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Research Article

A Fuzzy Regression Method Based on Monotone Nonparametric Least Squares Technique

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Abstract: Nonparametric fuzzy regression techniques have proven to be useful in addressing challenges related to modeling vague or imprecise variables, particularly in situations where data availability is constrained. These methods offer greater adaptability by eliminating the need for predefined functional forms, making them more practical compared to both machine-learning-based and parametric regression approaches. Machine learning methods, while effective, often require substantial data to produce reliable outputs, whereas parametric regression may encounter issues with small sample sizes, leading to reduced goodness of fit. The Fuzzy-Monotone Nonparametric Least Squares (MNLS) method represents an innovative advancement within this context. Specifically developed to manage triangular fuzzy outputs with crisp inputs, this method builds upon the fuzzy least squares framework developed by Diamond. It divides the regression task into three sub-components: Center, Left, and Right endpoint. Each component is subsequently processed using a Monotone Nonparametric Least Squares (MNLS) framework. This framework permits Fuzzy-MNLS to seamlessly combine convex and concave aspects within the regression model, leading to greater accuracy and versatility. Unlike machine learning-based fuzzy regression methods, Fuzzy-MNLS avoids the need for regularization while maintaining effectiveness even when data is sparse. Illustrative examples demonstrate that Fuzzy-MNLS consistently yields higher similarity scores and more accurate forecasts compared to other least squares methods. This robustness makes it an optimal choice for scenarios where data availability is limited.

Keywords: fuzzy regression, nonparametric, least squares

MSC: 62G08, 62J86, 62J05

1. Introduction

Fuzzy regression analysis is a valuable method for tackling problems involving imprecise or uncertain variables, especially within complex systems. The foundational work by Tanaka et al. [1] introduced this approach, which has since evolved to include various techniques such as possibilistic regression, fuzzy least squares, and machine learning-based methods.

As outlined in [2], these techniques present diverse ways to manage the inherent uncertainties encountered in regression analysis. Possibilistic regression, one of the initial approaches, was introduced by [1] utilizing linear programming to reduce the spread of fuzzy variables. As research progressed, optimization approaches like nonlinear

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programming [3–9] and goal programming [10–14] emerged to address the limitations inherent in basic possibilistic regression.

The concept of fuzzy least squares, which was initially developed by [15, 16] and later refined by Diamond [17, 18], is designed to determine fuzzy parameters by minimizing the squared gap between observed and predicted results. Subsequent developments include the method by Xu [19, 20], which gives equal weighting to the vertices of triangular fuzzy numbers, and the approach by Diamond and Körner [21], which tackles the issue of negative spreads. Zeng et al. [22] introduced a model that employs the absolute loss function to manage crisp inputs and outputs, offering an alternative to traditional least squares. Additional advancements are documented in [20, 23].

In recent years, machine learning techniques have been increasingly adopted for nonparametric fuzzy regression. These methods include regression trees, kernel regression, local regression, smoothing splines, and neural networks. Prominent techniques such as the rank transform method [23] and kernel smoothing [24–27] have been explored, alongside fuzzy regression machines employing support vector techniques [28, 29], and neural network-based fuzzy regression frameworks [30, 31], fuzzy genetic algorithms [32]. More robust fuzzy regression models have been proposed in [33–37]. Despite their potential, it is important to note that machine learning-based fuzzy regression often requires substantial sample sizes to achieve reliable model performance.

This study aims to develop a fuzzy regression method based on Monotone Nonparametric Least Squares (MNLS). Due to the inherent challenges associated with traditional fuzzy least squares, such as sensitivity to outliers and decreased accuracy with increasing variables or their magnitudes [23, 38, 39], employing nonparametric techniques helps to overcome these issues by removing assumptions about the functional form of the regression function.

1.1 *Background of MNLS and CNLS*

Monotone Nonparametric Least Squares (MNLS) and Convex Nonparametric Least Squares (CNLS) are regression techniques designed to model relationships without imposing a predefined functional form. MNLS specifically addresses scenarios where the regression function is monotonic (strictly increasing or decreasing) while accommodating both convex and concave components. This flexibility allows MNLS to capture nonlinear monotonicity in data.

In contrast, CNLS is designed to model regression functions that are strictly convex or concave, making it particularly suitable when the data inherently follows a single curvature pattern. However, this restriction may limit effectiveness when the data contains mixed convex-concave trends.

Building on these concepts, the proposed Fuzzy-MNLS method extends MNLS to handle fuzzy triangular outputs. The Fuzzy-MNLS method combines monotonicity with fuzzy least squares to better model data characterized by imprecision and variability. At the same time, Fuzzy-CNLS remains confined to modeling strictly convex or concave fuzzy relationships.

It is important to clarify that the theoretical superiority of MNLS over CNLS, as established in Theorem 2 of [40], pertains specifically to crisp-input crisp-output regression. The improved goodness-of-fit of MNLS is fundamentally linked to its ability to handle both convex and concave components within a monotonic framework, which does not imply that Fuzzy-MNLS universally outperforms Fuzzy-CNLS in all fuzzy contexts. The effectiveness of Fuzzy-MNLS is particularly evident when the data exhibits a monotonic trend with local convex or concave variations, as seen in the [41] dataset.

1.2 *Form flexibility and structural monotonicity of MNLS*

Although nonparametric least squares techniques do not require specifying a fixed functional form (such as linear, quadratic, or polynomial), they may still incorporate structural constraints to guide the modeling process. In the case of MNLS, the method explicitly introduces a monotonicity constraint, ensuring that the estimated regression function consistently increases or decreases. This monotonicity does not dictate a specific algebraic expression but instead enforces a behavioral characteristic that aligns with the inherent trend of the data. Therefore, the statement that MNLS does not assume a predefined functional form while imposing monotonicity as a structural guideline is consistent and logical.

By distinguishing between form flexibility and structural monotonicity, we aim to clarify the theoretical foundation of the Fuzzy-MNLS method. Understanding this distinction is crucial for recognizing when Fuzzy-MNLS will likely outperform other fuzzy regression methods, particularly when the data exhibits a monotonic pattern with local convex-concave variations. This conceptual foundation motivates the development of Fuzzy-MNLS as a robust approach to modeling complex fuzzy relationships while addressing the challenges associated with small sample sizes and nonlinear patterns.

1.3 *Research gaps*

This study addresses two critical research gaps. First, machine-learning-based fuzzy regression methods often fail when applied to small datasets. Second, parametric fuzzy least squares methods may not achieve satisfactory goodness-of-fit when dealing with nonlinear fuzzy regression functions.

We propose a novel fuzzy nonparametric regression method-Fuzzy-MNLS to bridge these gaps. This approach integrates Diamond's fuzzy least squares with Monotone Nonparametric Least Squares (MNLS), as introduced by Chung and Chen in [40]. The MNLS technique is particularly attractive as it avoids the need for prior functional form specification or smoothing parameters, offering flexibility in regression analysis.

1.4 *Contributions*

The primary contributions of this paper are as follows:

1. Development of Fuzzy-MNLS.

The Fuzzy-MNLS approach extends Diamond's fuzzy least squares framework by integrating the MNLS method. This integration allows the method to effectively capture both convex and concave elements within a single monotonic framework. Such a combination is particularly useful for modeling data patterns that exhibit nonlinear monotonic behavior, thereby enhancing the model's ability to represent complex relationships in data.

2. Application to small sample sizes.

Unlike many other nonparametric methods that require large datasets due to their reliance on local data points, Fuzzy-MNLS can perform well with small sample sizes by leveraging monotonicity constraints. This property enhances its practical applicability when data is limited.

3. Reduction of overfitting through structural constraints.

Instead of relying on regularization parameters like in machine learning methods, Fuzzy-MNLS inherently avoids overfitting by imposing monotonicity. This structural characteristic enhances model stability without the need for additional regularization techniques.

4. Enhanced goodness-of-fit in least squares applications.

Due to its dual capability of modeling both convex and concave components, Fuzzy-MNLS demonstrates improved goodness-of-fit compared to parametric least squares methods, particularly when the data exhibits monotonic trends with mixed curvature.

Table 1 presents an overview of how dataset characteristics correspond to various fuzzy regression methods, highlighting their interrelations.

Table 1. Mapping dataset characteristics to fuzzy regression techniques

	Parametric		Nonparametric	
	Probabilistic	Least squares	Machine-learning	Least squares
	Tanaka	Diamond	Regression trees, kernel regression, local regression, smoothing splines, neural network	CNLS, MNLS (the current paper)
Functional form	Linear	Nonlinear	No assumption	Concave/Convex/ Monotone increase/decrease
Sample size	Small	Small	Large	Small

The proposed Fuzzy-MNLS method builds on Diamond's fuzzy least squares by segmenting the problem into three distinct sub-models: Center, Left-end, and Right-end. The MNLS technique is then applied to each sub-model, following a structured approach similar to that described in [42]. However, unlike the Fuzzy-CNLS method from [42], which requires solving eight models with various convexity and concavity constraints, the Fuzzy-MNLS method simplifies this process by consolidating both convex and concave characteristics within a single integrated model. Consequently, the fuzzy regression function produced by Fuzzy-MNLS inherently accommodates both convex and concave patterns. The sole assumption imposed on the fuzzy regression function is monotonicity, which can be verified through expert analysis or engineering evaluation.

By merging Diamond's fuzzy least squares approach with the MNLS framework, the Fuzzy-MNLS method eliminates the necessity for predefined functional forms or smoothing parameters, as typically required in kernel regression. This characteristic enhances estimation accuracy and yields a set of hyperplanes that provide interpretability rather than functioning as opaque models. Furthermore, the combination of convexity and concavity constraints contributes to the method's robustness against outliers compared to conventional methods based on Diamond's framework.

The paper is systematically structured as follows: Section 2 provides an overview of Diamond's fuzzy least squares and the MNLS technique. Section 3 elaborates on the Fuzzy-MNLS method. Section 4 details the forecasting processes, while Section 5 outlines the metrics used to evaluate the method's goodness-of-fit in comparison with other fuzzy least squares techniques. In Section 6, a numerical example is presented to illustrate the method and discuss the forecasting procedure. Finally, Sections 7 and 8 contain the discussion and the paper's conclusions.

2. Preliminaries

This study utilizes the notation $\mathbb{R}$ to represent the set of real numbers, while $\mathscr{F}(\mathbb{R})$ denotes the set containing fuzzy numbers within the real number space. In the context of this research, we examine both asymmetric and symmetric triangular fuzzy numbers. A fuzzy number is denoted by the symbol $\widetilde{a}$.

Definition 1 *L-R* Fuzzy Number.

An *L-R* fuzzy number, denoted as $\widetilde{y} = (y_C,\ y_L,\ y_R)$, is defined by a membership function $\mu_{\widetilde{y}}(x)$ that specifies how each value within the range $\mathbb{R}$ is mapped. The membership function can be expressed as follows:

$$\mu_{\widetilde{y}}(x) = \begin{cases} L\left(\dfrac{y_C - x}{y_C - y_L}\right) & y_L \le x \le y_C, \\ R\left(\dfrac{x - y_C}{y_R - y_C}\right) & y_C \le x \le y_R, \\ 0 & \text{otherwise,} \end{cases}$$

In this context, $y_C,\ y_L$, and y_R denote the center, left limit, and right limit of the fuzzy number, respectively. The functions L and R are characterized as continuous and monotonically decreasing over the range $[0,\ 1]$. These functions satisfy the conditions: $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$. The left and right shape functions, $L(x)$ and $R(x)$, capture the characteristic shape of the fuzzy number.

If the relationship $y_C - y_L = y_R - y_c = y_e$ holds, and the shape functions L and R are identical, the number $\widetilde{y}$ becomes a symmetrical *L-R* fuzzy number, represented as $\widetilde{y} = (y_C,\ y_e)$.

Definition 2 Triangular Fuzzy Number.

A triangular fuzzy number is expressed as $\widetilde{y} = (y_C,\ \ y_L,\ \ y_R)$. The associated membership function is structured as follows:

$$\mu_{\tilde{y}}(x) = \begin{cases} \dfrac{x - y_L}{y_C - y_L} & y_L \leq x \leq y_C, \\ \\ \dfrac{y_R - x}{y_R - y_C} & y_C \leq x \leq y_R, \\ \\ 0 & \text{otherwise.} \end{cases}$$

Remark In some academic contexts, the fuzzy number format may be presented as $\tilde{y} = (y_C,\ y_l,\ y_r)$ where y_l and y_r represent the left and right spreads, respectively. These spreads are calculated as $y_l = y_C - y_L$ and $y_r = y_R - y_C$. However, for the purpose of this study, we maintain the endpoint format $\tilde{y} = (y_C,\ y_L,\ y_R)$, which proves beneficial for identifying unique hyperplanes fundamental to the fuzzy CNLS approach.

2.1 *Fuzzy linear regression models*

The structure of the fuzzy linear regression model is presented as follows:

$$\tilde{y}_i = \tilde{a} + \tilde{b}_1 x_{i1} + \tilde{b}_2 x_{i2} + \ldots + \tilde{b}_m x_{im}, \quad i = 1,\ 2,\ \ldots,\ n \tag{1}$$

Here, each observation consists of pairs $(x_{i1},\ x_{i2},\ \ldots,\ x_{im},\ \tilde{y}_i)$ for $i = 1,\ \ldots,\ n$, where the goal is to determine the fuzzy parameters $\tilde{a}$, $\tilde{b}_j$ for each predictor variable $j = 1,\ \ldots,\ m$.

The fuzzy parameters are represented as triangular fuzzy numbers: $\tilde{a} = (a_C,\ a_L,\ a_R)$, $\tilde{b}_j = (b_{Cj},\ b_{Lj},\ b_{Rj})$ for each predictor j. The estimated fuzzy response is denoted as $\hat{\tilde{y}}_i = (\hat{y}_{Ci},\ \hat{y}_{Li},\ \hat{y}_{Ri})$.

Based on Zadeh's extension principle, the relationship is given by:

$$\tilde{a} + \sum_{j=0}^{m} \tilde{b}_j x_{ij} = \left(a_C + \sum_{j=1}^{m} b_{Cj} x_{ij},\ a_L + \sum_{j=1}^{m} b_{Lj} x_{ij},\ a_R + \sum_{j=1}^{m} b_{Rj} x_{ij} \right) = \tilde{y}_i = (y_{Ci},\ y_{Li},\ y_{Ri}),$$

where Zadeh's Extension Principle [43] provides a framework to extend mathematical functions from crisp domains to fuzzy domains, ensuring that the fuzziness of inputs is systematically carried over to the outputs. This principle is fundamental in Fuzzy-MNLS, as it allows the generation of fuzzy outputs from both crisp and fuzzy inputs, thereby maintaining uncertainty throughout the modeling process [43].

2.2 *Diamond's fuzzy least squares estimation*

For the purpose of identifying the fuzzy parameters $\tilde{a}$ and $\tilde{b}_j$ within a regression framework, three key methodologies are utilized: (i) possibilistic regression analysis, (ii) machine learning-based methods, and (iii) fuzzy least squares techniques. For a comprehensive review of these methods, see the work by Chukhrova and Johannssen [2].

In the present study, we focus on the Fuzzy-CNLS method, which is derived from Diamond's fuzzy least squares technique [18]. To facilitate the discussion, we first present the foundational principles of Diamond's fuzzy least squares method, followed by an introduction to the CNLS approach.

The method developed by Diamond [18] aims to determine the fuzzy parameters by minimizing the total squared error between the estimated fuzzy outputs and the observed values. The minimization problem is structured to reduce the discrepancy between the predicted and actual outputs while accounting for the fuzziness in the data.

The optimization problem is formulated as follows:

[Diamond]

$$\min_{a_C,\ a_L,\ a_R,\ b_{cj},\ b_{Lj},\ b_{Rj}} \quad \sum_{i=1}^{n}\left(\widehat{\widetilde{y}}_i-\widetilde{y}_i\right)^2=\sum_{i=1}^{n}\left(a_C+\sum_{j=1}^{m}b_{Cj}x_{ij}-y_{Ci}\right)^2$$

$$+\sum_{i=1}^{n}\left(a_L+\sum_{j=1}^{m}b_{Lj}x_{ij}-y_{Li}\right)^2+\sum_{i=1}^{n}\left(a_R+\sum_{j=1}^{m}b_{Rj}x_{ij}-y_{Ri}\right)^2$$

This formulation minimizes the sum of squared differences between the predicted and actual fuzzy outputs. The values $a_C,\ a_L,$ and a_R denote the center, left boundary, and right boundary of the fuzzy number, respectively. Similarly, the coefficients $b_{Cj},\ b_{Lj},$ and b_{Rj} correspond to the fuzzy slopes associated with each input variable.

It is important to note that the fuzzy regression coefficients $\widetilde{a}$ and $\widetilde{b}_j$ are initially expressed in the spread format. When dealing with fuzzy triangular numbers, we adopt the endpoint format as proposed in Diamond's method [44], ensuring consistency in parameter representation.

2.3 CNLS

As a nonparametric regression framework, Convex Nonparametric Least Squares (CNLS) is constructed to estimate relationships while maintaining shape limitations on the regression function. The method estimates the dependent variable as a combination of a shape-restricted function and a random disturbance term:

$$y=f(x)+\varepsilon^{CNLS}$$

Here, $f(x)$ represents the shape-constrained function, x is the vector of input variables, y is the crisp output, and ε^{CNLS} denotes the error term, satisfying the condition $E(\varepsilon^{CNLS}|x)=0$.

The method was formulated by [45] to provide a robust approach for estimating nonparametric functions with monotonicity and convexity assumptions.

[CNLS]

$$\min_{\alpha,\ \beta,\ \varepsilon}\sum_{i=1}^{n}(\varepsilon_i^{CNLS})^2$$

$$y_i=\alpha_i^v+\sum_{j=1}^{m}\beta_{ij}^v x_{ij}+\varepsilon_i^{CNLS}\quad \forall i$$

$$\alpha_i^v+\sum_{j=1}^{m}\beta_{ij}^v x_{ij}\geq\alpha_h^v+\sum_{j=1}^{m}$$

$$\beta_{hj}^v x_{ij}\quad \forall i,\ h;\ i\neq h$$

$$\beta_{ij}^v\geq 0.$$

2.4 *MNLS*

By extending the [CNLS] of [45], Chung and Chen [40] derived Monotone Nonparametric Least Squares (MNLS) for estimating nonparametric functions with monotonicity.

[MNLS]

$$\min_{\alpha,\,\beta,\,\varepsilon} \sum_{i=1}^{n} \left(\varepsilon_i^{MNLS}\right)^2$$

s.t.

$$y_i = g_i^c + g_i^v + \varepsilon_i^{MNLS} \quad \forall i \tag{2}$$

$$g_i^c = \alpha_i^c + \sum_{j=1}^{m} \beta_{ij}^c x_{ij} \quad \forall i \tag{3}$$

$$\alpha_i^c + \sum_{j=1}^{m} \beta_{ij}^c x_{ij} \leq \alpha_h^c + \sum_{j=1}^{m} \beta_{hj}^c x_{ij} \quad \forall i,\ h;\ i \neq h \tag{4}$$

$$g_i^v = \alpha_i^v + \sum_{j=1}^{m} \beta_{ij}^v x_{ij} \quad \forall i \tag{5}$$

$$\alpha_i^v + \sum_{j=1}^{m} \beta_{ij}^v x_{ij} \geq \alpha_h^v + \sum_{j=1}^{m} \beta_{hj}^v x_{ij} \quad \forall i,\ h;\ i \neq h \tag{6}$$

$$\beta_{ij}^c \geq 0 \quad \forall i \tag{7}$$

$$\beta_{ij}^v \geq 0 \quad \forall i \tag{8}$$

where y_i denotes the crisp output, x_{ij} represents the j^{th} crisp input, and ε_i^{MNLS} indicates the error term capturing the deviation of the i^{th} observation from the predicted function.

Here, Equation (2) is the regression equation, Equations (4) and (6) are concavity and convexity constraints, Equations (7) and (8) are monotonicity constraints.

According to Chung [42], the derived results can be extended to handle negative monotone regression functions. This is achieved by modifying constraints Equations (7) and (8), replacing them with $\beta_{ij}^c \leq 0$ and $\beta_{ij}^v \leq 0$, respectively. This adjustment allows the MNLS method to accommodate scenarios where the monotonicity of the function is decreasing rather than increasing.

MNLS exhibits several notable benefits over conventional methods like CNLS and Ordinary Least Squares (OLS). Its nonparametric and nonlinear nature enables better performance in scenarios where the underlying data patterns are complex. Additionally, the flexibility to model both convex and concave components within a single framework contributes to its superior estimation accuracy, as highlighted in studies [33, 34, 46]. These characteristics make MNLS a robust choice for applications requiring precise modeling of monotonic relationships.

2.5 *Goodness of fit of MNLS and CNLS for fuzzy regression*

The MNLS method was introduced as a solution to address the fundamental shortcomings of the CNLS technique. One of the notable strengths of MNLS is its ability to seamlessly integrate convex and concave patterns into a single regression approach. This capability is essential when the data exhibits a generally monotonic trend but contains localized variations in curvature. By allowing for such mixed curvature within a single model, MNLS enhances the goodness-of-fit compared to conventional CNLS methods. The proof of MNLS's superiority in terms of goodness-of-fit is provided in Theorem 2 of [40], applying strictly to crisp-input crisp-output regression problems. However, the manuscript does not claim that Fuzzy-MNLS universally outperforms Fuzzy-CNLS. In fuzzy regression, we compare different goodness of fit metrics, such as RMSE, MAE, and Similarity Index. See Section 5 for more details on using RMS, MAE, and similarity Index.

2.6 *Deriving an explicit represent function of MNLS*

The represent function in regression is utilized to map the relationship between input variables and outputs. In the context of MNLS, this function captures both convex and concave components while preserving the monotonic trend of the data.

Given the estimated coefficients $\left(\alpha_i^c,\ \beta_{ij}^c,\ \alpha_i^v,\ \beta_{ij}^v\right)$ from [MNLS], we can construct the following explicit represent functions:

$\widehat{g}^c(x) = \min_{i=(i,\ \ldots,\ n)} \{\widehat{\alpha}_i^c + \sum_{j=1}^m \widehat{\beta}_{ij}^c x_{ij}\}$,

$\widehat{g}^v(x) = \max_{i=(i,\ \ldots,\ n)} \{\widehat{\alpha}_i^v + \sum_{j=1}^m \widehat{\beta}_{ij}^v x_{ij}\}$, and

$y = \widehat{g}(x) = \widehat{g}^c(x) + \widehat{g}^c(x)$ (3).

2.6.1 *Uniqueness of fitted values*

While the MNLS method may yield multiple optimal solutions for the individual components $\widehat{g}_i^c$ and $\widehat{g}_i^v$, the combined fitted values $\widehat{g}_i = \widehat{g}_i^c + \widehat{g}_i^v$ are unique. This consistency ensures that the final output accurately reflects the combined convex and concave trends within the data.

2.6.2 *Calculation of hyperplane envelopes*

To achieve the unique set of hyperplanes, we calculate the concave and convex envelopes separately.

[MNLS-Lcave$_i$]

$$\min_{\tilde{\alpha},\ \tilde{\beta}} \left\{ \tilde{\alpha}_i^c + \sum_{j=1}^m \tilde{\beta}_{ij}^c x_{ij} \mid \tilde{\alpha}_i^c + \sum_{j=1}^m \tilde{\beta}_{ij}^c x_{ij} \geq \hat{g}_i^c \quad \forall i \right\} \tag{9}$$

Similarly, for g_i^v, the convex part, we have

[MNLS-Lvex$_i$]

$$\max_{\tilde{\alpha},\ \tilde{\beta}} \left\{ \tilde{\alpha}_i^v + \sum_{j=1}^m \tilde{\beta}_{ij}^v x_{ij} \mid \tilde{\alpha}_i^v + \sum_{j=1}^m \tilde{\beta}_{ij}^v x_{ij} \leq \hat{g}_i^v \forall i \right\} \tag{10}$$

By calculating the lower concave envelope and the upper convex envelope separately, the Fuzzy-MNLS method ensures that the regression function correctly reflects both convex and concave patterns while maintaining monotonicity.

The MNLS method results in two distinct sets of hyperplane segments: $\mathbf{k^c}$ and $\mathbf{k^v}$. These segments correspond to the concave and convex components derived from the **[MNLS-Lcave$_i$]** and **[MNLS-Lvex$_i$]** formulations, respectively.

In practical applications, the hyperplane segments from **[MNLS-Lcave$_i$]** are utilized for the concave component, while those from **[MNLS-Lvex$_i$]** are applied for the convex component during forecasting processes, as detailed in [40].

3. Fuzzy-MNLS estimation

3.1 *Diamond's method*

Diamond's method provides a framework for estimating fuzzy regression coefficients when the parameters are represented as fuzzy triangular numbers. The model uses the endpoint format, as described in [35], to structure the fuzzy numbers within the regression framework. The [Diamond] method can be found in subsection 2.2.

3.1.1 *Decomposition of Diamond's method*

To simplify the estimation process, Diamond's method can be decomposed into three separate OLS-based submodels: [Diamond-L], [Diamond-C], and [Diamond-R], each corresponding to the left endpoint, center, and right endpoint, respectively.

[Diamond-L]

$$\min_{a_L,\, b_{Lj},\, \varepsilon_{Li}} \left\{ \sum_{i=1}^{n} (\varepsilon_{Li})^2 | y_{Li} = a_L + \sum_{j=1}^{m} b_{Lj} x_{ij} + \varepsilon_{Li} \qquad \forall i \right\}$$

[Diamond-C]

$$\min_{a_C,\, b_{Cj},\, \varepsilon_{Ci}} \left\{ \sum_{i=1}^{n} (\varepsilon_{Ci})^2 | y_{Ci} = a_C + \sum_{j=1}^{m} b_{Cj} x_{ij} + \varepsilon_{Ci} \qquad \forall i \right\}$$

[Diamond-R]

$$\min_{a_R,\, b_{Rj},\, \varepsilon_{Ri}} \left\{ \sum_{i=1}^{n} (\varepsilon_{Ri})^2 | y_{Ri} = a_R + \sum_{j=1}^{m} b_{Rj} x_{ij} + \varepsilon_{Ri} \qquad \forall i \right\}$$

The [Diamond-C] model estimates the regression function at the center point, while [Diamond-L] and [Diamond-R] handle the left and right endpoints, respectively. Despite being based on OLS, Diamond's method is limited in its ability to model monotonic or nonlinear relationships. This limitation often results in suboptimal goodness-of-fit when applied to data characterized by monotonic trends or convex-concave variations.

Ordinary Least Squares (OLS) is typically adopted within Diamond's method to analyze the fuzzy regression functions at the center, left boundary, and right boundary as separate elements. Despite its effectiveness in modeling linear correlations, OLS is not equipped to address monotonic or nonlinear characteristics commonly found in fuzzy data. As a result, OLS-based fuzzy regression methods may suffer from poor goodness-of-fit when the underlying data exhibits monotonic patterns with local convex-concave variations.

We propose replacing OLS with Monotone Nonparametric Least Squares (MNLS) to address this limitation. Unlike OLS, MNLS accommodates both convex and concave components within a monotonic framework, offering greater flexibility in capturing complex data patterns. By leveraging MNLS, the proposed Fuzzy-MNLS method improves the goodness-of-fit while maintaining the monotonicity constraint, which is essential for accurately modeling nonlinear monotonic relationships in fuzzy data.

3.2 *Fuzzy-MNLS method*

Diamond's approach originally utilizes Ordinary Least Squares (OLS) to approximate the fuzzy regression functions at the center, left boundary, and right boundary. However, to better capture monotonic and non-linear patterns in fuzzy data, we propose using Monotone Nonparametric Least Squares (MNLS) instead.

To formulate the Fuzzy-MNLS method, we leverage the results from [40, 42] to derive the MNLS formulations for each of the three sub-models: [Diamond-L], [Diamond-C], and [Diamond-R]. We begin by deriving the MNLS formulation for [Diamond-L], and subsequently, similar formulations for [Diamond-C] and [Diamond-R] can be obtained.

We consider the [Diamond-L] as $y_L = f_L(x) + \varepsilon_L$. Then, we apply MNLS to [Damond-L] and have the following quadratic programming like [MNLS].

[Fuzzy-MNLS-L]

$$\min_{a,\ b,\ \varepsilon} \sum_{i=1}^{n} (\varepsilon_{Li})^2$$

s.t.

$$y_{Li} = g^c_{Li} + g^v_{Li} + \varepsilon_{Li} \text{ for } i = 1, \ldots, n \tag{11}$$

$$g^c_{Li} = a^c_{Li} + \sum_{j=1}^{m} b^c_{L_{ij}} x_{ij} \ \forall i \tag{12}$$

$$a^c_{Li} + \sum_{j=1}^{m} b^c_{L_{ij}} x_{ij} \leq a^c_{Lh} + \sum_{j=1}^{m} b^c_{L_{hj}} x_{ij} \text{ for } i,\ h = 1, \ldots, n \text{ and } i \neq h \tag{13}$$

$$b^c_{Lij} \geq 0 \text{ for } i = 1, \ldots, n;\ j = 1, \ldots, m \tag{14}$$

$$g^v_{Li} = a^v_{Li} + \sum_{j=1}^{m} b^v_{Lij} x_{ij} \ \forall i \tag{15}$$

$$a^v_{Li} + \sum_{j=1}^{m} b^v_{Lij} x_{ij} \geq a^v_{Lh} + \sum_{j=1}^{m} b^v_{Lhj} x_{ij} \ \forall i,\ h;\ i \neq h \tag{16}$$

$$b^v_{Lij} \geq 0 \ \forall i,\ h \tag{17}$$

Then, we can derive an explicit represent function of y_L as follows. Given the estimated coefficients $\left(\widehat{a}^c_{Li},\ \widehat{b}^c_{Lj},\ \widehat{a}^v_{Li},\ \widehat{b}^v_{Lij}\right)$ from **[Fuzzy-MNLS-L]**, we construct the following explicit represent functions:

$\widehat{g}^c_{Li}(x) = \widehat{a}^c_{Li} + \sum_{j=1}^{m} \widehat{b}^c_{Lij} x_{ij}$, $\widehat{g}^v_{Li}(x) = \widehat{a}^v_{Li} + \sum_{j=1}^{m} \widehat{b}^v_{Lij} x_{ij}$, and $\widehat{y}_{Li}(x) = \widehat{g}_{Li}(x) = \widehat{g}^c_{Li}(x) + \widehat{g}^v_{Li}(x)$.

Due to the structure of the **[Fuzzy-MNLS-L]** model, there may be multiple optimal solutions. To address this, we calculate two distinct envelopes: A lower concave envelope for g^c_{Li} and an upper convex envelope for g^v_{Li}. These envelopes ensure that the estimated regression function remains consistent and interpretable.

3.2.1 *Calculating the lower concave and convex envelope*

The concave part of the function, g_{Li}^{c}, requires solving the following optimization problem to obtain the lower envelope:

[Fuzzy-MNLS-L-Lcave$_i$]

$$\min_{\tilde{a},\, \tilde{b}} \left\{ \tilde{a}_{Li}^{c} + \sum_{j=1}^{m} \tilde{b}_{Lij}^{c} x_{ij} \mid \tilde{a}_{Li}^{c} + \sum_{j=1}^{m} \tilde{b}_{Lij}^{c} x_{ij} \geq \hat{g}_{Li}^{c} \quad \forall i \right\} \tag{18}$$

Similarly, for g_{Li}^{v}, the convex part, we have

[Fuzzy-MNLS-L-Lvex$_i$]

$$\max_{\tilde{a},\, \tilde{b}} \left\{ a_{Li}^{v} + \sum_{j=1}^{m} \tilde{b}_{Lij}^{v} x_{ij} \mid \tilde{a}_{Li}^{v} + \sum_{j=1}^{m} \tilde{b}_{Lij}^{v} x_{ij} \leq \hat{g}_{Li}^{v} \quad \forall i \right\} \tag{19}$$

3.2.2 *Forecasting with hyperplane segments*

The resulting hyperplane segments, denoted as $\mathbf{k}_{\mathbf{L}}^{\mathbf{c}}$ and $\mathbf{k}_{\mathbf{L}}^{\mathbf{v}}$, are obtained from the solutions of **[Fuzzy-MNLS-L-Lcave$_i$]** and **[Fuzzy-MNLS-L-Lvex$_i$]**, respectively. These hyperplane segments are subsequently utilized in the forecasting processes as outlined in [42]. Figure 1 displays the regression model's predicted outcomes, which are structured using one input and four distinct hyperplanes.

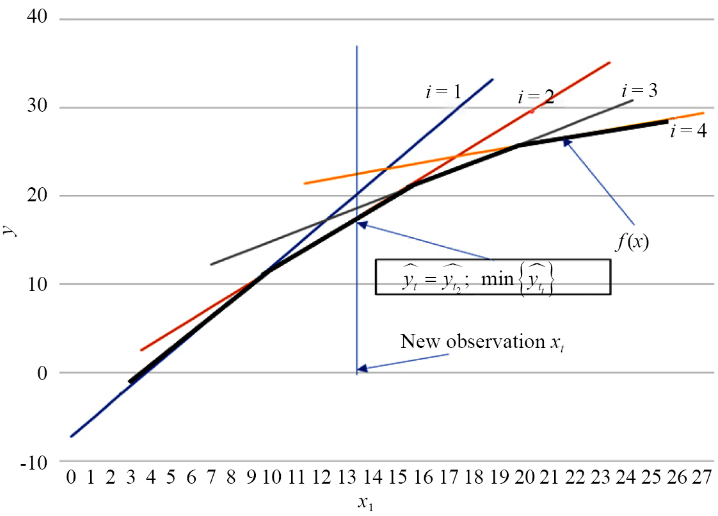

Figure 1. Illustration of the forecast $\widehat{y_t}$ for one x input and 4 hyperplanes of CNLS

Similarly, we can have **[Fuzzy-MNLS-C]** and **[Fuzzy-MNLS-C-Lcave$_i$]** for [Diamond-C], and **[Fuzzy-MNLS-R]** and **[Fuzzy-MNLS-R-Lvex$_i$]** for [Diamond-R].

To streamline the representation, we introduce the notation **[Diamond-(K)]** for $K = \{C,\ L,\ R\}$, which collectively denotes the three Diamond models associated with the center, left, and right endpoints. Consequently, the general models are defined as **[Fuzzy-MNLS-(K)]**, **[Fuzzy-MNLS-(K)-Lcave$_i$], and [Fuzzy-MNLS-(K)-Lvex$_i$]**, covering all three configurations within a unified framework.

[Fuzzy-MNLS-(K)]

$$\min_{\alpha,\ \beta,\ \varepsilon} \sum_{i=1}^{n} \left(\varepsilon_i^K\right)^2$$

s.t.

$$y_{Ki} = g_{Ki}^c + g_{Ki}^v + \varepsilon_i^K \quad \text{for } i = 1,\ \ldots,\ n \tag{20}$$

$$g_{Ki}^c = a_{Ki}^c + \sum_{j=1}^{m} b_{Kij}^c x_{ij} \quad \forall i \tag{21}$$

$$a_{Ki}^c + \sum_{j=1}^{m} b_{Kij}^c x_{ij} \le a_{Kh}^c + \sum_{j=1}^{m} b_{Khj}^c x_{ij} \quad \text{for } i,\ h = 1,\ \ldots,\ n \text{ and } i \neq h \tag{22}$$

$$g_{Ki}^v = a_{Ki}^v + \sum_{j=1}^{m} b_{Kij}^v x_{ij} \quad \forall i \tag{23}$$

$$a_{Ki}^v + \sum_{j=1}^{m} b_{Kij}^v x_{ij} \le a_{Kh}^v + \sum_{j=1}^{m} b_{Khj}^v x_{ij} \quad \forall i,\ h;\ i \neq h \tag{24}$$

$$b_{Kij}^c,\ b_{Kij}^v \ge 0 \quad \text{for } i = 1,\ \ldots,\ n;\ j = 1,\ \ldots,\ m \tag{25}$$

[Fuzzy-MNLS-(K)-Lcave$_i$]

$$\min_{\tilde{a},\ \tilde{b}} \left\{ \tilde{a}_{Ki}^c + \sum_{j=1}^{m} \tilde{b}_{Kij}^c x_{ij} \mid \tilde{a}_{Ki}^c + \sum_{j=1}^{m} \tilde{b}_{Kij}^c x_{ij} \ge \hat{g}_{Ki}^c \forall i \right\} \tag{26}$$

[Fuzzy-MNLS-(K)-Lvex$_i$]

$$\max_{\tilde{a},\ \tilde{b}} \left\{ a_{Ki}^v + \sum_{j=1}^{m} \tilde{b}_{Kij}^v x_{ij} \mid \tilde{a}_{Ki}^v + \sum_{j=1}^{m} \tilde{b}_{Kij}^v x_{ij} \le \hat{g}_{Ki}^v \forall i \right\} \tag{27}$$

4. Fuzzy-MNLS and forecasting

To perform forecasting using the Fuzzy-MNLS method, we start by defining the new observed input vector $\mathbf{x_t}$ for a given observation $\mathbf{t}$. Assume that **[Fuzzy-MNLS-(K)-Lcave$_i$]** and **[Fuzzy-MNLS-(K)-Lvex$_i$]** yield the unique hyperplanes obtains $\mathbf{k_K^c}$ and $\mathbf{k_K^v}$, respectively, for each endpoint $\mathbf{K} = \{C,\ L,\ R\}$. Then, for each $\mathbf{K}$, we calculate the fitted values as follows: for the concave component, $\widehat{g}_{Kti}^c = \widetilde{a}_{Ki}^c + \sum_{j=1}^{m} \widetilde{b}_{Kij}^c x_{tj}$, for $i \in \mathbf{k_K^c}$; and for the convex component, $\widehat{g}_{Kti}^v = \widetilde{a}_{Ki}^v + \sum_{j=1}^{m} \widetilde{b}_{Kij}^v x_{tj}$, for $i \in \mathbf{k_K^v}$.

Then, the forecasted

$$\hat{g}^c_{Kt} = \min\left\{\hat{g}^c_{Kti} \mid \hat{g}^c_{Kti} = \tilde{a}^c_{Ki} + \sum_{j=1}^{m} \tilde{b}^c_{Kij} x_{tj}, \ \text{ for } i \in \boldsymbol{k}^c_K\right\} \tag{28}$$

Similarly,

$$\hat{g}^v_{Kt} = \max\left\{\hat{g}^v_{Kti} \mid \hat{g}^v_{Kti} = \tilde{a}^v_{Ki} + \sum_{j=1}^{m} \tilde{b}^v_{Kij} x_{tj}, \ \text{ for } i \in \boldsymbol{k}^v_K\right\} \tag{29}$$

Then,

$$\hat{y}_{Kt} = \hat{g}^c_{Kt} + \hat{g}^v_{Kt} \text{ for } \boldsymbol{K} = \{C,\ L,\ R\} \tag{30}$$

A similar approach can be found in Chung [42] for MNLS.

Figure 2 illustrates the structured flow of information, from constructing the **[Fuzzy-MNLS-(K)]** model to obtaining forecasted outputs for each endpoint. This systematic approach ensures accurate forecasting while leveraging the unique properties of the concave and convex components.

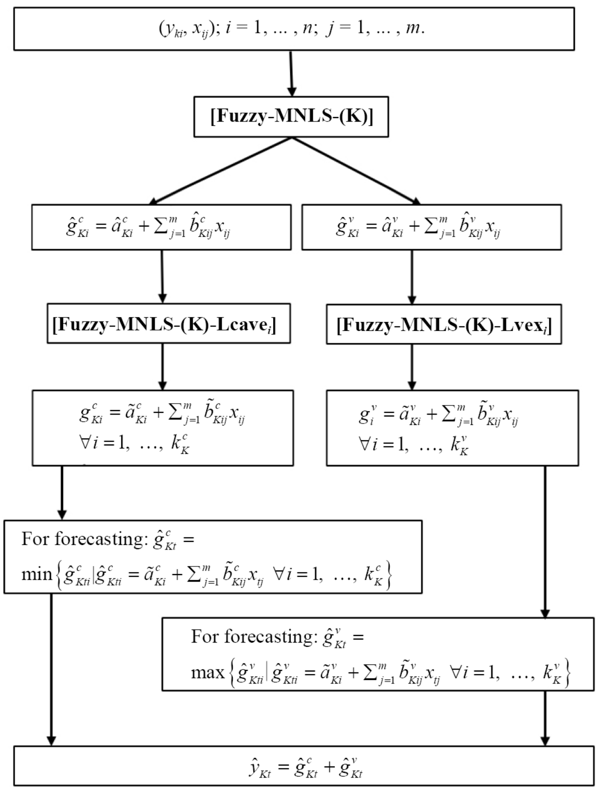

Figure 2. Workflow of Fuzzy-MNLS from model development to forecasting

Remarks The methodology outlined above can be extended to accommodate negative monotone regression functions. This is achieved by modifying the monotonicity constraints (Equation (25)) to reflect decreasing trends by setting $b_{K_{ij}^{c}}$, $b_{K_{ij}^{v}} \leq 0$. Such adjustments enable the Fuzzy-MNLS method to model functions where the relationship between inputs and outputs is characterized by a negative monotonic pattern.

5. Performance metrics for fuzzy regression

To evaluate the performance of fuzzy regression models, quantitative metrics must be used to capture the accuracy and similarity between the predicted and actual fuzzy outputs. In the current study, we utilize three primary metrics: Root Mean Square Error (RMSE), Mean Absolute Error (MAE), and the Similarity Index (S). These metrics offer a comprehensive assessment of the model's predictive performance by considering both numerical accuracy and the similarity of fuzzy numbers.

5.1 *RMSE*

Root Mean Square Error (RMSE) is a widely used metric for evaluating prediction accuracy. It quantifies the average magnitude of error between the predicted and actual values, giving higher weight to larger errors due to the squaring process. The RMSE is calculated as follows:

$$\text{RMSE} = \sqrt{\frac{1}{n}\sum_{i=1}^{n} d^2\left(y_i - \widehat{y}_i\right)},$$

where:

$d^2\left(y_i - \widehat{y}_i\right)$ denotes the discrepancy between the predicted and actual fuzzy outputs, often calculated as the average difference between the center, left, and right endpoints of the fuzzy triangular numbers.

n represents the number of observations.

RMSE is particularly useful when large errors are less acceptable, as it penalizes them more heavily compared to other error metrics. RMSE effectively captures the variability between the estimated and observed fuzzy numbers in fuzzy regression.

5.2 *MAE*

Mean Absolute Error (MAE) is another common metric for evaluating the accuracy of regression models. Unlike RMSE, it calculates the average of the absolute differences between the predicted and actual values, making it less sensitive to large deviations. The formula for MAE is:

$$\text{MAE} = \frac{1}{n}\sum_{i=1}^{n} d\left(y_i - \widehat{y}_i\right)$$

MAE provides an intuitive measure of the average error, which is especially relevant when the goal is to minimize the mean discrepancy without disproportionately focusing on outliers.

5.3 *S*

In addition to RMSE and MAE, we employ the Similarity Index (S) to assess how closely the predicted fuzzy number matches the actual fuzzy number. The similarity index of [22], used in [41], measures between two fuzzy triangular numbers, $\widetilde{a}$ and $\widetilde{b}$.

$$S(\tilde{a},\ \tilde{b}) = 1 - \frac{|a_c - b_c| + |a_l - b_l| + |a_r - b_r|}{\max\left(a_c + a_r,\ b_c + b_r\right) - \min\left(a_c - a_l,\ b_c - b_l\right)} \tag{31}$$

Noted that $S\left(\widetilde{a},\ \widetilde{b}\right)$ satisfies

(P1) $\widetilde{a} = \widetilde{b} \Longleftrightarrow S\left(\widetilde{a},\ \widetilde{b}\right) = 1$,

(P2) $S\left(\widetilde{a},\ \widetilde{b}\right) = S(\widetilde{b},\ \widetilde{a})$,

(P3) $\widetilde{a} \subseteq \widetilde{b} \subseteq \widetilde{c} \Rightarrow S\left(\widetilde{a},\ \widetilde{c}\right) \leq min\left\{S\left(\widetilde{a},\ \widetilde{b}\right),\ S\left(\widetilde{b},\ \widetilde{c}\right)\right\}$,

(P4) $S\left(\widetilde{a},\ \widetilde{a}^c\right) = 0$ if $\widetilde{a}$ is a crisp set, and

(P5) $0 \leq S(\widetilde{a},\ \widetilde{b}) \leq 1$.

$S\left(\widetilde{a},\ \widetilde{b}\right)$ also is a distance measure with $|a_c - b_c| + |a_l - b_l| + |a_r - b_r|$, which does not require the intersection properties. The Similarity Index is particularly valuable for fuzzy regression because it captures both the distance and similarity properties of fuzzy numbers, making it versatile for comparing fuzzy outputs. Unlike purely numerical error measures, the Similarity Index directly reflects how well the fuzzy predictions overlap with the observed values.

5.4 *Justification for metric selection*

Combining RMSE, MAE, and the Similarity Index ensures a holistic evaluation of the fuzzy regression model. RMSE captures the variability and penalizes large errors, while MAE provides a straightforward average error measure. The Similarity Index complements these metrics by evaluating the overlap between fuzzy sets, which is crucial for models with inherently fuzzy output. This combination provides a robust and balanced assessment of the model's performance.

5.5 *Alignment between objective function and performance metric*

In the proposed Fuzzy-MNLS method, we use Diamond's objective function (RSS) to train the model, as it ensures mathematical consistency and computational feasibility. This choice guarantees that Fuzzy-MNLS achieves a better or equal RSS than Fuzzy-CNLS, given that MNLS is a relaxation of CNLS. However, we also evaluate the model using the Similarity Index (S) to measure the closeness between the predicted and actual fuzzy numbers.

It is important to note that the Similarity Index and RSS capture different aspects of model performance. While RSS quantifies the magnitude of error, S measures the closeness and overlap of fuzzy numbers influenced by distance and range. Due to this difference, a model that minimizes RSS does not necessarily achieve higher similarity. This mismatch highlights that the choice of performance metric should align with the specific application requirements. While RSS may better reflect error minimization, similarity is more relevant when the closeness of fuzzy predictions is the primary concern.

6. Illustrative example

To demonstrate the application of the proposed **[Fuzzy-MNLS-(K)]** method, we present an illustrative example. The results obtained using Fuzzy-MNLS-(K) are compared against those produced by the [Diamond] method, the FLAR method as described in [22], and Fuzzy-CNLS-(K) from [42].

All computational implementations were carried out using GAMS (version 49.1.0) [36], employing the CONOPT solver. The simulations were executed on an iMac system (M1, 16 GB, Sequoia 15.4.1). The inclusion of the FLAR method is due to its growing popularity, and we hypothesize that it may demonstrate enhanced performance relative to Diamond's method.

6.1 *Example overview*

The example data is sourced from [41] and and has been previously utilized in [42]. The dataset contains a single input variable (x_1). Figure 3 is the scatter diagram. Figure 3 presents the scatter plot of the data, while Table 2 provides the detailed dataset used for the comparison.

Table 2. Example data [41]

i	x_1	y_L	y_c	y_R
1	5	4	11	19
2	8	11	16	20
3	11	15	18	21
4	14	21	24	26
5	17	23	25	27
6	19	26	30	34
7	22	27	31	39
8	24	28	37	48

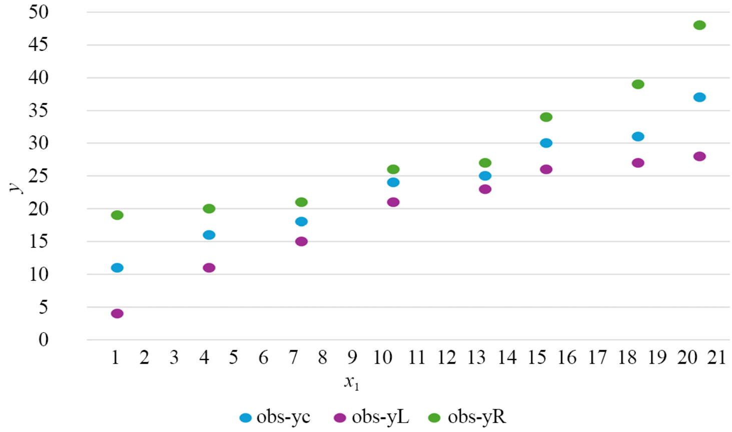

Figure 3. Scatter plot of the example

First, using **[Fuzzy-MNLS-(K)]** (10) to find g^c_{Ki} (concave component of **[Diamond-(K)]**) and g^v_{Ki} (convex component of **[Diamond-(K)]**). The corresponding hyperplanes obtained by **[Fuzzy-MNLS-(L)]**, **[Fuzzy-MNLS-(C)]**, and **[Fuzzy-MNLS-(R)]** are presented in the 4^{th} to 7^{th} columns of Table 3, 4, 5, respectively.

Table 3. Hyperplanes results from **[Fuzzy-MNLS-(L)]** in the example

i	x_i	y_L	$\widehat{g}^c_{Li}(x)$		$\widehat{g}^v_{Li}(x)$		$\hat{y}_L = \hat{g}_L(x) = \hat{g}^c_L(x) + \hat{g}^v_L(x)$		
			$\widehat{a}^c_{Li}$	$\widehat{b}^c_{Lij}$	$\widehat{a}^v_{Li}$	$\widehat{b}^v_{Lij}$	$\widehat{a}^*_{Li}$	$\widehat{b}^*_{Lij}$	$\widehat{y}_L$
1	5	4	−10.16	2.28	2.76	0.00	−7.40	2.28	4.00
2	8	11	−3.89	1.50	2.76	0.00	−1.13	1.50	10.84
3	11	15	−3.89	1.50	2.76	0.00	−1.13	1.50	15.33
4	14	21	−3.89	1.50	−0.39	0.29	−4.28	1.78	20.67
5	17	23	8.31	0.62	−3.76	0.48	4.55	1.11	23.41
6	19	26	8.31	0.62	−3.76	0.48	4.55	1.11	25.63
7	22	27	20.18	0.00	−3.76	0.48	16.42	0.48	27.08
8	24	28	20.18	0.00	−3.76	0.48	16.42	0.48	28.05

* $\widehat{a}_{Li} = \widehat{a}^c_{Li} + \widehat{a}^v_{Li}$; $\widehat{b}_{Lij} = \widehat{b}^c_{Lij} + \widehat{b}^v_{Lij}$

Table 4. Hyperplanes results from **[Fuzzy-MNLS-(C)]** in the example

i	x_i	y_C	$\widehat{g}^c_{Ci}(x)$		$\widehat{g}^v_{Ci}(x)$		$\hat{y}_C = \hat{g}_C(x) = \hat{g}^c_C(x) + \hat{g}^v_C(x)$		
			$\widehat{a}^c_{Ci}$	$\widehat{b}^c_{Cij}$	$\widehat{a}^v_{Ci}$	$\widehat{b}^v_{Cij}$	$\widehat{a}^*_{Ci}$	$\widehat{b}^*_{Cij}$	$\widehat{y}_C$
1	5	11	−7.50	1.53	10.87	0.00	3.37	1.53	11.00
2	8	16	−4.00	1.09	10.87	0.00	6.87	1.09	15.58
3	11	18	−4.00	1.09	7.04	0.35	3.04	1.44	18.84
4	14	24	−4.00	1.09	7.04	0.35	3.04	1.44	23.16
5	17	25	2.58	0.62	7.04	0.35	9.62	0.97	26.06
6	19	30	2.58	0.62	−1.08	0.83	1.50	1.44	28.94
7	22	31	14.33	0.00	−32.37	2.25	−18.03	2.25	31.42
8	24	37	14.33	0.00	−44.27	2.79	−29.93	2.79	37.00

*$\widehat{a}_{Ci} = \widehat{a}^c_{Ci} + \widehat{a}^v_{Ci}$; $\widehat{b}_{Cij} = \widehat{b}^c_{Cij} + \widehat{b}^v_{Cij}$

Table 5. Hyperplanes results from **[Fuzzy-MNLS-(R)]** in the example

i	x_i	y_R	$\widehat{g}^c_{Ri}(x)$		$\widehat{g}^v_{Ri}(x)$		$\hat{y}_R = \hat{g}_R(x) = \hat{g}^c_R(x) + \hat{g}^v_R(x)$		
			$\widehat{a}^c_{Ri}$	$\widehat{b}^c_{Rij}$	$\widehat{a}^v_{Ri}$	$\widehat{b}^v_{Rij}$	$\widehat{a}^*_{Ri}$	$\widehat{b}^*_{Rij}$	$\widehat{y}_R$
1	5	19	−15.58	0.48	31.91	0.00	16.33	0.48	18.71
2	8	20	−15.58	0.48	31.91	0.00	16.33	0.48	20.14
3	11	21	−15.58	0.48	31.91	0.00	16.33	0.48	21.57
4	14	26	−13.45	0.32	24.05	0.71	10.60	1.04	25.14
5	17	27	−12.58	0.26	24.05	0.71	11.48	0.98	28.07
6	19	34	−12.58	0.26	−0.64	2.17	−13.21	2.43	32.93
7	22	39	−7.60	0.00	−11.65	2.67	−19.25	2.67	39.43
8	24	48	−7.60	0.00	−49.83	4.39	−57.43	4.39	48.00

*$\widehat{a}_{Ri} = \widehat{a}^c_{Ri} + \widehat{a}^v_{Ri}$; $\widehat{b}_{Rij} = \widehat{b}^c_{Rij} + \widehat{b}^v_{Rij}$

The second step involves employing **[Fuzzy-MNLS-(K)-Lcave$_i$]** and **[Fuzzy-MNLS-(K)-Lvex$_i$]** to determine the hyperplanes that constitute the corresponding regression model for forecasting purposes in Tables 3, 4, 5. These hyperplanes are derived specifically for the Left, Center, and Right models to ensure accurate prediction. Tables 6, 7, 8 provide a summary of the hyperplane configurations for each of these models, highlighting the distinct structure associated with each component.

Table 6. Hyperplanes for left endpoint of the example

	[Fuzzy-MNLS-(L)-Lcave$_i$] $\hat{g}^c_{Li}(x)$		[Fuzzy-MNLS-(L)-Lvex$_i$] $\hat{g}^v_{Li}(x)$		Forecast $x_t = 13$	
i	$\widetilde{a}^c_{Li}$	$\widetilde{b}^c_{Lij}$	$\widetilde{a}^v_{Li}$	$\widetilde{b}^v_{Lij}$	$\widetilde{g}^c_{Lti}$	$\widetilde{g}^v_{Lti}$
1	−10.16	2.28	2.76	0.00	19.47	2.76
2	−3.89	1.50	−0.39	0.29	15.56	3.33*
3	8.31	0.62	−3.76	0.48	16.43	2.54
4	20.18	0.00	−3.76	0.48	20.18	-

*by Equation (29) $\max_i\{\widetilde{g}^v_{Lti}\}$ where $\widetilde{g}^v_{Lti} = \widetilde{a}^v_{Li} + \widetilde{b}^v_{Lij} * (x_t)$

Table 7. Hyperplanes for center endpoint of the example

i	[Fuzzy-MNLS-(C)-Lcave$_i$] $\hat{g}^c_{Ci}(x)$		[Fuzzy-MNLS-(C)-Lvex$_i$] $\hat{g}^v_{Ci}(x)$	
	$\widetilde{a}^c_{Ci}$	$\widetilde{b}^c_{Cij}$	$\widetilde{a}^v_{Ci}$	$\widetilde{b}^v_{Cij}$
1	−7.50	1.53	10.87	0.00
2	−4.00	1.09	7.04	0.35
3	2.58	0.62	−1.08	0.83
4	14.33	0.00	−32.37	2.25
5	-	-	−44.27	2.79

Table 8. Hyperplanes for right endpoint of the example

i	[Fuzzy-MNLS-(R)-Lcave$_i$] $\hat{g}^c_{Ri}(x)$		[Fuzzy-MNLS-(R)-Lvex$_i$] $\hat{g}^v_{Ri}(x)$	
	$\widetilde{a}^c_{Ri}$	$\widetilde{b}^c_{Rij}$	$\widetilde{a}^v_{Ri}$	$\widetilde{b}^v_{Rij}$
1	−15.58	0.48	31.91	0.00
2	−13.45	0.32	24.05	0.71
3	−12.58	0.26	−0.64	2.17
4	−7.60	0.00	−11.65	2.67
5	-	-	−49.83	4.39

6.2 *The number of hyperplanes of the Fuzzy-MNLS*

Figure 4 illustrates the resulting hyperplanes of Fuzzy-MNLS for Left, Center, and Right endpoints and the observations. Note that the hyperplanes shown in Figure 4 are the results of Equation (30) in Section 4. The hyperplanes shown in Tables 6, 7, 8 are used to conduct the forecasting process. Hence, we consider that the number of hyperplanes by Fuzzy-MNLS is the sum of the hyperplanes in Tables 6, 7, 8, which is 25, as shown in the last column of Table 9.

Table 9. The number of hyperplanes of the Fuzzy-MNLS and other methods

Hyperplane (i)	Center		Left endpoint		Right endpoint		No of Hyperplanes
	aC	bC	aL	bL	aR	bR	
Fuzzy-CNLS							11
1	5.67	1.2	−7.17	2.23	17.33	0.33	
2	−21.89	2.45	−2.32	1.62	15.98	0.51	
3			6.01	1.03	9.18	1.12	
4			16.76	0.46	−9.83	2.24	
5					−54.08	4.25	
Diamond	4.918	1.272	0.72	1.244	7.652	1.44	3
FLAR		1.542		0.214		0.273	3
Fuzzy-MNLS	Hyperplanes (see Table 5a, b, c)						25

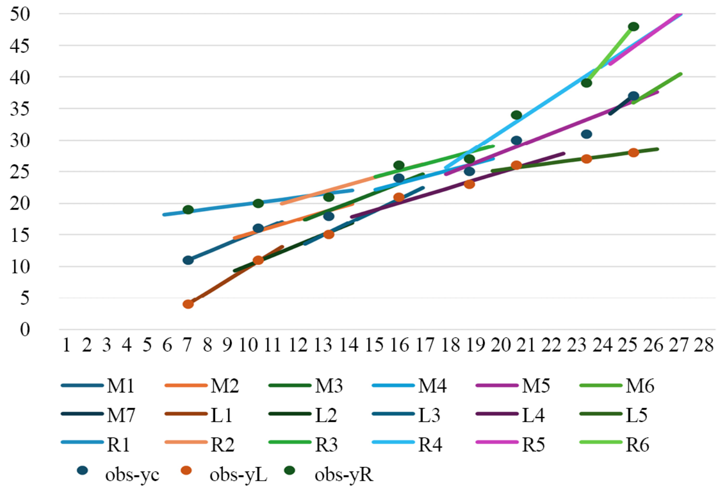

Figure 4. Fuzzy-MNLS hyperplanes and the data of the example

6.3 *Comparative forecasting accuracy analysis of the Fuzzy-MNLS*

To evaluate the effectiveness of the proposed Fuzzy-MNLS method, we conducted a comparative analysis with three other established fuzzy regression methods: Fuzzy-CNLS, Diamond method, and Fuzzy Least Absolute Regression (FLAR). The comparison focused on three key accuracy metrics: Similarity Measure (SM), Root Mean Square Error (RMSE), and Mean Absolute Error (MAE).

The Similarity Measure (SM) evaluates how closely the predicted fuzzy number matches the actual fuzzy number, with a higher value indicating better similarity. RMSE and MAE assess the prediction accuracy by quantifying the average difference between the predicted and actual fuzzy numbers. While RMSE is more sensitive to larger errors due to the squared difference, MAE provides a more direct interpretation of average error.

The results of the comparative analysis are summarized in Table 10. According to [42], the similarity score of Fuzzy-CNLS (0.749) significantly surpasses that of Diamond (0.543) and FLAR (0.593) in this instance, as indicated in the second column of Table 10. Notably, the proposed Fuzzy-MNLS method consistently demonstrates superior performance compared to the other techniques, showing the lowest RMSE and MAE values. This superior performance can be attributed to Fuzzy-MNLS's ability to incorporate both convex and concave components within a single optimization framework, thereby capturing complex patterns in the data more effectively.

Table 10. Comparative forecasting accuracy of fuzzy regression methods

	Similarity	RMSE	MAE
Fuzzy-CNLS	0.749*	0.787	0.694
Diamond	0.543*	2.328	2.097
FLAR	0.593*	3.169	2.689
Fuzzy-MNLS	0.841	0.588	0.494

*From [42]

The results indicate that Fuzzy-MNLS achieves a significantly higher similarity score than Fuzzy-CNLS, Diamond, and FLAR methods. The RMSE and MAE values demonstrate that Fuzzy-MNLS minimizes prediction errors more effectively. The performance improvement highlights the advantage of integrating both convex and concave patterns within the fuzzy regression function instead of relying solely on one type of pattern, as seen in the other methods.

This comparison validates the robustness and accuracy of the Fuzzy-MNLS method, reinforcing its potential for applications where accurate fuzzy regression modeling is critical.

6.4 *Forecasting process of the example*

Suppose we have a new observation t, where $x_t = 13$. Using the four hyperplanes of $\widehat{g}^c_{Li}(x)$ in Table 6 and Equation (28), we can find

$$\widehat{g}^c_{Lt} = \min\left\{ \widehat{g}^c_{Lti} \mid \widehat{g}^c_{Kti} = \widetilde{a}^c_{Li} + \sum_{j=1}^{m} \widetilde{b}^c_{Lij} x_{tj} \quad \forall i = 1, \ldots, 4 \right\}$$

$$= \min\{19.47,\ 15.56,\ 16.43,\ 20.18\} = 15.56.$$

Similarly, using Equation (29), $\widehat{g}^v_{Lt} = \max\{2.76,\ 3.33,\ 2.54\} = 3.33$, see the illustration in the last column of Table 6. Then, $\widehat{y}_{Lt} = 15.56 + 3.33 = 18.89$. Noted that the resulting Fuzzy-MNLS hyperplanes $(\widehat{a}_{Li},\ \widehat{b}_{Lij})$ in Table 6 (the 8^{th} and 9^{th} column) are not used for forecasting.

Similarly, using the hyperplanes in Table 7, we can have $\widehat{g}^c_{Ct} = 10.15$, $\widehat{g}^v_{Ct} = 11.57$, and $\widehat{y}_{Ct} = 10.15 + 11.57 = 21.72$. Using hyperplanes in Table 8, we have $\widehat{y}_{Rt} = -9.39 + 33.34 = 23.95$.

Hence, the estimated $\widehat{\widehat{y}}_t(13) = (\widehat{y}_{Lt},\ \widehat{y}_{Ct},\ \widehat{y}_{Rt}) = (18.89,\ 21.72,\ 23.95)$.

7. Discussion

In nonparametric fuzzy regression, machine-learning-based methods are popular due to their ability to model complex relationships. However, these methods often require a large sample size to train the regression models effectively, which may not be feasible in applications where data availability is limited. For instance, developing benchmarking models for building energy performance [36] may not always allow access to a large number of observations. On the other hand, parametric regression methods, which can operate with smaller sample sizes, may suffer from poor goodness-of-fit or low similarity when the model structure does not align well with the data.

We propose a new Fuzzy Monotone Nonparametric Least Squares (Fuzzy-MNLS) method designed explicitly for triangular fuzzy output with crisp input to address these challenges. This method effectively addresses the challenge of modeling complex, nonlinear monotonic relationships without the need for predefined functional forms. By eliminating rigid model specifications, Fuzzy-MNLS offers flexibility in capturing diverse data patterns.

The proposed Fuzzy-MNLS technique is rooted in the fuzzy least squares of Diamond. It extends the original model by dividing it into three distinct sub-models: Center, Left, and Right endpoint. This division aligns with Diamond's conceptual structure, where the Center, Left, and Right models are each formulated using OLS. The inherent structure of Diamond's method naturally supports this three-part decomposition. To further enhance the estimation process, the MNLS technique is applied to each sub-model individually. This integration allows for estimating fuzzy regression functions that capture both convex and concave components within a single framework. Hence, Fuzzy-MNLS outperforms the Fuzzy-CNLS method cited in [42] in terms of goodness-of-fit, especially when the dataset demonstrates monotonic behavior accompanied by curvature changes.

An illustrative example demonstrates that Fuzzy-MNLS consistently achieves higher similarity scores and improved forecasting accuracy compared to other least squares methods. The method's ability to handle convex and concave patterns within a monotonic framework makes it particularly effective when the data exhibits nonlinear monotonic trends.

However, the convex and concave constraints that enhance model flexibility also introduce computational challenges, especially as the number of observations (n) increases. The runtime of GAMS in the illustrative example is 0.109 seconds when $n = 8$. However, when the number of observations becomes large, the computational complexity of Fuzzy-MNLS significantly increases due to its nonparametric nature and the requirement to maintain monotonicity constraints. The complexity primarily arises from the pairwise comparisons needed to establish order relations among data points, which scales as $O(n^2)$, and the optimization cost of solving the nonlinear programming problem, which scales as $O(n^3)$.

Consequently, the overall computational burden can be approximated as $O(n^3)$, making Fuzzy-MNLS computationally intensive for large datasets. This cubic complexity is notably higher than Fuzzy-CNLS, which typically exhibits $O(n^2)$ complexity due to its simpler convexity or concavity constraints. According to our experience with crisp MNLS, when $n > 400$, it may take more than an hour to obtain the solution using the Minos solver. Sometimes, the Conopt solver fails to provide a solution due to the limitation of major iterations.

8. Conclusion and further research

The proposed Fuzzy-MNLS method addresses key limitations of machine-learning-based and parametric fuzzy regression methods. It demonstrates superior performance, particularly when small sample sizes are inevitable, by leveraging monotonicity constraints and allowing for convex and concave components. Unlike machine-learning models, it does not require regularization, and compared to Fuzzy-CNLS, it achieves better goodness of fit.

One notable limitation of Fuzzy-MNLS is its computational burden due to the numerous constraints introduced by combining convexity and concavity within a monotonic structure. The literature offers potential solutions to this challenge, as demonstrated by efficient algorithms proposed by Lee et al. [34] and Mazumder et al. [37]. Future research could focus on integrating these efficient algorithms into the Fuzzy-MNLS framework to enhance computational efficiency. Extending the method to include ridge-based approaches, as discussed in [47], or adapting it for fuzzy-input fuzzy-output models could further broaden its applicability.

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Conflict of interest

The author declares no competing interests.

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