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Research Article

Nonconforming Finite Elements and Multigrid Methods for Maxwell Eigenvalue Problem

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Abstract: The Maxwell eigenvalue problem refers to the task of solving the Maxwell equations under specific boundary conditions. In this paper, we primarily discuss nonconforming finite elements and multigrid methods for Maxwell eigenvalue Problem. By using an appropriate operator, the eigenvalue problem can be viewed as a curl-curl problem. We obtain the approximate optimal error estimates on graded mesh. We also prove the convergence of the W-cycle and full multigrid algorithms for the corresponding discrete problem.

Keywords: Maxwell eigenvalue problem, nonconforming finite element, multigrid method, curl-curl problem

MSC: 65M60

1. Introduction

Let $\mathscr{D}$ be a bounded polynomial domain in $\mathbb{R}^2$. We consider the following Maxwell eigenvalue problem:
Find $(u,\ \lambda) \in H_0(\text{curl};\ \mathscr{D}) \cap H(\text{div}^0;\ \mathscr{D})$ such that

$$(\nabla \times u,\ \nabla \times \mathbf{v}) = \lambda(u,\ \mathbf{v}) \qquad \forall\ v \in H_0(\text{curl};\ \mathscr{D}) \cap H(\text{div}^0;\ \mathscr{D}), \tag{1}$$

where $(\cdot,\ \cdot)$ denotes the inner product in $[L_2(\mathscr{D})]^2$, and the function spaces are defined as follows.

$$H(\text{curl};\ \mathscr{D}) = \left\{ \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in [L_2(\mathscr{D})]^2 : \nabla \times \mathbf{v} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \in L_2(\mathscr{D}) \right\},$$

$$H(\text{div};\ \mathscr{D}) = \left\{ \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in [L_2(\mathscr{D})]^2 : \nabla \cdot \mathbf{v} = \frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2} \in L_2(\mathscr{D}) \right\},$$

$$H_0(\text{curl};\ \mathscr{D}) = \{\mathbf{v} \in H(\text{curl};\ \mathscr{D}) : \mathbf{n} \times \mathbf{v} = 0 \text{ on } \partial\mathscr{D}\},$$

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and

$$H(\text{div}^0;\ \mathscr{D}) = \{\mathbf{v} \in H(\text{div};\ \mathscr{D}) : \nabla \cdot \mathbf{v} = 0\}.$$

Here, the vector $\mathbf{n}$ is the unit outer normal on $\partial\mathscr{D}$.

In recent years, with the rapid development of modern technology, Maxwell's equations have not only played a key role in traditional fields such as radio communication and radar systems, but also demonstrated new potential applications in cutting-edge areas such as new material science, biomedical engineering, and nano-optics [1, 2]. Maxwell eigenvalue problem refers to solving the Maxwell equations under specific boundary conditions rather than moving boundary conditions [3, 4]. In addition, it plays a crucial role in understanding the fundamental properties of electromagnetic fields in various physical systems.

In recent years, researchers have made a series of significant advancements in the Maxwell eigenvalue problem. For example, Brenner proposed the application of the finite element method to the eigenvalue problem of Maxwell's equations, demonstrating efficient computational capabilities and precise error analysis in the numerical solution of complex electromagnetic problems [5]. In addition, Boffi considered the finite element approximation of the Maxwell eigenvalue problem in two dimensions, and proved convergence piecewise linear elements on Powell-Sabin triangulations, piecewise quadratic elements on Clough-Tocher triangulations and piecewise quartics (and higher) elements on general shape-regular triangulations [6]. Besides, Bramble consider an approximation to the Maxwell's eigenvalue problem based on a very weak formulation of two div-curl systems with complementary boundary conditions [7]. Since the eigenfunction u has divergence-free constraint, it is not easy to achieve in numerical approximations. Researches [8–17] replace the Maxwell eigenvalue problem with the following by neglecting the divergence-free condition:

Find $(u,\ \lambda) \in H_0(\text{curl};\ \mathscr{D}) \times \mathbb{R}$ such that $u \neq 0$,

$$(\nabla \times u,\ \nabla \times \mathbf{v}) = \lambda(u, \mathbf{v}) \quad \forall v \in H_0(\text{curl};\ \mathscr{D}). \tag{2}$$

However, (2) introduces a non-physical zero eigenvalue into the spectrum. It will add more complexity when we analyze the eigensolvers.

In this paper, we present a numerical scheme by relating eigensolvers to a curl-curl problem. The scheme was proposed early in [5]. In order to simplify the problem into several scalar elliptic boundary value problems, we turn to introduce the Hodge decomposition, which has been applied to many problems. For example, the quad-curl Problem in [18], the Maxwell's equations in [19] and the two-dimensional time-harmonic Maxwell's equations with impedance boundary condition in [20]. Besides, [12] and [19] discuss the Hodge decomposition for three-dimensional vector fields.

In the conforming finite element method, the space V is required to contain the finite element space V_h, i.e., $V_h \subset V$. However, in the nonconforming finite element method, this condition is relaxed to $V_h \not\subset V$, thereby providing greater flexibility and adaptability for addressing more complex problems [21]. During the development of nonconforming finite element methods, Crouzeix and Raviart proposed the simplest nonconforming P_1 finite element method in the 1980s [22]. This method is based on the space of linear polynomials, with a notable characteristic that the degrees of freedom within an element do not need to maintain continuity across adjacent elements.

In contrast, Nedelec elements typically requires strict adherence to geometric conditions, and it has relatively poor adaptability to complex meshes; Discontinuous Galerkin finite element requires the construction of jump integrals and numerical fluxes. Both of them are resulting in high computational complexity.

Furthermore, the multigrid method is proposed for solving some boundary value problems in this work. Due to computational limitations, traditional methods often exhibit the disadvantage of slow convergence when faced with problems of high degrees of freedom and complex geometric shapes [23, 24]. However, multigrid methods offer fast convergence, versatility across Partial Differential Equation (PDE) types, and are ideal for parallel computing, making them efficient for large-scale, complex problems in various fields. It has been used in many works such as quantum eigenvalue problems [25], nonlinear eigenvalue problems based on Newton iteration [26] and coupled semilinear elliptic equation [27].

The rest of the paper is organized as follows. We analyze discrete problems by using graded meshes in Section 2. Then we introduce multigrid methods and derive related convergence rates in Section 3. In section 4, a summary is concluded.

2. Discrete problems based on graded meshes

In this section, we present an eigensolver which is related to a curl-curl problem. Furthermore, by applying the Hodge decomposition and the nonconforming finite elements, the convergence results are given.

2.1 *Construction of Maxwell eigensolver*

We introduce a bounded linear operator $T : [L_2(\mathscr{D})]^2 \to [L_2(\mathscr{D})]^2$ for the Maxwell eigenvalue problem (1). Given any function $f \in [L_2(\mathscr{D})]^2$, we define Tf satisfy

$$(\nabla \times Tf,\ \nabla \times v) + (Tf,\ v) = (f,\ v), \tag{3}$$

for all $v \in H_0(\text{curl};\ \mathscr{D}) \cap H(\text{div}^0;\ \mathscr{D})$. Obviously, T is a symmetric positive and compact operator from $[L_2(\mathscr{D})]^2$ to $[L_2(\mathscr{D})]^2$. In addition, $(u,\ \lambda)$ satisfies equation (1) if and only if $Tu = \dfrac{1}{1+\lambda}u$. Note that the eigenfunctions of T are exactly the eigenfunctions of the Maxwell equations.

2.2 *Hodge decomposition*

We define $\xi = \nabla \times Tf \in H^1(\mathscr{D})$, where ξ satisfies

$$(\nabla \times \xi,\ \nabla \times \psi) + (\xi,\ \psi) = (f,\ \nabla \times \psi) \quad \forall \psi \in H^1(\mathscr{D}). \tag{4}$$

Therefore, the Hodge decomposition of Tf is

$$Tf = \nabla \times \rho + \sum_{j=1}^{m} c_j \nabla \eta_j. \tag{5}$$

Here

$$\frac{\partial \rho}{\partial n} = 0 \quad \text{on } \partial\mathscr{D} \tag{6}$$

with the constraint

$$(\rho, 1) = \int_{\mathscr{D}} \rho\, dx = 0, \tag{7}$$

and m is a non-negative integer.

Suppose that $\partial\mathscr{D}$ has $m+1$ components. Γ_0 denotes the outward boundary of $\mathscr{D}$ and $\Gamma_1, \ldots, \Gamma_m$ denote the m parts of the interior boundaries. The functions $\eta_1, \ldots, \eta_m$ are defined as

$$\begin{aligned}
&(\nabla\eta_j, \nabla v) = 0 \quad \forall v \in H_0^1(\mathscr{D}),\\
&\eta_j\big|_{\Gamma_0} = 0,\\
&\eta_j\big|_{\Gamma_i} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad 1 \leq i \leq m.
\end{aligned} \tag{8}$$

The function ρ satisfies (7) and is determined by

$$(\nabla\times\rho, \nabla\times\psi) = (\xi, \psi) \quad \forall\psi \in H^1(\mathscr{D}). \tag{9}$$

The constants $c_1, \ldots, c_m$ in (5) are determined by

$$\sum_{j=1}^{m} c_j (\nabla\eta_j, \nabla\eta_j) = (f, \nabla\eta_i) \quad 1 \leq i \leq m. \tag{10}$$

Thus, (5) is solved by five steps:

(a) Compute the numerical approximation ξ_h of ξ by solving problem (4).
(b) Replace ξ with ξ_h and solve for the numerical approximation ρ_h of ρ by using (9).
(c) Compute the approximations $\eta_{1,\,h}, \ldots, \eta_{m,\,h}$ of $\eta_1, \ldots, \eta_m$ by solving the boundary value problems (8).
(d) Obtain the approximations $c_{1,\,h}, \ldots, c_{m,\,h}$ of $c_1, \ldots, c_m$ by solving the symmetric positive problem (10).
(e) Compute the numerical approximation $T_h f$ of Tf as

$$T_h f = \nabla\times\rho_h + \sum_{j=1}^{m} c_{j,\,h}\nabla\eta_{j,\,h}. \tag{11}$$

2.3 *A nonconforming finite element method*

Let τ_h be a family of triangulations of $\mathscr{D}$. We define the weight $\rho_\mu(T)$ associated with $T \in \tau_h$ as

$$\chi_\mu(T) = \prod_{i=1}^{L} |c_l - c_T|^{1-\mu_l}.$$

Here $c_1, \ldots, c_L$ are the corners of $\mathscr{D}$ with interior angles $\omega_1, \ldots, \omega_L$, and μ_l $(1 \leq l \leq L)$ is the grading parameter which is chosen by

$$\mu_l = 1 \quad \omega_l \leq \frac{\pi}{2}, \tag{12}$$

$$\mu_l < \frac{\pi}{2\omega_l} \quad \omega_l > \frac{\pi}{2}. \tag{13}$$

The graded mesh τ_h satisfies the following condition

$$h_T = \mathrm{diam}(T) \approx \chi_\mu(T)h \quad \forall T \in \tau_h, \tag{14}$$

where h is the mesh parameter.

Define a weighted Sobolev space

$$L_{2,\ \mu}(\mathscr{D}) = \left\{ \zeta \in L_{2,\ \mathrm{loc}}(\mathscr{D}) : \|\zeta\|^2_{L_{2,\ \mu}(\mathscr{D})} = \int_{\mathscr{D}} \chi^2_\mu(x)\zeta^2(x)\ dx < \infty \right\},$$

where the weight function χ_μ is determined by

$$\chi_\mu(x) = \prod_{l=1}^{L} |x - c_l|^{1-\mu_l}.$$

Clearly $L_2(\mathscr{D}) \subset L_{2,\ \mu}(\mathscr{D})$ and

$$\|\zeta\|_{L_{2,\ \mu}(\mathscr{D})} \leq C_{\mathscr{D}} \|\zeta\|_{L_2(\mathscr{D})} \quad \forall \zeta \in L_2(\mathscr{D}). \tag{15}$$

Hence (5) has a unique solution for any $f \in L_{2,\ \mu}(\mathscr{D})$. Moreover, the norm of the dual space $L_{2,\ -\mu}(\mathscr{D})$ of $L_{2,\ \mu}(\mathscr{D})$ is defined by

$$\|\xi\|^2_{L_{2,\ -\mu}(\mathscr{D})} = \int_{\mathscr{D}} \chi^{-2}_\mu(x)\xi^2(x)\ dx.$$

The nonconforming P_1 finite element space V_h associated with T_h is defined by

$$V_h = v \in L_2(\mathscr{D}) : v_T = v|_T \in P_1(T) \quad \forall T \in \tau_h,\ v = 0 \text{ at the midpoints of } \partial\mathscr{D}$$

v maintains continuity at the midpoints of the edges of τ_h.

Let $\Pi_h : C(\mathscr{D}) \to V_h$ be a weak interpolation operator for the nonconforming P_1 finite element. Therefore, Π_h satisfies the following interpolation error estimate for the Neumann problem (9) and the Dirichlet problem (8), which are similar to [28–30]. We have

$$\|\eta - \Pi_h \eta\|_{L_2(\mathscr{D})} + h|\eta - \Pi_h \eta|_{H^1(\mathscr{D})} \leq Ch^2. \tag{16}$$

Moreover,

$$\|\beta - \Pi_h \beta\|_{L_2(\mathscr{D})} + h|\beta - \Pi_h \beta|_{H^1(\mathscr{D})} \leq Ch^2 \|g\|_{L_{2,\ \mu}(\mathscr{D})}, \tag{17}$$

where β represents the solution to the Laplace equation subject to Neumann boundary conditions, and g denotes the function on the right-hand side (cf. [19]).

Let E_h be a set of all edges in T_h. We define $E_h^i = E_h \setminus \partial\mathscr{D}$ be the set of all interior edges. Let $e \in E_h^i$ be the edge shared by two triangles T_1, $T_2 \in T_h$ and $v_j = v|_{T_j},\ j = 1,\ 2$. Define the jump on e by $[\![v]\!] = n_1 v_1 + n_2 v_2$, where $n_1,\ n_2$ are the unit outward normal vector. If e is a boundary edge of $\mathscr{D}$, then $[\![v]\!] = vn$. Next we consider the nonconforming P_1 finite element method for (4), which is to find $\xi_h \in V_h$ such that

$$a_h(\xi_h,\ v) = F(v) \quad \forall v \in V_h, \tag{18}$$

where

$$a_h(\xi_h,\ v) = \sum_{T \in \tau_h} \int_T \nabla \times \xi_h \cdot \nabla \times v\ dx + (\xi_h,\ v), \tag{19}$$

$$F(v) = (f,\ \nabla \times v) \quad \forall v \in V_h. \tag{20}$$

The nonconforming P_1 finite element method for (9) is to find $\rho_h \in V_h$ such that

$$\hat{a}_h(\rho_h,\ v) = (\xi_h,\ v) \quad \forall v \in V_h, \tag{21}$$

where

$$\hat{a}_h(\rho_h,\ v) = \sum_{T \in \tau_h} \int_T \nabla \rho_h \cdot \nabla v\ dx,$$

$$(\rho_h,\ 1) = 0.$$

When $m \geq 1$, the approximation $\eta_{j,\ h} \in V_h$ of the harmonic function η_j in (8) is defined by

$$
\begin{aligned}
&\left(\nabla\eta_{j,\,h},\, \nabla v\right) = 0 \quad \forall v \in H_0^1(\mathscr{D}),\\
&\eta_{j,\,h}\big|_{\Gamma_0} = 0,\\
&\eta_{j,\,h}\big|_{\Gamma_i} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad 1 \le i \le m.
\end{aligned} \tag{22}
$$

To compute $c_{j,\,h}$, we introduce the following system:

$$
\sum_{j=1}^{m} c_{j,\,h}(\nabla\eta_{j,\,h}, \nabla\eta_{i,\,h}) = (f,\, \nabla\eta_{i,\,h}) \quad 1 \le i \le m. \tag{23}
$$

Thus, the piecewise constant vector field $T_h f$ of Tf is defined as

$$
T_h f = \nabla \times \rho_h + \sum_{j=1}^{m} c_{j,\,h} \nabla\eta_{j,\,h}. \tag{24}
$$

2.4 *Error analysis*

We start this section by defining a mesh-dependent energy norm $\|\cdot\|_h$ for any $v \in H^1(\mathscr{D}) + V_h$ as follows

$$
\|v\|_h = \sqrt{a_h(v,\, v)}.
$$

Combining with the Cauchy-Schwarz inequality, we observe that the form $a_h(v,\, v)$ is restricted with respect to with respect to $\|\cdot\|_h$, i.e.,

$$
|a_h(\omega,\, v)| \le \|\omega\|_h \|v\|_h \quad \forall v,\, \omega \in H^1(\mathscr{D}) + V_h.
$$

Firstly, we introduce the following Lemmas (cf. [21]).

Lemma 1 Suppose $dimV_h < \infty$, $a_h(\cdot,\, \cdot)$ is a symmetric positive-definite bilinear form on $V + V_h$ and u_h solve

$$
a_h(u_h,\, v) = F(v) \quad \forall v \in V_h.
$$

Then

$$
\|u - u_h\|_h \le \inf_{v \in V_h} \|u - v\|_h + \sup_{w \in V_h \setminus \{0\}} \frac{|a_h(u - u_h,\, w)|}{\|w\|_h},
$$

where $\|\cdot\|_h = \sqrt{a_h(\cdot,\ \cdot)}$.

Lemma 2 Let $T_h,\ h \in (0,\ 1]$ be a non-degenerate family of subdivisions of a polyhedral domain $\mathscr{D}$, suppose $(\mathscr{K},\ \mathscr{P},\ \mathscr{N})$ is a reference element. $1 \le p \le \infty$ and either $m - ln/p > 0$ when $p > 1$ or $m - l - n \ge 0$ when $p = 1$. And $\mathscr{I}^h$ is the global interpolation operator, for all $T \in T_h,\ h \in (0,\ 1]$ then there exists a positive constant C depending on the reference element, $n,\ m,\ p$ such that for $0 \le s \le m$,

$$\left(\sum_{T \in T_h} \left\|v - \mathscr{I}^h v\right\|^p_{w^p_\infty(T)}\right)^{1/p} \le Ch^{m-s}|v|_{w^m_p(\mathscr{D})} \quad \forall v \in W^m_p(\mathscr{D}),$$

when $p = \infty$, $\max_{T\in T_h} \left\|v - \mathscr{I}^h v\right\|_{w^s_\infty(T)}$. For $0 \le s \le l$,

$$\max_{T \in T^h} \left\|v - \mathscr{I}^h v\right\|_{w^s_\infty(T)} \le Ch^{m-s-n/p}|v|_{w^m_p(\mathscr{D})} \quad \forall v \in W^m_p(\mathscr{D}).$$

Lemma 3 Suppose $\mathscr{D}$ with Lipschitz boundary, $1 \le p \le \infty$, then there exists a positive constant C such that

$$\|v\|_{L^p(\partial\mathscr{D})} \le C\|v\|^{1-1/p}_{L^p(\mathscr{D})}\|v\|^{1/p}_{W^1_p(\mathscr{D})} \quad \forall v \in W^1_p(\mathscr{D}).$$

Next we turn to the error estimate.

Theorem 1 Let ξ_h be the solution of (18). Then the following discrete error estimate holds

$$\|\xi - \xi_h\|_h \le Ch\|f\|_{L_2(\mathscr{D})}, \tag{25}$$

where C is a positive constant.

Proof. Let $\omega \in V_h$ be arbitrary. Combining with (4), (19) and the partial integration, we obtain

$$a_h(\xi - \xi_h,\ \omega) = \sum_{e \in \varepsilon^h} \int_e \nabla\xi \cdot [\![\omega]\!] ds.$$

By using the midpoint rule, Cauchy-Schwarz inequality, Lemma 1, Lemma 2 and Lemma 3, we have

$$|a_h(\xi - \xi_h,\ \omega)| \le Ch\|f\|_{L_2(\mathscr{D})}\|\omega\|_h, \tag{26}$$

$$\|\xi - \xi_h\|_h \le \inf_{\xi_h \in V_h} \|\xi - \xi_h\|_h + \sup_{\omega \in V_h\setminus\{0\}} \frac{|a_h(\xi - \xi_h,\ \omega)|}{\|\omega\|_h}, \tag{27}$$

$$\inf_{\xi_h \in V_h} \|\xi - \xi_h\|_h \le \|\xi - \Pi_h\xi\|_h \le Ch\|f\|_{L_2(\mathscr{D})}. \tag{28}$$

The estimate (25) follows from (26), (27) and (28).

Theorem 2 For the solution ξ_h of (18), the following discrete error estimate holds

$$\|\xi - \xi_h\|_{L_{2,\ -\mu}(\mathscr{D})} \leq Ch\|f\|_{L_2(\mathscr{D})}. \tag{29}$$

Proof. Let the dual argument $\zeta \in H^1(\mathscr{D})$ be defined by

$$(\nabla \times \zeta,\ \nabla \times v) + (\zeta,\ v) = (\rho_\mu^{-2}(\xi - \xi_h),\ v) \quad \forall v \in H^1(\mathscr{D}).$$

Hence

$$\|\xi - \xi_h\|^2_{L_{2,\ -\mu}(\mathscr{D})} = (\nabla \times \zeta, \nabla \times (\xi - \xi_h)) + (\zeta,\ \xi - \xi_h)$$

$$\leq \|\zeta - \Pi_h \zeta\|_h \|\xi - \xi_h\|_h.$$

In view of the definition of Π_h, combining with Lemma 2, we have

$$\|\zeta - \Pi_h \zeta\|_h \leq Ch\|f\|_{L_2(\mathscr{D})}. \tag{30}$$

The estimate (29) follows from (25) and (30).
Corollary 1 Suppose the condition in Theorem 2 holds, we have

$$\|\xi - \xi_h\|_{L_2(\mathscr{D})} \leq Ch\|f\|_{L_2(\mathscr{D})}. \tag{31}$$

Theorem 3 Assume $f \in [L_2(\mathscr{D})]^2$. Then

$$|\rho - \rho_h|_{H^1(\mathscr{D})} \leq Ch\|f\|_{L_2(\mathscr{D})}. \tag{32}$$

Proof. Combining the midpoint rule, Cauchy-Schwarz inequality, Lemma 1, Lemma 2 and Lemma 3, we have

$$\|\rho - \rho_h\|_{L_2(\mathscr{D})} \leq Ch\|\xi\|_{H^1(\mathscr{D})}. \tag{33}$$

Since $\|\xi\|_{H^1(\mathscr{D})} \leq C\|f\|_{L_2(\mathscr{D})}$, then

$$\|\rho - \rho_h\|_{L_2(\mathscr{D})} \leq Ch\|f\|_{L_2(\mathscr{D})}. \tag{34}$$

Combining (31), (34) and Cauchy-Schwarz inequality, we have

$$|\rho_h-\rho|^2_{H^1(\mathscr{D})}=\|\nabla\times(\rho-\rho_h)\|^2_{L_2(\mathscr{D})}=(\xi-\xi_h,\ \rho-\rho_h)$$

$$\leq\|\xi-\xi_h\|_{L_2(\mathscr{D})}\|\rho-\rho_h\|_{L_2(\mathscr{D})}\leq Ch\|f\|_{L_2(\mathscr{D})}Ch\|f\|_{L_2(\mathscr{D})},$$

which means

$$|\rho-\rho_h|_{H^1(\mathscr{D})}\leq Ch\|f\|_{L_2(\mathscr{D})}.$$

Next we compare η_j with $\eta_{j,\ h}$ in $H^1(\mathscr{D})$. Clearly, we obtain $\eta_{j,\ h}$ by solving the Dirichlet problem (22).

Lemma 4 For $1\leq j\leq m$, we have

$$|\eta_j-\eta_{j,\ h}|_{H^1(\mathscr{D})}\leq Ch. \tag{35}$$

Proof. By using (16), we know

$$\|\eta-\Pi_h\eta\|_{L_2(\mathscr{D})}+h|\eta-\Pi_h\eta|_{H^1(\mathscr{D})}\leq Ch^2. \tag{36}$$

Let $\zeta_2\in H^1(\mathscr{D})$ which is determined by $(\nabla\zeta_2,\ \nabla v)=(\nabla(\eta_h-\eta),\ v),\ \forall v\in H^1(\mathscr{D})$. There exists a unique solution $\tilde{\eta}_h$ of (8) such that

$$|\tilde{\eta}_h-\eta|^2_{H^1(\mathscr{D})}=\|\nabla\times(\tilde{\eta}_h-\eta_h)\|^2_{L_2(\mathscr{D})}=(\nabla\zeta_2,\ \nabla(\tilde{\eta}_h-\eta_h))$$

$$\leq\|\nabla\zeta_2\|_{L_2(\mathscr{D})}\|\nabla(\tilde{\eta}_h-\eta_h)\|_{L_2(\mathscr{D})}\leq Ch\|\tilde{\eta}_h-\eta\|_{H^1(\mathscr{D}),}$$

which means

$$|\tilde{\eta}_h-\eta|_{H^1(\mathscr{D})}\leq Ch. \tag{37}$$

By (36), we know

$$|\eta_j-\tilde{\eta}_h|_{H^1(\mathscr{D})}\leq|\eta_j-\Pi_h\eta_j|_{H^1(\mathscr{D})}\leq Ch. \tag{38}$$

The estimate (35) follows from (37) and (38).

Combining with (10), (23) and (35), we have the following lemma with respect to the error estimate of $c_{j,\ h}$.

Lemma 5 [30] For $1\leq j\leq m$, then

$$|c_j - c_{j,\,h}| \leq Ch\|f\|_{L_2(\mathscr{D})}. \tag{39}$$

Theorem 4 Suppose h is small enough and $T_h f$ is the solution of (24). Then

$$\|Tf - T_h f\|_{L_2(\mathscr{D})} \leq Ch\|f\|_{L_2(\mathscr{D})}. \tag{40}$$

Proof. Bases on Lemma 3-Lemma 5 and triangle inequality, the discrete error estimate holds.

3. Multigrid methods

In this section, we establish the multigrid algorithm for solving discrete problems (18) and (21) on graded meshes.

For the initial triangulation τ_0 on an L-shaped domain, we chose a properly grading factor μ_l according to (12) and consider the procedure to generate the triangulation τ_k $(k \geq 1)$ which is the same as [29–31].

(a) If any vertex of $T \in T_k$ is not a reentrant corner, then $T \in T_k$ is divided uniformly by connecting midpoints of the edges of T.

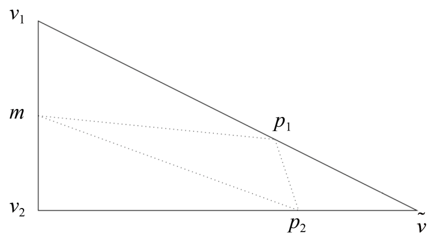

Figure 1. Refinement of a triangle at a reentrant corner

(b) Suppose $v_1,\ v_2,\ \tilde{v}$ are the vertexes of $T \in \tau_k$. For the midpoint of the edge v_1v_2, we denote as m. If $\tilde{v}$ is a reentrant corner, then $T \in T_k$ is divided by connecting $p_1,\ p_2$ and m, where p_i $(i = 1,\ 2)$ is a point on the edge $\tilde{v}v_i$ $(i = 1,\ 2)$ (cf. Figure 1) such that

$$\left|\frac{\tilde{v} - p_i}{\tilde{v} - v_i}\right| = 2^{-(1/\mu_l)} \quad i = 1,\ 2.$$

We take μ_l as $\frac{2}{3}$ when depicting the triangulation T_2 on the L-shaped domain in Figure 2.

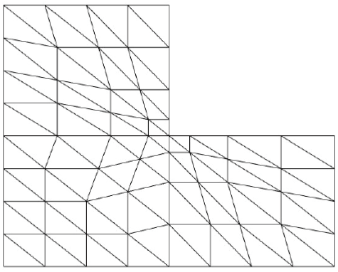

Figure 2. The triangulation T_2

3.1 *W-cycle multigrid algorithm*

3.1.1 *The k-th level multigrid algorithm*

Since these triangulations τ_k $(k \geq 0)$ satisfy the condition (14), we turn to suppose

$$h_k = \frac{1}{2} h_{k-1} \quad k \geq 1. \tag{41}$$

Let V_k be the nonconforming P_1 finite element space associated with T_k. For each k, the bilinear form $a_k(u, v)$ is defined on $V_k + H^1(\mathscr{D})$ as follows

$$a_k(u, v) = \sum_{T \in \tau_h} \int_T \nabla u \cdot \nabla v \ dx + (u, v). \tag{42}$$

The norm $\|v\|_k$ defines as the analog of $\|v\|_h$, i.e., $\|v\|_k = \sqrt{a_k(v, v)}$, and the analog of Π_h is defined by Π_k.

We introduce the operator $A_k: V_k \to V_k'$ as $\langle A_k \omega, v \rangle = a_k(\omega, v)$, $\forall \omega, v \in V_k$, where $\langle \cdot \rangle$ denotes the canonical bilinear form on $V_k' \times V_k$. The k-th level nonconforming finite element method for (4) is to find $\xi_k \in V_k$ such that

$$A_k \xi_k = f_k, \tag{43}$$

where $f_k \in V_k'$ satisfies $\langle f_k, v \rangle = (f, \nabla \times v)$ $\forall v \in V_k$. It is clear that (43) can be solved by the multigrid algorithms.

Since V_k is a nonconforming finite element space, $V_k \not\subset V_{k+1}$ and $V_k \not\subset H^1(\mathscr{D})$, we cannot directly use the natural injection transfer as in the finite element spaces. Moreover, we define a proper intergrid transfer operator $N_{k-1}^k : V_{k-1} \to V_k$ as a natural injection (cf. [32]). But the actual value of $(N_{k-1}^k v)(p)$ is determined by

$$(N_{k-1}^k v)(p) = \begin{cases} v(p) & p \in S_{k-1}, \\ \frac{1}{2}\left[v|_{T_1}(p) + v|_{T_2}(p)\right] & p \notin S_{k-1} \text{ and } p \text{ is shared by } T_1, T_2 \in \tau_k, \\ v(p) & \text{otherwise}, \end{cases}$$

where S_{k-1} is the vertices set of τ_{k-1} for any $p \in S_k$.

Define the fine to coarse intergrid transfer operator $N_k^{k-1} : V_k' \to V_{k-1}'$ to be the transpose of N_{k-1}^k which is related to $\langle \cdot, \cdot \rangle$, i.e.,

$$\langle N_k^{k-1}\omega,\ v\rangle = \langle \omega,\ N_{k-1}^k v\rangle \quad \forall \omega \in V_k',\ v \in V_{k-1}. \tag{44}$$

Nn order to analyze the error estimate, we define an operator $B_k : V_k \to V_k'$ such that

$$\langle B_k\omega,\ v\rangle = h_k^2 \sum_{T\in\tau_k}\sum_{m\in M_T} \omega(m)v(m), \tag{45}$$

where M_T is the set of vertices on the triangle T. It is easy to know that the spectral radius of $B_k^{-1}A_k$ satisfies $\rho(B_k^{-1}A_k) < Ch_k^{-2},\ \forall k \geq 0$. An appropriate damping factor λ is chosen such that the spectral radius $\rho(\lambda h_k^2 B_k^{-1}A_k)$ satisfies

$$\rho(\lambda h_k^2 B_k^{-1}A_k) < 1 \quad \forall k \geq 0. \tag{46}$$

Next we introduce a W-cycle algorithm for the equation

$$A_k z = g \quad \forall z \in V_k \quad \forall g \in V_k'. \tag{47}$$

Algorithm 1 $Mul_W(k,\ g,\ z_0,\ a_1,\ a_2)$ denote the output of the algorithm, where $z_0 \in V_k$ is the initial guess. Furthermore, the pre-smoothing and post-smoothing steps are denoted as a_1 and a_2. For $k = 0$, $Mul_W(0,\ g,\ z_0,\ a_1,\ a_2) = A_0^{-1}g$. For $k > 0$, $Mul_W(0,\ g,\ z_0,\ a_1,\ a_2)$ is compute by following procedure.

Pre-smoothing. With the condition $1 \leq l \leq a_1$, $z_l \in V_k$ is computed by

$$z_l = z_{l-1} + (\lambda h_k^2)B_k^{-1}(g - A_k z_{l-1}).$$

Error correction. Let $q_0 = 0$. For $1 \leq i \leq 2$, compute $z_{a_1} + 1$ recursively by

$$r_{k-1} = N_k^{k-1}(g - A_k z_{a_1}),\ q_i = Mul(k-1,\ r_{k-1},\ q_{i-1},\ a_1,\ a_2),\ z_{a_1} + 1 = z_{a_1} + N_{k-1}^k q_2.$$

Post-smoothing. For $a_1 + 2 \leq l \leq a_1 + a_2 + 1$, z_l is determined by

$$z_l = z_{l-1} + (\lambda h_k^2)B_k^{-1}(g - A_k z_{l-1}). \tag{48}$$

Finally, the output of the k-th level iteration is $Mul_W(k,\ g,\ z_0,\ a_1,\ a_2) = z_{a_1} + a_2 + 1$.

The multigrid Algorithm 1 can also be modified to solve the singular Neumann problem (9).

The space $\hat{V}_k$ is defined by $\hat{V}_k = \{v \in V_k : (v,\ 1) = 0\}$. We denote the orthogonal projection $\hat{P}_k : V_k \to \hat{V}_k$ with respect to $(\cdot,\ \cdot)_k$. Moreover, for any $v \in V_k$, $\hat{P}_k v \in \hat{V}_k$ satisfies

$$(w,\ \hat{P}_k v)_k = (w,\ v)_k \quad \forall \omega \in V_k. \tag{49}$$

We turn to compute $\hat{P}_k v$ explicitly as follows

$$\hat{P}_k v = v - \frac{(v,\ s_k)_k}{(s_k,\ s_k)_k} s_k, \tag{50}$$

where $s_k \in V_k$ spans the orthogonal complement of $\hat{V}_k$ with respect to $(\cdot,\ \cdot)_k$. In addition, we take N_k as the set of all the nodes associated with V_k and define s_k as the finite element function $s_k(p) = \dfrac{1}{3h_k^2 \cdot n(T_p)} \sum_{T \in T_p} |T|,\ \forall p \in N_k$, where T_p is the set of triangles in τ_k sharing p as a common vertex, $n(\tau_p)$ is the number of triangles in τ_p, and $|T|$ is the area of T.

The natural injection is denoted by $\hat{N}_k : \hat{V}_k \to V_k$. Moreover, an operator

$$\hat{A}_k = \hat{P}_k \circ B_k^{-1} \circ A_k \circ \hat{N}_k \tag{51}$$

is determined by$(\hat{A}_k w,\ v)_k = \sum_{T \in \tau_k} \int_T \nabla w \cdot \nabla v\ dx,\ \forall w,\ v \in \hat{V}_k$.

Now we define a W-cycle algorithm for

$$\hat{A}_k z = g \quad z \in \hat{V}_k, g \in \hat{V}_k'. \tag{52}$$

Algorithm 2 $Mul_W^1(k,\ g,\ z_0,\ a_1,\ a_2)$ denote the output of the algorithm, where $z_0 \in V_k$ is the initial guess. Furthermore, the pre-smoothing and post-smoothing steps are denoted as a_1 and a_2. For $k = 0$, $Mul_W^1(0,\ g,\ z_0,\ a_1,\ a_2) = (\hat{A_0})^{-1}(\hat{P}_0 B_0^{-1} g)$. For $k > 0$, $Mul_W^1(0,\ g,\ z_0,\ a_1,\ a_2)$ is compute by following procedure.

Pre-smoothing. With the condition $1 \le l \le a_1$, $z_l \in V_k$ is computed by

$$z_l = z_{l-1} + (\lambda h_k^2)(\hat{P}_k B_k^{-1} g - \hat{A}_k z_{l-1}). \tag{53}$$

Error correction. Let $q_0 = 0$. For $1 \le i \le 2$, compute $z_{a_1} + 1$ recursively by

$$r_{k-1} = N_k^{k-1}(\hat{P}_k B_k^{-1} g - \hat{A}_k z_{a_1}),\ q_i = Mul_W^1(k-1,\ r_{k-1},\ q_{i-1},\ a_1,\ a_2),\ z_{a_1} + 1 = z_{a_1} + N_{k-1}^k q_2.$$

Post-smoothing. For $a_1 + 2 \le l \le a_1 + a_2 + 1$, z_l is determined by

$$z_l = z_{l-1} + (\lambda h_k^2)(\hat{P}_k B_k^{-1} g - \hat{A}_k z_{l-1}). \tag{54}$$

Finally, the output of the k-th level iteration is $Mul_W^1(k,\ g,\ z_0,\ a_1,\ a_2) = z_{a_1} + a_2 + 1$.

The construction of the operators $\hat{P}_k$ and $\hat{N}_k$ are used to perform all the calculations in Algorithm 2 in the space V_k instead of $\hat{V}_k$. With (51), (53) and (54) can be rewritten as $z_l = z_{l-1} + (\lambda h_k^2)\hat{P}_k B_k^{-1}(g - A_k z_{l-1})$. Obviously, Algorithm 1 is identical with Algorithm 2.

3.1.2 *Full multigrid methods*

When applying the k-th iteration level to (18), the multigrid algorithm listed below is utilized. And we apply p times at each level.

Algorithm 3 (Full multigrid methods for (18)) For $k = 0,\ A_0\tilde{\xi}_0 = f_0$. For $k \geq 1$, the approximate solution $\tilde{\xi}_k \in \hat{v}_k$ is obtained by the following iterative procedure

$$\xi_0^k = N_{k-1}^k \tilde{\xi}_{k-1},$$

$$\xi_q^k = Mul_W(k,\ f_k,\ \xi_{q-1}^k,\ a_1,\ a_2) \quad 1 \leq q \leq p,$$

$$\tilde{\xi}_k = \xi_p^k.$$

Then we introduce the k-th level nonconforming finite element method for (9), which is to find $\rho_k \in \hat{V}_k$ such that

$$\hat{A}_k \rho_k = g_k, \tag{55}$$

where $g_k \in V_k'$ satisfies $\langle g_k,\ v\rangle = (\tilde{\xi}_k,\ v),\ \forall v \in V_k$. Here $\tilde{\xi}_k$ is obtained by Algorithm 3. In order to solve (55), we introduce the following Algorithm.

Algorithm 4 (Full multigrid methods for (55)) For $k = 0,\ \hat{A}_0\tilde{\rho}_0 = g_0$. For $k \geq 1$, the approximate solution $\tilde{\rho}_k$ is obtained by the following iterative process

$$\rho_0^k = N_{k-1}^k \tilde{\rho}_{k-1},$$

$$\rho_q^k = Mul_W^1(k,\ g_k,\ \rho_{q-1}^k,\ a_1,\ a_2) \quad 1 \leq q \leq p,$$

$$\tilde{\rho}_k = \rho_p^k.$$

3.2 *Error analysis*

We establish the error analysis of the W-cycle multigrid algorithm for discrete problems.

Firstly, we define the operator $R_k : V_k \to V_k$ which is used to measure the effect of smoothing steps as

$$R_k = N_d^k - (\lambda h_k^2) B_k^{-1} A_k, \tag{56}$$

and N_d^k is the identity operator on V_k. Then the k-th level error propagation operator $E_k : V_k \to V_k$ for Algorithm 3.1 is determined by the following famous recursive relation [17, 33]

$$E_k = R_k^{a_2}(N_d^k - N_{k-1}^k P_k^{k-1} + N_{k-1}^k E_{k-1}^2 P_k^{k-1}) R_k^{a_1}, \tag{57}$$

where $P_k^{k-1} : V_k \to V_{k-1}$ denotes the transpose of N_{k-1}^k in the variational form

$$a_{k-1}(P_k^{k-1}\omega,\ v) = a_k(\omega,\ N_{k-1}^k v) \quad \forall \omega \in V_k,\ v \in V_{k-1}. \tag{58}$$

Finally, the mesh-dependent norm is denoted as

$$|||v|||_{j,\ k} = \sqrt{\langle B_k (B_k^{-1} A_k)^j v,\ v\rangle} \quad \forall v \in V_k,\ k \geq 1,\ j = 0,\ 1,\ 2. \tag{59}$$

Obviously, we have

$$|||v|||_{0,\ k}^2 = \langle B_k v,\ v\rangle \approx \|v\|_{L_{2,\ -\mu}(\mathscr{D})} \quad \forall v \in V_k, \tag{60}$$

$$|||v|||_{1,\ k}^2 = \langle A_k v,\ v\rangle = a_k(v,\ v) \quad \forall v \in V_k. \tag{61}$$

By the Cauchy-Schwarz inequality

$$|||v|||_{2,\ k} = \max_{\omega \in V_k \setminus \{0\}} \frac{\langle A_k v,\ \omega\rangle}{|||\omega|||_{0,\ k}} \quad \forall v \in V_k. \tag{62}$$

Note that (46), (56) and (59) imply the following lemmas whose proof are standard in [17, 33].

Lemma 6 There exist constants C independent of k such that

$$|||R_k v|||_{j,\ k} \leq C |||v|||_{j,\ k}, \tag{63}$$

$$|||R_k^m v|||_{1,\ k} \leq C h_k^{-1} m^{-\frac{1}{2}} |||v|||_{0,\ k}, \tag{64}$$

$$|||R_k^m v|||_{2,\ k} \leq C h_k^{-1} m^{-\frac{1}{2}} |||v|||_{1,\ k}, \tag{65}$$

where $v \in V_k$, $k \geq 1$ and $j = 0,\ 1$.

We now use a duality argument to prove the following theorem.

Theorem 5 For any given $v \in V_k$ $(k \geq 1)$, there exists a constant C independent of k such that

$$\left|\left|\left|(N_d^k - N_{k-1}^k P_k^{k-1})v\right|\right|\right|_{0,\ k} \leq C h_k \left|\left|\left|(N_d^k - N_{k-1}^k P_k^{k-1})v\right|\right|\right|_{1,\ k} \leq C h_k^2 |||v|||_{2,\ k}. \tag{66}$$

Proof. For any given $v \in V_k$, let $\varepsilon = \rho_\mu^{-2}(N_d^k - N_{k-1}^k P_k^{k-1})v$, then we have

$$\|\varepsilon\|_{L_{2,\ \mu}(\mathscr{D})} = \|(N_d^k - N_{k-1}^k P_k^{k-1})v\|_{L_{2,\ -\mu}(\mathscr{D})}. \tag{67}$$

We introduce an argument $\zeta_3 \in H^1(\mathscr{D})$ which is determined by

$$(\nabla \times \zeta_3,\ \nabla \times v) + (\zeta_3,\ v) = (\varepsilon,\ v) \quad \forall v \in H^1(\mathscr{D}). \tag{68}$$

It is clear that ζ_3 also satisfies $a_k(\zeta_3,\ v) = (\varepsilon,\ v),\ \forall v \in V_k$. From (41), (17) and (67), we find that

$$\|\zeta_3 - N_{k-1}^k \Pi_{k-1}\zeta_3\|_k \leq C\|\zeta_3 - \Pi_{k-1}\zeta_3\|_{k-1} \leq Ch_{k-1}\|\varepsilon\|_{L_{2,\ \mu}(\mathscr{D})} \leq Ch_k\|(N_d^k - N_{k-1}^k P_k^{k-1})v\|_{L_{2,\ -\mu}(\mathscr{D})},$$

which means

$$\|\zeta_3 - N_{k-1}^k \Pi_{k-1}\zeta_3\|_k \leq Ch_k\|(N_d^k - N_{k-1}^k P_k^{k-1})v\|_{L_{2,\ -\mu}(\mathscr{D})}. \tag{69}$$

At first, we prove

$$\left|\!\left|\!\left|(N_d^k - N_{k-1}^k P_k^{k-1})v\right|\!\right|\!\right|_{0,\ k} \leq Ch_k \left|\!\left|\!\left|(N_d^k - N_{k-1}^k P_k^{k-1})v\right|\!\right|\!\right|_{1,\ k}. \tag{70}$$

It follows from (60) and (67) that

$$\begin{aligned}\left|\!\left|\!\left|(N_d^k - N_{k-1}^k P_k^{k-1})v\right|\!\right|\!\right|_{0,\ k}^2 &= \langle B_k(N_d^k - N_{k-1}^k P_k^{k-1})v,\ (N_d^k - N_{k-1}^k P_k^{k-1})v\rangle \approx \|(N_d^k - N_{k-1}^k P_k^{k-1})v\|_{L_{2,\ -\mu}(\mathscr{D})}^2 \\ &= \int_{\mathscr{D}} \rho_\mu^{-2}[(N_d^k - N_{k-1}^k P_k^{k-1})v]^2 dx = \int_{\mathscr{D}} \varepsilon(N_d^k - N_{k-1}^k P_k^{k-1})v dx = a_k(\zeta_3,\ (N_d^k - N_{k-1}^k P_k^{k-1})v).\end{aligned}$$

Moreover, (69) and (60) imply

$$\begin{aligned}a_k(\zeta_3,\ (N_d^k - N_{k-1}^k P_k^{k-1})v) &= a_k(\zeta_3 - N_{k-1}^k \Pi_{k-1}\zeta_3,\ (N_d^k - N_{k-1}^k P_k^{k-1})v) \\ &\leq \|\zeta_3 - N_{k-1}^k \Pi_{k-1}\zeta_3\|_k \|(N_d^k - N_{k-1}^k P_k^{k-1})v\|_k \\ &\leq Ch\|(N_d^k - N_{k-1}^k P_k^{k-1})v\|_{L_{2,\ -\mu}(\mathscr{D})} \left|\!\left|\!\left|(N_d^k - N_{k-1}^k P_k^{k-1})v\right|\!\right|\!\right|_{1,\ k} \\ &\approx Ch \left|\!\left|\!\left|(N_d^k - N_{k-1}^k P_k^{k-1})v\right|\!\right|\!\right|_{0,\ k} \left|\!\left|\!\left|(N_d^k - N_{k-1}^k P_k^{k-1})v\right|\!\right|\!\right|_{1,\ k}.\end{aligned}$$

Next we prove

$$\left|\!\left|\!\left|(N_d^k - N_{k-1}^k P_k^{k-1})v\right|\!\right|\!\right|_{1,\ k} \leq Ch_k|\!|\!|v|\!|\!|_{2,\ k}. \tag{71}$$

Combining with duality and (62), we obtain

$$\left|\left|\left|(N_d^k - N_{k-1}^k P_k^{k-1})v\right|\right|\right|_{1,\ k} = \sup_{w \in V_k \setminus \{0\}} \frac{a_k((N_d^k - N_{k-1}^k P_k^{k-1})v,\ w)}{|||w|||_{1,\ k}}. \tag{72}$$

Since

$$a_k((N_d^k - N_{k-1}^k P_k^{k-1} v),\ w) = a_k((N_d^k - N_{k-1}^k P_k^{k-1} w),\ v) \le \|(N_d^k - N_{k-1}^k P_k^{k-1})w\|_{0,\ k}\|v\|_{2,\ k} \le Ch|||w|||_{1,\ k}|||v|||_{2,\ k},$$

we finish the proof of (71). Finally, the Lemma 5 is a consequence of (70) and (71).

Two preliminary approximation properties with respect to the operators P_k^{k-1} and N_{k-1}^k are given in the following lemma [29].

Lemma 7

$$\left|\left|\left|P_k^{k-1} v\right|\right|\right|_{1,\ k-1} \le |||v|||_{1,\ k} \quad \forall v \in V_k, \tag{73}$$

$$\left|\left|\left|N_{k-1}^k v\right|\right|\right|_{1,\ k} \le |||v|||_{1,\ k-1} \quad \forall v \in V_{k-1}. \tag{74}$$

With a_1 pre-smoothing steps and a_2 post-smoothing steps on the two-grid algorithm, we introduce the following convergence.

Theorem 6 There exists a constant C independent of $k \ge 1$ such that the following holds

$$\left|\left|\left|R_k^{a_2}(N_d^k - N_{k-1}^k P_k^{k-1})R_k^{a_1} v\right|\right|\right|_{1,\ k} \le C[(1+a_1)(1+a_2)]^{-\frac{1}{2}}|||v|||_{1,\ k} \quad \forall v \in V_k. \tag{75}$$

Proof. If follows from Lemma 5 and 6, we have

$$\begin{aligned}
\left|\left|\left|R_k^{a_2}(N_d^k - N_{k-1}^k P_k^{k-1})R_k^{a_1} v\right|\right|\right|_{1,\ k} &\le C[(1+a_2)]^{-1/2}\left|\left|\left|(N_d^k - N_{k-1}^k P_k^{k-1})R_k^{a_1} v\right|\right|\right|_{0,\ k} \\
&\le C[(1+a_2)]^{-1/2}\left|\left|\left|R_k^{a_1} v\right|\right|\right|_{2,\ k} \\
&\le C[(1+a_1)(1+a_2)]^{-1/2}|||v|||_{1,\ k}.
\end{aligned}$$

Then we have the following convergence theorem for the W-cycle algorithm, which is based on Lemma 7, Theorem 6 and a perturbation argument [30].

Theorem 7 For any $\gamma \in (0,\ 1)$, there exists a positive integer m independent of k such that

$$|||z - Mul_w(k,\ g,\ z_0,\ a_1,\ a_2)|||_{1,\ k} \le \gamma|||z - z_0|||_{1,\ k}, \tag{76}$$

provided $a_1 + a_2 \geq m$.

Furthermore, (61) becomes

$$|||v|||_{1,\ k}^2 = \langle \hat{A}_k v,\ v\rangle \approx |v|_{H^1(\mathscr{D})} \quad \forall v \in \hat{V}_k,$$

which implies

$$|\rho - \rho_k|_{H^1(\mathscr{D})} = \inf_{v \in \hat{V}_K} |\rho - v|_{H^1(\mathscr{D})} \leq |\rho - \Pi_k \rho|_{H^1(\mathscr{D})}.$$

Therefore, if we replace V_k with $\hat{V}_k$, Theorem 7 also valid.

Now we analyze the error estimate of the k-th level iterations.

Lemma 8 [30] Suppose p is sufficiently large and h_1 is small enough, there exists a constant C such that

$$\|\xi - \tilde{\xi}_k\|_{L_2(\mathscr{D})} \leq C h_k \|f\|_{L_2(\mathscr{D})}.$$

The following theorem compares the exact solution ρ of (9) with the approximate solution $\tilde{\rho}_k$ obtained by Algorithm 4.4.

Theorem 8 Suppose p is sufficiently large and h_1 is small enough, we have

$$|\rho - \tilde{\rho}_k|_{H^1\mathscr{D})} \leq C h_k \|f\|_{L_2(\mathscr{D})}, \tag{77}$$

where C is a constant.

Proof. We find that $\rho_0 - \tilde{\rho}_0 = 0$ and suppose $\alpha^{r+1} < \dfrac{1}{2}$, then

$$\begin{aligned} |\rho_k - \tilde{\rho}_k|_{H^1(\mathscr{D})} =& |||\rho_k - \tilde{\rho}_k|||_{1,\ k} \leq \alpha^r |||\rho_k - \tilde{\rho}_{k-1}|||_{1,\ k} \leq C\alpha^r (|\rho_k - \rho|_{H^1(\mathscr{D})} + |\rho - \rho_{k-1}|_{H^1(\mathscr{D})} + |||\rho_{k-1} - \tilde{\rho}_k - 1|||_{1,\ k}) \\ \leq& C h_k \alpha^r \|f\|_{L_2(\mathscr{D})} + C^2 h_{k-1} \alpha^{2r} \|f\|_{L_2(\mathscr{D})} + \cdots + C^{k+1} h_0 \alpha^{(k+1)r} \|f\|_{L_2(\mathscr{D})} + |\rho_0 - \tilde{\rho}_0|_{H^1(\mathscr{D})} \\ \leq& C h_k \|f\|_{L_2(\mathscr{D})} \frac{\alpha^r}{1 - 2\alpha^r} + |\rho_0 - \tilde{\rho}_0|_{H^1(\mathscr{D})} \leq C h_k \|f\|_{L_2(\mathscr{D})} \end{aligned}$$

which means $|\rho_k - \tilde{\rho}_k|_{H^1(\mathscr{D})} \leq C h_k \|f\|_{L_2(\mathscr{D})}$. Combining with the triangle inequality and (21), we obtain

$$|\rho - \tilde{\rho}_k|_{H^1(\mathscr{D})} \leq |\rho_k - \tilde{\rho}_k|_{H^1(\mathscr{D})} + |\rho - \rho_k|_{H^1(\mathscr{D})} \leq C h_k \|f\|_{L_2(\mathscr{D})}.$$

In the case that $\mathscr{D}$ is not simply connected, we have the following lemmas.

Lemma 9 Suppose p is sufficiently large and h_1 is small enough, we have

$$\left|\eta_j - \tilde{\eta}_{j,\ k}\right|_{H^1(\mathscr{D})} \leq Ch_k, \tag{78}$$

where C is a constant.

Proof. The proof is similar to Theorem 8, and hence will be omitted.

Note that $\tilde{c}_{j,\ k}(1 \leq j \leq m)$ is computed by

$$\sum_{j=1}^{m} \tilde{c}_{j,\ k}(\nabla\tilde{\eta}_{j,\ k}, \nabla\tilde{\eta}_{i,\ j}) = (f,\ \nabla\tilde{\eta}_{i,\ k}) \quad 1 \leq i \leq m. \tag{79}$$

Moreover, the estimate of $\tilde{c}_{j,\ k}$ is shown by next lemma and the proof is similar to Theorem 2.4.

Lemma 10 Suppose p is sufficiently large and h_1 is small enough, we have

$$|c_j - \tilde{c}_{j,\ k}|_{H^1(\mathscr{D})} \leq Ch_k\|f\|_{L_2(\mathscr{D})}, \tag{80}$$

where C is a constant.

For any k-th level iteration, the approximate value $\tilde{T}_k f$ of Tf is determined as

$$\tilde{T}_k f = \nabla \times \tilde{\rho}_k + \sum_{j=1}^{m} \tilde{c}_{j,\ k}\nabla\tilde{\eta}_{j,\ k}. \tag{81}$$

According to (81), (77), (78) and (80), we are ready to compare Tf and $\tilde{T}_k f$. The proof is similar to Theorem 4.

Theorem 9 Suppose $\tilde{T}_k f$ is defined as (81), then

$$\left\|Tf - \tilde{T}_k f\right\|_{L_2(\mathscr{D})} \leq Ch_k\|f\|_{L_2(\mathscr{D})}.$$

For the problem (1), the following theorem holds provided $Tu = \dfrac{1}{1+\lambda}u$.

Theorem 10 Suppose $\tilde{u}_k$ is the approximation of u, we have

$$\|u - \tilde{u}_k\|_{L_2(\mathscr{D})} \leq Ch_k\|f\|_{L_2(\mathscr{D})}. \tag{82}$$

4. Conclusion

This paper presents an efficient numerical solution for the two-dimensional Maxwell eigenvalue problem by combining Hodge decomposition, nonconforming finite element methods and multigrid methods. The theoretical analysis show that the approximate solution $u_h$converges to the exact solution u in the L_2 norm with an order of $\mathscr{O}(h)$.The paper combines the advantages of different methods to better overcome the limitations of a single method, providing an effective solution for solving the Maxwell eigenvalue problem.

Conflict of interest

The authors declare no competing financial interest.

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