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Research Article

On Some New Milne-type Inequalities for Strongly Convex Functions

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Abstract: In this paper, we explore new Milne-type fractional integral inequalities, focusing on functions that are strongly convex and differentiable. We use tools like the Hölder's inequality and the power-mean inequality to derive these results. Our findings offer a notable improvement over earlier work in this area, providing more advanced and refined mathematical insights.

***Keywords*:** milne-type inequalities, strongly convex function, riemann-liouville fractional integrals, Hölder's inequality

MSC: 26D07, 26D10, 26D15

1. Introduction

Fractional calculus serves as a valuable tool in the modeling of physical and engineering processes, especially those that are most appropriately described by fractional differential equations. It is important to recognize that conventional mathematical models utilizing integer-order derivatives, even those that are nonlinear, often exhibit poor performance in numerous scenarios. In recent years, fractional calculus has emerged as a significant tool across multiple disciplines, including biology, chemistry, control theory, electricity, economics, mechanics, and image processing [1–3]. Furthermore, fractional calculus represents a significant area of study for elucidating physical phenomena and addressing practical issues. The Riemann-Liouville fractional integral is one of the most distinguished forms of fractional integrals [4], receiving considerable focus in various research studies.

The theory of inequality, one of the cornerstones of mathematics, is used in many fields of science have develop various formulas for numerical integration over time and look at their error boundaries. In the study of inequalities for fractional integrals, the authors also concentrated on generating new error bounds by utilizing other types of functions, such as convex functions and strongly convex functions. However, numerous researchers created additional bounds by applying the idea of fractional calculus. There are several significant integral inequalities in the literature, including Simpson, Trapezoid, Midpoint, and others. Extensions and generalizations of these integral inequalities are the subject of numerous works. For instance, several trapezoid-type inequalities were discovered for various differentiable convex functions. For more information about these topics, see [5–8] and references therein.

Since they both hold under the same conditions in terms of Newton-Cotes formulas, Milne's formula, which is of open type, is comparable to Simpson's formula, which is of close type. Assume that the mapping from $\mathfrak{F} : [\xi_1,\ \xi_2] \subset \mathbb{R} \to \mathbb{R}$

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is a four times continuously differentiable mapping on $(\xi_1,\ \xi_2)$. Let $\|\mathfrak{F}^4\|_\infty = \sup_{\nu\in(\xi_1,\ \xi_2)}|\mathfrak{F}^{(4)}(\nu)|\infty$. Then we have the following inequality:

$$\left|\frac{1}{3}\left[2\mathfrak{F}(\xi_1)-\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right)+2\mathfrak{F}(\xi_2)\right]-\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\mathfrak{F}(\nu)d\nu\right|$$

$$\leq\frac{7(\xi_2-\xi_1)^4}{23040}\|\mathfrak{F}^4\|_\infty. \tag{1}$$

In this paper, we will obtain a fractional version of the left-hand side of (1) and will consider several new bounds by using different mapping classes. Several mathematicians established Simpson-type inequalities for differentiable convex mappings [9], twice differentiable convex functions [10–12], bounded functions [13, 14], s-convex functions [15], extended (s, m)-convex mappings [16], and fractional integrals [4, 11, 17–24].

Polyak introduced the concept of strong convexity [25]. It has a wide range of applications, including many branches of mathematics, as well as optimization theory, mathematical economics, and approximation theory. It's important to note that all strongly convex functions are also convex, but not all convex functions are strongly convex. There are numerous advantages to using strongly convex functions (see [26]). Further, the idea of higher-order strongly convex functions was developed by Lin and Fukushima [27]. Hudzik and Maligranda [28] gave the idea of s-convex functions in the second sense. [29] generalized the notion of s-convexity and defined a new class of strongly s-convex functions. For more generalizations, we refer to [30, 31].

Recently, Budak et al. [32] established some new general Milne-type inequalities for fractional integrals for convex functions using Hölder's inequality and power-mean inequality. Motivated and inspired by the ongoing research in the field of integral inequalities, we derive some new Milne-type inequalities for differentiable strongly convex mappings in Riemann-Liouville fractional integrals using Hölder and power-mean inequalities. Furthermore, we explore the relationship between our results and the pertinent literature. We also include examples that exemplify our theoretical conclusions. The results established in this paper are stronger versions of the results given by Budak et al. [32].

The organization of this paper is as follows: In Section 2, we recall some basic notions, definitions, and lemmas that are necessary for our main results. In Section 3, using the aforementioned generalized lemma, we state and prove Milne-type inequalities for differentiable strongly convex mapping and also deduce several additional corollaries. Finally, in Section 4, we conclude and discuss future directions.

2. Preliminaries

In this part of the paper, we gather various notations, basic definitions, and critical results that will be essential for the following sections.

Definition 1 [33] Let Ω be a convex set on $\mathbb{R}$. The function $\mathfrak{F}:\Omega\subset\mathbb{R}\to\mathbb{R}$ is said to be convex function on Ω if,

$$\mathfrak{F}(\nu\xi_1+(1-\nu)\xi_2)\leq\nu\mathfrak{F}(\xi_1)+(1-\nu)\mathfrak{F}(\xi_2)$$

holds for all $\xi_1,\ \xi_2\in\Omega$ and $\nu\in[0,\ 1]$.

Definition 2 [25] A function $\mathfrak{F}:\Omega\subset\mathbb{R}\to\mathbb{R}$ is said to be strongly convex function on Ω with modulus $c>0$, if, it satisfies the following inequalities:

$$\mathfrak{F}(\nu\xi_1+(1-\nu)\xi_2)\leq\nu\mathfrak{F}(\xi_1)+(1-\nu)\mathfrak{F}(\xi_2)-c\nu(1-\nu)(\xi_1-\xi_2)^2$$

is valid for all $\xi_1, \xi_2 \in \Omega$ and $\nu \in [0, 1]$.

The Riemann-Liouville fractional integrals was defined in [16] as follows:

Definition 3 Let $\mathbb{L}_1([\xi_1, \xi_2])$ is the set of all real-valued functions whose absolute value is integrable in the interval $[\xi_1, \xi_2]$. If $\mathfrak{F} \in \mathbb{L}_1([\xi_1, \xi_2])$, the Riemann-Liouville fractional integrals $J^{\alpha}_{\xi_1+}\mathfrak{F}$ and $J^{\alpha}_{\xi_2-}\mathfrak{F}$ of order $\alpha > 0$ are defined by

$$J^{\alpha}_{\xi_1+}\mathfrak{F}(\nu) = \frac{1}{\Gamma(\alpha)}\int_{\xi_1}^{\nu}(\nu-\mu)^{(\alpha-1)}\mathfrak{F}(\mu)d\mu,\ \nu > \xi_1$$

and

$$J^{\alpha}_{\xi_2-}\mathfrak{F}(\nu) = \frac{1}{\Gamma(\alpha)}\int_{\nu}^{\xi_2}(\mu-\nu)^{(\alpha-1)}\mathfrak{F}(\mu)d\mu,\ \nu < \xi_2$$

respectively. where, $\Gamma(\alpha)$ is the Euler Gamma function defined by

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{(\alpha-1)}dt.$$

If, the order $\alpha = 0$ then $J^{0}_{\xi_1+}\mathfrak{F}(\nu) = J^{0}_{\xi_2-}\mathfrak{F}(\nu) = \mathfrak{F}(\nu)$, for further details on Riemann-Liouville fractional integrals, please refer to [4, 34, 35].

Lemma 1 [32] Let $\mathfrak{F} : [\xi_1, \xi_2] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on (ξ_1, ξ_2) such that $\mathfrak{F}' \in \mathbb{L}_1([\xi_1, \xi_2])$. Then the following inequality holds:

$$\begin{aligned}
&\frac{1}{3}\left[2\mathfrak{F}(\xi_1) - \mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right) + 2\mathfrak{F}(\xi_2)\right] \\
&- \frac{2^{(\alpha-1)}\Gamma(\alpha+1)}{(\xi_2-\xi_1)^{\alpha}}\left[J^{\alpha}_{\xi_1+}\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right) + J^{\alpha}_{\xi_2-}\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right)\right] \\
=&\frac{\xi_2-\xi_1}{4}\int_0^1\left(\nu^{\alpha}+\frac{1}{3}\right)\left[\mathfrak{F}'\left(\left(\frac{1-\nu}{2}\right)\xi_1 + \left(\frac{1+\nu}{2}\right)\xi_2\right)\right. \\
&\left.- \mathfrak{F}'\left(\left(\frac{1+\nu}{2}\right)\xi_1 + \left(\frac{1-\nu}{2}\right)\xi_2\right)\right]d\nu.
\end{aligned}$$

Milne-type inequalities are powerful tools for establishing bounds for symmetric expressions. When paired with strongly convex functions, these inequalities become more versatile and yields sharper estimates. In this work, we present a Milne-type inequality that incorporates the properties of strongly convex functions to extend and strengthen classical results.

3. Milne-type inequalities for differentiable strongly convex functions

In this section, we show several Milne-type inequalities for differentiable strongly convex mappings.

Theorem 2 Suppose that the assumptions of Lemma 1 hold. Let $|\mathfrak{F}'|$ be a strongly convex function on $[\xi_1,\ \xi_2]$ with modulus c. Then we have the following inequality:

$$\left|\frac{1}{3}\left[2\mathfrak{F}(\xi_1)-\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right)+2\mathfrak{F}(\xi_2)\right]\right.$$

$$\left.-\frac{2^{(\alpha-1)}\Gamma(\alpha+1)}{(\xi_2-\xi_1)^{\alpha}}\left[J^{\alpha}_{\xi_1^+}\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right)+J^{\alpha}_{\xi_2^-}\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right)\right]\right|$$

$$\leq\frac{(\xi_2-\xi_1)}{12}\left(\frac{\alpha+4}{\alpha+1}\right)(|\mathfrak{F}'(\xi_1)|+|\mathfrak{F}'(\xi_2)|)-c\frac{(\xi_2-\xi_1)^3}{36}\left(\frac{\alpha^2+4\alpha+12}{\alpha^2+4\alpha+3}\right). \tag{2}$$

Proof. Taking the modulus on Lemma 1 and utilizing the strong convexity of $|\mathfrak{F}'|$ with modulus c, we obtain the following:

$$\left|\frac{1}{3}\left[2\mathfrak{F}(\xi_1)-\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right)+2\mathfrak{F}(\xi_2)\right]\right.$$

$$\left.-\frac{2^{(\alpha-1)}\Gamma(\alpha+1)}{(\xi_2-\xi_1)^{\alpha}}\left[J^{\alpha}_{\xi_1^+}\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right)+J^{\alpha}_{\xi_2^-}\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right)\right]\right|$$

$$\leq\frac{\xi_2-\xi_1}{4}\int_0^1\left|v^{\alpha}+\frac{1}{3}\right|\left[\left|\mathfrak{F}'\left(\left(\frac{1-v}{2}\right)\xi_1+\left(\frac{1+v}{2}\right)\xi_2\right)\right|\right.$$

$$\left.+\left|\mathfrak{F}'\left(\left(\frac{1+v}{2}\right)\xi_1+\left(\frac{1-v}{2}\right)\xi_2\right)\right|\right]dv$$

$$\leq\frac{\xi_2-\xi_1}{4}\int_0^1\left|v^{\alpha}+\frac{1}{3}\right|\left[\left(\frac{1-v}{2}\right)|\mathfrak{F}'(\xi_1)|+\left(\frac{1+v}{2}\right)|\mathfrak{F}'(\xi_2)|\right.$$

$$-c\left(\frac{1+v}{2}\right)\left(\frac{1-v}{2}\right)(\xi_1-\xi_2)^2+\left(\frac{1+v}{2}\right)|\mathfrak{F}'(\xi_1)|$$

$$\left.+\left(\frac{1-v}{2}\right)|\mathfrak{F}'(\xi_2)|-c\left(\frac{1+v}{2}\right)\left(\frac{1-v}{2}\right)(\xi_2-\xi_1)^2\right]dv$$

$$\leq\frac{(\xi_2-\xi_1)}{4}\int_0^1\left(v^{\alpha}+\frac{1}{3}\right)\times\left[|\mathfrak{F}'(\xi_1)|+|\mathfrak{F}'(\xi_2)|-c\frac{(1-v^2)}{2}(\xi_2-\xi_1)^2\right]dv$$

$$\leq\frac{(\xi_2-\xi_1)}{4}(|\mathfrak{F}'(\xi_1)|+|\mathfrak{F}'(\xi_2)|)\int_0^1\left(v^{\alpha}+\frac{1}{3}\right)dv$$

$$
-c\frac{(\xi_2-\xi_1)^3}{8}\int_0^1\left(\nu^\alpha+\frac{1}{3}\right)(1-\nu^2)d\nu
$$

$$
\leq\frac{(\xi_2-\xi_1)}{4}(|\mathfrak{F}'(\xi_1)|+|\mathfrak{F}'(\xi_2)|)\left(\frac{\nu^{(\alpha+1)}}{\alpha+1}+\frac{\nu}{3}\right)_0^1
$$

$$
-c\frac{(\xi_2-\xi_1)^3}{8}\int_0^1\left(\nu^\alpha+\frac{1}{3}-\nu^{(\alpha+2)}-\frac{\nu^2}{3}\right)d\nu
$$

$$
\leq\frac{\xi_2-\xi_1}{4}(|\mathfrak{F}'(\xi_1)|+|\mathfrak{F}'(\xi_2)|)\left(\frac{1}{\alpha+1}+\frac{1}{3}\right)
$$

$$
-c\frac{(\xi_2-\xi_1)^3}{8}\left(\frac{\nu^{(\alpha+1)}}{\alpha+1}+\frac{\nu}{3}-\frac{\nu^{(\alpha+3)}}{\alpha+3}-\frac{\nu^3}{9}\right)_0^1
$$

$$
\leq\left(\frac{\xi_2-\xi_1}{12}\right)\left(\frac{\alpha+4}{\alpha+1}\right)(|\mathfrak{F}'(\xi_1)|+|\mathfrak{F}'(\xi_2)|)
$$

$$
-c\frac{(\xi_2-\xi_1)^3}{8}\left(\frac{1}{\alpha+1}+\frac{1}{3}-\frac{1}{\alpha+3}-\frac{1}{9}\right)
$$

$$
\leq\left(\frac{\xi_2-\xi_1}{12}\right)\left(\frac{\alpha+4}{\alpha+1}\right)(|\mathfrak{F}'(\xi_1)|+|\mathfrak{F}'(\xi_2)|)
$$

$$
-c\frac{(\xi_2-\xi_1)^3}{72}\left(\frac{(9\alpha+27)+3(\alpha^2+4\alpha+3)-9(\alpha+1)-(\alpha^2+4\alpha+3)}{(\alpha+1)(\alpha+3)}\right)
$$

$$
=\left(\frac{\xi_2-\xi_1}{12}\right)\left(\frac{\alpha+4}{\alpha+1}\right)(|\mathfrak{F}'(\xi_1)|+|\mathfrak{F}'(\xi_2)|)-c\frac{(\xi_2-\xi_1)^3}{36}\left(\frac{\alpha^2+4\alpha+12}{\alpha^2+4\alpha+3}\right).
$$

Thus, the proof is completed.

Corollary 1 If $\alpha=1$ is choosen in Theorem 1 reduces the following result:

$$
\left|\frac{1}{3}\left[2\mathfrak{F}(\xi_1)-\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right)+2\mathfrak{F}(\xi_2)\right]-\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\mathfrak{F}(\nu)d\nu\right|
$$

$$
\leq\frac{5(\xi_2-\xi_1)}{24}(|\mathfrak{F}'(\xi_1)|+|\mathfrak{F}'(\xi_2)|)-c\frac{17(\xi_2-\xi_1)^3}{288}.
$$

Remark 1 If we choose $c=0$ in Theorem 1, then it reduces to Theorem 1 of [32].

Notice that, the last term of the right-hand side of the inequality (2), i.e.,

$$c\frac{(\xi_2-\xi_1)^3}{36}\left(\frac{\alpha^2+4\alpha+12}{\alpha^2+4\alpha+3}\right),\ \xi_1<\xi_2 \text{ and } \alpha>0.$$

is always positive for strongly convex functions. Thus, the upper bound for the right-hand side of the Milne-type inequality obtained in inequality (2) is sharper than that of inequality (2.3) of Theorem 1 obtained in [32], which is demonstrated by the following example.

Example 1 Let $[\xi_1,\ \xi_2]=[0,\ 1]$ and define the function $\mathfrak{F}:[0,\ 1]\to\mathbb{R}$ as $\mathfrak{F}(\nu)=\nu^3+\nu^2$ so that $\mathfrak{F}'(\nu)=3\nu^2+2\nu$ and $|\mathfrak{F}'|$ is strongly convex on [0, 1] with modulus c, where $0<c<3$.

Under these assumptions, we have

$$\frac{1}{3}\left[2\mathfrak{F}(\xi_1)-\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right)+2\mathfrak{F}(\xi_2)\right]=\frac{29}{24}.$$

According to the definition of Riemann-Liouville fractional integrals, it follows that.

$$J^{\alpha}_{\xi_1+}\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right)=J^{\alpha}_{0+}\mathfrak{F}\left(\frac{1}{2}\right)=\frac{1}{\Gamma\alpha}\int_0^{\frac{1}{2}}\left(\frac{1}{2}-\nu\right)^{\alpha-1}\left(\nu^3+\nu^2\right)d\nu.$$

To solve the above equation using the integration by-parts method we get

$$J^{\alpha}_{\xi_1+}\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right)=\frac{2\alpha+9}{2^{(\alpha+2)}\Gamma(\alpha+4)}$$

and

$$J^{\alpha}_{\xi_2-}\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right)=J^{\alpha}_{1-}\mathfrak{F}\left(\frac{1}{2}\right)=\frac{1}{\Gamma\alpha}\int_{\frac{1}{2}}^{1}\left(\nu-\frac{1}{2}\right)^{(\alpha-1)}\left(\nu^3+\nu^2\right)d\nu$$

$$=\frac{8\alpha^3+38\alpha^2+46\alpha-9}{2^{(\alpha+2)}\Gamma(\alpha+4)}.$$

Thus we have

$$\frac{(2^{\alpha-1})\Gamma(\alpha+1)}{(\xi_2-\xi_1)^{\alpha}}\left[J^{\alpha}_{\xi_1^+}\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right)+J^{\alpha}_{\xi_2^-}\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right)\right]$$

$$=(2^{\alpha-1})\Gamma(\alpha+1)\left[\frac{2\alpha+9}{2^{(\alpha+2)}\Gamma(\alpha+4)}+\frac{8\alpha^3+38\alpha^2+46\alpha-9}{2^{(\alpha+2)}\Gamma(\alpha+4)}\right]$$

$$=\frac{8\alpha^3+38\alpha^2+48\alpha}{8(\alpha+1)(\alpha+2)(\alpha+3)}$$

$$=\frac{\alpha(4\alpha^2+19\alpha+24)}{4(\alpha+1)(\alpha+2)(\alpha+3)}.$$

As a results, the left hand side term (L) of inequality (2) reduces to

$$\frac{1}{3}\left[2\mathfrak{F}(\xi_1)-\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right)+2\mathfrak{F}(\xi_2)\right]-\frac{(2^{\alpha-1})\Gamma(\alpha+1)}{(\xi_2-\xi_1)^{\alpha}}\left[J^{\alpha}_{\xi_1^+}\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right)+J^{\alpha}_{\xi_2^-}\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right)\right]$$

$$=\frac{29}{24}-\frac{\alpha(4\alpha^2+19\alpha+24)}{4(\alpha+1)(\alpha+2)(\alpha+3)}=:L(\alpha)$$

Similarly, the right-hand side term (R_1) of inequality (2) turn out to be:

$$\left(\frac{\xi_2-\xi_1}{12}\right)\left(\frac{\alpha+4}{\alpha+1}\right)(|\mathfrak{F}'(\xi_1)|+|\mathfrak{F}'(\xi_2)|)-c\frac{(\xi_2-\xi_1)^3}{36}\left(\frac{\alpha^2+4\alpha+12}{\alpha^2+4\alpha+3}\right)$$

$$=\frac{5}{12}\left(\frac{\alpha+4}{\alpha+1}\right)-\frac{c}{36}\left(\frac{\alpha^2+4\alpha+12}{\alpha^2+4\alpha+3}\right)=:R_1(\alpha)$$

Next, we compare our bounds with existing bounds. In continuing with this, the right-hand side term (R_2) of inequality (2.3) of Theorem 1 obtained in [32] turns out to be:

$$\left(\frac{\xi_2-\xi_1}{12}\right)\left(\frac{\alpha+4}{\alpha+1}\right)(|\mathfrak{F}'(\xi_1)|+|\mathfrak{F}'(\xi_2)|)$$

$$=\frac{5}{12}\left(\frac{\alpha+4}{\alpha+1}\right)$$

$$R_2(\alpha)=\frac{5}{12}\left(\frac{\alpha+4}{\alpha+1}\right)$$

The results obtained from Example 1 are represented in Figure 1 for the specified value of $c=1$.

From Figure 1, we can see that $L(\alpha) \leq R_1(\alpha) \leq R_2(\alpha)$, which shows that $R_1(\alpha)$ is sharper error bound than $R_2(\alpha)$. Therefore, we can say that for the strongly convex functions, the result of Theorem 1 is stronger than that of Theorem 1 obtained in [32].

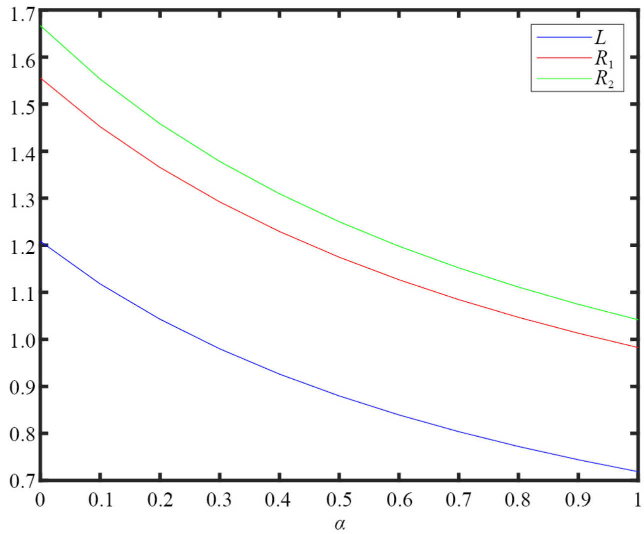

Figure 1. The diagram representing the example 2 that has been evaluated and calculated using the MATLAB program

Now, we compare the upper bounds for the right-hand side of the Milne-type inequality obtained in Theorem 1 by taking $c = 1$ and $\alpha = 1$ and with that of obtained in Corollary 1 of [6].

By taking $c = 1$ and $\alpha = 1$ in $R_1(\alpha)$, the right-hand side term turn out to be

$$\frac{5}{12}\left(\frac{5}{2}\right) - \frac{1}{36}\left(\frac{17}{8}\right) \approx 0.98263$$

and the right-hand side term obtained in Corollary 1 of [6] turn out to be

$$\frac{2}{3}\left[|f(\xi_2) - f(\xi_1)|\right]$$

$$= \frac{2}{3}(2) \approx 1.33333$$

It is clear that $0.98263 < 1.33333$, which shows that for strongly convex functions result of Theorem 1 gives better bounds than that of Corollary 1 in [6].

Theorem 2 Suppose that the assumptions of Lemma 1 holds. Also suppose that the mapping $|\mathfrak{F}'|^q$, $q > 1$, is strongly convex on $[\xi_1, \xi_2]$ with modulus c. Then the following inequality holds:

$$\left|\frac{1}{3}\left[2\mathfrak{F}(\xi_1)-\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right)+2\mathfrak{F}(\xi_2)\right]\right.$$

$$\left.-\frac{(2^{\alpha-1})\Gamma(\alpha+1)}{(\xi_2-\xi_1)^{\alpha}}\left[J^{\alpha}_{\xi_1^+}\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right)+J^{\alpha}_{\xi_2^-}\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right)\right]\right|$$

$$\leq\left(\frac{\xi_2-\xi_1}{4}\right)\left(\int_0^1\left(\nu^{\alpha}+\frac{1}{3}\right)^p d\nu\right)^{\frac{1}{p}}\left[\left(\frac{3|\mathfrak{F}'(\xi_1)|^q+|\mathfrak{F}'(\xi_2)|^q}{4}-\frac{c}{6}(\xi_2-\xi_1)^2\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{|\mathfrak{F}'(\xi_1)|^q+3|\mathfrak{F}'(\xi_2)|^q}{4}-\frac{c}{6}(\xi_2-\xi_1)^2\right)^{\frac{1}{q}}\right].$$

whenever $\frac{1}{p}+\frac{1}{q}=1$.

Proof. If we take the modulus on Lemma 1, we obtain

$$\left|\frac{1}{3}\left[2\mathfrak{F}(\xi_1)-\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right)+2\mathfrak{F}(\xi_2)\right]\right.$$

$$\left.-\frac{2^{(\alpha-1)}\Gamma(\alpha+1)}{(\xi_2-\xi_1)^{\alpha}}\left[J^{\alpha}_{\xi_1^+}\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right)+J^{\alpha}_{\xi_2^-}\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right)\right]\right|$$

$$\leq\frac{\xi_2-\xi_1}{4}\left[\int_0^1\left(\nu^{\alpha}+\frac{1}{3}\right)\left|\mathfrak{F}'\left(\left(\frac{1+\nu}{2}\right)\xi_1+\left(\frac{1-\nu}{2}\right)\xi_2\right)\right|d\nu\right.$$

$$\left.+\int_0^1\left(\nu^{\alpha}+\frac{1}{3}\right)\left|\mathfrak{F}'\left(\left(\frac{1+\nu}{2}\right)\xi_2+\left(\frac{1-\nu}{2}\right)\xi_1\right)\right|\right]d\nu. \tag{3}$$

Using the Hölder inequality in (3) and by utilizing the strong convexity of $|f'|^q$ with modulus c, we have

$$\int_0^1 \left(\nu^\alpha + \frac{1}{3}\right) \left| \mathfrak{F}'\left(\left(\frac{1+\nu}{2}\right)\xi_1 + \left(\frac{1-\nu}{2}\right)\xi_2\right)\right| d\nu$$

$$\leq \left(\int_0^1 \left(\nu^\alpha + \frac{1}{3}\right)^p d\nu\right)^{\frac{1}{p}} \left(\int_0^1 \left| \mathfrak{F}'\left(\left(\frac{1+\nu}{2}\right)\xi_1 + \left(\frac{1-\nu}{2}\right)\xi_2\right)\right|^q d\nu\right)^{\frac{1}{q}}$$

$$\leq \left(\int_0^1 \left(\nu^\alpha + \frac{1}{3}\right)^p d\nu\right)^{\frac{1}{p}} \left[\int_0^1 \left(\left(\frac{1+\nu}{2}\right)|\mathfrak{F}'(\xi_1)|^q + \left(\frac{1-\nu}{2}\right)|\mathfrak{F}'(\xi_2)|^q - c\left(\frac{1+\nu}{2}\right)\left(\frac{1-\nu}{2}\right)(\xi_1-\xi_2)^2\right) d\nu\right]^{\frac{1}{q}}$$

$$\leq \left(\int_0^1 \left(\nu^\alpha + \frac{1}{3}\right)^p d\nu\right)^{\frac{1}{p}} \left[\left(\frac{3|\mathfrak{F}'(\xi_1)|^q + |\mathfrak{F}'(\xi_2)|^q}{4}\right) - \frac{c(\xi_2-\xi_1)^2}{6}\right]^{\frac{1}{q}}. \tag{4}$$

Similarly, the following inequality can be derived:

$$\int_0^1 \left(\nu^\alpha + \frac{1}{3}\right) \left| \mathfrak{F}'\left(\left(\frac{1+\nu}{2}\right)\xi_2 + \left(\frac{1-\nu}{2}\right)\xi_1\right)\right| d\nu$$

$$\leq \left(\int_0^1 \left(\nu^\alpha + \frac{1}{3}\right)^p d\nu\right)^{\frac{1}{p}} \left[\left(\frac{|\mathfrak{F}'(\xi_1)|^q + 3|\mathfrak{F}'(\xi_2)|^q}{4}\right) - \frac{c(\xi_2-\xi_1)^2}{6}\right]^{\frac{1}{q}}. \tag{5}$$

Applying equations (4) and (5) in (3), we get

$$\left|\frac{1}{3}\left[2\mathfrak{F}(\xi_1) - \mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right) + 2\mathfrak{F}(\xi_2)\right]\right.$$

$$\left. - \frac{2^{(\alpha-1)}\Gamma(\alpha+1)}{(\xi_2-\xi_1)^\alpha}\left[J^\alpha_{\xi_1^+}\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right) + J^\alpha_{\xi_2^-}\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right)\right]\right|$$

$$\leq \frac{\xi_2-\xi_1}{4}\left(\int_0^1 \left(\nu^\alpha + \frac{1}{3}\right)^p d\nu\right)^{\frac{1}{p}} \left[\left(\frac{3|\mathfrak{F}'(\xi_1)|^q + |\mathfrak{F}'(\xi_2)|^q}{4} - \frac{c(\xi_2-\xi_1)^2}{6}\right)^{\frac{1}{q}}\right.$$

$$\left. + \left(\frac{|\mathfrak{F}'(\xi_1)|^q + 3|\mathfrak{F}'(\xi_2)|^q}{4} - \frac{c(\xi_2-\xi_1)^2}{6}\right)^{\frac{1}{q}}\right].$$

Hence, the theorem is proved.

Corollary 2 If $\alpha = 1$ we choosen in Theorem 2 reduces the following result:

$$\left|\frac{1}{3}\left[2\mathfrak{F}(\xi_1)-\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right)+2\mathfrak{F}(\xi_2)\right]-\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\mathfrak{F}(\nu)d\nu\right|$$

$$\leq\frac{\xi_2-\xi_1}{12}\left(\frac{4^{p+1}-1}{3(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{3|\mathfrak{F}'(\xi_1)|^q+|\mathfrak{F}'(\xi_2)|^q}{4}-\frac{c(\xi_2-\xi_1)^2}{6}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{|\mathfrak{F}'(\xi_1)|^q+3|\mathfrak{F}'(\xi_2)|^q}{4}-\frac{c(\xi_2-\xi_1)^2}{6}\right)^{\frac{1}{q}}\right].$$

Remark 2 If we choose $c=0$ in Theorem 2, then it reduces to Theorem 2 of [32].

Theorem 3 Assume that all the assumptions of Lemma 1 are met. if the mapping $|\mathfrak{F}'|^q$, $q\geq 1$, is strongly convex on $[\xi_1,\ \xi_2]$ with modulus c, Then we have the following inequality:

$$\left|\frac{1}{3}\left[2\mathfrak{F}(\xi_1)-\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right)+2\mathfrak{F}(\xi_2)\right]\right.$$

$$\left.-\frac{(2^{\alpha-1})\Gamma(\alpha+1)}{(\xi_2-\xi_1)^\alpha}\left[J^\alpha_{\xi_1^+}\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right)+J^\alpha_{\xi_2^-}\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right)\right]\right|$$

$$\leq\frac{\xi_2-\xi_1}{4}\left(\frac{\alpha+4}{3(\alpha+1)}\right)^{1-\frac{1}{q}}\left(\left[\left(\frac{1}{4}+\frac{2\alpha+3}{2(\alpha+1)(\alpha+2)}\right)|\mathfrak{F}'(\xi_1)|^q\right.\right.$$

$$\left.+\left(\frac{1}{12}+\frac{1}{2(\alpha+1)(\alpha+2)}\right)|\mathfrak{F}'(\xi_2)|^q-\frac{c(\xi_1-\xi_2)^2}{18}\frac{(\alpha^2+4\alpha+12)}{(\alpha^2+4\alpha+3)}\right]^{\frac{1}{q}}$$

$$+\left[\left(\frac{1}{4}+\frac{2\alpha+3}{2(\alpha+1)(\alpha+2)}\right)|\mathfrak{F}'(\xi_2)|^q+\left(\frac{1}{12}+\frac{1}{2(\alpha+1)(\alpha+2)}\right)|\mathfrak{F}'(\xi_1)|^q\right.$$

$$\left.\left.-\frac{c(\xi_1-\xi_2)^2}{18}\frac{(\alpha^2+4\alpha+12)}{(\alpha^2+4\alpha+3)}\right]^{\frac{1}{q}}\right).$$

Proof. Using the power-mean inequality in (3) and considering the strong convexity of $|\mathfrak{F}'|^q$ with the modulus c, we obtain the following:

$$\int_0^1 \left(\nu^\alpha + \frac{1}{3}\right) \left|\mathfrak{F}'\left(\left(\frac{1+\nu}{2}\right)\xi_1 + \left(\frac{1-\nu}{2}\right)\xi_2\right)\right| d\nu$$

$$\leq \left(\int_0^1 \left(\nu^\alpha + \frac{1}{3}\right) d\nu\right)^{1-\frac{1}{q}}$$

$$\times \left(\int_0^1 \left(\nu^\alpha + \frac{1}{3}\right) \left|\mathfrak{F}'\left(\left(\frac{1+\nu}{2}\right)\xi_1 + \left(\frac{1-\nu}{2}\right)\xi_2\right)\right|^q d\nu\right)^{\frac{1}{q}}$$

$$\leq \left(\frac{\alpha+4}{3(\alpha+1)}\right)^{1-\frac{1}{q}} \left[\int_0^1 \left(\nu^\alpha + \frac{1}{3}\right) \left(\left(\frac{1+\nu}{2}\right) |\mathfrak{F}'(\xi_1)|^q\right.\right.$$

$$\left.\left. + \left(\frac{1-\nu}{2}\right) |\mathfrak{F}'(\xi_2)|^q - c\left(\frac{1+\nu}{2}\right)\left(\frac{1-\nu}{2}\right)(\xi_1-\xi_2)^2\right) d\nu\right]^{\frac{1}{q}}$$

$$= \left(\frac{\alpha+4}{3(\alpha+1)}\right)^{1-\frac{1}{q}} \left[\left(\frac{1}{4} + \frac{2\alpha+3}{2(\alpha+1)(\alpha+2)}\right) |\mathfrak{F}'(\xi_1)|^q\right.$$

$$\left. + \left(\frac{1}{12} + \frac{1}{2(\alpha+1)(\alpha+2)}\right) |\mathfrak{F}'(\xi_2)|^q - \frac{c(\xi_1-\xi_2)^2}{18} \frac{(\alpha^2+4\alpha+12)}{(\alpha^2+4\alpha+3)}\right]^{\frac{1}{q}}. \tag{6}$$

Similarly as in getting (6), we have

$$\int_0^1 \left(\nu^\alpha + \frac{1}{3}\right) \left|\mathfrak{F}'\left(\left(\frac{1+\nu}{2}\right)\xi_2 + \left(\frac{1-\nu}{2}\right)\xi_1\right)\right| d\nu$$

$$\leq \left(\frac{\alpha+4}{3(\alpha+1)}\right)^{1-\frac{1}{q}} \left[\left(\frac{1}{4} + \frac{2\alpha+3}{2(\alpha+1)(\alpha+2)}\right) |\mathfrak{F}'(\xi_2)|^q\right.$$

$$\left. + \left(\frac{1}{12} + \frac{1}{2(\alpha+1)(\alpha+2)}\right) |\mathfrak{F}'(\xi_1)|^q - \frac{c(\xi_1-\xi_2)^2}{18} \frac{(\alpha^2+4\alpha+12)}{(\alpha^2+4\alpha+3)}\right]^{\frac{1}{q}}. \tag{7}$$

Substituting (6) and (7) into (3), we get

$$\Bigg|\frac{1}{3}\left[2\mathfrak{F}(\xi_1)-\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right)+2\mathfrak{F}(\xi_2)\right]$$

$$-\frac{(2^{\alpha-1})\Gamma\alpha+1}{(\xi_2-\xi_1)^{\alpha}}\left[J^{\alpha}_{\xi_1^+}\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right)+J^{\alpha}_{\xi_2^-}\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right)\right]\Bigg|$$

$$\leq\frac{\xi_2-\xi_1}{4}\left(\frac{\alpha+4}{3(\alpha+1)}\right)^{1-\frac{1}{q}}\left(\left[\left(\frac{1}{4}+\frac{2\alpha+3}{2(\alpha+1)(\alpha+2)}\right)|\mathfrak{F}'(\xi_1)|^q\right.\right.$$

$$\left.+\left(\frac{1}{12}+\frac{1}{2(\alpha+1)(\alpha+2)}\right)|\mathfrak{F}'(\xi_2)|^q-\frac{c(\xi_1-\xi_2)^2}{18}\frac{(\alpha^2+4\alpha+12)}{(\alpha^2+4\alpha+3)}\right]^{\frac{1}{q}}$$

$$+\left[\left(\frac{1}{4}+\frac{2\alpha+3}{2(\alpha+1)(\alpha+2)}\right)|\mathfrak{F}'(\xi_2)|^q+\left(\frac{1}{12}+\frac{1}{2(\alpha+1)(\alpha+2)}\right)|\mathfrak{F}'(\xi_1)|^q\right.$$

$$\left.\left.-\frac{c(\xi_1-\xi_2)^2}{18}\frac{(\alpha^2+4\alpha+12)}{(\alpha^2+4\alpha+3)}\right]^{\frac{1}{q}}\right).$$

Hence, the theorem is proved.

Corollary 3 If $\alpha=1$ we choosen in Theorem 3 reduces the following result:

$$\left|\frac{1}{3}\left[2\mathfrak{F}(\xi_1)-\mathfrak{F}\left(\frac{\xi_1+\xi_2}{2}\right)+2\mathfrak{F}(\xi_2)\right]-\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\mathfrak{F}(\nu)d\nu\right|$$

$$\leq\frac{5(\xi_2-\xi_1)}{24}\left(\left[\frac{(4|\mathfrak{F}'(\xi_1|)^q+|\mathfrak{F}'(\xi_2)|^q)}{5}-\frac{17c(\xi_1-\xi_2)^2}{120}\right]^{\frac{1}{q}}\right.$$

$$\left.+\left[\frac{(4|\mathfrak{F}'(\xi_2|)^q+|\mathfrak{F}'(\xi_1)|^q)}{5}-\frac{17c(\xi_1-\xi_2)^2}{120}\right]^{\frac{1}{q}}\right).$$

Remark 3 If we choose $c=0$ in Theorem 3, then it reduces to Theorem 3 of [32].

4. Conclusion

In this study, we established new Milne-type fractional inequalities tailored for differentiable strongly convex mappings. By weaving together key mathematical tools including Hölder's inequality, the power-mean inequality, and insights from strong convexity, we expanded the scope of existing results in fractional analysis. Notably, our findings not only generalize prior work, such as Budak's contributions [32], but also open doors to fresh applications across theoretical and applied mathematics. The Milne-type inequalities introduced here are more than theoretical advancements. For instance, these inequalities can refine models in visco-elastic material behavior, signal processing, and anomalous diffusion, where traditional calculus often falls short. By bridging fractional methods with Milne-type bounds, we gain precision in both solving complex equations and describing phenomena that defy conventional frameworks [36–38]. The fusion of these techniques promises to deepen theoretical exploration while equipping researchers and engineers with robust strategies for practical problems.

Author's contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

Availability of data and materials

Data sharing is not applicable to this paper as no datasets were generated or analyzed during the current study.

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Conflict of interests

The authers declare that they have no competing interests.

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