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Identification of Delay-Tolerant Networking by Employing MABAC Technique Based on Bipolar Complex Fuzzy Dombi Heronian Mean Operators

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Abstract: This work proposes a hybrid decision making model for dynamic and irregularly connected communication systems called Delay-Tolerant Networks (DTNs). A resilient and adaptable network allows for communication in an environment where traditional networks may fail to operate effectively. The main significance of this system is that it is commonly utilized in such scenarios where traditional networks are impractical, such as remote areas, disaster-stricken regions, space missions, and military operations. The proposed model includes the "Multi-Attributive Border Approximation Area Comparison" (MABAC) method, together with Bipolar Complex Fuzzy Dombi Heronian Mean (BCFDHM) operators. To take the positive as well as negative attributes' evaluations into consideration in complicated fuzzy environments, we use an enriched aggregation structure for the criteria, which incorporates the relationship between criteria through the Heronian mean function. Due to this, the MABAC technique within BCF information is more advanced and better than classical MABAC techniques in various models. After that, with the help of these enriched aggregation structures, we successfully identify and rank alternatives for DTN in an uncertain, imprecise, and bipolar condition. By employing the MABAC technique for the DTN systems, we find the best and better alternative to the DTN systems, which is $\tilde{A}_4$ as mentioned below in section 4. At last, we compare our initiated work with many existing theories to prove the authenticity of the suggested work.

***Keywords*:** DTN, Bipolar complex fuzzy set, BCFDHM aggregation operators, MABAC technique

MSC: 03E72, 68M11

Abbreviation

FS	Fuzzy Set
BFS	Bipolar Fuzzy Set
CFS	Complex Fuzzy Set
BCFS	Bipolar Complex Fuzzy Set
DTT	Dombi T-norm and T-conorm

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DTN	Delay-Tolerant Networking
MABAC	Multi-Attributive Border Approximation Area Comparison
BCFDHM AOs	Bipolar Complex Fuzzy Dombi Heronian Mean Aggregation Operators
BCFDWHM AOs	Bipolar Complex Fuzzy Dombi Weighted Heronian Mean Aggregation Operators

1. Introduction

Networking refers to the process of connecting and establishing relationships with individuals or groups to exchange information, ideas, resources, and support. It involves building and nurturing a network of contacts within a specific field, industry, or community. It plays a huge role in the present society. The potential of networking might be brought forth to realize opportunities, evolution, augmentation of the self, and broadened horizons about knowledge and even visibility. It is a primary aspect of development for the individual and the organization as it allows the user to stretch his/her experience and facilitates access to resources for success in a given area of interest. Various methods of networking are the ones used nowadays. But in this writing, we talk about DTN, as it is more useful and critical compared with other networking systems. It is because DTN is specifically made for providing communication purposes under some difficult and unpredictable situations in which regular networks do not perform reliably, or aren't present at all. Overcoming the obstacles of high latency, intermittent, low-bandwidth connectivity, and many other problems has been dedicated to the DTN system. DTN is a sort of computer networking infrastructure created to offer connectivity to networks in severe conditions where typical networking technologies may not be successful, such as in space, far away, and rural areas, disaster-stricken zones, and military activities. DTN is a networking intended for scenarios where there may be considerable delays. Data transfer takes place in the system through the store-and-forward scheme among the nodes. It means that the data will be stored at one node when the connection becomes available, and then it is transmitted to the next node. This process might continue until the data reaches its desired destination. The system can handle delays and breakages in the networks without losing data. For instance, if the nodes lose connection and cannot transfer data before the connection is restored, the data could be held back in such a case. In other words, this means that even if the network is completely unavailable, data protection may effectively be maintained. The vast majority of DTN protocols implement the Bundle Protocol (BP), which explains how the data will be organized and transmitted among the nodes. BP can be used for breaking up data into fragments delivered separately and will later be reassembled at the destination node. Such systems can be employed in various applications, including remote monitoring, disaster relief, and space exploration. Furthermore, these protocols may include security provisions against loss and manipulation of data. Different DTN systems, including store-and-forward networking, delay-tolerant mesh networking, opportunistic networking, intermittently connected networks, etc., are used for addressing different networking challenges. In these networks, messages would remain stored in nodes within the network until reaching their ultimate destination. The use of DTN is based on scenarios where many nodes are idle or relatively slow-moving. Nodes in the mesh can communicate with each other and self-heal from disruptions caused by temporary interference. Routing protocols, custody transfer, security, and application support are some of the key attributes of a DTN system. Security is an important aspect of DTN as it incorporates several features related to data packet access control from unauthorized access or tampering. These features include encryption, authentication, and access control mechanisms. DTN supports a large number of applications, including messaging, file sharing, and sensor data collection. In short, DTN systems enable communication and data transfer in situations where continuous connectivity cannot be guaranteed. Many researchers used DTN systems for different fields and areas, such as Torgerson et al. [1] established the foundational role of DTN systems in space systems. Perumal et al. [2] conducted a thorough literature study on the DTN framework for enhancing the effectiveness of internet connections in rural areas. Rashidi et al. [3] proposed an analysis of the performance of sparse DTNs. In DTN systems, geographically based energy-efficient restricted routing was implemented by Alone and Mangrulkar to [4].

A mathematical structure that goes beyond the notion of a typical crisp set by permitting degrees of membership as opposed to rigid binary membership. The idea of FS was presented by Zadeh [5] in 1965. In FS, each element of the set is assigned a membership value between 0 and 1, indicating to which set the element belongs to the set. This allows for a more flexible representation of uncertainties and ambiguity. In a crisp set, an element either fully belongs to the set (Membership

Degree (MD = 1)) or does not belong at all (MD = 0). But in FS, MD can range between 0 and 1, representing the MD or possibility of inclusion. For example, consider the example of "Tall People", which represents the height of individuals. Instead of categorizing individuals as tall or not tall, a FS allows assigning a membership value to each person, indicating their degree of tallness. After that, many researchers showed their attention to FS and its applications. So, in this context, fuzzy logic and its applications in medicine were introduced by Phuong and Kreinovich [6]. FS theory and its application were developed by Zimmermann [7]. Another important concept known as the Bipolar FS (BFS) is an extension of the concept of a traditional FS in fuzzy logic. In a BFS, each element of the set is associated with two different degrees of membership, one representing positive membership and the other representing negative membership. This allows for a more nuanced representation of uncertainty and ambiguity in the data. In BFS, the membership values can consist of the $[-1, 1]$, with -1 denoting a completely negative MD, 0 indicating a neutral or indeterminate membership, and 1 indicating a completely positive MD. The numbers between 0 and 1 denote different degrees of positive membership, whereas numbers between -1 and 0 denote different degrees of negative membership. Zhang [8] thus articulated the concept of BFSs and relations in the context of BFS, which serve as a mathematical foundation for decision analysis. Further, an extension of BFSs was suggested by Chen et al. [9] in their study. In the context of Aggregation Operators (AOs), Bipolar Fuzzy (BF) Hamacher AOs in Multi-Attribute Decision Making (MADM) were given by Wei et al. [10]. BF Dombi AOs in MADM were introduced by Jana et al. [11].

Another significant concept is Complex FS (CFS), which is also the generalization of classical FS. The membership degree in CFS can be a complex number rather than just a single scalar integer. By allowing complex numbers as membership degrees, CFSs can represent more complex and sophisticated information than traditional FSs. The complex membership degrees in a CFS can encode both the magnitude and phase information. The magnitude represents the strength of membership, similar to the membership degree in a traditional FS. The phase represents the directional information or the angle associated with the membership. Thus, Ramot et al. [12] developed the concept of CFS in polar form. After that, Tamir et al. [13] proposed the CFSs in the cartesian form, which is a modification of the concept of Ramot et al. [12]. Yazdanbakhsh and Dick [14] established a methodical assessment of CFSs and logic. In the context of AOs, Bi et al. [15] and Bi et al. [16] proposed Complex Fuzzy (CF) arithmetic and CF geometric AOs, respectively. A more generalized and advanced concept is Bipolar CFS (BCFS), which is used to handle uncertainty and contradictory information in a more advanced manner. It combines the concepts of bipolarity and complexity to handle both positive and negative evaluations as well as interdependencies between different elements. Each element is associated with a complex number that consists of a pair of values representing positive and negative membership, respectively. BCFS is more advanced and generalized to handle complexity and ambiguity as compared to the FS, BFS, and CFS. Mahmood and Ur Rehman [17] provided the most innovative and well-supported concept of BCFS. Based on Dombi AOs under Bipolar CF (BCF) data and their application in Multi-Attribute Decision Making (MADM) were encountered by Mahmood and Ur Rehman [18]. Mahmood et al. [19] investigated the BCF Hamacher AOs and their use in MADM. Mahmood et al. [20] utilized the geometric Aczel-Alsina (AA) AOs in the selection of operating systems.

The MADM methodology is used for Decision Making (DM) that assesses and ranks alternative actions based on multiple criteria. It is a systematic approach to integrate various factors at one time for the rational choice by decision-makers. MADM is usually used in business, engineering, project management, strategic planning, etc. Essentially, MADM defines a selection problem whereby one has to select the most suitable choice from among several alternatives. Each alternative is assessed on several attributes or criteria, which embody different facets of the choice. MADM provides a systematic way of dealing with difficult decisions with multiple criteria. It furthers the process that DM relies on and also promotes consistency and transparency. The methodology of MADM and Multi-Criteria DM (MCDM) is employed by most researchers for tackling many issues in reality. For instance, Sahoo et al. [21] established MCDM applications for solving energy management problems. Kumar and Pamucar [22] prepared an exhaustive review covering MCDM methods for addressing different issues in DM. Among several methods, the most prevalent and considered one is the MABAC technique, which is employed for MADM and MAGDM applications. The scope of this technique is to rank numerous alternatives according to various criteria and to evaluate them. It is commonly employed in MCDM to support complex decision problems. A comparative evaluation of alternatives is provided by this approach, which creates the Border Approximation Area (BAA) for each alternative on a criterion space. Many researchers pay a lot of attention to the utilization of MABAC techniques for DM. So, in this context, Pamucar and Cirovic [23] introduced the MABAC technique in their article to solve various real-life problems. For the FS, Verma [24] proposed a Fuzzy MABAC (F-MABAC) method. Zhao et al. [25] introduced the Intuitionistic Fuzzy MABAC (IF-MABAC) strategy founded on the progressive theory of prospect for Multi-Attribute Group DM (MAGDM). Mandal and Seikh [26] invented the interval-valued spherical fuzzy MABAC technique and its application to the plastic waste management process. Jana [27] proposed an extended BF-MABAC approach for MAGDM.

Dombi [28] first proposed the Dombi T-norm and T-conorm (DTT) concepts and put out these operators in FS theory. The strength of these t-norms over the other operators is in their versatility in capturing various kinds of uncertainty and fuzziness. It permits a potential membership between FSs that overlap. Dombi operators may express FSs with unclear or uncertain bounds, unlike other operators like the min and max operators. In summary, DTT provides non-exclusively parameterized control, asymmetry, and continuous transitions, which promote flexibility in managing FSs. They are useful tools in many fuzzy logic applications because of these characteristics, particularly when working with ambiguous or unclear input. FSs, BFSs, CFSs, and BCFSs environments are used to build Dombi AOs by utilizing the DTT. In the context of Dombi AOs, Seikh and Mandal [29] proposed the idea of IF dombi AOs. Seikh and Mandal [30] investigated the interval-valued Fermatean fuzzy dombi AOs. Researchers are now highly aware of the aggregated results due to the relevance of the correlation between criteria values. In light of this, the Heronian Mean (HM) operator is the most effective for this condition, given the correlation between the input data. The concept of the HM operator was first given by Sykora [31] in 2009. By taking into account the interactions between many criteria, closely related to the aggregated input data, and capturing the relationship between a criterion and itself, the HM operator provides a useful tool in MCDM. Because of these characteristics, it is a practical mathematical function for combining and evaluating many criteria in DM situations. So, Wei et al. [32] provided PF HM AOs for the MADM.

1.1 *Motivation of the paper*

Traditional DM frameworks often fall short when modeling complex human opinions involving dual perspectives (positive and negative) simultaneously. In complex DM environments, especially those characterized by high uncertainty, ambiguity, and conflicting criteria, conventional models often struggle to capture the full range of human perception and system behavior. Traditional FSs handle imprecision, and BFSs introduce duality by modeling positive and negative aspects separately. However, they cannot still represent nuanced cognitive information such as cyclic preferences or dynamic feedback. BCFS offers a richer structure by integrating bipolarity, complex values, and fuzzy logic, enabling a more nuanced representation of uncertainty. In practical scenarios, such as emergency response communication systems or satellite-based data transmission, decision-making must account for both favorable (e.g., bandwidth availability) and unfavorable (e.g., packet loss) conditions in a unified model. Further, Space missions often face extreme delays and signal distortion. Choosing optimal transmission paths involves fuzzy and imprecise data, as well as the evaluation of trade-offs between cost, delay, and risk. The BCFS model can handle the dual nature of these criteria more realistically. In oceanic environments, sensors collect marine data but face high latency and harsh transmission conditions. The decision to route or prioritize data streams can benefit from modeling both positive influences (e.g., signal clarity) and negative ones (e.g., noise, energy loss) using the proposed method. This is particularly useful in network environments, such as DTN systems, where decision parameters often exhibit conflicting tendencies (e.g., reliability vs. delay) and fuzzy imprecise data. Previous studies have addressed individual elements like fuzziness or bipolarity, but rarely their integration. This paper extends the literature by employing the BCFS framework in conjunction with the MABAC method to better reflect the intricacies of DTN DM. In addition to this, by comparing to the FS, CFS, and BFS, the BCFS is a more diverse and comprehensive structure. It is more nuanced and reliable in handling ambiguous and complex types of information. The data in the form of BCFS cannot be handled by FS, CFS, and BFS due to their shortcomings. However, BCFS can handle the FS, CFS, and BFS formats. For this reason, BCFS is the FS, CFS, and BFS generalization. DTT provides non-exclusively parameterized control, asymmetry, and continuous transitions, which promote flexibility in managing BCF information. By taking into account the interactions between many criteria, closely related to the aggregated input data, and capturing the relationship between a criterion and itself, the HM operator provides a useful tool in MCDM. Dombi HM AOs in the BCF context are novel and have not been described yet. These operators have more ability to capture the ambiguity and complexity in the data. There are certain limits and drawbacks with the AOs in the FS, BFS, and CFS, and they are unable to handle the data provided in the BCF structure. The data provided in the FS, BFS, and CFS is handled by our suggested AOs in addition to BCF information. Our proposed AOs are now more powerful instruments than the previous AOs because of this improvement and generality. Due to the mixture of Dombi with the HM operator, our concept has grown more comprehensive and can deal with confusing and unclear information, assisting DM in finding the optimal solution. Further, the MABAC technique is also very reliable in the context of the MAGDM approach. The strengths of MABAC within the context of BCFS make it more applicable for capturing the complexity and ambiguities in data. For example, consider a company that wants to assess the performance of custody transfer attributes in DTN systems. For this, they hire a team of experts to assess the performance of the custody transfer attribute in the DTN systems and find the optimal solution. The four key components of these attribute the positive aspects (i.e. energy efficiency) and negative aspects (i.e. potential data loss) and their effects (i.e. effects of energy efficiency on DTN systems) and side effects (i.e. side effects of potential data loss on DTN systems) must take into account by the experts. For this, an expert rates 0.7 to the positive aspects (i.e., energy efficiency), -0.5 to the negative aspects (i.e., potential data loss), 0.4 to the effect of energy efficiency on DTN systems, and -0.6 to the side effects of potential data loss on DTN systems. The experts must use BCFS to cope with such kind of data and determine the optimal solution. For this reason, we employ the BCFS in the DTN systems.

1.2 *Structure of the paper*

Following is the order in which the entire manuscript is organized: Section 2 of this article provided BCFS's primary notion, along with the score and accuracy function. Next, we construct the BCF operation by relying on DTT. Further, we provide the main notion of the HM operator at the end of this section. The innovative AOs known as the BCFDHM and BCF Dombi Weighted HM (BCFDWHM) operators are created in section 3. We also discuss various significant findings and theorems, including idempotency, monotonicity, and boundedness, to demonstrate the importance and superiority of the investigated work. In section 4, we introduced a novel MAGDM technique identified as the MABAC technique under the model of BCF information. To support the MABAC strategy, we also provided a numerical illustration of the DTN system. The comparison of our investigated theory with several prevalent theories is discussed in section 5. In section 6, we devise the implications of this study. At last, we conclude the whole manuscript in section 7.

2. Literature review

We cover the fundamental concepts of the BCF set, score, and accuracy functions, BCF Dombi operations, and the Heronian mean operator in this section.

Mahmood and Ur Rehman [17] provided the most innovative and well-supported concept of BCFS. It is the fusion of BFS and CFS. BFS contains only positive membership belonging and negative membership belonging, whereas CFS contains membership belonging in a two-dimensional way. However, BCFS contains positive and negative membership belonging in a two-dimensional way. That's why BCFS is the generalization of FS, BFS, and CFS, and it is more advanced and generalized to handle complexity and ambiguity in the data.

Definition 1 [17] A BCFS $\ddot{\theta}$ is defined as a triplet:

$$\ddot{\theta} = \left(o, W^{+}(o), W^{-}(o)\,;\ \forall o \in U\right)$$

Where U is the universal set representing the domain of discourse, and

$$W^{+}(o) = \pi^{+}(o) + i\vartheta^{+}(o) \quad \text{and} \quad W^{-}(o) = \pi^{-}(o) + i\vartheta^{-}(o)$$

are the positive and negative membership functions, respectively, with

$$\pi^{+}(o),\ \vartheta^{+}(o) \in [0,\ 1] \quad \text{and} \quad \pi^{-}(o),\ \vartheta^{-}(o) \in [-1,\ 0].$$

The membership functions map each element in U to a degree of positive and negative membership, indicating the level of support or rejection, respectively.

Score and accuracy functions are used for the ranking of different numbers and alternatives. It is difficult to compare two or more BCF Numbers (BCFNs) without a score and accuracy function. So, these function helps to compare and rank two or more numbers. So, score and accuracy functions play a vital role in research by providing objective and quantitative measures of model performance, facilitating model selection, and enabling performance comparison. So, based on this significance and attribute, we revise the definition of sore and accuracy function as follows:

Definition 2 [18] The score function is represented by the symbol δ_{Score} and is defined as follows:

$$\delta_{\text{Score}}(\ddot{\theta}) = \frac{1}{4}\left(2+\pi^{+}(o)+\vartheta^{+}(o)+\pi^{-}(o)+\vartheta^{-}(o)\right),\quad \delta_{\text{Score}} \in [0,\ 1]$$

Definition 3 [18] The accuracy function is represented by the symbol Δ_{Accuracy} and is defined as follows:

$$\Delta_{\text{Accuracy}}(\ddot{\theta}) = \frac{\pi^{+}(o)+\vartheta^{+}(o)-\pi^{-}(o)-\vartheta^{-}(o)}{4},\quad \Delta_{\text{Accuracy}} \in [0,\ 1]$$

Remark 1 Note that when the score function fails to operate, we use the accuracy function for the comparison and ranking of different BCFNs.

Definition 4 [28] Mathematically, the DTT is stated as:

$$\Gamma_{(D,\ \daleth)}(c,\ g) = \frac{1}{1+\left(\left(\frac{1-c}{c}\right)^{\daleth}+\left(\frac{1-g}{g}\right)^{\daleth}\right)^{1/\daleth}}$$

$$\Gamma^{*}_{(D,\ \daleth)}(c,\ g) = \frac{1}{1+\left(\left(\frac{c}{1-c}\right)^{\daleth}+\left(\frac{g}{1-g}\right)^{\daleth}\right)^{1/\daleth}}$$

Where c and g are any two real numbers belonging to $[0,\ 1]$, and $\daleth \geq 1$ controls the variability of the t-norm and t-conorm.

The algebraic operations are vital for aggregating the different values of AOs, which are as follows:

Definition 5 [18] Suppose

$$\ddot{\theta}_1 = \left(\pi_1^{+}+i\vartheta_1^{+},\ \pi_1^{-}+i\vartheta_1^{-}\right) \quad \text{and} \quad \ddot{\theta}_2 = \left(\pi_2^{+}+i\vartheta_2^{+},\ \pi_2^{-}+i\vartheta_2^{-}\right)$$

are two BCFNs, and $\daleth \geq 1$, $w > 0$, then the operation for BCFNs based on DTT is as follows:

(1)

$$\ddot{\theta}_1 \oplus \ddot{\theta}_2 = \left(\begin{array}{l} \left(1 - \dfrac{1}{1 + \left(\left(\dfrac{\pi_1^+}{1-\pi_1^+} \right)^{\daleth} + \left(\dfrac{\pi_2^+}{1-\pi_2^+} \right)^{\daleth} \right)^{1/\daleth}} \right) + i \left(1 - \dfrac{1}{1 + \left(\left(\dfrac{\vartheta_1^+}{1-\vartheta_1^+} \right)^{\daleth} + \left(\dfrac{\vartheta_2^+}{1-\vartheta_2^+} \right)^{\daleth} \right)^{1/\daleth}} \right), \\ \left(- \dfrac{1}{1 + \left(\left(\dfrac{1+\pi_1^-}{|\pi_1^-|} \right)^{\daleth} + \left(\dfrac{1+\pi_2^-}{|\pi_2^-|} \right)^{\daleth} \right)^{1/\daleth}} \right) + i \left(- \dfrac{1}{1 + \left(\left(\dfrac{1+\vartheta_1^-}{|\vartheta_1^-|} \right)^{\daleth} + \left(\dfrac{1+\vartheta_2^-}{|\vartheta_2^-|} \right)^{\daleth} \right)^{1/\daleth}} \right) \end{array} \right).$$

(2)

$$\ddot{\theta}_1 \otimes \ddot{\theta}_2 = \left(\begin{array}{l} \left(\dfrac{1}{1 + \left(\left(\dfrac{1-\pi_1^+}{\pi_1^+} \right)^{\daleth} + \left(\dfrac{1-\pi_2^+}{\pi_2^+} \right)^{\daleth} \right)^{1/\daleth}} + i \dfrac{1}{1 + \left(\left(\dfrac{1-\vartheta_1^+}{\vartheta_1^+} \right)^{\daleth} + \left(\dfrac{1-\vartheta_2^+}{\vartheta_2^+} \right)^{\daleth} \right)^{1/\daleth}} \right), \\ -1 + \dfrac{1}{1 + \left(\left(\dfrac{|\pi_1^-|}{1+\pi_1^-} \right)^{\daleth} + \left(\dfrac{|\pi_2^-|}{1+\pi_2^-} \right)^{\daleth} \right)^{1/\daleth}} + i \left(-1 + \dfrac{1}{1 + \left(\left(\dfrac{|\vartheta_1^-|}{1+\vartheta_1^-} \right)^{\daleth} + \left(\dfrac{|\vartheta_2^-|}{1+\vartheta_2^-} \right)^{\daleth} \right)^{1/\daleth}} \right) \end{array} \right).$$

(3)

$$w\ddot{\theta}_1 = \left(\begin{array}{l} 1 - \dfrac{1}{1 + \left(w \left(\dfrac{\pi_1^+}{1-\pi_1^+} \right)^{\daleth} \right)^{1/\daleth}} + i \left(1 - \dfrac{1}{1 + \left(w \left(\dfrac{\vartheta_1^+}{1-\vartheta_1^+} \right)^{\daleth} \right)^{1/\daleth}} \right), \\ - \dfrac{1}{1 + \left(w \left(\dfrac{1+\pi_1^-}{|\pi_1^-|} \right)^{\daleth} \right)^{1/\daleth}} + i \left(- \dfrac{1}{1 + \left(w \left(\dfrac{1+\vartheta_1^-}{|\vartheta_1^-|} \right)^{\daleth} \right)^{1/\daleth}} \right) \end{array} \right).$$

(4)

$$\ddot{\theta}_1^w = \left(\begin{array}{c} \dfrac{1}{1+\left(w\left(\dfrac{1-\pi_1^+}{\pi_1^+}\right)^{\daleth}\right)^{1/\daleth}} + i\left(1-\dfrac{1}{1+\left(w\left(\dfrac{1-\vartheta_1^+}{\vartheta_1^+}\right)^{\daleth}\right)^{1/\daleth}}\right), \\ -1+\dfrac{1}{1+\left(w\left(\dfrac{|\pi_1^-|}{1+\pi_1^-}\right)^{\daleth}\right)^{1/\daleth}} + i\left(-1+\dfrac{1}{1+\left(w\left(\dfrac{|\vartheta_1^-|}{1+\vartheta_1^-}\right)^{\daleth}\right)^{1/\daleth}}\right) \end{array} \right).$$

Definition 6 [31] Let $(c_1, c_2, \ldots, c_\eta)$ be a collection of real numbers. Then, mathematically, the HM operator is stated as:

$$\mathrm{HM}^{(p,\,q)}(c_1, c_2, \ldots, c_\eta) = \left(\frac{2}{\eta(\eta+1)}\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta} c_\kappa^p c_j^q\right)^{\frac{1}{p+q}}$$

where $p, q \geq 0$.

3. The BCFDHM operators

In this section, we extend the HM to the BCF environment and propose some novel BCF AOs known as BCFDHM and BCFDWHM operators by using DTT. The main reason for selecting and using the Dombi Heronian AOs in comparison with Dombi, Frank, Archimedean, Aczel-Alsina, and Hamacher operators is shown in Table 1, which is as under:

Table 1. Importance of utilizing the Dombi Heronian mean operator in comparison with various existing operators

Operators	Strengths	Weaknesses	Parameterized	Interdependence handling	Complexity	BCFS suitability
Dombi	High flexibility; models smooth transitions	Requires parameter tuning	Yes	No	Moderate	High
Dombi Heronian mean	Models nonlinear and dependent criteria; highly flexible	Slightly higher computation	Yes	Yes	Moderate	Very high
Frank	Smooth behavior from min to max	Complex inverse functions; hard to interpret	Yes	No	Moderate	Medium
Hamacher	Easy to compute; interpretable in basic fuzzy logic	Fixed form limits flexibility	No	No	Low	Medium
Aczel-Alsina	Useful in modeling smooth preference changes	Sensitive to input scale and boundary issues	Yes	No	Moderate	Medium
Archimedean	General form covers many t-norms	Implementation complexity	Yes	No	Moderate to high	Medium

3.1 *The BCFDHM operator*

We discuss the definition of the BCFDHM operator in which HM operators are extended to the BCF environment based on DTT in this section. The definition of the BCFDHM operator is as under:

Definition 7 Let

$$\ddot{\theta}_\kappa = \left(\pi_\kappa^+ + i\vartheta_\kappa^+,\ \pi_\kappa^- + i\vartheta_\kappa^-\right) \quad \text{for} \quad \kappa = 1, 2, \ldots, \eta$$

be a collection of BCFNs. Also, let $w = (w_1, w_2, \ldots, w_\eta)^T$ be the associated Weight Vector (W-V) satisfying $w_\kappa \in [0, 1]$ for all κ, and $\sum_{\kappa=1}^{\eta} w_\kappa = 1$. Then, the BCFDHM operator is denoted and defined as:

$$\mathrm{BCFDHM}^{(p,\,q)}(\ddot{\theta}_1, \ddot{\theta}_2, \ldots, \ddot{\theta}_\eta) = \left(\frac{2}{\eta(\eta+1)} \sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\left(\ddot{\theta}_\kappa^p \otimes_D \ddot{\theta}_j^q\right)\right)^{\frac{1}{p+q}}$$

Theorem 1 Suppose $\ddot{\theta}_\kappa = (\pi_\kappa^+ + i\vartheta_\kappa^+,\ \pi_\kappa^- + i\vartheta_\kappa^-)$ is a set of BCFNs. Then the aggregated value of the BCFDHM operator is still a BCFN and:

$$\mathrm{BCFDHM}^{(p,\,q)}(\ddot{\theta}_1, \ddot{\theta}_2, \ldots, \ddot{\theta}_\eta)$$

$$= \left(\begin{array}{c} \dfrac{1}{1+\left(\dfrac{\eta(\eta+1)}{2(p+q)}\cdot\dfrac{1}{\left(\displaystyle\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\dfrac{1}{p\left(\dfrac{1-\pi_\kappa^+}{\pi_\kappa^+}\right)^{\daleth}+q\left(\dfrac{1-\pi_j^+}{\pi_j^+}\right)^{\daleth}}\right)}\right)^{\frac{1}{\daleth}}} \\ +i\dfrac{1}{1+\left(\dfrac{\eta(\eta+1)}{2(p+q)}\cdot\dfrac{1}{\left(\displaystyle\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\dfrac{1}{p\left(\dfrac{1-\vartheta_\kappa^+}{\vartheta_\kappa^+}\right)^{\daleth}+q\left(\dfrac{1-\vartheta_j^+}{\vartheta_j^+}\right)^{\daleth}}\right)}\right)^{\frac{1}{\daleth}}} \end{array}\right.,$$

$$\left.\begin{array}{l} -1+\dfrac{1}{\left(1+\left(\dfrac{\eta(\eta+1)}{2(p+q)}\cdot\dfrac{1}{\left(\displaystyle\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\dfrac{1}{\left|\left|p\left(\dfrac{|\pi_{\kappa}^{-}|}{1+\pi_{\kappa}^{-}}\right)^{\urcorner}\right|+\left|q\left(\dfrac{|\pi_{j}^{-}|}{1+\pi_{j}^{-}}\right)^{\urcorner}\right|\right|}\right)}\right)\right)^{\frac{1}{\urcorner}}} \\ +i\left(-1+\dfrac{1}{\left(1+\left(\dfrac{\eta(\eta+1)}{2(p+q)}\cdot\dfrac{1}{\left(\displaystyle\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\dfrac{1}{\left|\left|p\left(\dfrac{|\vartheta_{\kappa}^{-}|}{1+\vartheta_{\kappa}^{-}}\right)^{\urcorner}\right|+\left|q\left(\dfrac{|\vartheta_{j}^{-}|}{1+\vartheta_{j}^{-}}\right)^{\urcorner}\right|\right|}\right)}\right)\right)^{\frac{1}{\urcorner}}}\right) \end{array}\right) \tag{1}$$

Where $p,\, q \geq 0$, $\urcorner > 0$, and $\kappa = 1,\, 2,\, \ldots,\, \eta$.

Proof. For the proof, see Appendix A. □

Theorem 2 [Idempotency] Let

$$\ddot{\theta}_{\kappa} = \left(\pi_{\kappa}^{+} + i\vartheta_{\kappa}^{+},\, \pi_{\kappa}^{-} + i\vartheta_{\kappa}^{-}\right)$$

be a set of BCFNs. If $\ddot{\theta}_{\kappa} = \ddot{\theta} = (\pi^{+} + i\vartheta^{+},\, \pi^{-} + i\vartheta^{-})$ for all κ, then

$$\mathrm{BCFDHM}^{(p,\, q)}(\ddot{\theta}_1,\, \ddot{\theta}_2,\, \ldots,\, \ddot{\theta}_{\eta}) = \ddot{\theta}.$$

Proof. This theorem's proof is related to the proof of the previously mentioned Theorem 1. □

Theorem 3 [Monotonicity] Suppose

$$\ddot{\theta}_{\kappa} = \left(\pi_{\kappa}^{+} + i\vartheta_{\kappa}^{+},\, \pi_{\kappa}^{-} + i\vartheta_{\kappa}^{-}\right) \qquad \text{and} \qquad \ddot{\theta}'_{\kappa} = \left(\pi_{\kappa}^{'+} + i\vartheta_{\kappa}^{'+},\, \pi_{\kappa}^{'-} + i\vartheta_{\kappa}^{'-}\right)$$

are two collections of BCFNs. If $\ddot{\theta}_\kappa \leq \ddot{\theta}'_\kappa$ for all κ, i.e.,

$$\pi_\kappa^+ \leq \pi_\kappa^{'+}, \quad \vartheta_\kappa^+ \leq \vartheta_\kappa^{'+}, \quad \pi_\kappa^- \leq \pi_\kappa^{'-}, \quad \vartheta_\kappa^- \leq \vartheta_\kappa^{'-},$$

then

$$\mathrm{BCFDHM}^{(p,\, q)}(\ddot{\theta}_1,\, \ddot{\theta}_2,\, \ldots,\, \ddot{\theta}_\eta) \leq \mathrm{BCFDHM}^{(p,\, q)}(\ddot{\theta}'_1,\, \ddot{\theta}'_2,\, \ldots,\, \ddot{\theta}'_\eta).$$

Proof. This theorem's proof is related to the proof of the previously mentioned Theorem 1. □

Theorem 4 [Boundedness] Let

$$\ddot{\theta}_\kappa = \left(\pi_\kappa^+ + i\vartheta_\kappa^+,\, \pi_\kappa^- + i\vartheta_\kappa^-\right)$$

be a set of BCFNs. If

$$\ddot{\theta}^+ = \max\left(\ddot{\theta}_1,\, \ddot{\theta}_2,\, \ldots,\, \ddot{\theta}_\eta\right) \quad \text{and} \quad \ddot{\theta}^- = \min\left(\ddot{\theta}_1,\, \ddot{\theta}_2,\, \ldots,\, \ddot{\theta}_\eta\right),$$

then

$$\ddot{\theta}^- \leq \mathrm{BCFDHM}^{(p,\, q)}(\ddot{\theta}_1,\, \ddot{\theta}_2,\, \ldots,\, \ddot{\theta}_\eta) \leq \ddot{\theta}^+.$$

Proof. This theorem's proof is related to the proof of the previously mentioned Theorem 1. □

3.2 *The BCFDWHM operator*

We discuss the definition of the BCFDWHM operator in which HM operators are extended to the BCF environment based on DTT in this section. The definition of the BCFDWHM operator is as under:

Definition 8 Let

$$\ddot{\theta}_\kappa = \left(\pi_\kappa^+ + i\vartheta_\kappa^+,\, \pi_\kappa^- + i\vartheta_\kappa^-\right), \quad \text{for} \quad \kappa = 1,\, 2,\, \ldots,\, \eta,$$

be a collection of BCFNs. And, let $w = (w_1,\, w_2,\, \ldots,\, w_\eta)^T$ be the associated W-V satisfying $w_\kappa \in [0,\, 1]$ for all κ, with $\sum_{\kappa=1}^{\eta} w_\kappa = 1$. Then, the BCFDWHM operator is denoted and defined as:

$$\mathrm{BCFDWHM}^{(p,\, q)}(\ddot{\theta}_1,\, \ddot{\theta}_2,\, \ldots,\, \ddot{\theta}_\eta) = \left(\frac{2}{\eta(\eta+1)} \sum_{\kappa=1}^{\eta} \sum_{j=1}^{\eta} \left((w_\kappa \ddot{\theta}_\kappa)^p \otimes_D (w_j \ddot{\theta}_j)^q\right)\right)^{\frac{1}{p+q}}$$

Theorem 5 Suppose $\ddot{\theta}_\kappa = (\pi_\kappa^+ + i\vartheta_\kappa^+,\, \pi_\kappa^- + i\vartheta_\kappa^-)$ for $\kappa = 1,\, 2,\, \ldots,\, \eta$ be a collection of BCFNs. Then the aggregated values of the BCFDWHM operator are still a BCFN, and

$$
\mathrm{BCFDWHM}^{(p,\,q)}(\ddot{\theta}_1,\, \ddot{\theta}_2,\, \ldots,\, \ddot{\theta}_\eta)
$$

$$
= \left(\begin{array}{l} \dfrac{1}{1+\left(\dfrac{\eta(\eta+1)}{2(p+q)} \cdot \dfrac{1}{\left(\sum\limits_{\kappa=1}^{\eta} \sum\limits_{j=1}^{\eta} \dfrac{1}{\left(\dfrac{p}{w_\kappa \left(\dfrac{\pi_\kappa^+}{1-\pi_\kappa^+} \right)^{\daleth}} + \dfrac{q}{w_j \left(\dfrac{\pi_j^+}{1-\pi_j^+} \right)^{\daleth}} \right)} \right)} \right)^{1/\daleth}} \\ +i \dfrac{1}{1+\left(\dfrac{\eta(\eta+1)}{2(p+q)} \cdot \dfrac{1}{\left(\sum\limits_{\kappa=1}^{\eta} \sum\limits_{j=1}^{\eta} \dfrac{1}{\left(\dfrac{p}{w_\kappa \left(\dfrac{\vartheta_\kappa^+}{1-\vartheta_\kappa^+} \right)^{\daleth}} + \dfrac{q}{w_j \left(\dfrac{\vartheta_j^+}{1-\vartheta_j^+} \right)^{\daleth}} \right)} \right)} \right)^{1/\daleth}} \end{array} \right),
$$

$$\left. -1+\frac{1}{\left(1+\left(\frac{\eta(\eta+1)}{2(p+q)}\cdot\frac{1}{\left(\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\frac{1}{\left\|\left|\frac{p}{w_\kappa\left(\frac{1+\pi_\kappa^-}{|\pi_\kappa^-|}\right)^{\daleth}}\right|+\left|\frac{q}{w_j\left(\frac{1+\pi_j^-}{|\pi_j^-|}\right)^{\daleth}}\right|\right\|}\right)}\right)\right)^{1/\daleth}} \right. \\ \left. +i\left(-1+\frac{1}{\left(1+\left(\frac{\eta(\eta+1)}{2(p+q)}\cdot\frac{1}{\left(\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\frac{1}{\left\|\left|\frac{p}{w_\kappa\left(\frac{1+\vartheta_\kappa^-}{|\vartheta_\kappa^-|}\right)^{\daleth}}\right|+\left|\frac{q}{w_j\left(\frac{1+\vartheta_j^-}{|\vartheta_j^-|}\right)^{\daleth}}\right|\right\|}\right)}\right)\right)^{1/\daleth}}\right)\right) \tag{2}$$

Where p, $q \geq 0$, $\daleth > 0$, and $\kappa = 1, 2, \ldots, \eta$.

Proof. For the proof, see Appendix A. □

Theorem 6 [Idempotency] Let

$$\ddot{\theta}_\kappa = \left(\pi_\kappa^+ + i\vartheta_\kappa^+,\ \pi_\kappa^- + i\vartheta_\kappa^-\right)$$

be a set of BCFNs. If $\ddot{\theta}_\kappa = \ddot{\theta} = (\pi^+ + i\vartheta^+, \pi^- + i\vartheta^-)$ for all κ, then

$$\mathrm{BCFDWHM}^{(p,\,q)}(\ddot{\theta}_1, \ddot{\theta}_2, \ldots, \ddot{\theta}_\eta) = \ddot{\theta}.$$

Proof. This theorem's proof is related to the proof of the previously mentioned Theorem 5. □

Theorem 7 [Monotonicity] Suppose

$$\ddot{\theta}_\kappa = \left(\pi_\kappa^+ + i\vartheta_\kappa^+,\ \pi_\kappa^- + i\vartheta_\kappa^-\right) \qquad \text{and} \qquad \ddot{\theta}'_\kappa = \left(\pi_\kappa^{;+} + i\vartheta_\kappa^{;+},\ \pi_\kappa^{;-} + i\vartheta_\kappa^{;-}\right)$$

are two collections of BCFNs. If $\ddot{\theta}_\kappa \le \ddot{\theta}'_\kappa$ for all κ, i.e.,

$$\pi_\kappa^+ \le \pi_\kappa^{;+}, \quad \vartheta_\kappa^+ \le \vartheta_\kappa^{;+}, \quad \pi_\kappa^- \le \pi_\kappa^{;-}, \quad \vartheta_\kappa^- \le \vartheta_\kappa^{;-},$$

then

$$\mathrm{BCFDWHM}^{(p,\,q)}(\ddot{\theta}_1, \ddot{\theta}_2, \ldots, \ddot{\theta}_\eta) \le \mathrm{BCFDWHM}^{(p,\,q)}(\ddot{\theta}'_1, \ddot{\theta}'_2, \ldots, \ddot{\theta}'_\eta).$$

Proof. This theorem's proof is related to the proof of the previously mentioned Theorem 5. □

Theorem 8 [Boundedness] Let

$$\ddot{\theta}_\kappa = \left(\pi_\kappa^+ + i\vartheta_\kappa^+,\ \pi_\kappa^- + i\vartheta_\kappa^-\right)$$

be a set of BCFNs. If

$$\ddot{\theta}^+ = \max\left(\ddot{\theta}_1, \ddot{\theta}_2, \ldots, \ddot{\theta}_\eta\right) \quad \text{and} \quad \ddot{\theta}^- = \min\left(\ddot{\theta}_1, \ddot{\theta}_2, \ldots, \ddot{\theta}_\eta\right),$$

then

$$\ddot{\theta}^- \le \mathrm{BCFDWHM}^{(p,\,q)}(\ddot{\theta}_1, \ddot{\theta}_2, \ldots, \ddot{\theta}_\eta) \le \ddot{\theta}^+.$$

Proof. This theorem's proof is related to the proof of the previously mentioned Theorem 5. □

4. The MABAC method within the BCF framework

Suppose there is a set of η attributes $\{\mathbb{Y}_1, \mathbb{Y}_2, \ldots, \mathbb{Y}_\eta\}$ and m alternatives $\{\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_m\}$ with an associated set of W-Vs $\{h_1, h_2, \ldots, h_\eta\}$, and R experts $\{\mathbb{S}_1, \mathbb{S}_2, \ldots, \mathbb{S}_R\}$ with associated W-Vs $\{\breve{e}_1, \breve{e}_2, \ldots, \breve{e}_R\}$. Then, the BCF evaluation matrix is given by:

$$\mathbf{MT} = \left[\hat{A}^R_{\rho\tau}\right]_{m\times\eta} = \left((\pi^+_\kappa + i\vartheta^+_\kappa)^R, (\pi^-_\kappa + i\vartheta^-_\kappa)^R\right)_{m\times\eta}$$

where $\rho = 1, 2, \ldots, m$, $\tau = 1, 2, \ldots, \eta$, $(\pi^+_\kappa)^R$, $(\vartheta^+_\kappa)^R \in [0, 1]$, and $(\pi^-_\kappa)^R$, $(\vartheta^-_\kappa)^R \in [-1, 0]$.

And, the expression is followed by the BCF MABAC technique as:

Step 1: Examining a BCF matrix $\mathbf{MT} = [\hat{A}^R_{\rho\tau}]_{m\times\eta} = \left((\pi^+_\kappa + i\vartheta^+_\kappa)^R, (\pi^-_\kappa + i\vartheta^-_\kappa)^R\right)_{m\times\eta}$, $\rho = 1, 2, \ldots, m$, $\tau = 1, 2, \ldots, \eta$ is constructed as:

$$\mathbf{MT} = \begin{bmatrix} \left(\pi^{+R}_{11} + i\vartheta^{+R}_{11}, \ \pi^{-R}_{11} + i\vartheta^{-R}_{11}\right) & \left(\pi^{+R}_{12} + i\vartheta^{+R}_{12}, \ \pi^{-R}_{12} + i\vartheta^{-R}_{12}\right) & \cdots & \left(\pi^{+R}_{1\eta} + i\vartheta^{+R}_{1\eta}, \ \pi^{-R}_{1\eta} + i\vartheta^{-R}_{1\eta}\right) \\ \left(\pi^{+R}_{21} + i\vartheta^{+R}_{21}, \ \pi^{-R}_{21} + i\vartheta^{-R}_{21}\right) & \left(\pi^{+R}_{22} + i\vartheta^{+R}_{22}, \ \pi^{-R}_{22} + i\vartheta^{-R}_{22}\right) & \cdots & \left(\pi^{+R}_{2\eta} + i\vartheta^{+R}_{2\eta}, \ \pi^{-R}_{2\eta} + i\vartheta^{-R}_{2\eta}\right) \\ \vdots & \vdots & & \ddots \\ \left(\pi^{+R}_{m1} + i\vartheta^{+R}_{m1}, \ \pi^{-R}_{m1} + i\vartheta^{-R}_{m1}\right) & \left(\pi^{+R}_{m2} + i\vartheta^{+R}_{m2}, \ \pi^{-R}_{m2} + i\vartheta^{-R}_{m2}\right) & \cdots & \left(\pi^{+R}_{m\eta} + i\vartheta^{+R}_{m\eta}, \ \pi^{-R}_{m\eta} + i\vartheta^{-R}_{m\eta}\right) \end{bmatrix}$$

Where $\hat{A}^R_{\rho\tau} = \left((\pi^+_\kappa + i\vartheta^+_\kappa)^R, (\pi^-_\kappa + i\vartheta^-_\kappa)^R\right)$, $\rho = 1, 2, \ldots, m$, $\tau = 1, 2, \ldots, \eta$, represents a value provided by expert S_R based on attribute $\mathbb{Y}_\tau$ under the BCF framework.

Step 2: In Step 2, merge or combine the gathered values of $\hat{A}^R_{\rho\tau}$ into $\hat{A}_{\rho\tau}$ by utilizing BCFDWHM operators as illustrated below:

$$\mathbf{MT} = \begin{bmatrix} \left(\pi^+_{11} + i\vartheta^+_{11}, \ \pi^-_{11} + i\vartheta^-_{11}\right) & \left(\pi^+_{12} + i\vartheta^+_{12}, \ \pi^-_{12} + i\vartheta^-_{12}\right) & \cdots & \left(\pi^+_{1\eta} + i\vartheta^+_{1\eta}, \ \pi^-_{1\eta} + i\vartheta^-_{1\eta}\right) \\ \left(\pi^+_{21} + i\vartheta^+_{21}, \ \pi^-_{21} + i\vartheta^-_{21}\right) & \left(\pi^+_{22} + i\vartheta^+_{22}, \ \pi^-_{22} + i\vartheta^-_{22}\right) & \cdots & \left(\pi^+_{2\eta} + i\vartheta^+_{2\eta}, \ \pi^-_{2\eta} + i\vartheta^-_{2\eta}\right) \\ \vdots & \vdots & & \ddots \\ \left(\pi^+_{m1} + i\vartheta^+_{m1}, \ \pi^-_{m1} + i\vartheta^-_{m1}\right) & \left(\pi^+_{m2} + i\vartheta^+_{m2}, \ \pi^-_{m2} + i\vartheta^-_{m2}\right) & \cdots & \left(\pi^+_{m\eta} + i\vartheta^+_{m\eta}, \ \pi^-_{m\eta} + i\vartheta^-_{m\eta}\right) \end{bmatrix}$$

Step 3: We used the following method to normalize the resultant matrix $\mathbf{MT} = [\hat{A}_{\rho\tau}]_{m\times\eta}$, $\rho = 1, 2, \ldots, m$, $\tau = 1, 2, \ldots, \eta$ by considering the characteristics of each attribute:

For Benefit attributes:

$$\mathbf{MT} = \hat{A}_{\rho\tau} = \left(\pi^+_\kappa + i\vartheta^+_\kappa, \ \pi^-_\kappa + i\vartheta^-_\kappa\right), \quad \rho = 1, 2, \ldots, m, \quad \tau = 1, 2, \ldots, \eta$$

For Cost attributes:

$$\mathbf{MT} = \left(\hat{A}_{\rho\tau}\right)^c = \left(\pi^-_\kappa + i\vartheta^-_\kappa, \ \pi^+_\kappa + i\vartheta^+_\kappa\right), \quad \rho = 1, 2, \ldots, m, \quad \tau = 1, 2, \ldots, \eta$$

Step 4: For the normalized matrix $\mathbf{MT} = (\hat{A}_{\rho\tau})^c = (\pi_\kappa^+ + i\vartheta_\kappa^+,\ \pi_\kappa^- + i\vartheta_\kappa^-)$ and the weight of attributes h_τ, $(\tau = 1, 2, \ldots, \eta)$, the normalized BCF weighting matrix

$$\mathbf{w}\left(\mathbf{MT}_{\rho\tau}\right) = \left((\pi_\kappa^+ + i\vartheta_\kappa^+)^\circ,\ (\pi_k^- + i\vartheta_\kappa^-)^\circ\right)$$

is computed using the following formula:

$$\mathbf{w}\left(\mathbf{MT}_{\rho\tau}\right) = \underset{\tau}{h} \oplus \mathbf{MT}_{\rho\tau}, \quad \rho = 1, 2, \ldots, m, \quad \tau = 1, 2, \ldots, \eta$$

$$\mathbf{w}\left(\mathbf{MT}_{\rho\tau}\right) = \left(\begin{array}{l} 1 - \dfrac{1}{1 + \left(h_\tau \left(\dfrac{\pi_{\rho\tau}^+}{1 - \pi_{\rho\tau}^+}\right)^\daleth\right)^{1/\daleth}} + i \left(1 - \dfrac{1}{1 + \left(h_\tau \left(\dfrac{\vartheta_{\rho\tau}^+}{1 - \vartheta_{\rho\tau}^+}\right)^\daleth\right)^{1/\daleth}}\right), \\ \dfrac{-1}{1 + \left(h_\tau \left(\dfrac{1 + \pi_{\rho\tau}^-}{|\pi_{\rho\tau}^-|}\right)^\daleth\right)^{1/\daleth}} + i \left(\dfrac{-1}{1 + \left(h_\tau \left(\dfrac{1 + \vartheta_{\rho\tau}^-}{|\vartheta_{\rho\tau}^-|}\right)^\daleth\right)^{1/\daleth}}\right) \end{array} \right) \tag{3}$$

Step 5: Step 5 calculates the BAA values of the BAA matrix $T = [t_\tau]_{1\times\eta}$ which is developed by:

$$t_\tau = \left(\sum_{\rho=1}^{m} \mathbf{MT}_{\rho\tau}\right)^{1/m}$$

$$= \left(\begin{array}{l} \dfrac{1}{1 + \left(\sum_{\rho=1}^{m} \dfrac{1}{m} \left(\dfrac{1 - \pi_{\rho\tau}^+}{\pi_{\rho\tau}^+}\right)^\daleth\right)^{1/\daleth}} + i \dfrac{1}{1 + \left(\sum_{\rho=1}^{m} \dfrac{1}{m} \left(\dfrac{1 - \vartheta_{\rho\tau}^+}{\vartheta_{\rho\tau}^+}\right)^\daleth\right)^{1/\daleth}}, \\ -1 + \dfrac{1}{1 + \left(\sum_{\rho=1}^{m} \dfrac{1}{m} \left(\dfrac{|\pi_{\rho\tau}^-|}{1 + \pi_{\rho\tau}^-}\right)^\daleth\right)^{1/\daleth}} + i \left(-1 + \dfrac{1}{1 + \left(\sum_{\rho=1}^{m} \dfrac{1}{m} \left(\dfrac{|\vartheta_{\rho\tau}^-|}{1 + \vartheta_{\rho\tau}^-}\right)^\daleth\right)^{1/\daleth}}\right) \end{array} \right) \tag{4}$$

Step 6: The distance matrix $D = [d_{\rho\tau}]_{m\times\eta}$ is calculated in Step 6 using the following equation:

$$
d_{\rho\tau} = \begin{cases} d(\mathbf{w}(\mathbf{MT}_{\rho\tau}),\ t_\tau), & \text{if } \ \mathbf{w}(\mathbf{MT}_{\rho\tau}) > t_\tau \\ 0, & \text{if } \ \mathbf{w}(\mathbf{MT}_{\rho\tau}) = t_\tau \\ -d(\mathbf{w}(\mathbf{MT}_{\rho\tau}),\ t_\tau), & \text{if } \ \mathbf{w}(\mathbf{MT}_{\rho\tau}) < t_\tau \end{cases}
$$

Where $d(\mathbf{w}(\mathbf{MT}_{\rho\tau}), t_\tau)$ is the mean distance between $\mathbf{w}(\mathbf{MT}_{\rho\tau})$ and t_τ.

Step 7: The values of $d_{\rho\tau}$ must be computed using the formula shown below:

$$
S_\rho = \sum_{\tau=1}^{\eta} d_{\rho\tau}
$$

4.1 *Application*

Many networking systems are used in today's era. Here, we discussed the DTN system, a system that is more valuable and important than other networking systems. This is because DTN is specially designed for communication under unpredictable and challenging conditions where traditional networks may not be reliable or available. It is specially developed to eradicate problems with high latency, intermittent connectivity, and limited available bandwidth. Also, in the areas of remote conditions, regions that have been struck by disasters, space missions, and, in particular, military operations. DTN is an advancement in computing networking infrastructures that provides linking to networks under extreme conditions such as space, remote and disaster-stricken areas, and military operations where normal networking technologies fail. It is a network that works well in such situations in which there may be long delays, sending data between its nodes through the store-and-forward technique. This process goes on until the destination is reached. Initially, data is stored in one node and sent once the connection is available to another node. This system can handle network delays without data loss. For example, say if a node lost connectivity, the data would be stored until the node regained connectivity and could transfer it. In other words, data protection can prevail even when the network suddenly vanishes. Most DTN protocols mostly laid out upon BP, which is the protocol to dictates how data is organized and forwarded among nodes. BP splits the data into separate bundles that reassemble at the destination node. This system may enable a multitude of applications, including remote sensing, disaster relief, and space exploration. These protocols may also implement some security features to protect against data loss or data manipulation. There are several different types of DTN systems, including:

(1) **Store and Forward Networking:**

Often known as store-and-forward systems, this type of DTN system stores messages at the nodes, waiting for an opportunity to forward them to a final destination. Essentially, it is the same as email, where messages are stored in servers until they can be delivered to the recipient.

(2) **Delay-Tolerant Mesh Networking:**

This kind of DTN is applied in environments where many nodes are stationary or move slowly. The nodes in the network form a mesh through which they can communicate with each other, however temporary the disconnection might be.

(3) **Opportunistic Networking:**

Opportunistic networks operate in such a way as to use the brief intervals of connectivity established between nodes to exchange messages. In a mobile network, for example, nodes may only be within range of each other for a short time duration while on the move, and thus messages need to be exchanged rapidly during those opportunistic moments.

(4) **Intermittently Connected Networking:**

Intermittent networks, also termed interruption-connected networks, are disconnected networks that exploit opportunistic connectivity. The nodes in such networks stay connected for shorter durations as compared to the connection duration available in normal networks (for example, when they are within the range of a Wi-Fi network). Information is stored by nodes until it is possible to transfer it to the final destination.

DTN is an important term because it forms the basis for many applications, such as space missions, disaster management, and remote sensing applications. Also, DTN does have several features and attributes, along with different types. Here are some important attributes of DTN:

(1) **Routing Protocols:**

DTN employs routing protocols designed specifically to face the unique challenges that delay networking poses. These protocols allow the data packets to find the best path to reach their destination across the network, considering network topologies, mobility of nodes, and resources available.

(2) **Custody Transfer:**

There is an inbuilt custody transfer mechanism in DTN, which is responsible for transferring the data packets to other nodes and assuring their integrity and protection. This will be particularly relevant in a scenario where the nodes may malfunction or have errors.

(3) **Security:**

The security features of DTN cover a very vast area and ensure protection against unauthorized access or data packet tampering. The scope of these features includes encryption, authentication, and access control mechanisms.

(4) **Application Support:**

DTN supports an endless list of applications such as messaging, file transfer, and sensor data collection. DTN can thus be molded to fit the needs of applications in a wide range of environments and scenarios.

4.2 *Numerical example*

For instance, a network provider wishes to compare the performance of four distinct DTN system types to select the best possible system. These 4 DTN systems are listed below:

- $\hat{A}_1$: Store and Forward Networking.
- $\hat{A}_2$: Delay-Tolerant Mesh Networking.
- $\hat{A}_3$: Opportunistic Networking.
- $\hat{A}_4$: Intermittently Connected Networking.

They are assessed using the fundamental attributes:

- $\mathbb{Y}_1$: Routing Protocols.
- $\mathbb{Y}_2$: Custody Transfer.
- $\mathbb{Y}_3$: Security.
- $\mathbb{Y}_4$: Application Support.

The network company hired a team of three experts $\mathbb{S}_R$ $(R = 1, 2, 3)$, with respective weights of $(0.45, 0.35, 0.20)$ for this evaluation. Additionally, the attribute weights for the four DTN systems are given as: $(0.45, 0.30, 0.15, 0.10)$. The structure of the BCFN contains the evaluated values of the four DTN systems.

Step 1: Each expert's BCF matrices are displayed in Tables 2, 3, and 4.

Table 2. The assessed values that Expert 1 described

	$\mathbb{Y}_1$	$\mathbb{Y}_2$	$\mathbb{Y}_3$	$\mathbb{Y}_4$
$\hat{A}_1$	$\begin{pmatrix} 0.25+i0.49 \\ -0.69-i0.51 \end{pmatrix}$	$\begin{pmatrix} 0.10+i0.61 \\ -0.92-i0.86 \end{pmatrix}$	$\begin{pmatrix} 0.43+i0.49 \\ -0.90-i0.91 \end{pmatrix}$	$\begin{pmatrix} 0.08+i0.86 \\ -0.76-i0.63 \end{pmatrix}$
$\hat{A}_2$	$\begin{pmatrix} 0.27+i0.59 \\ -0.92-i0.65 \end{pmatrix}$	$\begin{pmatrix} 0.39+i0.59 \\ -0.71-i0.09 \end{pmatrix}$	$\begin{pmatrix} 0.15+i0.61 \\ -0.75-i0.79 \end{pmatrix}$	$\begin{pmatrix} 0.12+i0.24 \\ -0.98-i0.68 \end{pmatrix}$
$\hat{A}_3$	$\begin{pmatrix} 0.34+i0.53 \\ -0.97-i0.78 \end{pmatrix}$	$\begin{pmatrix} 0.24+i0.52 \\ -0.64-i0.87 \end{pmatrix}$	$\begin{pmatrix} 0.29+i0.16 \\ -0.85-i0.39 \end{pmatrix}$	$\begin{pmatrix} 0.44+i0.56 \\ -0.19-i0.79 \end{pmatrix}$
$\hat{A}_4$	$\begin{pmatrix} 0.23+i0.57 \\ -0.78-i0.89 \end{pmatrix}$	$\begin{pmatrix} 0.19+i0.67 \\ -0.13-i0.97 \end{pmatrix}$	$\begin{pmatrix} 0.76+i0.81 \\ -0.93-i0.42 \end{pmatrix}$	$\begin{pmatrix} 0.81+i0.23 \\ -0.35-i0.47 \end{pmatrix}$

Table 3. The assessed values that Expert 2 described

	$\mathbb{Y}_1$	$\mathbb{Y}_2$	$\mathbb{Y}_3$	$\mathbb{Y}_4$
$\hat{A}_1$	$\begin{pmatrix} 0.60+i0.32 \\ -0.78-i0.09 \end{pmatrix}$	$\begin{pmatrix} 0.45+i0.51 \\ -0.87-i0.61 \end{pmatrix}$	$\begin{pmatrix} 0.42+i0.67 \\ -0.12-i0.88 \end{pmatrix}$	$\begin{pmatrix} 0.67+i0.09 \\ -0.54-i0.66 \end{pmatrix}$
$\hat{A}_2$	$\begin{pmatrix} 0.16+i0.45 \\ -0.77-i0.90 \end{pmatrix}$	$\begin{pmatrix} 0.73+i0.39 \\ -0.11-i0.71 \end{pmatrix}$	$\begin{pmatrix} 0.37+i0.59 \\ -0.75-i0.90 \end{pmatrix}$	$\begin{pmatrix} 0.61+i0.59 \\ -0.27-i0.72 \end{pmatrix}$
$\hat{A}_3$	$\begin{pmatrix} 0.22+i0.47 \\ -0.55-i0.31 \end{pmatrix}$	$\begin{pmatrix} 0.18+i0.20 \\ -0.29-i0.11 \end{pmatrix}$	$\begin{pmatrix} 0.54+i0.19 \\ -0.83-i0.57 \end{pmatrix}$	$\begin{pmatrix} 0.33+i0.66 \\ -0.15-i0.85 \end{pmatrix}$
$\hat{A}_4$	$\begin{pmatrix} 0.29+i0.77 \\ -0.68-i0.99 \end{pmatrix}$	$\begin{pmatrix} 0.81+i0.89 \\ -0.18-i0.93 \end{pmatrix}$	$\begin{pmatrix} 0.88+i0.26 \\ -0.77-i0.44 \end{pmatrix}$	$\begin{pmatrix} 0.83+i0.31 \\ -0.90-i0.47 \end{pmatrix}$

Table 4. The assessed values that Expert 3 described

	$\mathbb{Y}_1$	$\mathbb{Y}_2$	$\mathbb{Y}_3$	$\mathbb{Y}_4$
$\hat{A}_1$	$\begin{pmatrix} 0.81+i0.37 \\ -0.69-i0.71 \end{pmatrix}$	$\begin{pmatrix} 0.54+i0.08 \\ -0.65-i0.97 \end{pmatrix}$	$\begin{pmatrix} 0.77+i0.20 \\ -0.22-i0.10 \end{pmatrix}$	$\begin{pmatrix} 0.55+i0.43 \\ -0.21-i0.17 \end{pmatrix}$
$\hat{A}_2$	$\begin{pmatrix} 0.36+i0.83 \\ -0.77-i0.90 \end{pmatrix}$	$\begin{pmatrix} 0.60+i0.50 \\ -0.91-i0.89 \end{pmatrix}$	$\begin{pmatrix} 0.67+i0.90 \\ -0.78-i0.25 \end{pmatrix}$	$\begin{pmatrix} 0.26+i0.65 \\ -0.87-i0.22 \end{pmatrix}$
$\hat{A}_3$	$\begin{pmatrix} 0.11+i0.19 \\ -0.58-i0.71 \end{pmatrix}$	$\begin{pmatrix} 0.62+i0.83 \\ -0.10-i0.94 \end{pmatrix}$	$\begin{pmatrix} 0.22+i0.23 \\ -0.85-i0.27 \end{pmatrix}$	$\begin{pmatrix} 0.27+i0.45 \\ -0.77-i0.54 \end{pmatrix}$
$\hat{A}_4$	$\begin{pmatrix} 0.66+i0.46 \\ -0.49-i0.39 \end{pmatrix}$	$\begin{pmatrix} 0.33+i0.41 \\ -0.71-i0.59 \end{pmatrix}$	$\begin{pmatrix} 0.70+i0.89 \\ -0.89-i0.26 \end{pmatrix}$	$\begin{pmatrix} 0.23+i0.79 \\ -0.79-i0.51 \end{pmatrix}$

Step 2: We use the BCFDWHM operator to aggregate the values of $\mathbf{MT}_1$, $\mathbf{MT}_2$, and $\mathbf{MT}_3$ into $\mathbf{MT}$, so we get $\hat{A}^R_{\rho\tau}$ aggregated to $\hat{A}_{\rho\tau}$. We may view the aggregate matrix in Table 5.

Step 3: Given that the information in matrix $\mathbf{MT} = [\hat{A}_{\rho\tau}]_{m\times\eta}$ is of a beneficial type. So, Table 5 matrix would be the same as the normalized matrix.

Table 5. Aggregated matrix

	$\mathbb{Y}_1$	$\mathbb{Y}_2$	$\mathbb{Y}_3$	$\mathbb{Y}_4$
$\hat{A}_1$	$\begin{pmatrix} 0.88+i0.43 \\ -0.53-i0.66 \end{pmatrix}$	$\begin{pmatrix} 0.98+i0.44 \\ -0.31-i0.16 \end{pmatrix}$	$\begin{pmatrix} 0.68+i0.50 \\ -0.40-i0.32 \end{pmatrix}$	$\begin{pmatrix} 0.99+i0.70 \\ -0.61-i0.66 \end{pmatrix}$
$\hat{A}_2$	$\begin{pmatrix} 0.84+i0.44 \\ -0.32-i0.32 \end{pmatrix}$	$\begin{pmatrix} 0.80+i0.43 \\ -0.37-i0.41 \end{pmatrix}$	$\begin{pmatrix} 0.96+i0.47 \\ -0.50-i0.38 \end{pmatrix}$	$\begin{pmatrix} 0.97+i0.66 \\ -0.11-i0.61 \end{pmatrix}$
$\hat{A}_3$	$\begin{pmatrix} 0.75+i0.43 \\ -0.16-i0.56 \end{pmatrix}$	$\begin{pmatrix} 0.88+i0.42 \\ -0.73-i0.26 \end{pmatrix}$	$\begin{pmatrix} 0.83+i0.75 \\ -0.38-i0.75 \end{pmatrix}$	$\begin{pmatrix} 0.63+i0.49 \\ -0.62-i0.45 \end{pmatrix}$
$\hat{A}_4$	$\begin{pmatrix} 0.89+i0.57 \\ -0.56-i0.06 \end{pmatrix}$	$\begin{pmatrix} 0.94+i0.74 \\ -0.68-i0.15 \end{pmatrix}$	$\begin{pmatrix} 0.94+i0.63 \\ -0.28-i0.78 \end{pmatrix}$	$\begin{pmatrix} 0.92+i0.66 \\ -0.38-i0.73 \end{pmatrix}$

Step 4: The normalized BCF weighting matrix $\mathbf{w}\left(\mathbf{MT}_{\rho\tau}\right) = \left((\pi^+_\kappa + i\vartheta^+_\kappa)^\circ, (\pi^-_\kappa + i\vartheta^-_\kappa)^\circ\right)$ and the attribute's weight h_τ for $\tau = 1, 2, \ldots, \eta$ is calculated using Equation (3), which is shown below in Table 6.

Table 6. Standardized weighted matrix

	$\mathbb{Y}_1$	$\mathbb{Y}_2$	$\mathbb{Y}_3$	$\mathbb{Y}_4$
$\hat{A}_1$	$\begin{pmatrix} 0.83+i0.34 \\ -0.63-i0.74 \end{pmatrix}$	$\begin{pmatrix} 0.99+i0.08 \\ -0.45-i0.26 \end{pmatrix}$	$\begin{pmatrix} 0.25+i0.07 \\ -0.63-i0.55 \end{pmatrix}$	$\begin{pmatrix} 0.99+i0.21 \\ -0.83-i0.86 \end{pmatrix}$
$\hat{A}_2$	$\begin{pmatrix} 0.78+i0.35 \\ -0.41-i0.41 \end{pmatrix}$	$\begin{pmatrix} 0.71+i0.08 \\ -0.52-i0.56 \end{pmatrix}$	$\begin{pmatrix} 0.98+i0.06 \\ -0.72-i0.61 \end{pmatrix}$	$\begin{pmatrix} 0.98+i0.16 \\ -0.28-i0.83 \end{pmatrix}$
$\hat{A}_3$	$\begin{pmatrix} 0.67+i0.34 \\ -0.22-i0.65 \end{pmatrix}$	$\begin{pmatrix} 0.89+i0.07 \\ -0.83-i0.39 \end{pmatrix}$	$\begin{pmatrix} 0.64+i0.40 \\ -0.61-i0.89 \end{pmatrix}$	$\begin{pmatrix} 0.13+i0.04 \\ -0.84-i0.72 \end{pmatrix}$
$\hat{A}_4$	$\begin{pmatrix} 0.84+i0.47 \\ -0.65-i0.09 \end{pmatrix}$	$\begin{pmatrix} 0.97+i0.55 \\ -0.80-i0.24 \end{pmatrix}$	$\begin{pmatrix} 0.95+i0.18 \\ -0.50-i0.90 \end{pmatrix}$	$\begin{pmatrix} 0.87+i0.16 \\ -0.66-i0.90 \end{pmatrix}$

Step 5: We assess the BAA matrix $T = [t_\tau]_{1\times\eta}$ by using Equation (4), and the obtained BAA matrix $T = [t_\tau]_{1\times\eta}$ is shown as follows:

$$t_1 = (0.95+i0.39,\ -0.46-i0.60), \quad t_2 = (0.98+i0.02,\ -0.84-i0.22),$$

$$t_3 = (0.46+i0.02,\ -0.62-i0.95), \quad t_4 = (0.15+i0.01,\ -0.87-i0.95)$$

Step 6: Table 7 shows the result of our evaluation of the distance $D = [d_{\rho\tau}]_{m\times\eta}$ between each alternative and the BAA.

Table 7. Distance between alternative and BAA

	$\mathbb{Y}_1$	$\mathbb{Y}_2$	$\mathbb{Y}_3$	$\mathbb{Y}_4$
$\hat{A}_1$	(-0.12)	(0.13)	(0.17)	(0.29)
$\hat{A}_2$	(0.11)	(-0.25)	(0.25)	(0.42)
$\hat{A}_3$	(-0.20)	(-0.08)	(0.16)	(0.08)
$\hat{A}_4$	(0.22)	(0.15)	(0.21)	(0.28)

Step 7: In this step, we have:

$$S_1 = (-0.12) + (0.13) + (0.17) + (0.29) = 0.47$$

$$S_2 = 0.53$$

$$S_3 = -0.04$$

$$S_4 = 0.86$$

To assist us in making the optimal choice, the aforementioned examination yielded the following order list:

$$\hat{A}_4 \succ \hat{A}_2 \succ \hat{A}_1 \succ \hat{A}_3$$

It is because $\hat{A}_4$ has a value greater than any other option. So, it is considered as the best and most superior alternative. Following this, we draw Figure 1 to understand more clearly which one is the best alternative.

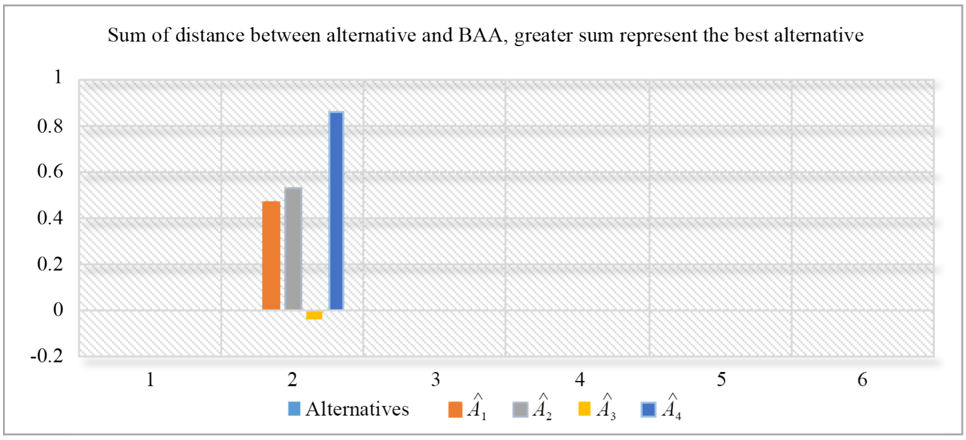

Figure 1. The sum of the distance between alternative and BAA

Note that if we change of values of BCFNs in Table 5 and weights as (0.25, 0.20, 0.35, 0.20), then we can see that it effects on overall ranking results, which is shown as follows Table 8.

Table 8. After changing the BCFNs

Alternatives/Attributes	$\mathbb{Y}_1$	$\mathbb{Y}_2$	$\mathbb{Y}_3$	$\mathbb{Y}_4$
$\hat{A}_1$	$\begin{pmatrix} 0.56+i0.21 \\ -0.62-i0.41 \end{pmatrix}$	$\begin{pmatrix} 0.31+i0.56 \\ -0.90-i0.52 \end{pmatrix}$	$\begin{pmatrix} 0.41+i0.58 \\ -0.65-i0.72 \end{pmatrix}$	$\begin{pmatrix} 0.91+i0.10 \\ -0.11-i0.19 \end{pmatrix}$
$\hat{A}_2$	$\begin{pmatrix} 0.01+i0.91 \\ -0.51-i0.61 \end{pmatrix}$	$\begin{pmatrix} 0.71+i0.63 \\ -0.93-i0.86 \end{pmatrix}$	$\begin{pmatrix} 0.71+i0.67 \\ -0.94-i0.87 \end{pmatrix}$	$\begin{pmatrix} 0.71+i0.87 \\ -0.72-i0.61 \end{pmatrix}$
$\hat{A}_3$	$\begin{pmatrix} 0.91+i0.98 \\ -0.74-i0.37 \end{pmatrix}$	$\begin{pmatrix} 0.86+i0.29 \\ -0.67-i0.59 \end{pmatrix}$	$\begin{pmatrix} 0.38+i0.71 \\ -0.54-i0.49 \end{pmatrix}$	$\begin{pmatrix} 0.51+i0.45 \\ -0.79-i0.19 \end{pmatrix}$
$\hat{A}_4$	$\begin{pmatrix} 0.42+i0.53 \\ -0.51-i0.72 \end{pmatrix}$	$\begin{pmatrix} 0.12+i0.23 \\ -0.79-i0.93 \end{pmatrix}$	$\begin{pmatrix} 0.65+i0.56 \\ -0.82-i0.91 \end{pmatrix}$	$\begin{pmatrix} 0.61+i0.87 \\ -0.10-i0.39 \end{pmatrix}$

After performing all the steps of the MABAC technique, we obtain the sum of each alternative as:

$$S_1 = 0.93, \quad S_2 = -0.39, \quad S_3 = -0.33, \quad S_4 = -0.76$$

After ranking, the aforementioned examination yielded the following order list:

$$\hat{A}_1 \succ \hat{A}_3 \succ \hat{A}_2 \succ \hat{A}_4$$

Since $\hat{A}_1$ has a value greater than any other option, it is considered as the best and superior alternative.

From the above result, we observe that if we change the BCFNs

$$\ddot{\theta}_1 = \left(\pi_1^+ + i\vartheta_1^+,\ \pi_1^- + i\vartheta_1^-\right)$$

and the weights to (0.25, 0.20, 0.35, 0.20), it affects the overall ranking result.

In the first case, $\hat{A}_4$ is the best alternative, and after changing the values of BCFNs and weights, we find that $\hat{A}_1$ becomes the best alternative. This implies that changes in the weights and BCFNs values significantly influence the ranking results.

5. Comparative analysis

As a way of demonstrating the value and advantages of our suggested MABAC approach, we are going to compare it to a few other prevalent methods in this section.

In a comparison analysis, we compare our established approach in the environment of BCFS with the existing MABAC techniques in various frameworks. In the above application, we utilized the MABAC method for MAGDM in the environment of BCFS. So, here are two significant points of the BCF MABAC method. One is the MABAC method, which is more effective in group DM and finding the finest alternative. The other is the BCF structure, which is more generalized, novel, advanced, and effective in finding ambiguity and complexities.

First, we will compare our proposed method with the F-MABAC technique. In the context of the MABAC method, Verma [24] established a F-MABAC method, which consists of fuzzy information. His investigated work can handle the fuzziness of the data. However, the F-MABAC method can only handle a single type of fuzzy information, which consists of possible values between 0 and 1. It neither handles two-dimensional information like a CFS nor handles positive and negative information like a BFS. But, on the other hand, our proposed BCF MABAC method can handle both types of information and give the ranking result for the selection of the best alternative as shown in the above example. Our proposed method not only handles fuzzy information but also handles two-dimensional data with positive and negative aspects of the information. Due to these attributes, the BCF MABAC approach is more generalized and advanced as compared to the F-MABAC approach.

Secondly, we now compare our BCF MABAC technique with the BF-MABAC technique. Jana [27] proposed the BF-MABAC method for MAGDM. The proposed method of Jana [27] is more generalized than Verma's [24] proposed method. The BF-MABAC method can handle positive and negative aspects of data which is not encountered by Verma's [24] proposed method. In this way, Jana's [27] proposed method is more generalized than Verma's [24] method. However, the Jana [27] approach is not successful when the information being gathered is in the shape of CFS. This is because the BF-MABAC method cannot handle the data with two-dimensional information, such as in CFS. However, our proposed approach not only handles the positive and negative aspects of the data but also can tackle the two-dimensional data. The proposed work can rank the different alternatives and choose the best one, as we do in the above example. So, our proposed method is more generalized and novel as compared to Jana's [27] proposed method.

Compared to previous approaches like the F-MABAC and the BF-MABAC methods, our suggested MABAC method is unique and more comprehensive. The main reason is that the problem in the environment of F-MABAC and BF-MABAC can also be solved by our proposed BCF MABAC method, which gives better ranking results for the selection of the best one. However, the problem in the environment of BCF MABAC cannot be solved by F-MABAC and BF-MABAC methods and does not give any ranking result for choosing the best alternative. The following Table 9 shows that the proposed method of Verma [24] and Jana [27] failed to work in the environment of BCFSs.

Table 9. Comparison of proposed method with existing methods

Sources	Method	Sum of the distance between alternative and BAA	Ranking
R-Verma [24]	F-MABAC	xxxxx	xxxxx
C-Jana [27]	BF-MABAC	xxxxx	xxxxx
Invented work	BCF-MABAC	$S_1 = 0.47$ $S_2 = 0.53$ $S_3 = -0.04$ $S_4 = 0.86$	$\hat{A}_4 \succ \hat{A}_2 \succ \hat{A}_1 \succ \hat{A}_3$

Now, we compare our proposed AOs, which are BCFDHM and BCFDWHM AOs, with existing AOs such as the AOs proposed by Wei et al. [10], Jana et al. [11], Bi et al. [15], and Bi et al. [16]. The following Table 10 shows the comparison of the intended work with several prevalent theories.

Table 10. Comparative study of proposed AOs with the prevalent AOs

Methods/Operators	$\delta_{\text{Score}}(\hat{A}_1)$	$\delta_{\text{Score}}(\hat{A}_2)$	$\delta_{\text{Score}}(\hat{A}_3)$	$\delta_{\text{Score}}(\hat{A}_4)$	Ranking
Wei et al. [10]	xxxxx	xxxxx	xxxxx	xxxxx	xxxxx
Jana et al. [11]	xxxxx	xxxxx	xxxxx	xxxxx	xxxxx
Bi et al. [15]	xxxxx	xxxxx	xxxxx	xxxxx	xxxxx
Bi et al. [16]	xxxxx	xxxxx	xxxxx	xxxxx	xxxxx
BCFDHM AO	0.813	0.812	0.796	0.862	$\hat{A}_4 \succ \hat{A}_1 \succ \hat{A}_2 \succ \hat{A}_3$
BCFDWHM AO	0.295	0.283	0.275	0.341	$\hat{A}_4 \succ \hat{A}_1 \succ \hat{A}_2 \succ \hat{A}_3$

Table 10 shows the comparative analysis of proposed AOs with the existing AOs developed by different researchers at their time. But the one thing demonstrated by the above Table 10 is that all the previous theories have some drawbacks and flaws, and can't handle the data in the BCF structure. No doubt these theories are valid and have some advantages. However, our proposed AOs in the environment of the BCF structure are more nuanced and reliable as compared to the AOs developed in the environments of FS, BFS, and CFS structures. Also, Dombi HM operators are more generalized than simple Dombi and Heronian mean operators. So, due to these facts, our described AOs are more generalized. The point-wise and detailed discussion of the above comparison of Table 10 is as under, which demonstrates the advancement, generalization compatibility, and reliability of our proposed AOs.

(1) We begin by drawing a comparison with Wei et al. [10] AOs. In the context of the BF structure, Wei et al. [10] presented BF Hamacher AOs. Hamacher t-norm and t-conorm are superior to the other operators of well-capturing abilities of different types of fuzziness and uncertainty. However, the Hamacher AOs cannot capture the relative importance and diversity of the input values. But our proposed AOs not only capture various kinds of uncertainty and fuzziness, but also capture the relative importance of the input values. Also, Wei et al. [10] proposed that AOs in the BF environment can only tackle the positive and negative aspects of a certain thing in one dimension. Nevertheless, the information provided in Table 5 is two-dimensional. Therefore, it is invalid for the data in Table 5 and is unable to manage the information in a two-dimensional manner. However, our suggested AOs address the data in two dimensions in addition to managing the positive and negative elements. Our proposed AOs in the environment of BCFS can also be valid for the data given in the BF environment by taking an imaginary part equal to zero. So, due to these facts, our proposed AOs are more generalized and advanced as compared to the AOs established by Wei et al. [10].

(2) Secondly, BF Dombi AOs have been suggested by Jana et al. [11] in the context of the BF environment. Dombi DTT provides non-exclusively parameterized control, asymmetry, and continuous transitions, which promote flexibility in managing FSs. They are useful tools in many fuzzy logic applications because of these characteristics, particularly when working with ambiguous or unclear input. However, DTT cannot find the interrelationship between the input data. Our proposed AOs consist of DTT with an HM operator. So, it gains both the features of DTT and HM operators. Also, Jana et al. [11] proposed Dombi AOs in the environment of BF environment. It can handle the evaluation of people in positive and negative senses in one dimension. However, the data given in Table 5 is two-dimensional, so the proposed work of Jana et al. [11] cannot handle the data given in Table 5. However, our proposed AOs can handle the data given in Table 5 and are also applicable in the BF environment. So, due to these facts, our proposed AOs in the environment of BCFS are more generalized, reliable, and advanced as compared to the AOs established by Jana et al. [11].

(3) Thirdly, we compare our proposed work with the established work of Bi et al. [15], and Bi et al. [16]. Bi et al. [15], and Bi et al. [16] proposed CF arithmetic and CF geometric AOs, respectively, in the environment of CFS. Simple arithmetic and geometric AOs can capture the fuzziness and ambiguity in the data. These AOs are more basic and easier to understand as compared to the other AOs. However, these AOs cannot provide the non-exclusively parameterized control, asymmetry, and continuous transitions in managing the Fuzzy data. Also, they cannot find the interrelationship among the input data. However, our proposed AOs are more advanced and authentic in capturing the fuzziness and uncertainty in the data. Also, Bi et al. [15], and Bi et al. [16] proposed arithmetic and geometric AOs, respectively, in the environment

of CF structure that can handle the two-dimensional data. It can handle the information given in two dimensions. But Table 5 data includes both positive and negative belonging. Therefore, the facts shown in Table 5 cannot be addressed by CF AOs. However, our suggested AOs address the data in two dimensions in addition to managing the positive and negative elements. So, due to these facts, our suggested AOs are more generalized, reliable, and advanced as compared to the AOs established by Bi et al. [15], and Bi et al. [16].

(4) As all aforementioned AOs have some weaknesses and drawbacks. They are consequently unable to rank the alternatives and are unable to identify the optimal option. However, the two-dimensional data can be handled by our suggested AOs in both positive and negative aspects. It can determine the best solution by ranking the values of several choices. The information provided in Table 5 is aggregated by using the BCFDHM and BCFDWHM operators. Next, we ranked the values of the various alternatives using the score and accuracy function. After ranking, we observe that the value of alternative $\hat{A}_4$ is higher in both situations than the values of all other alternatives. For this reason, when compared to the existing AOs, our suggested AOs are more dominant, powerful, and exceptional. So, we conclude with this point that our proposed method is more generalized, up-to-date, and novel as compared to the existing work in the FS, BFS, and CFS.

6. Implications

There are numerous positive implications of BCFS in real-life situations. As the BCF structure contains positive and negative aspects along with two dimensions means extra fuzzy information. Due to the fusion of CFS and BFS, this structure captures the ambiguity, uncertainty, and complexity in the data. The operators described in this study can help us aggregate the ambiguity and uncertainty in the data. We may also aggregate the data in fuzzy, BF, and CF environments due to this structure. When information or data is organized in the formats of FS, BFS, CFS, and BCFS, the DM method used in this study can handle any real-world DM challenges. BCF structure is used to tackle complex issues in various fields such as engineering, economics, environmental science, image recognition, medical diagnosis, industrialization, and digital networking. Experts can identify the regions where the DTN networking installation is required for sustainable growth through this study. It also helps the experts and researchers to determine the way of artificial intelligence that has more influence in the field of digital technology and the health sector. Additionally, this study allowed the researchers to collaborate with HM operators on DTT.

7. Conclusion

The application that we used in this paper is about networking, which is famously known as the DTN system. A resilient and adaptable network allows for communication in an environment where traditional networks may fail to operate effectively. The main significance of this system is that it is commonly utilized in such scenarios where traditional networks are impractical, such as remote areas, disaster-stricken regions, space missions, and military operations. This networking is advanced and more applicable as compared to the traditional network. It is specially developed to overcome issues such as high latency, intermittent connectivity, and limited bandwidth. Further, we introduced some novel AOs, which are known as BCFDHM and BCFDWHM, in the environment of BCFS. The data in the form of BCFS cannot be handled by FS, CFS, and BFS due to their limitations. However, BCFS can manage the data stored in FS, CFS, and BFS formats. We utilized the operation of Dombi with HM in the environment of BCFS. Further, Dombi HM operators are more generalized than simple Dombi and HM operators. Unlike the traditional means such as the arithmetic mean and geometric mean, the HM considers not only the individual values but also the difference between them. So, due to these facts, our described AOs are more generalized. In addition to this, we established the main properties, such as monotonicity, idempotency, and boundedness, to show the worth and efficiency of our proposed work. Under the BCF information model, we proposed the MABAC method together with BCFDHM operators. The MABAC technique is a DM technique that ranks and evaluates a choice of options against several criteria. Compared with simple MADM, MABAC is a more sophisticated and superior assessment for DM. The AOs known as BCFDHM and BCFDWHM help in the development of the MABAC technique. After that, with the help of these enriched aggregation structures, we successfully identify and rank alternatives for DTN in an uncertain, imprecise, and bipolar condition. By employing the MABAC technique for the DTN system, we find the best and better alternative to the DTN system. At last, we compared our initiated work with many existing theories to prove the authenticity of the suggested work.

7.1 *Limitations*

As we know, structures such as BCF rough set, BCF soft set, and hesitant BCFS require more sophisticated conditions than our established notion. Thus, the current idea fails when the decision maker provides the BCF rough set, BCF soft set, and hesitant BCFS.

7.2 *Future direction*

In the future, we aim to expand the concept of the MABAC technique in various other mathematical structures such as quasirung orthopair FSs [33], complex picture FSs [34], T-spherical FSs [35], and cubic complex hesitant FSs [36], etc.

Ethical statements

The authors declare that this is their original work and that it is not being considered or submitted to any other journal at the same time.

Data availability

The data will be available on reasonable request to the corresponding author.

Human and animal participants

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflict of interest

The authors state that they have no conflicts of interest for the publication of this work.

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Appendix A

Proof of Theorem 1

Proof. In light of the Definition 7, we obtain the following:

$$\ddot{\theta}_{\kappa}^{p} = \begin{pmatrix} \dfrac{1}{1+\left(p\left(\dfrac{1-\pi_{\kappa}^{+}}{\pi_{\kappa}^{+}}\right)^{\daleth}\right)^{1/\daleth}} + i\dfrac{1}{1+\left(p\left(\dfrac{1-\vartheta_{\kappa}^{+}}{\vartheta_{\kappa}^{+}}\right)^{\daleth}\right)^{1/\daleth}}, \\ -1+\dfrac{1}{1+\left(p\left(\dfrac{|\pi_{\kappa}^{-}|}{1+\pi_{\kappa}^{-}}\right)^{\daleth}\right)^{1/\daleth}} + i\left(-1+\dfrac{1}{1+\left(p\left(\dfrac{|\vartheta_{\kappa}^{-}|}{1+\vartheta_{\kappa}^{-}}\right)^{\daleth}\right)^{1/\daleth}}\right) \end{pmatrix}$$

$$\ddot{\theta}_{j}^{q} = \begin{pmatrix} \dfrac{1}{1+\left(q\left(\dfrac{1-\pi_{j}^{+}}{\pi_{j}^{+}}\right)^{\daleth}\right)^{1/\daleth}} + i\dfrac{1}{1+\left(q\left(\dfrac{1-\vartheta_{j}^{+}}{\vartheta_{j}^{+}}\right)^{\daleth}\right)^{1/\daleth}}, \\ -1+\dfrac{1}{1+\left(q\left(\dfrac{|\pi_{j}^{-}|}{1+\pi_{j}^{-}}\right)^{\daleth}\right)^{1/\daleth}} + i\left(-1+\dfrac{1}{1+\left(q\left(\dfrac{|\vartheta_{j}^{-}|}{1+\vartheta_{j}^{-}}\right)^{\daleth}\right)^{1/\daleth}}\right) \end{pmatrix}$$

For the sake of ease and simplicity in the calculation, we assume

$$\frac{1-\pi_{\kappa}^{+}}{\pi_{\kappa}^{+}} = A_{\kappa}, \quad \frac{1-\pi_{j}^{+}}{\pi_{j}^{+}} = A_{j}, \quad \frac{1-\vartheta_{\kappa}^{+}}{\vartheta_{\kappa}^{+}} = B_{\kappa}, \quad \frac{1-\vartheta_{j}^{+}}{\vartheta_{j}^{+}} = B_{j},$$

$$\frac{|\pi_{\kappa}^{-}|}{1+\pi_{\kappa}^{-}} = C_{\kappa}, \quad \frac{|\pi_{j}^{-}|}{1+\pi_{j}^{-}} = C_{j}, \quad \frac{|\vartheta_{\kappa}^{-}|}{1+\vartheta_{\kappa}^{-}} = D_{\kappa}, \quad \frac{|\vartheta_{j}^{-}|}{1+\vartheta_{j}^{-}} = D_{j}$$

Then,

$$\ddot{\theta}_{\kappa}^{p} = \begin{pmatrix} \dfrac{1}{1+\left(p\left(A_{\kappa}\right)^{\daleth}\right)^{1/\daleth}} + i\left(\dfrac{1}{1+\left(p\left(B_{\kappa}\right)^{\daleth}\right)^{1/\daleth}}\right), \\ -1+\dfrac{1}{1+\left(p\left(C_{\kappa}\right)^{\daleth}\right)^{1/\daleth}} + i\left(-1+\dfrac{1}{1+\left(p\left(D_{\kappa}\right)^{\daleth}\right)^{1/\daleth}}\right) \end{pmatrix}$$

$$\ddot{\theta}_j^q = \left(\begin{array}{l} \dfrac{1}{1+\left(q\left(A_j\right)^{\daleth}\right)^{1/\daleth}} + i\left(\dfrac{1}{1+\left(q\left(B_j\right)^{\daleth}\right)^{1/\daleth}}\right), \\ -1+\dfrac{1}{1+\left(q\left(C_j\right)^{\daleth}\right)^{1/\daleth}} + i\left(-1+\dfrac{1}{1+\left(q\left(D_j\right)^{\daleth}\right)^{1/\daleth}}\right) \end{array} \right)$$

$$\ddot{\theta}_\kappa^p \otimes_D \ddot{\theta}_j^q$$

$$= \left(\begin{array}{l} \dfrac{1}{1+\left(\left(\dfrac{1-\dfrac{1}{1+(p(A_\kappa)^{\daleth})^{1/\daleth}}}{\dfrac{1}{1+(p(A_\kappa)^{\daleth})^{1/\daleth}}}\right)^{\daleth}+\left(\dfrac{1-\dfrac{1}{1+(q(A_j)^{\daleth})^{1/\daleth}}}{\dfrac{1}{1+(q(A_j)^{\daleth})^{1/\daleth}}}\right)^{\daleth}\right)^{1/\daleth}} \\ +i\dfrac{1}{1+\left(\left(\dfrac{1-\dfrac{1}{1+(p(B_\kappa)^{\daleth})^{1/\daleth}}}{\dfrac{1}{1+(p(B_\kappa)^{\daleth})^{1/\daleth}}}\right)^{\daleth}+\left(\dfrac{1-\dfrac{1}{1+(q(B_j)^{\daleth})^{1/\daleth}}}{\dfrac{1}{1+(q(B_j)^{\daleth})^{1/\daleth}}}\right)^{\daleth}\right)^{1/\daleth}}, \\ -1+\dfrac{1}{1+\left(\left(\dfrac{\left|-1+\dfrac{1}{1+(p(C_\kappa)^{\daleth})^{1/\daleth}}\right|}{1-1+\dfrac{1}{1+(p(C_\kappa)^{\daleth})^{1/\daleth}}}\right)^{\daleth}+\left(\dfrac{\left|-1+\dfrac{1}{1+(q(C_j)^{\daleth})^{1/\daleth}}\right|}{1-1+\dfrac{1}{1+(q(C_j)^{\daleth})^{1/\daleth}}}\right)^{\daleth}\right)^{1/\daleth}} \\ +i\left(-1+\dfrac{1}{1+\left(\left(\dfrac{\left|-1+\dfrac{1}{1+(p(D_\kappa)^{\daleth})^{1/\daleth}}\right|}{1-1+\dfrac{1}{1+(p(_\kappa)^{\daleth})^{1/\daleth}}}\right)^{\daleth}+\left(\dfrac{\left|-1+\dfrac{1}{1+(q(_j)^{\daleth})^{1/\daleth}}\right|}{1-1+\dfrac{1}{1+(q(_j)^{\daleth})^{1/\daleth}}}\right)^{\daleth}\right)^{1/\daleth}}\right) \end{array} \right)$$

$$
= \begin{pmatrix}
\dfrac{1}{1+\left(\left(1+(p(A_\kappa)^{\daleth})^{1/\daleth}-1\right)^{\daleth}+\left(1+(q(A_j)^{\daleth})^{1/\daleth}-1\right)^{\daleth}\right)^{1/\daleth}} \\
+i\dfrac{1}{1+\left(\left(1+(p(B_\kappa)^{\daleth})^{1/\daleth}-1\right)^{\daleth}+\left(1+(q(B_j)^{\daleth})^{1/\daleth}-1\right)^{\daleth}\right)^{1/\daleth}}, \\
-1+\dfrac{1}{1+\left(\left|-1-(p(C_\kappa)^{\daleth})^{1/\daleth}+1\right|^{\daleth}+\left|-1-(q(C_j)^{\daleth})^{1/\daleth}+1\right|^{\daleth}\right)^{1/\daleth}} \\
+i\left(-1+\dfrac{1}{1+\left(\left|-1-(p(D_\kappa)^{\daleth})^{1/\daleth}+1\right|^{\daleth}+\left|-1-(q(D_j)^{\daleth})^{1/\daleth}+1\right|^{\daleth}\right)^{1/\daleth}}\right)
\end{pmatrix}
$$

$$
= \begin{pmatrix}
\dfrac{1}{1+(p(A_\kappa)^{\daleth}+q(A_j)^{\daleth})^{1/\daleth}}+i\dfrac{1}{1+(p(B_\kappa)^{\daleth}+q(B_j)^{\daleth})^{1/\daleth}}, \\
-1+\dfrac{1}{1+(|p(C_\kappa)^{\daleth}|+|q(C_j)^{\daleth}|)^{1/\daleth}}+i\left(-1+\dfrac{1}{1+(|p(D_\kappa)^{\daleth}|+|q(D_j)^{\daleth}|)^{1/\daleth}}\right)
\end{pmatrix}
$$

$$
\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\ddot{\theta}_\kappa^p \otimes_D \ddot{\theta}_j^q
$$

$$
= \begin{pmatrix}
1-\dfrac{1}{1+\left(\displaystyle\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\frac{1}{p(A_\kappa)^{\daleth}+q(A_j)^{\daleth}}\right)^{1/\daleth}}+i\left(1-\dfrac{1}{1+\left(\displaystyle\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\frac{1}{p(B_\kappa)^{\daleth}+q(B_j)^{\daleth}}\right)^{1/\daleth}}\right), \\
\dfrac{-1}{1+\left(\displaystyle\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\frac{1}{\left||p(C_\kappa)^{\daleth}|+|q(C_j)^{\daleth}|\right|}\right)^{1/\daleth}}+i\dfrac{-1}{1+\left(\displaystyle\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\frac{1}{\left||p(D_\kappa)^{\daleth}|+|q(D_j)^{\daleth}|\right|}\right)^{1/\daleth}}
\end{pmatrix}
$$

$$
\frac{2}{\eta(\eta+1)} \sum_{\kappa=1}^{\eta} \sum_{j=1}^{\eta} \ddot{\theta}_{\kappa}^{p} \otimes_D \ddot{\theta}_{j}^{q}
$$

$$
= \left(\begin{array}{l} 1-\dfrac{1}{1+\left(\dfrac{2}{\eta(\eta+1)}\left(\dfrac{1-\dfrac{1}{1+\left(\sum\limits_{\kappa=1}^{\eta}\sum\limits_{j=1}^{\eta}\dfrac{1}{p(A_{\kappa})^{\daleth}+q(A_j)^{\daleth}}\right)^{1/\daleth}}}{1-1+\dfrac{1}{1+\left(\sum\limits_{\kappa=1}^{\eta}\sum\limits_{j=1}^{\eta}\dfrac{1}{p(A_{\kappa})^{\daleth}+q(A_j)^{\daleth}}\right)^{1/\daleth}}}\right)^{\daleth}\right)^{1/\daleth}} \\ +i\left(1-\dfrac{1}{1+\left(\dfrac{2}{\eta(\eta+1)}\left(\dfrac{1-\dfrac{1}{1+\left(\sum\limits_{\kappa=1}^{\eta}\sum\limits_{j=1}^{\eta}\dfrac{1}{p(B_{\kappa})^{\daleth}+q(B_j)^{\daleth}}\right)^{1/\daleth}}}{1-1+\dfrac{1}{1+\left(\sum\limits_{\kappa=1}^{\eta}\sum\limits_{j=1}^{\eta}\dfrac{1}{p(B_{\kappa})^{\daleth}+q(B_j)^{\daleth}}\right)^{1/\daleth}}}\right)^{\daleth}\right)^{1/\daleth}} \right) \end{array} \right),
$$

$$
\left.\frac{-1}{1+\left(\frac{2}{\eta(\eta+1)}\left(\frac{1-\frac{1}{1+\left(\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\frac{1}{\left|\left|p(C_\kappa)^{\daleth}\right|+\left|q(C_j)^{\daleth}\right|\right|}\right)^{1/\daleth}}}{\left|\frac{-1}{1+\left(\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\frac{1}{\left|\left|p(C_\kappa)^{\daleth}\right|+\left|q(C_j)^{\daleth}\right|\right|}\right)^{1/\daleth}}\right|}\right)^{\daleth}\right)^{1/\daleth}}
+i\frac{-1}{1+\left(\frac{2}{\eta(\eta+1)}\left(\frac{1-\frac{1}{1+\left(\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\frac{1}{\left|\left|p(D_\kappa)^{\daleth}\right|+\left|q(D_j)^{\daleth}\right|\right|}\right)^{1/\daleth}}}{\left|\frac{-1}{1+\left(\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\frac{1}{\left|\left|p(D_\kappa)^{\daleth}\right|+\left|q(D_j)^{\daleth}\right|\right|}\right)^{1/\daleth}}\right|}\right)^{\daleth}\right)^{1/\daleth}}\right)
$$

$$
=\left(\begin{array}{l}
1-\frac{1}{1+\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\frac{1}{p(A_\kappa)^{\daleth}+q(A_j)^{\daleth}}\right)\right)^{1/\daleth}} \\
+i\left(1-\frac{1}{1+\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\frac{1}{p(B_\kappa)^{\daleth}+q(B_j)^{\daleth}}\right)\right)^{1/\daleth}}\right), \\
\frac{-1}{1+\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\frac{1}{\left|\left|p(C_\kappa)^{\daleth}\right|+\left|q(C_j)^{\daleth}\right|\right|}\right)\right)^{1/\daleth}} \\
+i\frac{-1}{1+\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\frac{1}{\left|\left|p(D_\kappa)^{\daleth}\right|+\left|q(D_j)^{\daleth}\right|\right|}\right)\right)^{1/\daleth}}
\end{array}\right)
$$

$$
\left(\frac{2}{\eta(\eta+1)}\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\ddot{\theta}_{\kappa}^{p}\otimes_{D}\ddot{\theta}_{j}^{q}\right)^{\frac{1}{p+q}}
$$

$$
= \left(\begin{array}{l}
\dfrac{1}{1+\left(\dfrac{1}{p+q}\left(\dfrac{1-1+\dfrac{1}{1+\left(\dfrac{2}{\eta(\eta+1)}\cdot\displaystyle\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\dfrac{1}{p(A_{\kappa})^{\daleth}+q(A_{j})^{\daleth}}\right)^{1/\daleth}}}{1-\dfrac{1}{1+\left(\dfrac{2}{\eta(\eta+1)}\cdot\sum_{\kappa=1}^{\eta}\displaystyle\sum_{j=1}^{\eta}\dfrac{1}{p(A_{\kappa})^{\daleth}+q(A_{j})^{\daleth}}\right)^{1/\daleth}}}\right)\right)^{\daleth}\right)^{1/\daleth}} \\
+i\dfrac{1}{1+\left(\dfrac{1}{p+q}\left(\dfrac{1-1+\dfrac{1}{1+\left(\dfrac{2}{\eta(\eta+1)}\cdot\displaystyle\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\dfrac{1}{p(B_{\kappa})^{\daleth}+q(B_{j})^{\daleth}}\right)^{1/\daleth}}}{1-\dfrac{1}{1+\left(\dfrac{2}{\eta(\eta+1)}\cdot\displaystyle\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\dfrac{1}{p(B_{\kappa})^{\daleth}+q(B_{j})^{\daleth}}\right)^{1/\daleth}}}\right)\right)^{\daleth}\right)^{1/\daleth}}
\end{array}\right),
$$

$$
\left. -1+\frac{1}{\left(1+\left(\frac{1}{p+q}\left(\frac{\left|\frac{-1}{1+\left(\frac{2}{\eta(\eta+1)}\cdot\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\frac{1}{\left|\left|p(C_\kappa)^{\daleth}\right|+\left|q(C_j)^{\daleth}\right|\right|}\right)^{1/\daleth}}\right|}{1-\frac{1}{1+\left(\frac{2}{\eta(\eta+1)}\cdot\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\frac{1}{\left|\left|p(C_\kappa)^{\daleth}\right|+\left|q(C_j)^{\daleth}\right|\right|}\right)^{1/\daleth}}}\right)\right)^{\daleth}\right)^{1/\daleth}}\right.
$$

$$
\left. +i\left(-1+\frac{1}{\left(1+\left(\frac{1}{p+q}\left(\frac{\left|\frac{-1}{1+\left(\frac{2}{\eta(\eta+1)}\cdot\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\frac{1}{\left|\left|p(D_\kappa)^{\daleth}\right|+\left|q(D_j)^{\daleth}\right|\right|}\right)^{1/\daleth}}\right|}{1-\frac{1}{1+\left(\frac{2}{\eta(\eta+1)}\cdot\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\frac{1}{\left|\left|p(D_\kappa)^{\daleth}\right|+\left|q(D_j)^{\daleth}\right|\right|}\right)^{1/\daleth}}}\right)\right)^{\daleth}\right)^{1/\daleth}}\right)\right)
$$

$$
= \left(\begin{array}{l}
\dfrac{1}{1+\left(\dfrac{\eta(\eta+1)}{2(p+q)} \cdot \dfrac{1}{\sum\limits_{\kappa=1}^{\eta}\sum\limits_{j=1}^{\eta} \dfrac{1}{p(A_\kappa)^{\daleth}+q(A_j)^{\daleth}}}\right)^{1/\daleth}} \\
+i \dfrac{1}{1+\left(\dfrac{\eta(\eta+1)}{2(p+q)} \cdot \dfrac{1}{\sum\limits_{\kappa=1}^{\eta}\sum\limits_{j=1}^{\eta} \dfrac{1}{p(B_\kappa)^{\daleth}+q(B_j)^{\daleth}}}\right)^{1/\daleth}}, \\
-1+\dfrac{1}{1+\left(\dfrac{\eta(\eta+1)}{2(p+q)} \cdot \dfrac{1}{\sum\limits_{\kappa=1}^{\eta}\sum\limits_{j=1}^{\eta} \dfrac{1}{\left|\left|p(C_\kappa)^{\daleth}\right|+\left|q(C_j)^{\daleth}\right|\right|}}\right)^{1/\daleth}} \\
+i\left(-1+\dfrac{1}{1+\left(\dfrac{\eta(\eta+1)}{2(p+q)} \cdot \dfrac{1}{\sum\limits_{\kappa=1}^{\eta}\sum\limits_{j=1}^{\eta} \dfrac{1}{\left|\left|p(D_\kappa)^{\daleth}\right|+\left|q(D_j)^{\daleth}\right|\right|}}\right)^{1/\daleth}} \right)
\end{array} \right) \tag{A1}
$$

We get the following result by putting:

$$
\frac{1-\pi_\kappa^+}{\pi_\kappa^+}=A_\kappa, \quad \frac{1-\pi_j^+}{\pi_j^+}=A_j, \quad \frac{1-\vartheta_\kappa^+}{\vartheta_\kappa^+}=B_\kappa, \quad \frac{1-\vartheta_j^+}{\vartheta_j^+}=B_j,
$$

$$
\frac{|\pi_\kappa^-|}{1+\pi_\kappa^-}=C_\kappa, \quad \frac{|\pi_j^-|}{1+\pi_j^-}=C_j, \quad \frac{|\vartheta_\kappa^-|}{1+\vartheta_\kappa^-}=D_\kappa, \quad \frac{|\vartheta_j^-|}{1+\vartheta_j^-}=D_j
$$

in the above Equation (A1).

$$
= \left(\frac{1}{1+\left(\dfrac{\eta(\eta+1)}{2(p+q)}\cdot\dfrac{1}{\left(\displaystyle\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\dfrac{1}{p\left(\dfrac{1-\pi_{\kappa}^{+}}{\pi_{\kappa}^{+}}\right)^{\daleth}+q\left(\dfrac{1-\pi_{j}^{+}}{\pi_{j}^{+}}\right)^{\daleth}}\right)}\right)^{\frac{1}{\daleth}}} \right.
$$

$$
\left. +i\frac{1}{1+\left(\dfrac{\eta(\eta+1)}{2(p+q)}\cdot\dfrac{1}{\left(\displaystyle\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\dfrac{1}{p\left(\dfrac{1-\vartheta_{\kappa}^{+}}{\vartheta_{\kappa}^{+}}\right)^{\daleth}+q\left(\dfrac{1-\vartheta_{j}^{+}}{\vartheta_{j}^{+}}\right)^{\daleth}}\right)}\right)^{\frac{1}{\daleth}}} \right. ,
$$

$$
\left. -1+\frac{1}{1+\left(\frac{\eta(\eta+1)}{2(p+q)}\cdot\frac{1}{\left(\sum\limits_{\kappa=1}^{\eta}\sum\limits_{j=1}^{\eta}\frac{1}{\left|\left|p\left(\frac{|\pi_{\kappa}^{-}|}{1+\pi_{\kappa}^{-}}\right)^{\daleth}\right|+\left|q\left(\frac{|\pi_{j}^{-}|}{1+\pi_{j}^{-}}\right)^{\daleth}\right|\right|}\right)}\right)^{\frac{1}{\daleth}}}\right.
$$

$$
\left. +i\left(-1+\frac{1}{1+\left(\frac{\eta(\eta+1)}{2(p+q)}\cdot\frac{1}{\left(\sum\limits_{\kappa=1}^{\eta}\sum\limits_{j=1}^{\eta}\frac{1}{\left|\left|p\left(\frac{|\vartheta_{\kappa}^{-}|}{1+\vartheta_{\kappa}^{-}}\right)^{\daleth}\right|+\left|q\left(\frac{|\vartheta_{j}^{-}|}{1+\vartheta_{j}^{-}}\right)^{\daleth}\right|\right|}\right)}\right)^{\frac{1}{\daleth}}}\right)\right)
$$

which is required for BCFDHM operator. □

Proof of Theorem 5 In light of the Definition 8, we obtain the following:

$$\left(w_\kappa\,\ddot{\theta}_\kappa\right)=\left(\begin{array}{c}1-\dfrac{1}{1+\left(w_\kappa\left(\dfrac{\pi_\kappa^+}{1-\pi_\kappa^+}\right)^{\urcorner}\right)^{1/\urcorner}}+i\left(1-\dfrac{1}{1+\left(w_\kappa\left(\dfrac{\vartheta_\kappa^+}{1-\vartheta_\kappa^+}\right)^{\urcorner}\right)^{1/\urcorner}}\right),\\ \dfrac{-1}{1+\left(w_\kappa\left(\dfrac{1+\pi_\kappa^-}{|\pi_\kappa^-|}\right)^{\urcorner}\right)^{1/\urcorner}}+i\left(\dfrac{-1}{1+\left(w_\kappa\left(\dfrac{1+\vartheta_\kappa^-}{|\vartheta_\kappa^-|}\right)^{\urcorner}\right)^{1/\urcorner}}\right)\end{array}\right)$$

$$\left(w_j\,\ddot{\theta}_j\right)=\left(\begin{array}{c}1-\dfrac{1}{1+\left(w_j\left(\dfrac{\pi_j^+}{1-\pi_j^+}\right)^{\urcorner}\right)^{1/\urcorner}}+i\left(1-\dfrac{1}{1+\left(w_j\left(\dfrac{\vartheta_j^+}{1-\vartheta_j^+}\right)^{\urcorner}\right)^{1/\urcorner}}\right),\\ \dfrac{-1}{1+\left(w_j\left(\dfrac{1+\pi_j^-}{|\pi_j^-|}\right)^{\urcorner}\right)^{1/\urcorner}}+i\left(\dfrac{-1}{1+\left(w_j\left(\dfrac{1+\vartheta_j^-}{|\vartheta_j^-|}\right)^{\urcorner}\right)^{1/\urcorner}}\right)\end{array}\right)$$

To facilitate and streamline the computation, we assume that

$$\frac{\pi_\kappa^+}{1-\pi_\kappa^+}=A_\kappa,\quad \frac{\pi_j^+}{1-\pi_j^+}=A_j,\quad \frac{\vartheta_\kappa^+}{1-\vartheta_\kappa^+}=B_\kappa,\quad \frac{\vartheta_j^+}{1-\vartheta_j^+}=B_j,$$

$$\frac{1+\pi_\kappa^-}{|\pi_\kappa^-|}=C_\kappa,\quad \frac{1+\pi_j^-}{|\pi_j^-|}=C_j,\quad \frac{1+\vartheta_\kappa^-}{|\vartheta_\kappa^-|}=D_\kappa,\quad \frac{1+\vartheta_j^-}{|\vartheta_j^-|}=D_j.$$

$$\left(w_\kappa\,\ddot{\theta}_\kappa\right)=\left(\begin{array}{c}1-\dfrac{1}{1+\left(w_\kappa\left(A_\kappa\right)^{\urcorner}\right)^{1/\urcorner}}+i\left(1-\dfrac{1}{1+\left(w_\kappa\left(B_\kappa\right)^{\urcorner}\right)^{1/\urcorner}}\right),\\ \dfrac{-1}{1+\left(w_\kappa\left(C_\kappa\right)^{\urcorner}\right)^{1/\urcorner}}+i\left(\dfrac{-1}{1+\left(w_\kappa\left(D_\kappa\right)^{\urcorner}\right)^{1/\urcorner}}\right)\end{array}\right)$$

$$(w_j\,\ddot{\theta}_j) = \left(\begin{array}{l} 1 - \dfrac{1}{1+\left(w_j\left(A_j\right)^{\daleth}\right)^{1/\daleth}} + i\left(1 - \dfrac{1}{1+\left(w_j\left(B_j\right)^{\daleth}\right)^{1/\daleth}}\right), \\ \dfrac{-1}{1+\left(w_j\left(C_j\right)^{\daleth}\right)^{1/\daleth}} + i\left(\dfrac{-1}{1+\left(w_j\left(D_j\right)^{\daleth}\right)^{1/\daleth}}\right) \end{array} \right)$$

$$(w_\kappa\,\ddot{\theta}_\kappa)^p = \left(\begin{array}{l} \dfrac{1}{1+\left(p\left(\dfrac{1-1+\dfrac{1}{1+\left(w_\kappa\left(A_\kappa\right)^{\daleth}\right)^{1/\daleth}}}{1-\dfrac{1}{1+\left(w_\kappa\left(A_\kappa\right)^{\daleth}\right)^{1/\daleth}}}\right)^{\daleth}\right)^{1/\daleth}} \\ +i\dfrac{1}{1+\left(p\left(\dfrac{1-1+\dfrac{1}{1+\left(w_\kappa\left(B_\kappa\right)^{\daleth}\right)^{1/\daleth}}}{1-\dfrac{1}{1+\left(w_\kappa\left(B_\kappa\right)^{\daleth}\right)^{1/\daleth}}}\right)^{\daleth}\right)^{1/\daleth}}, \\ -1+\dfrac{1}{1+\left(p\left(\dfrac{\left|\dfrac{-1}{1+\left(w_\kappa\left(C_\kappa\right)^{\daleth}\right)^{1/\daleth}}\right|}{1-\dfrac{1}{1+\left(w_\kappa\left(C_\kappa\right)^{\daleth}\right)^{1/\daleth}}}\right)^{\daleth}\right)^{1/\daleth}} \\ +i\left(-1+\dfrac{1}{1+\left(p\left(\dfrac{\left|\dfrac{-1}{1+\left(w_\kappa\left(D_\kappa\right)^{\daleth}\right)^{1/\daleth}}\right|}{1-\dfrac{1}{1+\left(w_\kappa\left(D_\kappa\right)^{\daleth}\right)^{1/\daleth}}}\right)^{\daleth}\right)^{1/\daleth}}\right) \end{array} \right)$$

$$(w_\kappa \ddot{\theta}_\kappa)^p = \left(\begin{array}{l} \dfrac{1}{1+\left(p\left(\dfrac{1}{w_\kappa(A_\kappa)^\daleth}\right)\right)^{1/\daleth}} + i\dfrac{1}{1+\left(p\left(\dfrac{1}{w_\kappa(B_\kappa)^\daleth}\right)\right)^{1/\daleth}}, \\ -1+\dfrac{1}{1+\left(p\left(\dfrac{1}{w_\kappa(C_\kappa)^\daleth}\right)\right)^{1/\daleth}} + i\left(-1+\dfrac{1}{1+\left(p\left(\dfrac{1}{w_\kappa(D_\kappa)^\daleth}\right)\right)^{1/\daleth}}\right) \end{array} \right)$$

$$(w_j \ddot{\theta}_j)^q = \left(\begin{array}{l} \dfrac{1}{1+\left(q\left(\dfrac{1}{w_j(A_j)^\daleth}\right)\right)^{1/\daleth}} + i\dfrac{1}{1+\left(q\left(\dfrac{1}{w_j(B_j)^\daleth}\right)\right)^{1/\daleth}}, \\ -1+\dfrac{1}{1+\left(q\left(\dfrac{1}{w_j(C_j)^\daleth}\right)\right)^{1/\daleth}} + i\left(-1+\dfrac{1}{1+\left(q\left(\dfrac{1}{w_j(D_j)^\daleth}\right)\right)^{1/\daleth}}\right) \end{array} \right)$$

$$(w_\kappa \ddot{\theta}_\kappa)^p \otimes_D (w_j \ddot{\theta}_j)^q$$

$$= \left(\begin{array}{l} \dfrac{1}{1+\left(\left(\dfrac{1-\dfrac{1}{1+\left(p\left(\dfrac{1}{w_\kappa(A_\kappa)^\daleth}\right)\right)^{1/\daleth}}}{\dfrac{1}{1+\left(p\left(\dfrac{1}{w_\kappa(A_\kappa)^\daleth}\right)\right)^{1/\daleth}}}\right)^\daleth + \left(\dfrac{1-\dfrac{1}{1+\left(q\left(\dfrac{1}{w_j(A_j)^\daleth}\right)\right)^{1/\daleth}}}{\dfrac{1}{1+\left(q\left(\dfrac{1}{w_j(A_j)^\daleth}\right)\right)^{1/\daleth}}}\right)^\daleth\right)^{1/\daleth}} \\ +i\dfrac{1}{1+\left(\left(\dfrac{1-\dfrac{1}{1+\left(p\left(\dfrac{1}{w_\kappa(B_\kappa)^\daleth}\right)\right)^{1/\daleth}}}{\dfrac{1}{1+\left(p\left(\dfrac{1}{w_\kappa(B_\kappa)^\daleth}\right)\right)^{1/\daleth}}}\right)^\daleth + \left(\dfrac{1-\dfrac{1}{1+\left(q\left(\dfrac{1}{w_j(B_j)^\daleth}\right)\right)^{1/\daleth}}}{\dfrac{1}{1+\left(q\left(\dfrac{1}{w_j(B_j)^\daleth}\right)\right)^{1/\daleth}}}\right)^\daleth\right)^{1/\daleth}} \end{array} \right),$$

$$
\begin{aligned}
& -1+\frac{1}{1+\left(\left(\dfrac{\left|-1+\dfrac{1}{1+\left(p\left(\dfrac{1}{w_\kappa(C_\kappa)^{\daleth}}\right)\right)^{1/\daleth}}\right|}{1-1+\dfrac{1}{1+\left(p\left(\dfrac{1}{w_\kappa(C_\kappa)^{\daleth}}\right)\right)^{1/\daleth}}}\right)^{\daleth}+\left(\dfrac{\left|-1+\dfrac{1}{1+\left(q\left(\dfrac{1}{(w_jC_j)^{\daleth}}\right)\right)^{1/\daleth}}\right|}{1-1+\dfrac{1}{1+\left(q\left(\dfrac{1}{(w_jC_j)^{\daleth}}\right)\right)^{1/\daleth}}}\right)^{\daleth}\right)^{1/\daleth}} \\
& \left.+i\left(-1+\frac{1}{1+\left(\left(\dfrac{\left|-1+\dfrac{1}{1+\left(p\left(\dfrac{1}{w_\kappa(D_\kappa)^{\daleth}}\right)\right)^{1/\daleth}}\right|}{1-1+\dfrac{1}{1+\left(p\left(\dfrac{1}{w_\kappa(D_\kappa)^{\daleth}}\right)\right)^{1/\daleth}}}\right)^{\daleth}+\left(\dfrac{\left|-1+\dfrac{1}{1+\left(q\left(\dfrac{1}{(w_jD_j)^{\daleth}}\right)\right)^{1/\daleth}}\right|}{1-1+\dfrac{1}{1+\left(q\left(\dfrac{1}{(w_jD_j)^{\daleth}}\right)\right)^{1/\daleth}}}\right)^{\daleth}\right)^{1/\daleth}}\right)\right) \\
& =\left(\begin{array}{l}
\dfrac{1}{1+\left(p\left(\dfrac{1}{w_\kappa(A_\kappa)^{\daleth}}\right)+q\left(\dfrac{1}{w_j(A_j)^{\daleth}}\right)\right)^{1/\daleth}}+i\dfrac{1}{1+\left(p\left(\dfrac{1}{w_\kappa(B_\kappa)^{\daleth}}\right)+q\left(\dfrac{1}{w_j(B_j)^{\daleth}}\right)\right)^{1/\daleth}}, \\
-1+\dfrac{1}{1+\left(\left|p\left(\dfrac{1}{w_\kappa(C_\kappa)^{\daleth}}\right)\right|+\left|q\left(\dfrac{1}{w_j(C_j)^{\daleth}}\right)\right|\right)^{1/\daleth}} \\
+i\left(-1+\dfrac{1}{1+\left(\left|p\left(\dfrac{1}{w_\kappa(D_\kappa)^{\daleth}}\right)\right|+\left|q\left(\dfrac{1}{w_j(D_j)^{\daleth}}\right)\right|\right)^{1/\daleth}}\right)
\end{array}\right)
\end{aligned}
$$

$$\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\left((w_\kappa\ddot{\theta}_\kappa)^p \otimes_D (w_j\ddot{\theta}_j)^q\right)$$

$$= \left(\begin{array}{l} 1-\dfrac{1}{1+\left(\displaystyle\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\dfrac{1}{\dfrac{p}{w_\kappa(A_\kappa)^{\daleth}}+\dfrac{q}{w_j(A_j)^{\daleth}}}\right)^{1/\daleth}} \\ +i\left(1-\dfrac{1}{1+\left(\displaystyle\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\dfrac{1}{\dfrac{p}{w_\kappa(B_\kappa)^{\daleth}}+\dfrac{q}{w_j(B_j)^{\daleth}}}\right)^{1/\daleth}}\right), \\ \dfrac{-1}{1+\left(\displaystyle\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\dfrac{1}{\left|\left|\dfrac{p}{w_\kappa(C_\kappa)^{\daleth}}\right|+\left|\dfrac{q}{w_j(C_j)^{\daleth}}\right|\right|}\right)^{1/\daleth}} \\ +i\dfrac{-1}{1+\left(\displaystyle\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\dfrac{1}{\left|\left|\dfrac{p}{w_\kappa(D_\kappa)^{\daleth}}\right|+\left|\dfrac{q}{w_j(Dj)^{\daleth}}\right|\right|}\right)^{1/\daleth}} \end{array} \right)$$

$$\frac{2}{\eta(\eta+1)}\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\left((w_\kappa\ddot{\theta}_\kappa)^p\otimes_D(w_j\ddot{\theta}_j)^q\right)$$

$$=\left(\begin{array}{l} 1-\dfrac{1}{1+\left(\dfrac{2}{\eta(\eta+1)}\left(\dfrac{1-\dfrac{1}{1+\left(\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\dfrac{1}{\dfrac{p}{w_\kappa(A_\kappa)^\daleth}+\dfrac{q}{w_j(A_j)^\daleth}}\right)^{1/\daleth}}}{1-1+\dfrac{1}{1+\left(\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\dfrac{1}{\dfrac{p}{w_\kappa(A_\kappa)^\daleth}+\dfrac{q}{w_j(A_j)^\daleth}}\right)^{1/\daleth}}}\right)^\daleth\right)^{1/\daleth}} \\ +i\left(1-\dfrac{1}{1+\left(\dfrac{2}{\eta(\eta+1)}\left(\dfrac{1-\dfrac{1}{1+\left(\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\dfrac{1}{\dfrac{p}{w_\kappa(B_\kappa)^\daleth}+\dfrac{q}{w_j(B_j)^\daleth}}\right)^{1/\daleth}}}{1-1+\dfrac{1}{1+\left(\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\dfrac{1}{\dfrac{p}{w_\kappa(B_\kappa)^\daleth}+\dfrac{q}{w_j(B_j)^\daleth}}\right)^{1/\daleth}}}\right)^\daleth\right)^{1/\daleth}}\right) \end{array}\right),$$

$$
\left.\frac{-1}{1+\left(\frac{2}{\eta(\eta+1)}\left(\frac{1-\frac{1}{1+\left(\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\frac{1}{\left|\left|\frac{p}{w_\kappa(C_\kappa)^\daleth}\right|+\left|\frac{q}{w_j(C_j)^\daleth}\right|\right|}\right)^{1/\daleth}}}{\frac{-1}{\left|1+\left(\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\frac{1}{\left|\left|\frac{p}{w_\kappa(C_\kappa)^\daleth}\right|+\left|\frac{q}{w_j(C_j)^\daleth}\right|\right|}\right)^{\frac{1}{\daleth}}\right|}}\right)^{\daleth}\right)^{1/\daleth}}
+i\frac{-1}{1+\left(\frac{2}{\eta(\eta+1)}\left(\frac{1-\frac{1}{1+\left(\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\frac{1}{\left|\left|\frac{p}{w_\kappa(D_\kappa)^\daleth}\right|+\left|\frac{q}{w_j(D_j)^\daleth}\right|\right|}\right)^{1/\daleth}}}{\frac{-1}{\left|1+\left(\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\frac{1}{\left|\left|\frac{p}{w_\kappa(D_\kappa)^\daleth}\right|+\left|\frac{q}{w_j(D_j)^\daleth}\right|\right|}\right)^{\frac{1}{\daleth}}\right|}}\right)^{\daleth}\right)^{1/\daleth}}\right)
$$

$$
=\left(\begin{array}{l}1-\frac{1}{1+\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\frac{1}{\frac{p}{w_\kappa(A_\kappa)^\daleth}+\frac{q}{w_j(A_j)^\daleth}}\right)\right)^{\frac{1}{\daleth}}}\\ +i\left(1-\frac{1}{1+\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\frac{1}{\frac{p}{w_\kappa(B_\kappa)^\daleth}+\frac{q}{w_j(B_j)^\daleth}}\right)\right)^{\frac{1}{\daleth}}}\right)\end{array}\right),
$$

$$
\left.\begin{array}{l}
\dfrac{-1}{1+\left(\dfrac{2}{\eta(\eta+1)}\left(\displaystyle\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\dfrac{1}{\left|\left|\dfrac{p}{w_{\kappa}(C_{\kappa})^{\urcorner}}\right|+\left|\dfrac{q}{w_{j}(C_{j})^{\urcorner}}\right|\right|}\right)\right)^{\frac{1}{\urcorner}}} \\
+i\dfrac{-1}{1+\left(\dfrac{2}{\eta(\eta+1)}\left(\displaystyle\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\dfrac{1}{\left|\left|\dfrac{p}{w_{\kappa}(D_{\kappa})^{\urcorner}}\right|+\left|\dfrac{q}{w_{j}(D_{j})^{\urcorner}}\right|\right|}\right)\right)^{\frac{1}{\urcorner}}}
\end{array}\right)
$$

$$
\left(\frac{2}{\eta(\eta+1)}\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\left((w_{\kappa}\ddot{\theta}_{\kappa})^{p}\otimes_{D}(w_{j}\ddot{\theta}_{j})^{q}\right)\right)^{\frac{1}{p+q}}
$$

$$
=\left(\begin{array}{l}
\dfrac{1}{1+\left(\dfrac{1}{p+q}\left(\dfrac{1-1+\dfrac{1}{1+\left(\dfrac{2}{\eta(\eta+1)}\left(\displaystyle\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\dfrac{1}{\dfrac{p}{w_{\kappa}(A_{\kappa})^{\urcorner}}+\dfrac{q}{w_{j}(A_{j})^{\urcorner}}}\right)\right)^{\frac{1}{\urcorner}}}}{1-\dfrac{1}{1+\left(\dfrac{2}{\eta(\eta+1)}\left(\displaystyle\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\dfrac{1}{\dfrac{p}{w_{\kappa}(A_{\kappa})^{\urcorner}}+\dfrac{q}{w_{j}(A_{j})^{\urcorner}}}\right)\right)^{\frac{1}{\urcorner}}}}\right)^{\urcorner}\right)^{\frac{1}{\urcorner}}} \\
+i\dfrac{1}{1+\left(\dfrac{1}{p+q}\left(\dfrac{1-1+\dfrac{1}{1+\left(\dfrac{2}{\eta(\eta+1)}\left(\displaystyle\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\dfrac{1}{\dfrac{p}{w_{\kappa}(B_{\kappa})^{\urcorner}}+\dfrac{q}{w_{j}(B_{j})^{\urcorner}}}\right)\right)^{\frac{1}{\urcorner}}}}{1-\dfrac{1}{1+\left(\dfrac{2}{\eta(\eta+1)}\left(\displaystyle\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\dfrac{1}{\dfrac{p}{w_{\kappa}(B_{\kappa})^{\urcorner}}+\dfrac{q}{w_{j}(B_{j})^{\urcorner}}}\right)\right)^{\frac{1}{\urcorner}}}}\right)^{\urcorner}\right)^{\frac{1}{\urcorner}}}
\end{array}\right),
$$

$$
\left. -1+\frac{1}{\left(1+\left(\frac{1}{p+q}\left(\frac{\left|\frac{-1}{\left(1+\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\frac{1}{\left|\left|\frac{p}{w_{\kappa}(C_{\kappa})^{\daleth}}\right|+\left|\frac{q}{w_{j}(C_{j})^{\daleth}}\right|\right|}\right)\right)\right)^{1/\daleth}}\right|}{1-\frac{1}{\left(1+\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\frac{1}{\left|\left|\frac{p}{w_{\kappa}(C_{\kappa})^{\daleth}}\right|+\left|\frac{q}{w_{j}(C_{j})^{\daleth}}\right|\right|}\right)\right)\right)^{1/\daleth}}}\right)\right)^{\daleth}\right)^{1/\daleth}} \right.
$$

$$
\left. +i\left(-1+\frac{1}{\left(1+\left(\frac{1}{p+q}\left(\frac{\left|\frac{-1}{\left(1+\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\frac{1}{\left|\left|\frac{p}{w_{\kappa}(D_{\kappa})^{\daleth}}\right|+\left|\frac{q}{w_{j}(D_{j})^{\daleth}}\right|\right|}\right)\right)\right)^{1/\daleth}}\right|}{1-\frac{1}{\left(1+\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\frac{1}{\left|\left|\frac{p}{w_{\kappa}(D_{\kappa})^{\daleth}}\right|+\left|\frac{q}{w_{j}(D_{j})^{\daleth}}\right|\right|}\right)\right)\right)^{1/\daleth}}}\right)\right)^{\daleth}\right)^{1/\daleth}}\right)\right)
$$

$$
= \left(\begin{array}{l}
\dfrac{1}{1+\left(\dfrac{\eta(\eta+1)}{2(p+q)}\left(\dfrac{1}{\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\dfrac{1}{\dfrac{p}{w_\kappa(A_\kappa)^\daleth}+\dfrac{q}{w_j(A_j)^\daleth}}}\right)\right)^{1/\daleth}} \\
+i\dfrac{1}{1+\left(\dfrac{\eta(\eta+1)}{2(p+q)}\left(\dfrac{1}{\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\dfrac{1}{\dfrac{p}{w_\kappa(B_\kappa)^\daleth}+\dfrac{q}{w_j(B_j)^\daleth}}}\right)\right)^{1/\daleth}}, \\
-1+\dfrac{1}{1+\left(\dfrac{\eta(\eta+1)}{2(p+q)}\left(\dfrac{1}{\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\dfrac{1}{\left|\left|\dfrac{p}{w_\kappa(C_\kappa)^\daleth}\right|+\left|\dfrac{q}{w_j(C_j)^\daleth}\right|\right|}}\right)\right)^{1/\daleth}} \\
+i\left(-1+\dfrac{1}{1+\left(\dfrac{\eta(\eta+1)}{2(p+q)}\left(\dfrac{1}{\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\dfrac{1}{\left|\left|\dfrac{p}{w_\kappa(D_\kappa)^\daleth}\right|+\left|\dfrac{q}{w_j(D_j)^\daleth}\right|\right|}}\right)\right)^{1/\daleth}}\right)
\end{array} \right) \tag{A2}
$$

We get the following result by putting:

$$
\frac{\pi_\kappa^+}{1-\pi_\kappa^+}=A_\kappa, \quad \frac{\pi_j^+}{1-\pi_j^+}=A_j, \quad \frac{\vartheta_\kappa^+}{1-\vartheta_\kappa^+}=B_\kappa, \quad \frac{\vartheta_j^+}{1-\vartheta_j^+}=B_j,
$$

$$
\frac{1+\pi_\kappa^-}{\left|\pi_\kappa^-\right|}=C_\kappa, \quad \frac{1+\pi_j^-}{\left|\pi_j^-\right|}=C_j, \quad \frac{1+\vartheta_\kappa^-}{\left|\vartheta_\kappa^-\right|}=D_\kappa, \quad \frac{1+\vartheta_j^-}{\left|\vartheta_j^-\right|}=D_j
$$

in the above Equation (A2).

$$= \left(\frac{1}{\left(1+\left(\frac{\eta(\eta+1)}{2(p+q)}\cdot\frac{1}{\left(\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\frac{1}{\left(\frac{p}{w_\kappa\left(\frac{\pi_\kappa^+}{1-\pi_\kappa^+}\right)^{\daleth}}+\frac{q}{w_j\left(\frac{\pi_j^+}{1-\pi_j^+}\right)^{\daleth}}\right)}\right)}\right)\right)^{1/\daleth}} + i\frac{1}{\left(1+\left(\frac{\eta(\eta+1)}{2(p+q)}\cdot\frac{1}{\left(\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\frac{1}{\left(\frac{p}{w_\kappa\left(\frac{\vartheta_\kappa^+}{1-\vartheta_\kappa^+}\right)^{\daleth}}+\frac{q}{w_j\left(\frac{\vartheta_j^+}{1-\vartheta_j^+}\right)^{\daleth}}\right)}\right)}\right)\right)^{1/\daleth}} \right),$$

$$\left(-1+\frac{1}{\left(1+\left(\frac{\eta(\eta+1)}{2(p+q)}\cdot\frac{1}{\left(\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\frac{1}{\left|\left|\frac{p}{w_\kappa\left(\frac{1+\pi_\kappa^-}{|\pi_\kappa^-|}\right)^\daleth}\right|+\left|\frac{q}{w_j\left(\frac{1+\pi_j^-}{|\pi_j^-|}\right)^\daleth}\right|\right|}\right)}\right)\right)^{1/\daleth}} + i\left(-1+\frac{1}{\left(1+\left(\frac{\eta(\eta+1)}{2(p+q)}\cdot\frac{1}{\left(\sum_{\kappa=1}^{\eta}\sum_{j=1}^{\eta}\frac{1}{\left|\left|\frac{p}{w_\kappa\left(\frac{1+\vartheta_\kappa^-}{|\vartheta_\kappa^-|}\right)^\daleth}\right|+\left|\frac{q}{w_j\left(\frac{1+\vartheta_j^-}{|\vartheta_j^-|}\right)^\daleth}\right|\right|}\right)}\right)\right)^{1/\daleth}}\right)\right)$$

which is required for BCFDWHM operator.