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Research Article

Analytical Solutions of Two Space-Time Fractional Nonlinear Models Using Jacobi Elliptic Function Expansion Method

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Abstract: In this manuscript, the space-time fractional Equal-width (s-tfEW) and the space-time fractional Wazwaz-Benjamin-Bona-Mahony (s-tfWBBM) models have been investigated which frequently arises in nonlinear optics, solid states, fluid mechanics and shallow water. Jacobi elliptic function expansion integral technique has been used to build more innovative exact solutions of the s-tfEW and s-tfWBBM nonlinear partial models. In this research, fractional beta-derivatives are applied to convert the partial models to ordinary models. Several types of solutions have been derived for the models and performed some new solitary wave phenomena. The derived solutions have been presented in the form of Jacobi elliptic functions initially. Persevering different conditions on a parameter, we have achieved hyperbolic and trigonometric functions solutions from the Jacobi elliptic function solutions. Besides the scientific derivation of the analytical findings, the results have been illustrated graphically for clear identification of the dynamical properties. It is noticeable that the integral scheme is simplest, most conventional and convenient in handling many nonlinear models arising in applied mathematics and the applied physics to derive diverse structural precise solutions.

Keywords: space-time fractional equal width equation, space-time fractional Wazwaz-Benjamin-Bona-Mahony, balance number, fractional beta-derivative, Jacobi elliptic function expansion method, analytical solutions

1. Introduction

In the current world, fractional derivatives have been applied to study the calculus of arbitrary order for modelling of nonlinear happening in different fields like fluid mechanics, signal processing, control theory, astrophysics, dynamical systems, plasma physics, non-linear biological systems, nanotechnology, and engineering. Many real-life problems of the above areas can be modelled by Partial Differential Equation (PDE) relating to the fractional derivatives. The concept of solitons, the top decisive way in applications to such models has played an important role to identify the complex incident in various fields of sciences. Up to days, many techniques have been introduced for deriving exact wave solutions of nonlinear models but the innovation reached is deficient. The precise mathematical methods to derive different classes of exact solutions namely; the inverse variational methods [1], the Darboux Transformation [2], the Exp-function technique [3], tanh method [4], the $\exp(-(\Phi)\eta)$-expansion method [5], first integral scheme [6], the $\tan(\Theta/2)$-expansion approach [7], the Hirota bilinear method [8-9], the sine-cosine analysis [10], the new extended $(G'/$

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G)-expansion method [11], the modified double sub-equation method [12], the mapping and ansatz methods [13-14], the Jacobi elliptic function expansion method [15-16] as well.

Moreover, it is very problematic to derive the exact solution of nonlinear fractional PDE via the best possible method. So, it is significant to arise the explicit solutions which are exact for advanced study of these nonlinear fractional models and have to realize the nonlinear phenomena. Many powerful and useful ways have been introduced to solve the exact solution of nonlinear fractional equations [17-18]. The Jacobi elliptic function expansion method [15-16] is an excellent way to integrate fractional nonlinear differential models.

In this research work, we start the research with s-tfEW [18] and s-WBBM [19-21] models to analyse the nonlinear phenomena Hosseini and Ayati [18] presented exact solutions of the s-tfEW with the help of Kudrayshov method. Benjamin-Bona-Mohony introduces the BBM equation [19]. Then Wazwaz modified this equation to WBBM [20]. This script considers the Jacobi elliptic function expansion method to integrate the s-tfEW and s-tfWBBM models for deriving exact solutions. This technique also bases on the homogeneous balance method which is an influential procedure for achieving solutions of fractional PDE introduced by Zhang and Zhang [17]. According to this method, fractional complex transform and some useful formulas of fractional beta-derivative [21-25] are applied to transform the nonlinear s-tfEW equation to ordinary differential equation.

2. Beta-fractional derivative

Let us review the beta-derivative [21-25] as follows:

Definition 1 Let $\phi : [a, \infty) \to \Re$, then the fractional beta-derivative of ϕ of order β is defined as

$D^{\eta}(\phi)(x) = \lim_{\varepsilon \to 0} \dfrac{\phi(x + \varepsilon(x + \frac{1}{\Gamma(\beta)})^{1-\beta}) - \phi(x)}{\varepsilon}$, for all $x \geq a$, $\beta \in (0, 1]$. If the limit of the above exists, then $\phi(x)$ is said to be beta-differentiable.

Some properties of the derivative for the functions $\phi(x)$ and $\psi(x)$

(i). $D^{\beta}(m\phi(x) + n\psi(x)) = mD^{\beta}\phi(x) + nD^{\beta}\psi(x)$, where a and b are constants.

(ii). $D^{\beta} x^{\alpha} = \alpha(x + \frac{1}{\Gamma(\beta)})^{\alpha-\beta}$. $\alpha \in \Re$.

(iii). $D^{\beta}(\phi\psi) = \phi D^{\beta}(\psi) + \psi D^{\beta}(\phi)$.

(iv). $D^{\beta}(\frac{\phi}{\psi}) = \dfrac{\psi D^{\beta}(\phi) - \phi D^{\beta}(\psi)}{\psi^2}$, where $\psi \neq 0$.

(v). $D^{\beta}(c) = 0$, where c is a constant.

Here $D^{\beta}(\psi(x)) = (x + \frac{1}{\Gamma(\beta)})^{1-\beta} \frac{d\psi}{dx}$.

Definition 2 Let $\phi : [0, \infty) \to \Re$ such that ϕ is differentiable. Let $\psi(x)$ be another function defined the same range of $\phi(x)$ and also differentiable. Then, the two functions satisfied the following rule [19]:

$$D^{\beta}(\phi o \psi) = (x + \frac{1}{\Gamma(\beta)})^{1-\beta} \psi(x)' \phi'(\psi(x)).$$

3. The Jacobi elliptic function expansion method

Consider a given nonlinear wave equation

$$\mathrm{N}(\varphi, D_t^{\gamma_2}\varphi, D_x^{\gamma_1}\varphi, D_t^{2\gamma_2}\varphi, D_x^{2\gamma_1}\varphi....) = 0. \tag{1}$$

The function $\varphi = \varphi(x, t)$ is unknown wave surface and N is a function of $\varphi = \varphi(x, t)$ and its highest order fractional derivatives.

We seek its wave transformation

$$\varphi = \varphi(\xi),\ \xi = \frac{k}{\Gamma(\gamma_1)}x^{\gamma_1} - \frac{c}{\Gamma(\gamma_2)}t^{\gamma_2}. \quad (2)$$

The symbols k the wave number and c wave speed.

By using the above transformation Eq. (2), the fractional nonlinear Eq. (1) is converted to the following ordinary differential equation;

$$P(\varphi,\ \varphi',\ \varphi'',\ \varphi'''.........). \quad (3)$$

In [17], $\varphi(\xi)$ is a trail solution in the form of Jacobi elliptic sine function $sn(\xi)$,

$$\varphi(\xi) = a_0 + \sum_{i=1}^{n} a_i sn^i(\xi) + \sum_{i=1}^{n} b_i sn^{-i}(\xi). \quad (4)$$

$sn(\xi)$ is Jacobi elliptic sine function, and its highest degree is

$$\mathrm{P}(\varphi(\xi)) = n. \quad (5)$$

$$\mathrm{P}(\frac{d\varphi}{d\xi}) = n+1,\ \mathrm{P}(\varphi\frac{d\varphi}{d\xi}) = 2n+1,\ \mathrm{P}(\frac{d^2\varphi}{d\xi^2}) = n+2,\ \text{and}\ \mathrm{P}(\frac{d^3\varphi}{d\xi^3}) = n+3. \quad (6)$$

Thus, we can consider n in Eq. (4) to homogenous balance from the terms of the highest order of derivative term and nonlinear.

Here, $cn(\xi)$ and $dn(\xi)$ are the Jacobi elliptic cosine function and the Jacobi elliptic functions respectively.

And

$$cn^2(\xi) = 1 - sn^2(\xi),\ dn^2(\xi) = 1 - m^2 sn^2(\xi),\ \text{where}\ m\,(0 < m < 1). \quad (7)$$

$$\frac{d}{d\xi}(sn(\xi)) = cn(\xi)dn(\xi),\ \frac{d}{d\xi}(cn(\xi)) = -sn(\xi)dn(\xi). \quad (8)$$

$$\frac{d}{d\xi}(dn(\xi)) = -m^2 sn(\xi)cn(\xi). \quad (9)$$

We know that, when $m \to 1$, and $m \to 0$, then $sn(\xi) \to \tanh(\xi)$ and $sn(\xi) \to \sin(\xi)$ respectively. Thus, using Eq. (4) and its derivatives along with Eq. (7) and Eq. (8) into Eq. (3), we achieve a set of equations with unknown parameters. Solve the system for the unknown parameters. Using the parameters, the series solution of Eq. (4) is determined in terms of Jacobi elliptic functions.

We can convert the Jacobi elliptic sine function to solitonic and periodic function by selecting the conditions $m \to 1$, and $m \to 0$ respectively.

4. Application of the method

In this section, we apply Jacobi Elliptic Expansion function method to the s-tfEW and the s-tfWBBM models.

4.1 *Solutions of s-tfEW equation*

The space-time fractional EW(s-tfEW) equation [18] read as:

$$D_t^{\beta}\varphi(x,t)+\varepsilon D_x^{\beta}\varphi^2(x,t)-\delta D_{xxt}^{3\beta}\varphi(x,t)=0,\ t>0,\ 0<\beta\le 1. \tag{10}$$

Introducing a travelling wave transformation for s-tfEW model Eq. (10)

$$\varphi(x,t)=f(\xi),\ \xi=\frac{k}{\Gamma(\beta)}x^{\beta}-\frac{c}{\Gamma(\beta)}t^{\beta}. \tag{11}$$

Eq. (11) converts nonlinear partial differential Eq. (10) to the following nonlinear ordinary differential equation (ODE),

$$-cf'+\varepsilon k(f^2)'+\delta ck^2 f'''=0. \tag{12}$$

Integrating Eq. (12) with respect to ξ, then the equation converted to the nonlinear ODE Eq. (13),

$$-cf+\varepsilon kf^2+\delta ck^2 f''=0. \tag{13}$$

Using the balancing role (f^2 with f'') in Eq. (13) gives $n=2$. Now, choose an auxiliary solution for the balance number.

$$f(\xi)=a_0+a_1 sn(\xi)+a_2 sn^2(\xi)+b_1 sn^{-1}(\xi)+b_2 sn^{-2}(\xi). \tag{14}$$

Inserting $f(\xi)$ from Eq. (14) to the Eq. (13), then equating adjacent terms of $sn^i(\xi)$ to zero and solve these terms for a_0, a_1, a_2, b_1 and b_2, we get

Case-1:

$$k=\frac{1}{2\sqrt{d}\sqrt[4]{m^4+14m^2+1}},\ a_0=\frac{c\sqrt{d}\,(m^2+1-2\sqrt{(m^4+14m^2+1)})}{\varepsilon\sqrt[4]{m^4+14m^2+1}},$$

$$a_2=-\frac{3c\sqrt{d}m^2}{\sqrt[4]{m^4+14m^2+1}},\ b_2=-\frac{3c\sqrt{d}}{\varepsilon\sqrt[4]{m^4+14m^2+1}},\ a_1=0,\ b_1=0.$$

Case-2:

$$k=-\frac{1}{2\sqrt{d}\sqrt[4]{m^4+14m^2+1}},\ a_0=-\frac{c\sqrt{d}\,(m^2+1-2\sqrt{(m^4+14m^2+1)})}{\varepsilon\sqrt[4]{m^4+14m^2+1}},$$

$$a_2 = \frac{3c\sqrt{d}m^2}{\sqrt[4]{m^4 + 14m^2 + 1}},\ b_2 = \frac{3c\sqrt{d}}{\varepsilon\sqrt[4]{m^4 - m^2 + 1}},\ a_1 = 0,\ b_1 = 0.$$

Case-3:

$$k = \frac{1}{2\sqrt{d}\sqrt[4]{m^4 - m^2 + 1}},\ a_0 = \frac{c\sqrt{d}(m^2 + 1 - 2\sqrt{(m^4 - m^2 + 1)})}{\varepsilon\sqrt[4]{m^4 - m^2 + 1}},$$

$$b_2 = -\frac{3c\sqrt{d}}{\varepsilon\sqrt[4]{m^4 - m^2 + 1}},\ a_1 = 0,\ a_2 = 0,\ b_1 = 0.$$

Case-4:

$$k = -\frac{1}{2\sqrt{d}\sqrt[4]{m^4 - m^2 + 1}},\ a_0 = -\frac{c\sqrt{d}(m^2 + 1 - 2\sqrt{(m^4 - m^2 + 1)})}{\varepsilon\sqrt[4]{m^4 - m^2 + 1}},$$

$$b_2 = \frac{3c\sqrt{d}}{\varepsilon\sqrt[4]{m^4 - m^2 + 1}},\ a_1 = 0,\ a_2 = 0,\ b_1 = 0.$$

Case:-5:

$$k = \frac{1}{2\sqrt{d}\sqrt[4]{m^4 - m^2 + 1}},\ a_0 = \frac{c\sqrt{d}(m^2 + 1 - 2\sqrt{(m^4 - m^2 + 1)})}{\varepsilon\sqrt[4]{m^4 - m^2 + 1}},$$

$$a_2 = -\frac{3c\sqrt{d}m^2}{\varepsilon\sqrt[4]{m^4 - m^2 + 1}},\ a_1 = 0,\ b_1 = 0,\ b_2 = 0.$$

Case-6:

$$k = -\frac{1}{2\sqrt{d}\sqrt[4]{m^4 - m^2 + 1}},\ a_0 = -\frac{c\sqrt{d}(m^2 + 1 - 2\sqrt{(m^4 - m^2 + 1)})}{\varepsilon\sqrt[4]{m^4 - m^2 + 1}},$$

$$a_2 = \frac{3c\sqrt{d}m^2}{\varepsilon\sqrt[4]{m^4 - m^2 + 1}},\ a_1 = 0,\ b_1 = 0,\ b_2 = 0.$$

Eq. (10) are reduced the following exact solutions by using (case-1-6)

$$\varphi(x,t)=\frac{c\sqrt{d}(m^2+1-2\sqrt{m^4+14m^2+1})}{\varepsilon\sqrt[4]{m^4+14m^2+1}}-\frac{3c\sqrt{d}m^2}{\sqrt[4]{m^4+14m^2+1}}sn^2(\frac{1}{2\sqrt{d}\sqrt[4]{m^4+14m^2+1}}\frac{1}{\Gamma(\beta)}x^\beta-\frac{c}{\Gamma(\beta)}t^\beta)$$

$$-\frac{3c\sqrt{d}}{\varepsilon\sqrt[4]{m^4+14m^2+1}}sn^{-2}(\frac{1}{2\sqrt{d}\sqrt[4]{m^4+14m^2+1}}\frac{1}{\Gamma(\beta)}x^\beta-\frac{c}{\Gamma(\beta)}t^\beta). \tag{15}$$

$$\varphi(x,t)=-\frac{c\sqrt{d}(m^2+1-2\sqrt{m^4+14m^2+1})}{\varepsilon\sqrt[4]{m^4+14m^2+1}}+\frac{3c\sqrt{d}m^2}{\sqrt[4]{m^4+14m^2+1}}sn^2(-\frac{1}{2\sqrt{d}\sqrt[4]{m^4+14m^2+1}}\frac{1}{\Gamma(\beta)}x^\beta-\frac{c}{\Gamma(\beta)}t^\beta)$$

$$+\frac{3c\sqrt{d}}{\varepsilon\sqrt[4]{m^4+14m^2+1}}sn^{-2}(-\frac{1}{2\sqrt{d}\sqrt[4]{m^4+14m^2+1}}\frac{1}{\Gamma(\beta)}x^\beta-\frac{c}{\Gamma(\beta)}t^\beta). \tag{16}$$

$$\varphi(x,t)=\frac{c\sqrt{d}(m^2+1-2\sqrt{m^4-m^2+1})}{\varepsilon\sqrt[4]{m^4-m^2+1}}-\frac{3c\sqrt{d}}{\varepsilon\sqrt[4]{m^4-m^2+1}}sn^{-2}(\frac{1}{2\sqrt{d}\sqrt[4]{m^4-m^2+1}}\frac{1}{\Gamma(\beta)}x^\beta-\frac{c}{\Gamma(\beta)}t^\beta). \tag{17}$$

$$\varphi(x,t)=-\frac{c\sqrt{d}(m^2+1-2\sqrt{m^4-m^2+1})}{\varepsilon\sqrt[4]{m^4-m^2+1}}+\frac{3c\sqrt{d}}{\varepsilon\sqrt[4]{m^4-m^2+1}}sn^{-2}(-\frac{1}{2\sqrt{d}\sqrt[4]{m^4-m^2+1}}\frac{1}{\Gamma(\beta)}x^\beta-\frac{c}{\Gamma(\beta)}t^\beta). \tag{18}$$

$$\varphi(x,t)=\frac{c\sqrt{d}(m^2+1+\sqrt{m^4-m^2+1})}{\varepsilon\sqrt[4]{m^4-m^2+1}}-\frac{3c\sqrt{d}m^2}{\varepsilon\sqrt[4]{m^4-m^2+1}}sn^{2}(\frac{1}{2\sqrt{d}\sqrt[4]{m^4-m^2+1}}\frac{1}{\Gamma(\beta)}x^\beta-\frac{c}{\Gamma(\beta)}t^\beta). \tag{19}$$

$$\varphi(x,t)=-\frac{c\sqrt{d}(m^2+1+\sqrt{m^4-m^2+1})}{\varepsilon\sqrt[4]{m^4-m^2+1}}+\frac{3c\sqrt{d}m^2}{\varepsilon\sqrt[4]{m^4-m^2+1}}sn^{2}(-\frac{1}{2\sqrt{d}\sqrt[4]{m^4-m^2+1}}\frac{1}{\Gamma(\beta)}x^\beta-\frac{c}{\Gamma(\beta)}t^\beta). \tag{20}$$

Eq. (15-20) represent the solutions in term of Jacobi elliptic function.
When $m\to 1$, the solutions Eq. (15-20) convert in the form,

$$\varphi(x,t)=-\frac{3c\sqrt{d}}{\varepsilon}-\frac{3c\sqrt{d}}{2}\tanh^2(\frac{1}{4\sqrt{d}}\frac{1}{\Gamma(\beta)}x^\beta-\frac{c}{\Gamma(\beta)}t^\beta)-\frac{3c\sqrt{d}}{2\varepsilon}\tanh^{-2}(\frac{1}{4\sqrt{d}}\frac{1}{\Gamma(\beta)}x^\beta-\frac{c}{\Gamma(\beta)}t^\beta). \tag{21}$$

$$\varphi(x,t)=\frac{3c\sqrt{d}}{\varepsilon}+\frac{3c\sqrt{d}}{2}\tanh^2(-\frac{1}{4}\frac{1}{\Gamma(\beta)}x^\beta-\frac{c}{\Gamma(\beta)}t^\beta)+\frac{3c\sqrt{d}}{2\varepsilon}\tanh^{-2}(-\frac{1}{4\sqrt{d}}\frac{1}{\Gamma(\beta)}x^\beta-\frac{c}{\Gamma(\beta)}t^\beta). \tag{22}$$

$$\varphi(x,t) = -\frac{3c\sqrt{d}}{\varepsilon}\tanh^{-2}(\frac{1}{2\sqrt{d}}\frac{1}{\Gamma(\beta)}x^{\beta} - \frac{c}{\Gamma(\beta)}t^{\beta}). \tag{23}$$

$$\varphi(x,t) = \frac{3c\sqrt{d}}{\varepsilon}\tanh^{-2}(-\frac{1}{2\sqrt{d}}\frac{1}{\Gamma(\beta)}x^{\beta} - \frac{c}{\Gamma(\beta)}t^{\beta}). \tag{24}$$

$$\varphi(x,t) = -\frac{3c\sqrt{d}}{\varepsilon}\tanh^{2}(\frac{1}{2\sqrt{d}}\frac{1}{\Gamma(\beta)}x^{\beta} - \frac{c}{\Gamma(\beta)}t^{\beta}). \tag{25}$$

$$\varphi(x,t) = \frac{3c\sqrt{d}}{\varepsilon}\tanh^{2}(-\frac{1}{2\sqrt{d}}\frac{1}{\Gamma(\beta)}x^{\beta} - \frac{c}{\Gamma(\beta)}t^{\beta}). \tag{26}$$

Solitary wave solutions Eq. (21-26) come in terms of hyperbolic functions form.
When $m \to 0$, the solutions Eq. (15-20) convert in the form,

$$\varphi(x,t) = -\frac{c\sqrt{d}}{\varepsilon} - \frac{3c\sqrt{d}}{\varepsilon}\sin^{-2}(\frac{1}{2\sqrt{d}}\frac{1}{\Gamma(\beta)}x^{\beta} - \frac{c}{\Gamma(\beta)}t^{\beta}). \tag{27}$$

$$\varphi(x,t) = \frac{c\sqrt{d}}{\varepsilon} + \frac{3c\sqrt{d}}{\varepsilon}\sin^{-2}(-\frac{1}{2\sqrt{d}}\frac{1}{\Gamma(\eta)}x^{\eta} - \frac{c}{\Gamma('\eta)}t^{\eta}). \tag{28}$$

$$\varphi(x,t) = -\frac{c\sqrt{d}}{\varepsilon} - \frac{3c\sqrt{d}}{\varepsilon}\sin^{-2}(\frac{1}{2\sqrt{d}}\frac{1}{\Gamma(\beta)}x^{\beta} - \frac{c}{\Gamma(\beta)}t^{\beta}). \tag{29}$$

$$\varphi(x,t) = \frac{c\sqrt{d}}{\varepsilon} + \frac{3c\sqrt{d}}{\varepsilon}\sin^{-2}(-\frac{1}{2\sqrt{d}}\frac{1}{\Gamma(\beta)}x^{\beta} - \frac{c}{\Gamma(\beta)}t^{\beta}). \tag{30}$$

These are periodic wave solutions of the nonlinear-tfEW model and the other two solutions (19), (20) give constants only.

4.2 *Solutions of the WBBM model*

The space-time fractional WBBM equation [21] read as:

$$D_t^{\beta}\varphi(x,y,z,t) + D_x^{\beta}\varphi(x,y,z,t) + D_y^{\beta}\varphi(x,y,z,t) - D_{xzt}^{3\beta}\varphi(x,y,z.t) = 0,\ t > 0,\ 0 < \beta \le 1. \tag{31}$$

Considering a travelling wave transformation for space-time fractional 3D WBBM model Eq. (31)

$$\varphi(x,t) = \varphi(\zeta),\ \zeta = \frac{1}{\Gamma(\beta)}(\ell x^{\beta} + \wp y^{\beta} + c z^{\beta} - w t^{\beta}). \tag{32}$$

Eq. (32) transform the WBBM Eq. (31) to the following nonlinear ODE,

$$(-w+\ell)\phi' + \wp(\phi^3)' + \ell cw\phi''' = 0. \tag{33}$$

Integrating Eq. (33) with respect to ζ, then Eq. (31) converted to the nonlinear ODE Eq. (34),

$$(-w+\ell)\phi + \wp\,\phi^3 + \ell cw\phi'' = 0. \tag{34}$$

Using the balancing role (ϕ^2 with ϕ'') in Eq. (34) gives $n = 1$. Now, choose an auxiliary solution for the balance number.

$$\phi(\zeta) = a_0 + a_1 sn(\zeta) + b_1 sn^{-1}(\zeta). \tag{35}$$

Plugging $\phi(\zeta)$ from Eq. (35) to the Eq. (34), then comparing the adjacent terms of $sn^i(\zeta)$ to zero and solving these algebraic equations for a_0, a_1, w and b_1, we get four sets of solutions.

Case-1:

$$w = \frac{\ell}{\ell cm^2 - 6\ell cm + \ell c + 1},\ a_0 = 0,\ a_1 = \pm \ell m\sqrt{\frac{-2c}{\wp(\ell cm^2 - 6\ell cm + \ell c + 1)}},\ b_2 = \pm \ell\sqrt{\frac{-2c}{\wp(\ell cm^2 - 6\ell cm + \ell c + 1)}}.$$

Case-2:

$$w = \frac{\ell}{\ell cm^2 + 6\ell cm + \ell c + 1},\ a_0 = 0,\ a_1 = \pm \ell m\sqrt{\frac{-2c}{\wp(\ell cm^2 + 6\ell cm + \ell c + 1)}},\ b_2 = \pm \ell\sqrt{\frac{-2c}{\wp(\ell cm^2 + 6\ell cm + \ell c + 1)}}.$$

Case-3:

$$w = \frac{\ell}{\ell cm^2 + \ell c + 1},\ a_0 = 0,\ a_1 = 0,\ b_2 = \pm \ell\sqrt{\frac{-2c}{\wp(\ell cm^2 + \ell c + 1)}}.$$

Case-4:

$$w = \frac{\ell}{\ell cm^2 + \ell c + 1},\ a_0 = 0,\ b_1 = 0,\ a_1 = \pm \ell m\sqrt{\frac{-2c}{\wp(\ell cm^2 + \ell c + 1)}}.$$

The exact solutions of Eq. (31) by using (case-1-4)

$$\varphi_{11}(x,t) = \ell\sqrt{\frac{-2c}{\wp(\ell cm^2 - 6\ell cm + \ell c + 1)}}\left\{\begin{array}{l} -msn\left(\frac{1}{\Gamma(\beta)}(\ell x^\beta + \wp y^\beta + cz^\beta - wt^\beta)\right) \\ +sn^{-1}\left(\frac{1}{\Gamma(\beta)}(\ell x^\beta + \wp y^\beta + cz^\beta - wt^\beta)\right) \end{array}\right\}. \tag{36}$$

$$\varphi_{12}(x,t)=\ell\sqrt{\frac{-2c}{\wp(\ell cm^2-6\ell cm+\ell c+1)}}\begin{Bmatrix} msn(\frac{1}{\Gamma(\beta)}(\ell x^\beta+\wp y^\beta+cz^\beta-wt^\beta)) \\ -sn^{-1}(\frac{1}{\Gamma(\beta)}(\ell x^\beta+\wp y^\beta+cz^\beta-wt^\beta)) \end{Bmatrix}. \tag{37}$$

In Eq. (36) and Eq. (37), $w=\frac{\ell}{\ell cm^2-6\ell cm+\ell c+1}$.

$$\varphi_{13}(x,t)=\ell\sqrt{\frac{-2c}{\wp(\ell cm^2+6\ell cm+\ell c+1)}}\begin{Bmatrix} msn(\frac{1}{\Gamma(\beta)}(\ell x^\beta+\wp y^\beta+cz^\beta-wt^\beta)) \\ +sn^{-1}(\frac{1}{\Gamma(\beta)}(\ell x^\beta+\wp y^\beta+cz^\beta-wt^\beta)) \end{Bmatrix}. \tag{38}$$

$$\varphi_{14}(x,t)=-\ell\sqrt{\frac{-2c}{\wp(\ell cm^2+6\ell cm+\ell c+1)}}\begin{Bmatrix} msn(\frac{1}{\Gamma(\beta)}(\ell x^\beta+\wp y^\beta+cz^\beta-wt^\beta)) \\ +sn^{-1}(\frac{1}{\Gamma(\beta)}(\ell x^\beta+\wp y^\beta+cz^\beta-wt^\beta)) \end{Bmatrix}. \tag{39}$$

In Eq. (38) and Eq. (39), $w=\frac{\ell}{\ell cm^2+6\ell cm+\ell c+1}$.

$$\varphi_{15}(x,t)=\ell\sqrt{\frac{-2c}{\wp(\ell cm^2+\ell c+1)}}sn^{-1}(\frac{1}{\Gamma(\beta)}(\ell x^\beta+\wp y^\beta+cz^\beta-wt^\beta)). \tag{40}$$

$$\varphi_{16}(x,t)=-\ell\sqrt{\frac{-2c}{\wp(\ell cm^2+\ell c+1)}}sn^{-1}(\frac{1}{\Gamma(\beta)}(\ell x^\beta+\wp y^\beta+cz^\beta-wt^\beta)). \tag{41}$$

$$\varphi_{17}(x,t)=\ell m\sqrt{\frac{-2c}{\wp(\ell cm^2+\ell c+1)}}sn(\frac{1}{\Gamma(\beta)}(\ell x^\beta+\wp y^\beta+cz^\beta-wt^\beta)). \tag{42}$$

$$\varphi_{18}(x,t)=-\ell m\sqrt{\frac{-2c}{\wp(\ell cm^2+\ell c+1)}}sn(\frac{1}{\Gamma(\beta)}(\ell x^\beta+\wp y^\beta+cz^\beta-wt^\beta)). \tag{43}$$

In Eq. (40), Eq. (41), Eq. (42) and Eq. (43), $w=\frac{\ell}{\ell cm^2+\ell c+1}$.

Eq. (36-43) represent the Jacobi elliptic function solutions of Eq. (31).

When $m\to 1$, the solutions Eq. (36-43) convert to the following form,

$$\varphi_{19}(x,t)=\ell\sqrt{\frac{-2c}{\wp(1-4\ell c)}}\begin{Bmatrix}-\tanh(\frac{1}{\Gamma(\beta)}(\ell x^{\beta}+\wp y^{\beta}+cz^{\beta}-wt^{\beta}))\\+\coth(\frac{1}{\Gamma(\beta)}(\ell x^{\beta}+\wp y^{\beta}+cz^{\beta}-wt^{\beta}))\end{Bmatrix}. \tag{44}$$

$$\varphi_{19}(x,t)=\ell\sqrt{\frac{-2c}{\wp(1-4\ell c)}}\begin{Bmatrix}\tanh(\frac{1}{\Gamma(\beta)}(\ell x^{\beta}+\wp y^{\beta}+cz^{\beta}-wt^{\beta}))\\-\coth(\frac{1}{\Gamma(\beta)}(\ell x^{\beta}+\wp y^{\beta}+cz^{\beta}-wt^{\beta}))\end{Bmatrix}. \tag{45}$$

In Eq. (44) and Eq. (45), $w=\frac{\ell}{(1-4\ell c)}$.

$$\varphi_{20}(x,t)=\ell\sqrt{\frac{-2c}{\wp(1+8\ell cm)}}\begin{Bmatrix}\tanh(\frac{1}{\Gamma(\beta)}(\ell x^{\beta}+\wp y^{\beta}+cz^{\beta}-wt^{\beta}))\\+\coth(\frac{1}{\Gamma(\beta)}(\ell x^{\beta}+\wp y^{\beta}+cz^{\beta}-wt^{\beta}))\end{Bmatrix}. \tag{46}$$

$$\varphi_{21}(x,t)=-\ell\sqrt{\frac{-2c}{\wp(1+8\ell c)}}\begin{Bmatrix}\tanh(\frac{1}{\Gamma(\beta)}(\ell x^{\beta}+\wp y^{\beta}+cz^{\beta}-wt^{\beta}))\\+\coth(\frac{1}{\Gamma(\beta)}(\ell x^{\beta}+\wp y^{\beta}+cz^{\beta}-wt^{\beta}))\end{Bmatrix}. \tag{47}$$

In Eq. (46) and Eq. (47) carry the value of $w=\frac{\ell}{(1+8\ell c)}$.

$$\varphi_{22}(x,t)=\ell\sqrt{\frac{-2c}{\wp(1+2\ell c)}}\coth(\frac{1}{\Gamma(\beta)}(\ell x^{\beta}+\wp y^{\beta}+cz^{\beta}-wt^{\beta})). \tag{48}$$

$$\varphi_{23}(x,t)=-\ell\sqrt{\frac{-2c}{\wp(1+2\ell c)}}\coth(\frac{1}{\Gamma(\beta)}(\ell x^{\beta}+\wp y^{\beta}+cz^{\beta}-wt^{\beta})). \tag{49}$$

$$\varphi_{25}(x,t)=-\ell\sqrt{\frac{-2c}{\wp(1+2\ell c)}}\tanh(\frac{1}{\Gamma(\beta)}(\ell x^{\beta}+\wp y^{\beta}+cz^{\beta}-wt^{\beta})). \tag{50}$$

In Eq. (48), Eq. (49) and Eq. (50), $w=\frac{\ell}{(1+2\ell c)}$.

Solitary wave solutions come from the hyperbolic functions Eq. (44-50).

When $m\rightarrow 0$, the solutions Eq. (36-43) convert to the form,

$$\varphi_{25}(x,t)=\ell\sqrt{\frac{-2c}{\wp(1+\ell c)}}\left\{\cos ec(\frac{1}{\Gamma(\beta)}(\ell x^{\beta}+\wp y^{\beta}+cz^{\beta}-wt^{\beta}))\right\}. \quad (51)$$

$$\varphi_{26}(x,t)=\ell\sqrt{\frac{-2c}{\wp(1+\ell c)}}\left\{-\cos ec(\frac{1}{\Gamma(\beta)}(\ell x^{\beta}+\wp y^{\beta}+cz^{\beta}-wt^{\beta}))\right\}. \quad (52)$$

$$\varphi_{27}(x,t)=-\ell\sqrt{\frac{-2c}{\wp(1+\ell c)}}\left\{\cos ec(\frac{1}{\Gamma(\beta)}(\ell x^{\beta}+\wp y^{\beta}+cz^{\beta}-wt^{\beta}))\right\}. \quad (53)$$

In Eq. (51), Eq. (52) and Eq. (53), $w=\frac{\ell}{(1+\ell c)}$.

Eq. (36)-Eq. (43) are Jacobi functions solution of the nonlinear WBBM model. Out of the eight Jacobi elliptic functions, three of them are repeated and two results give zero solution. So, these five solutions are neglected.

5. Graphical representation

In this section, we will provide some graphical representations of the exact solutions of the space-time fractional Equal Width(s-tfEW) equation (Eq. (10)) and the space-time fractional Wazwaz-Benjamin-Bona-Mahony (s-tfWBBM) model (Eq. (31)). Graphical representations are portrayed below using the selected exact solutions of EW and WBBM model.

5.1 *Graphics of the solutions of s-tfEW equation*

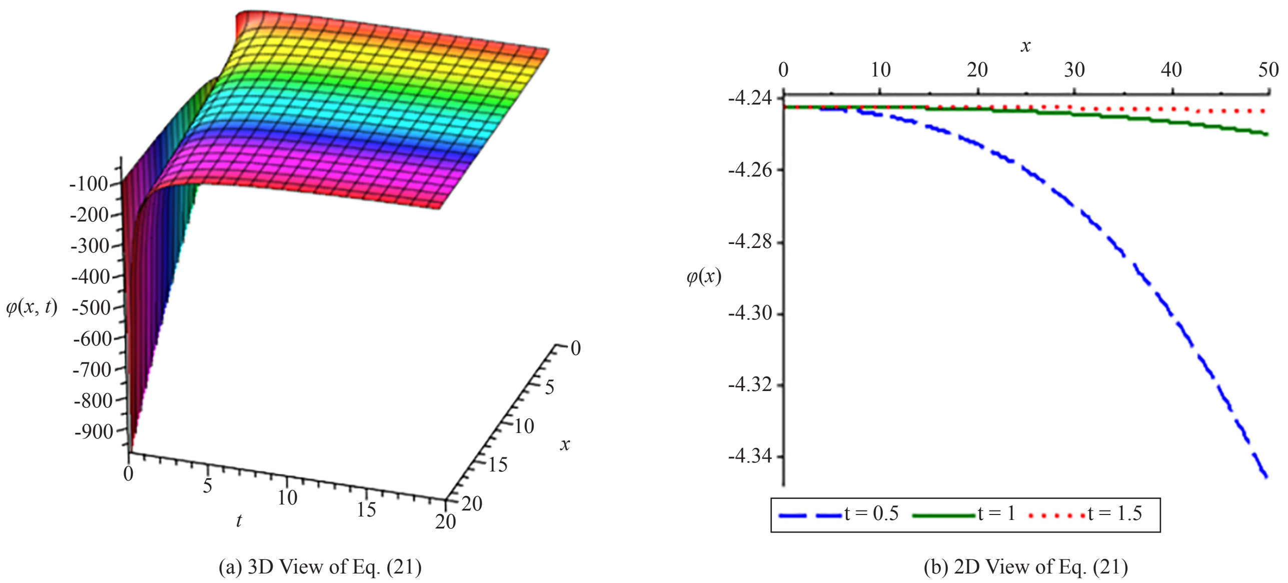
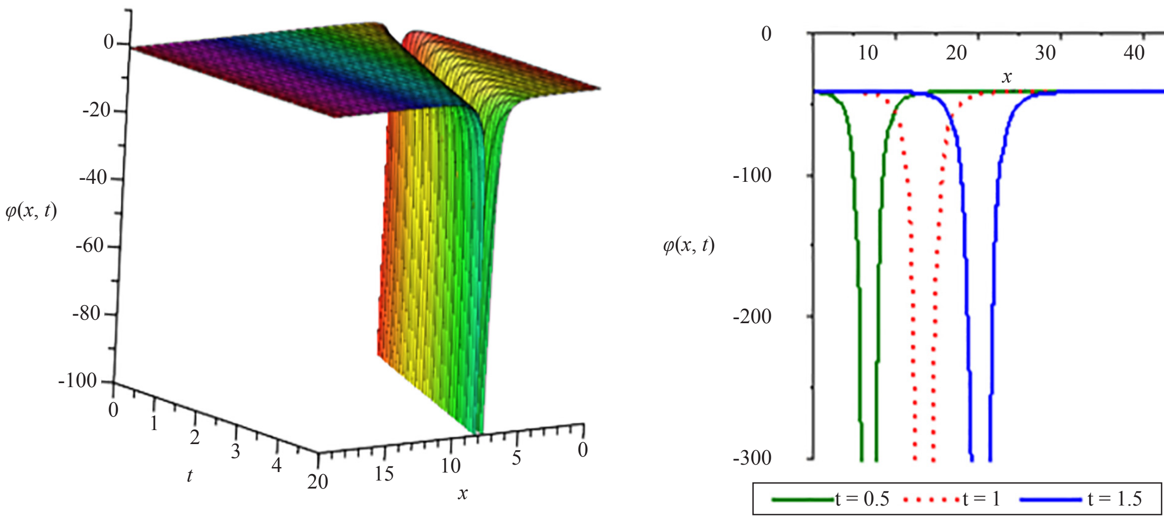


Figure 1. Represent the solitary wave $\varphi(x, t)$ in Eq. (21) for the physical parametric values, $d = 0.5$, $\beta = 1/6$, $c = 1$, $\varepsilon = 1$: (a) 3D surface, (b) 2D graphs at $t = 0.5, 1, 1.5$.

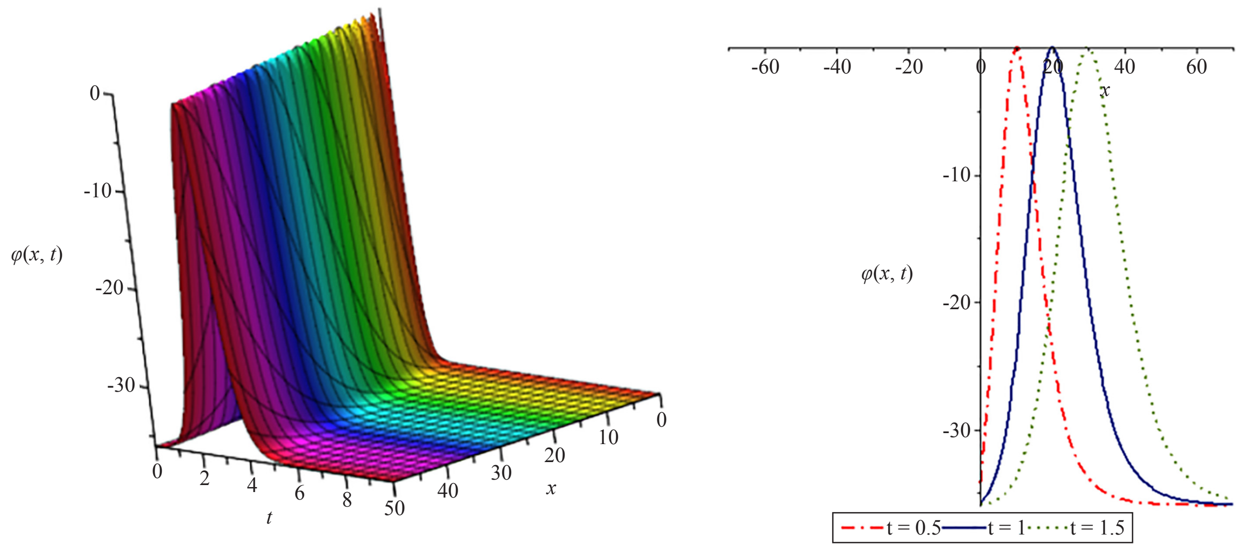


(a) 3D View of Eq. (23)

(b) 2D View of Eq. (23)

Figure 2. Represent the solitary wave $\varphi(x, t)$ in Eq. (23) for the physical parametric values, $d = 0.5$, $\beta = 3/4$, $c = 5$, $\varepsilon = 2$: (a) 3D surface and (b) 2D graphs at $t = 0.5, 1, 1.5$.

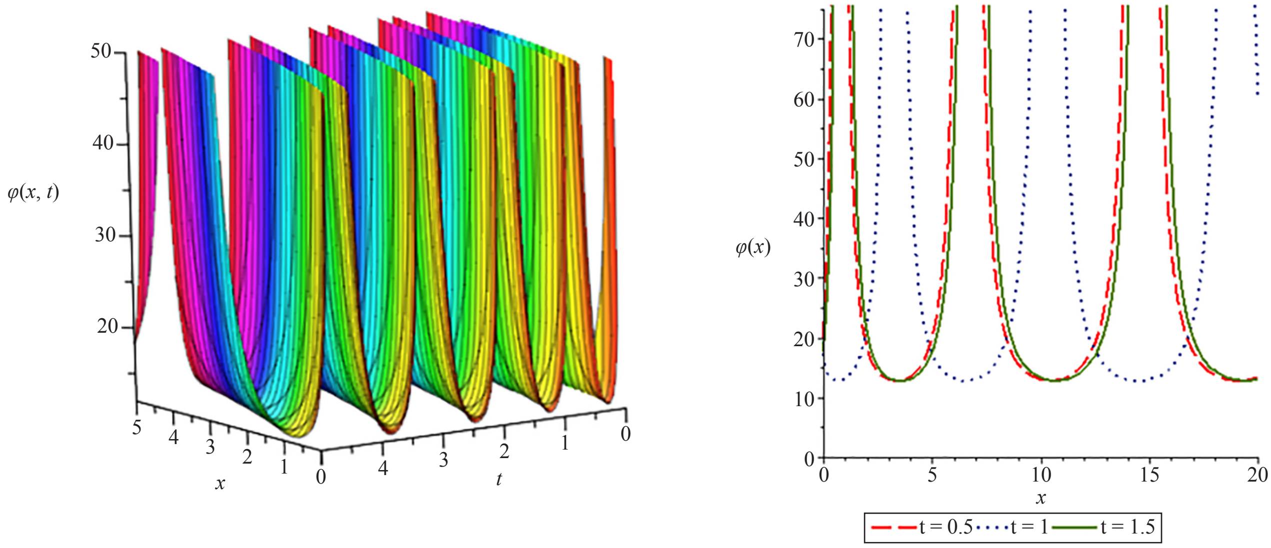
Three types of results are achieved for EW equation. All of the results are analysed and some of them are depicted in Figures (1-4). The graphs signify the change of amplitude, direction, shape of the derived wave solutions to identify the intrinsic nature of the model. The solution $\varphi(x, t)$ in Eq. (15-20) represents the Jacobi elliptic functions Eq. (21-26) shows the solitonic nature comes from hyperbolic function and Eq. (27-30) are trigonometric function exhibit as periodic waves.



(a) 3D View of Eq. (25)

(b) 2D View of Eq. (25)

Figure 3. Represent the bell type solitary wave $\varphi(x, t)$ in Eq. (25) for the physical parametric values, $d = 1$, $\beta = 3/5$, $c = 3$, $\varepsilon = 0.25$: (a) 3D surface and (b) 2D graphs for and $t = 0.5, 1, 1.5$.



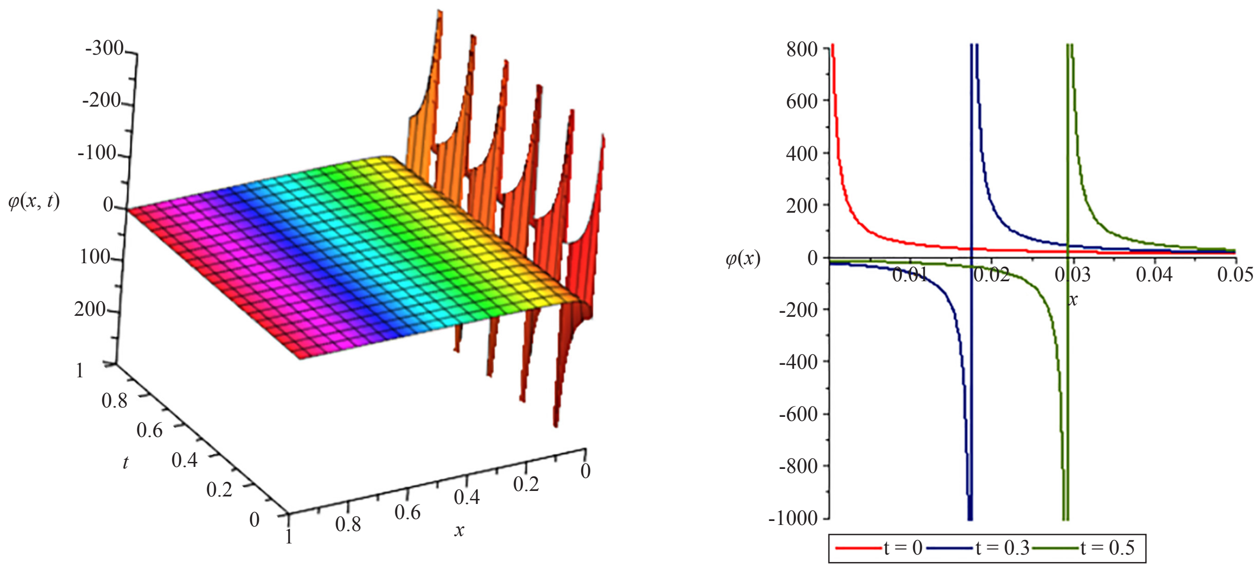
(a) 3D View of Eq. (27)

(b) 2D View of Eq. (27)

Figure 4. Represent the periodic wave of $\varphi(x, t)$ in Eq. (27) for the physical parametric values, $d = 0.5$, $\beta = 3/4$, $c = -3$, $\varepsilon = 1$: (a) 3D surface and (b) 2D graphs at $t = 0.5, 1, 1.5$.

5.2 *Graphics of the equation WBBM*

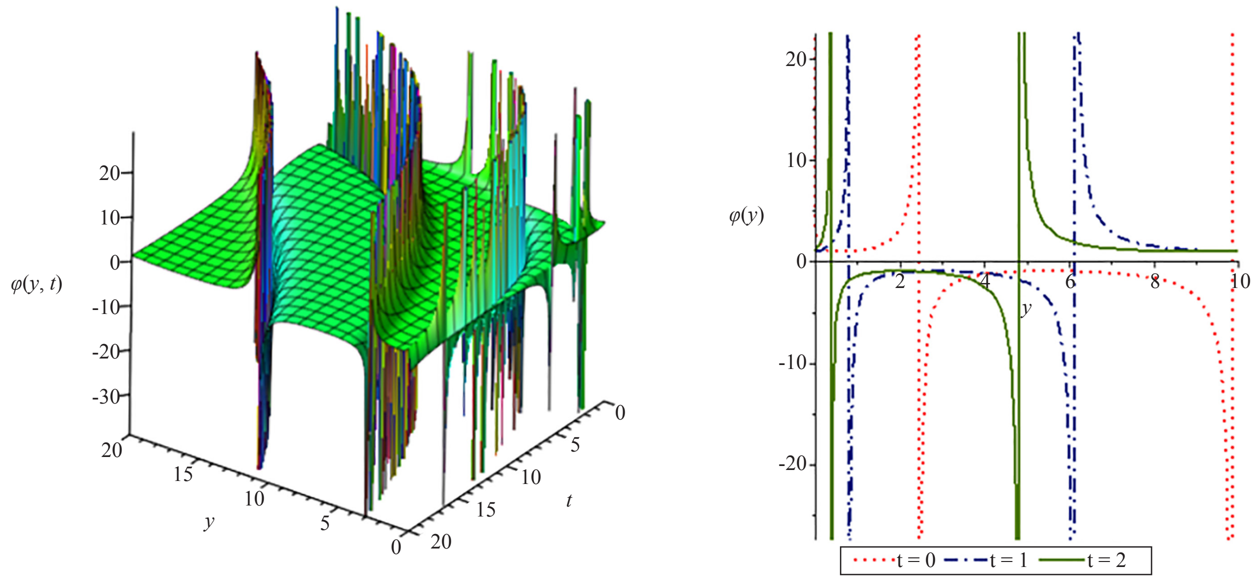
The findings of the research on WBBM model are in the types of hyperbolic (Eq. (44-51)) and trigonometric (Eq. (52-55)) functions. Hyperbolic and trigonometric functions represent solitonic and periodic solutions. All the results are analysed and two types of function have been shown graphically in Figure 5 to Figure 6.



(a) 3D View of Eq. (48)

(b) 2DView of Eq. (48)

Figure 5. Represent the solitary periodic wave $\varphi(x, t)$ in Eq. (48) for the physical parametric values, $\beta = 0.99$, $l = 2$, $c = -2$, $\wp = 1$, $z = 0$, $y = 0$: (a) 3D surface, (b) 2D graphs at $t = 0, 1.03, 0.5$.



(a) 3D View of Eq. (52)

(b) 3D View of Eq. (52)

Figure 6. Represent the periodic wave $\varphi(x, t)$ in Eq. (52) for the physical parametric values, $\beta = 0.5$, $l = 2$, $c = -2$, $\wp = 1$, $z = 0$, $x = 0$: (a) 3D surface, (b) 2D graphs at $t = 0, 1.2$.

Remarks More other Jacobi function solutions to the s-tfEW and WBBM equation are derivable by keeping the trial solution in terms of the Jacobi functions $cn(\xi)$ and $dn(\xi)$ as below;

$$u(\xi) = a_0 + \sum_{i \to 1}^{n} a_i cn^i(\xi) + \sum_{i \to 1}^{n} a_{-i} cn^{-i}(\xi). \tag{54}$$

And

$$u(\xi) = a_0 + \sum_{i \to 1}^{n} a_i dn^i(\xi) + \sum_{i \to 1}^{n} a_{-i} dn^{-i}(\xi). \tag{55}$$

In view of Eq. (54) and Eq. (55), we can add soliton and non-solitonic solutions describe via cnoidal, dnoidal waves and trigonometric functions.

6. Concluding remarks

In this portion, the space-time fractional EW and WBBM equation has successfully integrated via Jacobi elliptic function expansion technique with beta-derivatives. By introducing a fractional transformation, the considered nonlinear partial travelling wave equation was reduced to ordinary differential model. Then we successfully used Jacobi elliptic expansion method to integrate the model. At the end of our procedure, three types of solutions are achieved namely, Jacobi elliptic, hyperbolic and trigonometric function with unknown parameters, which indicates that Jacobi elliptic expansion technique is very fruitful as well as appropriate to find the exact solutions of nonlinear fractional models. Here we, successfully derived cnoidal and dnoidal waves solutions to the fractional models which were not found in the previous literature. In addition, the graphical illustration of some different types of solutions has been plotted with unknown parameters in Figures (1-4) and Figures (5-6) for s-tfEW and WBBM respectively. Researchers can undoubtedly use the technique to analyse the internal mechanism of nonlinear physical systems.

Conflict of interest

Authors declared that they have no conflict of interest.

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