}} \left\{ \dfrac{1}{1 + \left(\sum_{z=1}^{\Upsilon} \tilde{w}_z \left(\dfrac{1 + \tilde{h}^{-}_{\Delta z}}{|\tilde{h}^{-}_{\Delta z}|} \right)^{\xi} \right)^{\frac{1}{\xi}}} \right\} \end{array} \right) \tag{10}$$

If the select the HBFDWG operator then we have

$$\mathrm{HBFDWG}(\beta_{\Delta 1},\ \beta_{\Delta 2},\ \ldots,\ \beta_{\Delta \Upsilon}) = \left(\begin{array}{c} \bigcup_{\tilde{h}_z^+ \in \tilde{X}_{\Delta z}^+} \left\{ \dfrac{1}{1+\left(\sum_{z=1}^{\Upsilon} \tilde{w}_z \left(\dfrac{1-\tilde{h}_{\Delta z}^+}{\tilde{h}_{\Delta z}^+}\right)^{\xi}\right)^{\frac{1}{\xi}}} \right\}, \\ \bigcup_{\tilde{h}_z^- \in \tilde{X}_{\Delta z}^-} \left\{ 1-\dfrac{1}{1+\left(\sum_{z=1}^{\Upsilon} \tilde{w}_z \left(\dfrac{\left|\tilde{h}_{\Delta z}^-\right|}{1+\tilde{h}_{\Delta z}^-}\right)^{\xi}\right)^{\frac{1}{\xi}}} \right\} \end{array} \right) \tag{11}$$

Step 3: Compute the score values $\mathrm{scor}(\beta_{\Delta})\ \left(\Delta = 1,\ 2,\ \ldots,\ \bar{\Upsilon}\right)$ by utilizing the formula in the fundamental section.
Step 4: Rank all the alternatives.
Step 5: Choose the best alternative.

6.1 *Case study*

Problem 1 A geothermal energy system must be identified and installed by a medium-scale dairy processing factory in place of a thermal energy system that is presently utilizing fossil fuel in its operation.

Background 1 International Dairy Processing Plant with the capacity to produce 50,000 tons of dairy products annually utilizes fossil fuel for their thermal energy requirements. Ironically, concerning fuel price increases and environmental issues, the management of the plant was tasked with the sole responsibility of adopting renewable energy options. The plant requires heat energy with varying temperature ranges of 60-180 °C concerning ptimeseurization, sterilization, and drying processes. So far, they use 45,000 MWh per year and have peak daily demand for production hours. The decision-making team included the plant's chief engineer, sustainability manager, financial controller, and independent consultants in renewable energy. Geological surveys conducted early in the project suggested that they were situated in suitable geothermal zones with temperatures of 150-200 °C at reasonable depths. Following intensive technical feasibility studies and site surveys, the decision specialist selected four prospective geothermal systems based on the reasoning presented in Table 1.

Table 1. The geothermal energy systems

Symbols	Geothermal energy systems	Expert's explanation
$\mathscr{B}_1$	Direct Use Heat Exchange System (DUHES)	Selected due to its best application for direct utilization in the dairy industry for heating. The decision expert considered this system because the plant temperature range of 60-180 °C applied to the temperature of the available geothermal resource. This aspect of the system is highly desirable in the context of the available plant infrastructure because the system is simple to design and implement. Such a system, in the opinion of the specialist, can be integrated with current heating processes with minimal adjustments implemented.
$\mathscr{B}_2$	Binary Cycle Power System (BCPS)	This alternative was selected taking into account the necessity of the plant to consume both electricity and heat. The decision expert also noted that the explored geothermal resource temperature range of between 150-200 °C is suitable for the binary cycle. The fact that the system is used to produce electricity and also to supply process heat through the cooling cycle was the reason. The expert also noted that it could help minimize the reliance on grid electricity.
$\mathscr{B}_3$	Combined Heat and Power System (CHPS)	The expert selected this option since the ratio of total energy consumption was quite high for this option. System flexibility in transferring energy use beginning with electricity production and continued use of heat for other processes was perfect for the multi-energy needs of the plant. The expert indicated the ability to get the highest utilization of resources and gain operational flexibility.
$\mathscr{B}_4$	Enhanced Geothermal System (EGS)	The inclusion of this alternative was made by the expert concerning the possibility of higher temperature operation and higher power output. The expert pointed out that, although the system was highly technical, it had benefits in terms of the sustainable use of resources in the long term and the possibility of expanding the system in the future. The stability of the output of the system during the different seasons of the year was also an important factor.

Table 2 contains the criteria provided by the expert by considering serval aspects.

Table 2. The criteria of geothermal energy systems

Symbol	Criteria	Expert explanation
$\acute{g}_1$	Initial investment cost	The expert considered this criterion as the most important one because of the plant's financial limitation and the need to achieve a desired ROI. This includes: The four major cost categories are: (1) Drilling and well development costs (2) Surface equipment and installation (3) Integration with existing systems and (4) Associated infrtimesructure modifications. The expert stressed its significance as it affects the feasibility of projects and future profitability most of the time. This criterion was considered important for obtaining shareholders' approval and for financial anticipation.
$\acute{g}_2$	Thermal efficiency	Selected by the expert among the most significant technical parameters because it directly impacts the operating costs of the company and utilization of resources. The expert selected: Heat transfer effectiveness, energy conversion losses, part load efficiency, and resource use factor. This parameter was selected to obtain the best from geothermal resources and, at the same time, meet the plant's energy demand.
$\acute{g}_3$	Environmental impact	This criterion was added to the expert's list due to legal obligations and corporate and sustainable initiatives. It encompasses: There are four categories of impacts: (1) Greenhouse gas emissions reduction potential (2) Water consumption and management (3) Land use requirements (4) Local ecosystem effects. The expert pointed out that it is relevant for both compliance with environmental legislation and the implementation of CSR initiatives.
$\acute{g}_4$	Operational reliability	This criterion was chosen by the expert based on the fact that the dairy plant has to be in operation for most of the time. It includes: The four criteria include (1) System availability and uptime (2) Maintenance requirements and complexity (3) Technical maturity and spare parts availability (4) Long-term resource sustainability. The expert stressed the relevance of the measure to maintain uninterrupted production and to reduce operational risks.

Based on the criteria, the decision expert assesses the given geothermal energy systems, and the assessment values in the HBFNs are devised in Table 3.

Table 3. Hesitant bipolar fuzzy decision matrix

	$\acute{g}_1$	$\acute{g}_2$	$\acute{g}_3$	$\acute{g}_4$
$\mathscr{B}_1$	{(0.1248), (0.1200)}, {(-0.1120), (-0.5502)}	{(0.1188)}, {(-0.2072), (-0.0012)}	{(0.1001), (0.5554)}, {(-0.1172)}	{(0.1764), (0.3504)}, {(-0.1729)}
$\mathscr{B}_2$	{(0.1124), (0.5674), (0.1234)}, {(-0.8795), (-0.6757)}	{(0.1980), (0.4380)}, {(-0.1765), (-0.4235)}	{(0.1536), (-0.6478)}	{(0.3567), (-0.0980)}, {(-0.5438)}
$\mathscr{B}_3$	{(0.1987), (0.1769)}, {(-0.1456), (-0.1349)}	{(0.1364), (-0.7382)}	{(0.4738), (0.8900)}, {(-0.3456), (-0.9876)}	{(0.1637), (-0.1378)}
$\mathscr{B}_4$	{(0.7589), (-0.3456)}	{(0.1453), (0.0987)}, {(-0.2679), (-0.9874)}	{(0.5679), (0.1235)}, {(-0.8762)}	{(0.9087), (0.1234)}, {(-0.0987), (-0.1234)}

Step 1: There is no need to normalize the data in Table 3 because all attributes are of the benefit type.

Step 2: For $\xi = 7$, use the HBFDWA operator to determine all the preferences values β_Δ of the geothermal energy systems $\mathscr{B}_\Delta\,(\Delta = 1,\ 2,\ 3,\ 4)$.

$$\mathscr{B}_1 = \begin{pmatrix} \{(0.137861976),\ (0.279658329),\ (0.52287767),\ (0.522912656), \\ (0.137238225),\ (0.279656238),\ (0.522887765),\ (0.522912654)\}, \\ \{(-0.113623299),\ (-0.00150973),\ (-0.119650677)\} \end{pmatrix}$$

$$\mathscr{B}_2 = \begin{pmatrix} \{(0.285482404),\ (0.383965879),\ (0.524822264),\ (0.525437372), \\ (0.285485487),\ (0.383966038)\},\ \{(-0.146062318),\ (-0.21242547), \\ (-0.14681406),\ (-0.445052236),\ (-0.146062318),\ (-0.212425466), \\ (-0.14681406),\ (-0.445038993)\} \end{pmatrix}$$

$$\mathscr{B}_3 = \begin{pmatrix} \{0.441326392,\ 0.876515898,\ 0.441326392,\ 0.876515898, \\ 0.441324385,\ 0.876515898\},\ \{-0.16446542,\ -0.164565606, \\ -0.154212679,\ -0.154268255\} \end{pmatrix}$$

$$\mathscr{B}_4 = \begin{pmatrix} \{0.877510051,\ 0.726131476,\ 0.877510008,\ 0.726047781, \\ 0.877510051,\ 0.726131476,\ 0.877510008,\ 0.726047781\}, \\ \{-0.14778,\ -0.18216,\ -0.1478,\ -0.18229\} \end{pmatrix}$$

Step 3: The obtained score values of $\mathrm{scor}(\beta_\Delta)\,(\Delta = 1,\ 2,\ 3,\ 4)$ of the overall HBFNs $(\beta_\Delta)\,(\Delta = 1,\ 2,\ 3,\ 4)$.

$$\mathrm{scor}(\beta_1) = 0.212413, \quad \mathrm{scor}(\beta_2) = 0.317890, \quad \mathrm{scor}(\beta_3) = 0.409149, \quad \mathrm{scor}(\beta_4) = 0.483405$$

Step 4: Rank all the geothermal energy systems.

$\mathscr{B}_\Delta\,(\Delta = 1,\ 2,\ 3,\ 4)$ with the following score values $\mathrm{scor}(\beta_\Delta)\,(\Delta = 1,\ 2,\ 3,\ 4)$ of the overall HBFNs:

$$\mathscr{B}_4 > \mathscr{B}_3 > \mathscr{B}_2 > \mathscr{B}_1$$

Step 5: $\mathscr{B}_4$ is the best geothermal energy system.

The graphical ranking representation of geothermal energy systems based on HBFDWA operators is discussed in Figure 5.

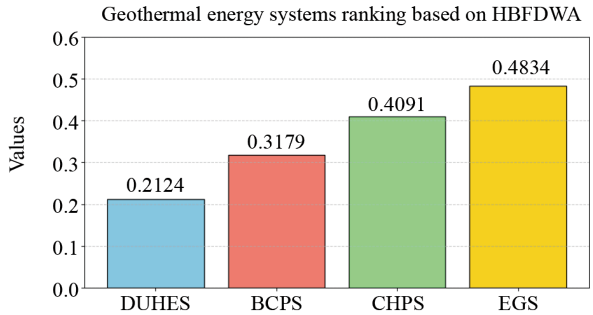

Figure 5. Ranking geothermal energy systems based on HBFDWA operators

If we use the HBFDWG operator instead of the HBFDWA operator then all the above steps are similar for the HBFDWG framework.

Step 1: The given data in Table 1 is benefit type, so there is no need to normalize it.

Step 2: For $\xi = 7$, apply the HBFDWG operator to determine all the preference values β_Δ of the geothermal energy systems $\mathscr{B}_\Delta\,(\Delta = 1,\ 2,\ 3,\ 4)$.

$$\beta_1 = \begin{pmatrix} \{(0.109219038),\ (0.109247028),\ (0.132936553),\ (0.866904257), \\ (0.108697906),\ (0.108724749),\ (0.130193143),\ (0.130325235)\}, \\ \{(-0.173310749),\ (-0.124028764),\ (-0.507370652),\ (-0.507370168)\} \end{pmatrix}$$

$$\beta_2 = \begin{pmatrix} \{(0.128998898),\ (0.129088906),\ (0.497065986),\ (0.171396995), \\ (0.139533304),\ (0.139715237)\}, \\ \{(-0.860051767),\ (-0.860051774),\ (-0.860051768),\ (-0.860051775), \\ (-0.651430969),\ (-0.651486644),\ (-0.651440372),\ (-0.65149603)\} \end{pmatrix}$$

$$\beta_3 = \begin{pmatrix} \{(0.1626488167),\ (0.162648336),\ (0.162648167),\ (0.162648336), \\ (0.160916889),\ (0.160917042)\}, \\ \{(-0.691409379),\ (-0.9858903),\ (-0.691409379),\ (-0.9858903)\} \end{pmatrix}$$

$$\beta_4 = \begin{pmatrix} \{(0.176240304),\ (0.155408556),\ (0.136248533),\ (0.132915262), \\ (0.878876481),\ (0.119871693),\ (0.116714467),\ (0.273952219)\}, \\ \{(-0.86129),\ (-0.98419)\} \end{pmatrix}$$

Step 3: The score values $\text{scor}(\beta_\Delta)$ for each geothermal energy system are calculated as follows.

$$\text{scor}(\beta_1) = 0.270026, \quad \text{scor}(\beta_2) = 0.478362, \quad \text{scor}(\beta_3) = 0.500360, \quad \text{scor}(\beta_4) = 0.585760$$

Step 4: Based on the score values, the ranking of the geothermal energy systems $\mathscr{B}_\Delta$ is.

$$\mathscr{B}_4 > \mathscr{B}_3 > \mathscr{B}_2 > \mathscr{B}_1$$

Step 5: $\mathscr{B}_4$ is selected as the best geothermal energy system.

The graphical ranking representation of geothermal energy systems based on HBFDWG operators is discussed in Figure 6.

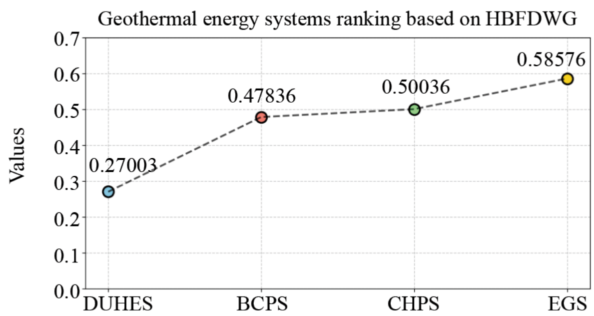

Figure 6. Ranking geothermal energy systems based on HBFDWG operators

6.2 *Sensitivity analysis*

Sensitivity analysis plays an important role in decision-making, particularly in ranking sophisticated geothermal energy systems. In the current research, we perform a sensitivity analysis based on Dombi aggregation operators to determine the stability and robustness of the developed operators. By adjusting the input parameters and weights, we investigate the effect of ranking sophisticated geothermal energy systems. It gives a stronger sense of how resistant the chosen methods are across various situations and, once more, increases the validity of our model.

The aggregated results of HBFDWA and HBFDWG operators are discussed in Table 4 and Table 5.

Table 4. The aggregated results for HBFDWA operators based on the variations of the parameter ξ

HBFDWA operator	$\text{scor}(\beta_1)$	$\text{scor}(\beta_2)$	$\text{scor}(\beta_3)$	$\text{scor}(\beta_4)$
$\text{HBFDWA}_{\xi=4}$	0.20139	0.318509	0.408215	0.495225
$\text{HBFDWA}_{\xi=6}$	0.20973	0.317626	0.408930	0.485842
$\text{HBFDWA}_{\xi=9}$	0.216226	0.318571	0.409452	0.480420
$\text{HBFDWA}_{\xi=12}$	0.219805	0.319443	0.409726	0.478069
$\text{HBFDWA}_{\xi=15}$	0.222043	0.320062	0.409890	0.476775
$\text{HBFDWA}_{\xi=19}$	0.233963	0.320615	0.410017	0.475751
$\text{HBFDWA}_{\xi=22}$	0.224952	0.320900	0.410071	0.475251
$\text{HBFDWA}_{\xi=23}$	0.225225	0.320978	0.410084	0.475115
$\text{HBFDWA}_{\xi=25}$	0.225705	0.321115	0.410103	0.474880
$\text{HBFDWA}_{\xi=42}$	0.227941	0.321742	0.410075	0.473832

Table 5. The aggregated results for HBFDWG operators based on the variations of the parameter ξ

HBFDWG operator	scor(β_1)	scor(β_2)	scor(β_3)	scor(β_4)
HBFDWG$_{\xi=4}$	0.260339	0.476271	0.497210	0.587607
HBFDWG$_{\xi=6}$	0.267680	0.477959	0.499880	0.586209
HBFDWG$_{\xi=9}$	0.273420	0.478812	0.500710	0.585099
HBFDWG$_{\xi=12}$	0.276713	0.479111	0.500727	0.584434
HBFDWG$_{\xi=15}$	0.278804	0.479261	0.500603	0.583982
HBFDWG$_{\xi=19}$	0.280602	0.479398	0.500436	0.583565
HBFDWG$_{\xi=22}$	0.281528	0.479486	0.500334	0.583341
HBFDWG$_{\xi=23}$	0.281783	0.479514	0.500305	0.583278
HBFDWG$_{\xi=25}$	0.282233	0.479567	0.500253	0.583166
HBFDWG$_{\xi=42}$	0.284342	0.479903	0.500003	0.582637

Ranking of all the alternatives are discussed in Table 6 and Table 7.

Table 6. Ranking alternatives based on HBFDWA operators

HBFDWA operator	Ranking of alternatives
HBFDWA$_{\xi=4}$	$\mathscr{B}_4 > \mathscr{B}_3 > \mathscr{B}_2 > \mathscr{B}_1$
HBFDWA$_{\xi=6}$	$\mathscr{B}_4 > \mathscr{B}_3 > \mathscr{B}_2 > \mathscr{B}_1$
HBFDWA$_{\xi=9}$	$\mathscr{B}_4 > \mathscr{B}_3 > \mathscr{B}_2 > \mathscr{B}_1$
HBFDWA$_{\xi=12}$	$\mathscr{B}_4 > \mathscr{B}_3 > \mathscr{B}_2 > \mathscr{B}_1$
HBFDWA$_{\xi=15}$	$\mathscr{B}_4 > \mathscr{B}_3 > \mathscr{B}_2 > \mathscr{B}_1$
HBFDWA$_{\xi=19}$	$\mathscr{B}_4 > \mathscr{B}_3 > \mathscr{B}_2 > \mathscr{B}_1$
HBFDWA$_{\xi=22}$	$\mathscr{B}_4 > \mathscr{B}_3 > \mathscr{B}_2 > \mathscr{B}_1$
HBFDWA$_{\xi=23}$	$\mathscr{B}_4 > \mathscr{B}_3 > \mathscr{B}_2 > \mathscr{B}_1$
HBFDWA$_{\xi=25}$	$\mathscr{B}_4 > \mathscr{B}_3 > \mathscr{B}_2 > \mathscr{B}_1$
HBFDWA$_{\xi=42}$	$\mathscr{B}_4 > \mathscr{B}_3 > \mathscr{B}_2 > \mathscr{B}_1$

Table 7. Ranking alternatives based on HBFDWG operators

HBFDWG operator	Ranking of alternatives
HBFDWG$_{\xi=4}$	$\mathscr{B}_4 > \mathscr{B}_3 > \mathscr{B}_2 > \mathscr{B}_1$
HBFDWG$_{\xi=6}$	$\mathscr{B}_4 > \mathscr{B}_3 > \mathscr{B}_2 > \mathscr{B}_1$
HBFDWG$_{\xi=9}$	$\mathscr{B}_4 > \mathscr{B}_3 > \mathscr{B}_2 > \mathscr{B}_1$
HBFDWG$_{\xi=12}$	$\mathscr{B}_4 > \mathscr{B}_3 > \mathscr{B}_2 > \mathscr{B}_1$
HBFDWG$_{\xi=15}$	$\mathscr{B}_4 > \mathscr{B}_3 > \mathscr{B}_2 > \mathscr{B}_1$
HBFDWG$_{\xi=19}$	$\mathscr{B}_4 > \mathscr{B}_3 > \mathscr{B}_2 > \mathscr{B}_1$
HBFDWG$_{\xi=22}$	$\mathscr{B}_4 > \mathscr{B}_3 > \mathscr{B}_2 > \mathscr{B}_1$
HBFDWG$_{\xi=23}$	$\mathscr{B}_4 > \mathscr{B}_3 > \mathscr{B}_2 > \mathscr{B}_1$
HBFDWG$_{\xi=25}$	$\mathscr{B}_4 > \mathscr{B}_3 > \mathscr{B}_2 > \mathscr{B}_1$
HBFDWG$_{\xi=42}$	$\mathscr{B}_4 > \mathscr{B}_3 > \mathscr{B}_2 > \mathscr{B}_1$

Graphical representation of variations by different parameters is discussed in Figure 7 and Figure 8.

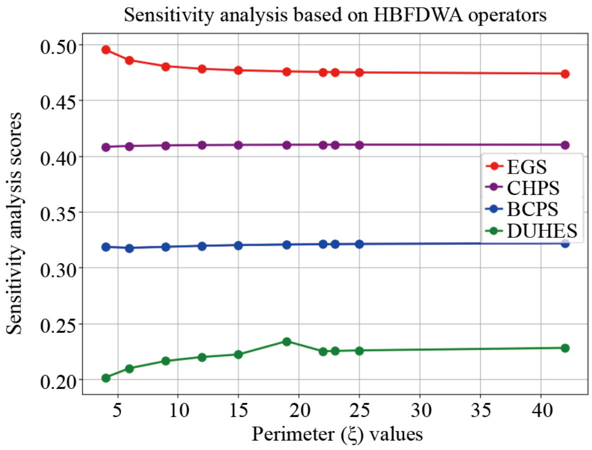

Figure 7. Sensitivity analysis with varying parameter values based on HBFDWA operators

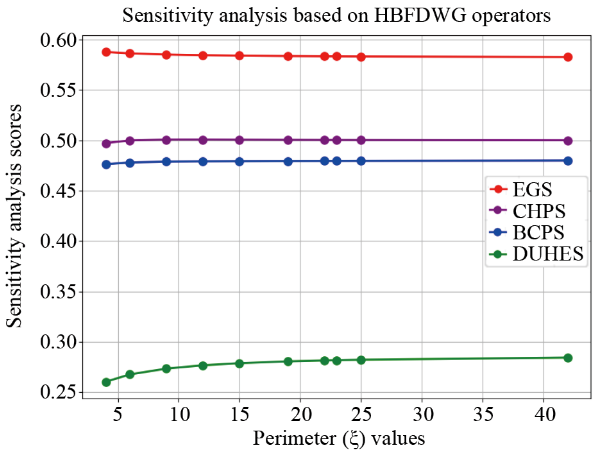

Figure 8. Sensitivity analysis with varying parameter values based on HBFDWG operators

The foregoing sensitivity analysis indicates that parameter value changes cannot alter the ranking of geothermal energy systems, which indicates the solidity and stability of the proposed operators. Further, the constructed operators indicate that the functionality in dealing with uncertainty and yielding credible rankings presents them as an important framework for decision-makers in the future.

7. Comparative analysis

It is easy to identify the pros and cons of different methods by performing comparative analysis, which helps in making decisions. In complex problems, it illustrate how new advancements overcome limitations and improve data management. For easy comparison and discussion, we assume the different theories of bipolar fuzzy sets and most assumed theories are connected to different Aggregation operators. Then we attempt to match these theories' outcomes with the predicted theory outcomes as mentioned in the following Table 8. The assumed theories are;

- The theory of Bipolar Fuzzy Dombi Aggregation Operators (BFDAOs) by Jana et al. [40].
- The theory of Bipolar Fuzzy Hamacher Aggregation Operators (BFHAOs) by Wei et al. [41].
- The theory of the Bipolar Fuzzy Probability Aggregation Operators (BFPAOs) by Chen et al. [42].

- The theory of Bipolar Fuzzy Aczel-Alsina Power Aggregation Operators (BFAAPAOs) by Garg et al. [43].

Table 8. Comparative analysis between proposed and existing theories

Theories	Score values	Ranking
BFDAOs by Jana et al. [40]	$\text{scor}(\beta_1) = \times\times\times\times$, $\text{scor}(\beta_2) = \times\times\times\times$, $\text{scor}(\beta_3) = \times\times\times\times$, $\text{scor}(\beta_4) = \times\times\times\times$	No ranking
BFHAOs by Wei et al. [41]	$\text{scor}(\beta_1) = \times\times\times\times$, $\text{scor}(\beta_2) = \times\times\times\times$, $\text{scor}(\beta_3) = \times\times\times\times$, $\text{scor}(\beta_4) = \times\times\times\times$	No ranking
BFPAOs by Chen et al. [42]	$\text{scor}(\beta_1) = \times\times\times\times$, $\text{scor}(\beta_2) = \times\times\times\times$, $\text{scor}(\beta_3) = \times\times\times\times$, $\text{scor}(\beta_4) = \times\times\times\times$	No ranking
BFAAPAOs by Garg et al. [43]	$\text{scor}(\beta_1) = \times\times\times\times$, $\text{scor}(\beta_2) = \times\times\times\times$, $\text{scor}(\beta_3) = \times\times\times\times$, $\text{scor}(\beta_4) = \times\times\times\times$	No ranking
HBFDWA (Proposed)	$\text{scor}(\beta_1) = 0.212413$, $\text{scor}(\beta_2) = 0.317890$, $\text{scor}(\beta_3) = 0.409149$, $\text{scor}(\beta_4) = 0.483405$	$\mathscr{B}_4 > \mathscr{B}_3 > \mathscr{B}_2 > \mathscr{B}_1$
HBFDWG (Proposed)	$\text{scor}(\beta_1) = 0.270026$, $\text{scor}(\beta_2) = 0.478362$, $\text{scor}(\beta_3) = 0.500360$, $\text{scor}(\beta_4) = 0.585760$	$\mathscr{B}_4 > \mathscr{B}_3 > \mathscr{B}_2 > \mathscr{B}_1$

• The work of Jana et al. [40] introduces a new way to connect positive and negative preferences in Dombi operations. It uses a convenient parameterized design that does well in basic bipolar situations. But this kind of evaluation depends on experts who can provide quick answers and don't accommodate doubts or hesitation. Consequently, its direct use is limited when the data includes uncertain or conflicting information.

• The theory of BFHAOs by Wei et al. [41] utilizes Hamacher functions for managing interactions in bipolar fuzzy data. It provides adjustable sensitivity in decision-making contexts. Nonetheless, it does not consider hesitant inputs, which are common in complex systems. This restricts its ability to reflect ambiguity in expert judgments.

• Chen et al. [42] use their BFPAO theory which combines the concepts of probability and bipolar fuzziness, to address randomness in information. However, it only works if accurate probability distributions are offered in these settings. This theory does not allow calculating several scores for the same criterion. For this reason, it does not handle hesitation from experts with accuracy.

• The theory of BFAAPAOs by Garg et al. [43] focuses on capturing nonlinear relationships using power mean functions. It enhances modeling capacity in interdependent decision factors. However, this theory does not support hesitant or interval-valued information. As a result, it lacks robustness when facing vague or imprecise inputs.

• For comparison, our emerging HBF Dombi aggregation operators theory goes beyond the above models by directly incorporating hesitancy into bipolar fuzzy settings. It allows for experts to submit several possible ratings per attribute, which is more in line with real-world uncertainty. The Dombi operations introduce adjustable control over aggregation behavior. The resulting framework is well-suited for robust assessment of geothermal energy systems, where multidimensional uncertainty is inevitable.

8. Conclusion

The research offers a sophisticated decision-making model for ranking geothermal energy systems specific to industrial processes. Through HBFSs, the model accounts for uncertainty and doubt in expert assessments to reflect a more realistic and detailed appraisal. The incorporation of Dombi aggregation operators introduces flexibility and accuracy to aggregation, accepting different levels of importance over multiple criteria. The presented approach beats current models by efficiently handling ambiguity and bipolar data, typical of energy-related choices. Comparative analysis showcases the superiority of the framework in precisely ranking geothermal options. The outcome provides useful information to policymakers and energy planners seeking sustainable and efficient industrial energy alternatives. Sensitivity analysis validates the strength and integrity of the presented approach. Additionally, the model promotes transparency and clarity in intricate decision-making surroundings. Its availability showcases the applicability of uncertain bipolar fuzzy techniques to multi-criteria energy planning. Further extensions of this framework can be conducted by future studies into other renewable power systems and dynamic decision-making contexts such as [44–47]. Furthermore, our proposed research has the following findings;

• Development of HBFDWA operators.

• Development of HBFDOWA operators.

• Development of HBFDWG operators.

• Development of HBFDOWG operators.

• A case study is performed based on the developed operators to rank geothermal energy systems.

• A comparative analysis and sensitivity analysis are performed to show the superiority and stability of the proposed approach.

• A sensitivity analysis is discussed to show the influences of the parameter.

8.1 *Limitation*

Although our proposed HBF Dombi aggregation method works well in many situations, it still has some limitations. These limitations are mostly due to the structure and complexity of the method.

• The use of HBFSs and Dombi operations increases the computational burden, especially for large datasets.

• The complex nature of the aggregation process may be less intuitive for non-expert decision-makers.

• As the number of alternatives or attributes grows, the method may become less efficient or harder to apply.

• The chosen evaluation factors might not be representative of all environmental, technical, and economic complexities of geothermal systems in various industrial areas.

• The decision-making process relies highly on expert judgment, and hence subjective bias could be introduced despite hesitant bipolar fuzzy modeling.

• The obtained results are sensitive to input data variation and membership degree, which could influence the stability of prioritization results under uncertain scenarios.

• The model is intended for a static decision context and could not easily be applied to dynamic or real-time shifting industrial or energy policy contexts.

Data availability

The data will be available on reasonable request to the corresponding author.

Ethics declaration statement

The authors state that this is their original work and it is neither submitted nor under consideration in any other journal simultaneously.

Human and animal participants

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflict of interest

About the publication of this manuscript, the authors declare that they have no conflict of interest.

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