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Research Article

Variance of the Product of *n* Random Variables

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Abstract: Identities for the variance of the sum of dependent/independent random variables are well known in undergraduate text books. But we are aware of no such identities for the product of dependent/independent random variables. In this note, we derive such identities in the most general forms.

***Keywords*:** covariance, product, sum, variance

MSC: 62E99

1. Introduction

Suppose $X_1, X_2, \ldots, X_n$ are random variables with finite variances and finite covariances. It is well known that

$$\operatorname{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \operatorname{Var}(X_i) + 2 \sum_{i,\ j=1,\ i<j}^{n} \operatorname{Cov}(X_i, X_j). \tag{1}$$

If $X_1, X_2, \ldots, X_n$ are independent random variables then

$$\operatorname{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \operatorname{Var}(X_i). \tag{2}$$

Both these results can be found in most undergraduate textbooks on mathematical statistics.

Often the interest is with product of two or more random variables. Apart from Lemma 1 in Skarupski and Wu [1], which gives an expression for the variance of the product of two independent random variables, we are not aware of any result similar to (1) or (2) for products of random variables.

Products of two or more (independent or dependent) random variables come up in various practical contexts, especially in fields like finance, physics, and engineering. Some real world examples are

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• When considering inflation as one variable and investment returns as another, their product can yield useful insights into potential future profits or losses [2].

• The joint distribution obtained from Bayesian networks exemplifies how the product of probabilistic outcomes across different random variables can be structured. The joint probability distribution can thus be derived as the product of individual probabilities of each random variable, as outlined by Saputro et al. [3]. This is especially significant in decision-making models where the dependencies between variables significantly influence outcomes.

• Product distributions have been extensively studied in relation to specific types of random variables, such as triangular and Gaussian distributions. For instance, Glickman and Xu demonstrated the derivation of probability density functions for the product of independent triangular random variables through integral calculus, highlighting unique properties embodied in this mathematical relationship [4]. Similarly, the distribution of the product of independent Gaussian random variables has been well characterized, which is crucial for applications in areas ranging from signal processing to statistics [5, 6].

• In communication systems, the influence of environmental factors on signal strength can lead to products of multiple random variables impacting the overall yield of the communication signal. Pizon discusses how the product of random variables associated with different communication channels can provide a comprehensive forecasting model [7]. Furthermore, computational methods established by Glen et al. facilitate the calculation of the probability density function for such products, allowing researchers and practitioners to model complex scenarios efficiently [8].

• Investigations into tail behaviors of products of non-negative random variables reveal significant applications, particularly in financial mathematics and reliability engineering. For example, the product of variables representing interest rates and portfolio values can exhibit different stability characteristics compared to the initial variable, which can influence long-term forecasts and risk measures [9, 10].

The purpose of this note is to extend (1) and (2) for products of random variables. The main results are derived in Section 3. Some of their particular cases are stated in Section 4. Some conclusions and future work are noted in Section 5. The proof of the results in Section 2 requires a technical lemma which is stated and proved in Section 2.

Throughout, we define

$$\operatorname{Cov}\left(X_{i_1}, X_{i_2}, \ldots, X_{i_\ell}\right) = E\left\{\left[X_{i_1} - E\left(X_{i_1}\right)\right]\left[X_{i_2} - E\left(X_{i_2}\right)\right]\cdots\left[X_{i_\ell} - E\left(X_{i_\ell}\right)\right]\right\}.$$

2. A technical lemma

The following lemma is needed for Section 3.

Lemma 2.1 We have

$$\prod_{i=1}^{n}(a_i+b_i) = \prod_{i=1}^{n} a_i + \prod_{k=1}^{n} b_i + \sum_{i_1=1}^{n} a_{i_1} \prod_{k=1,\, k\neq i_1}^{n} b_k$$

$$+ \sum_{i_1,\, i_2=1,\, i_1<i_2}^{n} \left[a_{i_1} a_{i_2} \prod_{k=1,\, k\neq i_1,\, i_2}^{n} b_k\right] + \cdots$$

$$+ \sum_{i_1,\, i_2,\, \ldots,\, i_{n-1}=1,\, i_1<i_2<\cdots<i_{n-1}}^{n} \left[a_{i_1} a_{i_2} \cdots a_{i_{n-1}} \prod_{k=1,\, k\neq i_1,\, i_2,\, \ldots,\, i_{n-1}}^{n} b_k\right].$$

Proof. By prove by induction. The result is obvious for $n = 1$. Assume the result for $n = m$. For $n = m+1$,

$$\prod_{i=1}^{m+1}(a_i+b_i)=(a_{m+1}+b_{m+1})\prod_{i=1}^{m}(a_i+b_i)$$

$$=a_{m+1}\prod_{i=1}^{m}a_i+a_{m+1}\prod_{k=1}^{m}b_i+a_{m+1}\sum_{i_1=1}^{m}a_{i_1}\prod_{k=1,\ k\neq i_1}^{m}b_k$$

$$+a_{m+1}\sum_{i_1,\ i_2=1,\ i_1<i_2}^{n}\left[a_{i_1}a_{i_2}\prod_{k=1,\ k\neq i_1,\ i_2}^{m}b_k\right]+\cdots$$

$$+a_{m+1}\sum_{i_1,\ i_2,\ \ldots,\ i_{m-1}=1,\ i_1<i_2<\cdots<i_{m-1}}^{m}\left[a_{i_1}a_{i_2}\cdots a_{i_{m-1}}\prod_{k=1,\ k\neq i_1,\ i_2,\ \ldots,\ i_{m-1}}^{m}b_k\right]$$

$$+b_{m+1}\prod_{i=1}^{n}a_i+b_{m+1}\prod_{k=1}^{m}b_i$$

$$+b_{m+1}\sum_{i_1=1}^{m}a_{i_1}\prod_{k=1,\ k\neq i_1}^{m}b_k$$

$$+b_{m+1}\sum_{i_1,\ i_2=1,\ i_1<i_2}^{m}\left[a_{i_1}a_{i_2}\prod_{k=1,\ k\neq i_1,\ i_2}^{m}b_k\right]+\cdots$$

$$+b_{m+1}\sum_{i_1,\ i_2,\ \ldots,\ i_{m-1}=1,\ i_1<i_2<\cdots<i_{m-1}}^{m}\left[a_{i_1}a_{i_2}\cdots a_{i_{m-1}}\prod_{k=1,\ k\neq i_1,\ i_2,\ \ldots,\ i_{m-1}}^{m}b_k\right].$$

Rearranging the terms, we see that the result is proved for $n=m+1$. □

3. Main results

Theorem 3.1 derives the analog of (1) for the product of n random variables. Corollary 3.1 derives the analog of (2) for the product of n random variables.

Theorem 3.1 Let $X_1,\ X_2,\ \ldots,\ X_n$ be random variables with finite means and finite variances. Then

$$\mathrm{Var}\left(\prod_{i=1}^{n}X_i\right)=E\left[\prod_{i=1}^{n}X_i^2\right]-\left\{E\left[\prod_{i=1}^{n}X_i\right]\right\}^2,$$

where

$$E\left[\prod_{i=1}^{n} X_i\right] = \operatorname{Cov}(X_1,\ X_2,\ \ldots,\ X_n) + \prod_{k=1}^{n} E(X_i)$$

$$+ \sum_{i_1,\ i_2=1,\ i_1<i_2}^{n} \left[\operatorname{Cov}(X_{i_1},\ X_{i_2}) \prod_{k=1,\ k\neq i_1,\ i_2}^{n} E(X_k)\right] + \cdots$$

$$+ \sum_{i_1,\ i_2,\ \ldots,\ i_{n-1}=1,\ i_1<i_2<\cdots<i_{n-1}}^{n} \left[\operatorname{Cov}\left(X_{i_1},\ X_{i_2},\ \cdots,\ X_{i_{n-1}}\right) \prod_{k=1,\ k\neq i_1,\ i_2,\ \ldots,\ i_{n-1}}^{n} E(X_k)\right]$$

and

$$E\left[\prod_{i=1}^{n} X_i^2\right] = \operatorname{Cov}\left(X_1^2,\ X_2^2,\ \ldots,\ X_n^2\right) + \prod_{k=1}^{n} \left\{\operatorname{Var}(X_i) + [E(X_i)]^2\right\}$$

$$+ \sum_{i_1,\ i_2=1,\ i_1<i_2}^{n} \left[\operatorname{Cov}\left(X_{i_1}^2,\ X_{i_2}^2\right) \prod_{k=1,\ k\neq i_1,\ i_2}^{n} \left\{\operatorname{Var}(X_k) + [E(X_k)]^2\right\}\right] + \cdots$$

$$+ \sum_{i_1,\ i_2,\ \ldots,\ i_{n-1}=1,\ i_1<i_2<\cdots<i_{n-1}}^{n} \left[\operatorname{Cov}\left(X_{i_1}^2,\ X_{i_2}^2,\ \cdots,\ X_{i_{n-1}}^2\right) \prod_{k=1,\ k\neq i_1,\ i_2,\ \ldots,\ i_{n-1}}^{n} \left\{\operatorname{Var}(X_k) + [E(X_k)]^2\right\}\right].$$

Proof. Let $\Delta X_i = X_i - E(X_i)$ and $\Omega X_i^2 = X_i^2 - E\left(X_i^2\right)$ for $i = 1,\ 2,\ \ldots,\ n$. By Lemma 2.1, we can write

$$\prod_{i=1}^{n} X_i = \prod_{i=1}^{n} [\Delta X_i + E(X_i)]$$

$$= \prod_{i=1}^{n} \Delta X_i + \prod_{k=1}^{n} E(X_i)$$

$$+ \sum_{i_1=1}^{n} \left[\Delta X_{i_1} \prod_{k=1,\ k\neq i_1}^{n} E(X_k)\right]$$

$$+ \sum_{i_1,\ i_2=1,\ i_1<i_2}^{n} \left[\Delta X_{i_1} \Delta X_{i_2} \prod_{k=1,\ k\neq i_1,\ i_2}^{n} E(X_k)\right] + \cdots$$

$$+ \sum_{i_1,\ i_2,\ \ldots,\ i_{n-1}=1,\ i_1<i_2<\cdots<i_{n-1}}^{n} \left[\Delta X_{i_1} \Delta X_{i_2} \cdots \Delta X_{i_{n-1}} \prod_{k=1,\ k\neq i_1,\ i_2,\ \ldots,\ i_{n-1}}^{n} E(X_k)\right]$$

and

$$\prod_{i=1}^{n} X_i^2 = \prod_{i=1}^{n} \left[\Omega X_i^2 + E\left(X_i^2\right)\right]$$

$$= \prod_{i=1}^{n} \Omega X_i^2 + \prod_{k=1}^{n} E\left(X_i^2\right)$$

$$+ \sum_{i_1=1}^{n} \left[\Omega X_{i_1}^2 \prod_{k=1,\ k\neq i_1}^{n} E\left(X_k^2\right)\right]$$

$$+ \sum_{i_1,\ i_2=1,\ i_1<i_2}^{n} \left[\Omega X_{i_1}^2 \Omega X_{i_2}^2 \prod_{k=1,\ k\neq i_1,\ i_2}^{n} E\left(X_k^2\right)\right] + \cdots$$

$$+ \sum_{i_1,\ i_2,\ \ldots,\ i_{n-1}=1,\ i_1<i_2<\cdots<i_{n-1}}^{n} \left[\Omega X_{i_1}^2 \Omega X_{i_2}^2 \cdots \Omega X_{i_{n-1}}^2 \prod_{k=1,\ k\neq i_1,\ i_2,\ \ldots,\ i_{n-1}}^{n} E\left(X_k^2\right)\right].$$

Taking expectations and noting that $E\Delta X_{i_1} = 0, E\Omega X_{i_1}^2 = 0$ and $E\left(X_k^2\right) = \operatorname{Var}\left(X_k\right) + \left[E\left(X_k\right)\right]^2$, we obtain the result. □

Corollary 3.1 Let $X_1,\ X_2,\ \ldots,\ X_n$ be independent random variables with finite means and finite variances. Then

$$\operatorname{Var}\left(\prod_{i=1}^{n} X_i\right) = \prod_{i=1}^{n} \left\{\operatorname{Var}\left(X_i\right) + \left[E\left(X_i\right)\right]^2\right\} - \prod_{i=1}^{n} \left[E\left(X_i\right)\right]^2.$$

4. Particular cases

Corollaries 4.1 to 4.4 are particular cases of Theorem 3.1 for $n = 2,\ 3,\ 4$.

Corollary 4.1 If $X_1,\ X_2$ are random variables with finite means and finite variances then

$$\operatorname{Var}\left(X_1 X_2\right) = \operatorname{Cov}\left(X_1^2,\ X_2^2\right) + \prod_{i=1}^{2} \left\{\operatorname{Var}\left(X_i\right) + \left[E\left(X_i\right)\right]^2\right\}$$

$$- \left[\operatorname{Cov}\left(X_1,\ X_2\right) + E\left(X_1\right) E\left(X_2\right)\right]^2.$$

If $X_1,\ X_2,\ X_3$ are random variables with finite means and finite variances then

$$
\begin{aligned}
\operatorname{Var}(X_1X_2X_3) = \operatorname{Cov}\left(X_1^2,\ X_2^2,\ X_3^2\right) + \prod_{i=1}^{3}\left\{\operatorname{Var}(X_i) + [E(X_i)]^2\right\} \\
+ \operatorname{Cov}\left(X_2^2,\ X_3^2\right)\left\{\operatorname{Var}(X_1) + [E(X_1)]^2\right\} \\
+ \operatorname{Cov}\left(X_1^2,\ X_3^2\right)\left\{\operatorname{Var}(X_2) + [E(X_2)]^2\right\} \\
+ \operatorname{Cov}\left(X_1^2,\ X_2^2\right)\left\{\operatorname{Var}(X_3) + [E(X_3)]^2\right\} \\
- \left\{\operatorname{Cov}(X_1,\ X_2,\ X_3) + \prod_{i=1}^{3} E(X_i)\right. \\
+ \operatorname{Cov}(X_2,\ X_3)E(X_1) + \operatorname{Cov}(X_1,\ X_3)E(X_2) \\
\left. + \operatorname{Cov}(X_1,\ X_2)E(X_3)\right\}^2.
\end{aligned}
$$

If $X_1,\ X_2,\ X_3,\ X_4$ are random variables with finite means and finite variances then

$$
\begin{aligned}
\operatorname{Var}(X_1X_2X_3X_4) = \operatorname{Cov}\left(X_1^2,\ X_2^2,\ X_3^2,\ X_4^2\right) + \prod_{i=1}^{4}\left\{\operatorname{Var}(X_i) + [E(X_i)]^2\right\} \\
+ \operatorname{Cov}\left(X_3^2,\ X_4^2\right)\left\{\operatorname{Var}(X_1) + [E(X_1)]^2\right\}\left\{\operatorname{Var}(X_2) + [E(X_2)]^2\right\} \\
+ \operatorname{Cov}\left(X_2^2,\ X_4^2\right)\left\{\operatorname{Var}(X_1) + [E(X_1)]^2\right\}\left\{\operatorname{Var}(X_3) + [E(X_3)]^2\right\} \\
+ \operatorname{Cov}\left(X_2^2,\ X_3^2\right)\left\{\operatorname{Var}(X_1) + [E(X_1)]^2\right\}\left\{\operatorname{Var}(X_4) + [E(X_4)]^2\right\} \\
+ \operatorname{Cov}\left(X_1^2,\ X_4^2\right)\left\{\operatorname{Var}(X_2) + [E(X_2)]^2\right\}\left\{\operatorname{Var}(X_3) + [E(X_3)]^2\right\} \\
+ \operatorname{Cov}\left(X_1^2,\ X_3^2\right)\left\{\operatorname{Var}(X_2) + [E(X_2)]^2\right\}\left\{\operatorname{Var}(X_4) + [E(X_4)]^2\right\} \\
+ \operatorname{Cov}\left(X_1^2,\ X_2^2\right)\left\{\operatorname{Var}(X_3) + [E(X_3)]^2\right\}\left\{\operatorname{Var}(X_4) + [E(X_4)]^2\right\}
\end{aligned}
$$

$$+\operatorname{Cov}\left(X_1^2,\ X_2^2,\ X_3^2\right)\left\{\operatorname{Var}(X_4)+[E(X_4)]^2\right\}$$

$$+\operatorname{Cov}\left(X_1^2,\ X_2^2,\ X_4^2\right)\left\{\operatorname{Var}(X_3)+[E(X_3)]^2\right\}$$

$$+\operatorname{Cov}\left(X_1^2,\ X_3^2,\ X_4^2\right)\left\{\operatorname{Var}(X_2)+[E(X_2)]^2\right\}$$

$$+\operatorname{Cov}\left(X_2^2,\ X_3^2,\ X_4^2\right)\left\{\operatorname{Var}(X_1)+[E(X_1)]^2\right\}$$

$$-\left\{\operatorname{Cov}(X_1,\ X_2,\ X_3,\ X_4)+\prod_{i=1}^{4}E(X_i)\right.$$

$$+\operatorname{Cov}(X_3,\ X_4)E(X_1)E(X_2)+\operatorname{Cov}(X_2,\ X_4)E(X_1)E(X_3)$$

$$+\operatorname{Cov}(X_2,\ X_3)E(X_1)E(X_4)+\operatorname{Cov}(X_1,\ X_4)E(X_2)E(X_3)$$

$$+\operatorname{Cov}(X_1,\ X_3)E(X_2)E(X_4)+\operatorname{Cov}(X_1,\ X_2)E(X_3)E(X_4)$$

$$+\operatorname{Cov}(X_1,\ X_2,\ X_3)E(X_4)+\operatorname{Cov}(X_1,\ X_2,\ X_4)E(X_3)$$

$$\left.+\operatorname{Cov}(X_1,\ X_3,\ X_4)E(X_2)+\operatorname{Cov}(X_2,\ X_3,\ X_4)E(X_1)\right\}^2.$$

Corollary 4.2 If $X_1,\ X_2,\ X_3$ are pairwise independent random variables with finite means and finite variances then

$$\operatorname{Var}(X_1X_2X_3)=\operatorname{Cov}\left(X_1^2,\ X_2^2,\ X_3^2\right)+\prod_{i=1}^{3}\left\{\operatorname{Var}(X_i)+[E(X_i)]^2\right\}$$

$$-\left\{\operatorname{Cov}(X_1,\ X_2,\ X_3)+\prod_{i=1}^{3}E(X_i)\right\}^2.$$

If $X_1,\ X_2,\ X_3,\ X_4$ are pairwise independent random variables with finite means and finite variances then

$$
\begin{aligned}
\operatorname{Var}(X_1X_2X_3X_4) = {} & \operatorname{Cov}\left(X_1^2,\ X_2^2,\ X_3^2,\ X_4^2\right) + \prod_{i=1}^{4}\left\{\operatorname{Var}(X_i) + [E(X_i)]^2\right\} \\
& + \operatorname{Cov}\left(X_1^2,\ X_2^2,\ X_3^2\right)\left\{\operatorname{Var}(X_4) + [E(X_4)]^2\right\} \\
& + \operatorname{Cov}\left(X_1^2,\ X_2^2,\ X_4^2\right)\left\{\operatorname{Var}(X_3) + [E(X_3)]^2\right\} \\
& + \operatorname{Cov}\left(X_1^2,\ X_3^2,\ X_4^2\right)\left\{\operatorname{Var}(X_2) + [E(X_2)]^2\right\} \\
& + \operatorname{Cov}\left(X_2^2,\ X_3^2,\ X_4^2\right)\left\{\operatorname{Var}(X_1) + [E(X_1)]^2\right\} \\
& - \left\{\operatorname{Cov}(X_1,\ X_2,\ X_3,\ X_4) + \prod_{i=1}^{4} E(X_i)\right. \\
& + \operatorname{Cov}(X_1,\ X_2,\ X_3)\, E(X_4) + \operatorname{Cov}(X_1,\ X_2,\ X_4)\, E(X_3) \\
& \left. + \operatorname{Cov}(X_1,\ X_3,\ X_4)\, E(X_2) + \operatorname{Cov}(X_2,\ X_3,\ X_4)\, E(X_1)\right\}^2.
\end{aligned}
$$

In the following corollary, we suppose $X_1,\ X_2,\ \ldots,\ X_n$ are Gaussian random variables with zero means and

$$\sigma_{i_1,\ i_2,\ \ldots,\ i_\ell} = \operatorname{Cov}\left(X_{i_1},\ X_{i_2},\ \ldots,\ X_{i_\ell}\right) = E\left(X_{i_1}X_{i_2}\cdots X_{i_\ell}\right)$$

and

$$\varsigma_{i_1,\ i_2,\ \ldots,\ i_\ell} = \operatorname{Cov}\left(X_{i_1}^2,\ X_{i_2}^2,\ \ldots,\ X_{i_\ell}^2\right) = E\left\{\left[X_{i_1}^2 - E\left(X_{i_1}^2\right)\right]\left[X_{i_2}^2 - E\left(X_{i_2}^2\right)\right]\cdots\left[X_{i_\ell}^2 - E\left(X_{i_\ell}^2\right)\right]\right\}.$$

Corollary 4.3 If $X_1,\ X_2$ are Gaussian random variables as stated then

$$\operatorname{Var}(X_1X_2) = \varsigma_{1,\ 2} + \sigma_{1,\ 1}\sigma_{2,\ 2} - \sigma_{1,\ 2}^2.$$

If $X_1,\ X_2,\ X_3$ are Gaussian random variables as stated then

$$\operatorname{Var}(X_1X_2X_3) = \varsigma_{1,\ 2,\ 3} + \prod_{i=1}^{3}\sigma_{i,\ i} + \varsigma 2,\ 3\sigma_{1,\ 1} + \varsigma_{1,\ 3}\sigma_{2,\ 2} + \varsigma_{1,\ 2}\sigma_{3,\ 3} - \sigma_{1,\ 2,\ 3}^2.$$

If $X_1,\ X_2,\ X_3,\ X_4$ are Gaussian random variables as stated then

$$\mathrm{Var}\left(X_1X_2X_3X_4\right) = \varsigma_{1,\ 2,\ 3,\ 4} + \prod_{i=1}^{4}\sigma_{i,\ i} + \varsigma_{3,\ 4}\sigma_{1,\ 1}\sigma_{2,\ 2} + \varsigma_{2,\ 4}\sigma_{1,\ 1}\sigma_{3,\ 3} + \varsigma_{2,\ 3}\sigma_{1,\ 1}\sigma_{4,\ 4}$$

$$+ \varsigma_{1,\ 4}\sigma_{2,\ 2}\sigma_{3,\ 3} + \varsigma_{1,\ 3}\sigma_{2,\ 2}\sigma_{4,\ 4} + \varsigma_{1,\ 2}\sigma_{3,\ 3}\sigma_{4,\ 4} + \varsigma_{1,\ 2,\ 3}\sigma_{4,\ 4} + \varsigma_{1,\ 2,\ 4}\sigma_{3,\ 3} + \varsigma_{1,\ 3,\ 4}\sigma_{2,\ 2}$$

$$+ \varsigma_{2,\ 3,\ 4}\sigma_{1,\ 1} - \sigma^2_{1,\ 2,\ 3,\ 4}.$$

The final corollary is motivated by the theory of normal mean-variance mixtures [11–15]. Suppose that Z_1 and Z_2 are correlated Gaussian random variables with the joint probability density function

$$f_{Z_1,\ Z_2}\left(z_1,\ z_2\right) = \frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left[-\frac{z_1^2+z_2^2-2\rho z_1z_2}{2\left(1-\rho^2\right)}\right]$$

for $-1 < \rho < 1$, $-\infty < z_1 < \infty$ and $-\infty < z_2 < \infty$. Suppose further that Z_3 is a gamma random variable independent of $\left(Z_1,\ Z_2\right)$ with the probability density function

$$f_{Z_3}\left(z_3\right) = \frac{z_3^{a-1}\exp\left(-z_3\right)}{\Gamma(a)}$$

for $a > 0$ and $z_3 > 0$. Define $X_1 = \left|Z_1\right|^{\frac{1}{2}-c}$, $X_2 = Z_2$ and $X_3 = Z_3^c$, where $c \in \left(0,\ \frac{1}{2}\right)$ is a constant.

The product $X_1X_2X_3$ arises in the theory of normal mean-variance mixtures with applications, see [11–15] and references therein for details. It is not difficult to see that

$$\mathrm{Cov}\left(X_1,\ X_2\right) = 0, \qquad \mathrm{Cov}\left(X_1,\ X_2,\ X_3\right) = 0, \qquad \mathrm{Cov}\left(X_1^2,\ X_2^2,\ X_3^2\right) = 0,$$

$$\mathrm{Cov}\left(X_1,\ X_3\right) = 0, \qquad \mathrm{Cov}\left(X_1^2,\ X_3^2\right) = 0, \qquad \mathrm{Cov}\left(X_2,\ X_3\right) = 0, \qquad \mathrm{Cov}\left(X_2^2,\ X_3^2\right) = 0,$$

$$\mathrm{Cov}\left(X_1^2,\ X_2^2\right) = \frac{\rho^2 2^{\frac{1}{2}-c}}{\sqrt{\pi}}\left[2\Gamma(2-c) - \Gamma(1-c)\right],$$

$$E\left(X_1\right) = \frac{2^{\frac{1}{4}-\frac{c}{2}}}{\sqrt{\pi}}\Gamma\left(\frac{3}{4}-\frac{c}{2}\right), \qquad E\left(X_2\right) = 0, \qquad E\left(X_3\right) = \frac{\Gamma(c+a)}{\Gamma(a)},$$

$$\mathrm{Var}\left(X_1\right) = \frac{2^{\frac{1}{2}-c}}{\sqrt{\pi}}\Gamma(1-c) - \frac{2^{\frac{1}{2}-c}}{\pi}\Gamma^2\left(\frac{3}{4}-\frac{c}{2}\right), \qquad \mathrm{Var}\left(X_2\right) = 1,$$

and

$$\operatorname{Var}(X_3)=\frac{\Gamma(2c+a)}{\Gamma(a)}-\left[\frac{\Gamma(c+a)}{\Gamma(a)}\right]^2.$$

Hence, we have the following corollary.

Corollary 4.4 Let $X_1,\ X_2,\ X_3$ be random variables as stated. Then

$$\operatorname{Var}(X_1X_2X_3)=\frac{2^{\frac{1}{2}-c}}{\sqrt{\pi}}\Gamma(1-c)+1+\frac{\Gamma(2c+a)}{\Gamma(a)}+\frac{\rho^2 2^{\frac{1}{2}-c}}{\sqrt{\pi}}(1-2c)\Gamma(1-c)\frac{\Gamma(2c+a)}{\Gamma(a)}.$$

To the best of our knowledge, the result presented in Corollary 4.4 is new and original.

5. Conclusions

We have derived general identities for the variance of the product of n dependent or independent random variables. We have also given particular cases for $n=2,\ 3,\ 4$. Future work is to derive similar identities for the element wise variances of the product of complex random variables or random matrices.

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Conflict of interest

Both authors have no conflicts of interest.

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