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Research Article

A New Category of Analytical Functions Constructed with Mittag-Leffler Function and Lambert Series

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Abstract: This investigation explores a novel subclass of analytical functions, designated as $T_{\mathscr{J}}(\xi, \rho, \mathrm{A}, \mathrm{B})$, constructed through the application of a linear operator incorporating both the Mittag-Leffler function and Lambert series. We provide sufficient conditions for an analytic function to be a member of the introduced class, we obtain the distortion theorems, the extreme points and the coefficient bounds. When applicable, additional findings are derived utilizing the established Robin's inequalities that assert upper bounds of the Lambert series coefficients.

***Keywords*:** univalent functions, starlike, convolution, Lambert series, Mittag-Leffler function

MSC: 30C45, 30C50

1. Introduction

The Mittag-Leffler function denoted as $E_\alpha(z)$ where $\alpha \in \mathbb{C}$, with $\Re(\alpha) > 0$ [1, 2] is expressed as:

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)},\ z \in \mathbb{C}.$$

A two-parameter extension investigated by Wiman [3] defines $E_{\alpha,\ \beta}(z)$ for all $\alpha,\ \beta \in \mathbb{C}$, with $\Re(\alpha,\ \beta > 0)$ as:

$$E_{\alpha,\ \beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)},\ z \in \mathbb{C}.$$

Numerous scholars have explored generalizations of the Mittag-Leffler function including their applications in the theory of geometric functions [4–9].

For this analysis, we focus on the generalization proposed by Salah and Darus [10]:

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$$qF_{\alpha,\ \beta}^{\theta,\ k} = \sum_{n=0}^{\infty}\prod_{j=1}^{q}\frac{(\theta_j)_{k_j n}}{(\beta_j)_{\alpha_j n}}\cdot\frac{z^n}{n!}. \tag{1}$$

Where $(\theta)_v$ represents the Pochhammer symbol, defined as:

$$(\theta)_v := \frac{\Gamma(\theta+v)}{\Gamma(\theta)} = \begin{cases} 1, & \text{if } v = 0,\ \theta \in \mathbb{C}\backslash\{0\} \\ \theta(\theta+1)\ldots(\theta+n-1), & \text{if } v = n \in N,\ \theta \in \mathbb{C} \end{cases}$$

$$(1)_n = n!,\ n \in \mathcal{N}_0,\ \mathcal{N}_0 = \mathcal{N} \cup \{0\},\ \mathcal{N} = \{1,\ 2,\ 3,\ \ldots\},$$

with $\left(q \in \mathcal{N},\ j = 1,\ 2,\ 3,\ \ldots\ q;\ \Re\left\{\theta_j,\ \beta_j\right\} > 0,\ \text{and}\ \Re(\alpha_j) > \max\left\{0,\ \Re(\ k_j) - 1;\ \Re(\ k_j)\right\};\ \Re(k_j) > 0\right)$.

In number theory [11–14], the Lambert series appears in connection with arithmetic functions:

$$\sum_{n=1}^{\infty}\sigma_0(n)x^n = \sum_{n=1}^{\infty}\frac{x^n}{1-x^n}, \tag{2}$$

where $\sigma_0(n) = d(n)$ is the number of positive divisors of n.

Additionally,

$$l(z) = \sum_{n=1}^{\infty}\sigma_\alpha(n)x^n = \sum_{n=1}^{\infty}\frac{n^\alpha x^n}{1-x^n} \tag{3}$$

where $\sigma_\alpha(n)$ represents the higher-order divisors' sum function of n. When $\alpha = 1$, $\sigma_1(n) = \sigma(n)$, denotes the divisor sum function relevant to the Riemann Hypothesis.

It is important to distinguish between the Lambert series and the Lambert W function, which arises naturally in diverse scientific and engineering problems [15].

In 1984, Guy Robin [16] established that:

$$\sigma(n)\ < e^\gamma n\log\log n + \frac{0.6483n}{\log\log n},\ \ n \geq 3. \tag{4}$$

Further, he demonstrated that the Riemann hypothesis equivalently states:

$$\sigma(n)\ < e^\gamma n\log\log n,\ \ n > 5{,}040, \tag{5}$$

where $\gamma = 0.7721\cdots$, is the Euler-Mascheroni constant.

This paper does not attempt to prove or disprove Robin's inequality or the Riemann hypothesis. Interested readers are directed to the references for further exploration [17–22].

The aim of this study is to incorporate the Mittag-Leffler function and the Lambert series in order to introduces a linear operator and then to define a new analytical functions subclass. In the next section, we recall some basic definitions and concepts.

2. Foundational concepts

Let $\mathscr{A}$ represent the class of analytic functions expressed as:

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n,\ \ z\in\ \mathbb{D}:=\{z\in\mathbb{C}:|z|<1\}, \tag{6}$$

and $\mathscr{T}$ denote the subclass of $\mathscr{A}$ comprising functions of the form:

$$f(z)=z-\sum_{n=2}^{\infty}a_n z^n,\ \ z\in\ \mathbb{D}:=\{z\in\mathbb{C}:|z|<1\}, \tag{7}$$

We recall the Hadamard product (convolution) definition: For functions $f\in\mathscr{A}$ of the form described above and $g\in\mathscr{A}$ expressed as:

$$g(z)=z+\sum_{n=2}^{\infty}b_n z^n,\ \ z\in\mathbb{D}, \tag{8}$$

the convolution $(*)$ is obtained through:

$$(f*g)(z):=z+\sum_{n=2}^{\infty}a_n b_n z^n,\ \ z\in\mathbb{D}. \tag{9}$$

We utilize the Lambert series $\mathscr{L}(z)$, with coefficients represented by the sum of divisors function $\sigma(n)$:

$$\mathscr{L}(z)=\sum_{n=1}^{\infty}\frac{nz^n}{1-z^n}=\sum_{n=1}^{\infty}\sigma(n)z^n=z+\sum_{n=2}^{\infty}\sigma(n)z^n,\ \ z\in\mathbb{D}$$

Since $qF_{\alpha,\,\beta}^{\theta,\,k}$ is not a member of class $\mathscr{A}$, we apply a normalization:

$$q\mathbb{F}_{\alpha,\,\beta}^{\theta,\,k}=\prod_{j=1}^{q}\frac{(\beta_j)_{\alpha_j}}{(\alpha_j)_{k_j}}\left(qF_{\alpha,\,\beta}^{\theta,\,k}-1\right)=z+\sum_{n=2}^{\infty}\prod_{j=1}^{q}\frac{(\beta_j)_{\alpha_j}}{(\alpha_j)_{k_j}}\,\frac{(\theta_j)_{k_j n}}{(\beta_j)_{\alpha_j n}}\cdot\frac{z^n}{n!}. \tag{10}$$

We consider a linear operator recently introduced and studies by Jamal Salah [23]. The linear operator $\mathscr{J}\,(\mathscr{L},\mathbb{F})\,(z):\mathscr{A}\longrightarrow\mathscr{A}$ for a function $f\in\mathscr{A}$ is:

$$\mathscr{J}(\mathscr{L}, \mathbb{F})(z) := \left(q\mathbb{F}^{\theta,\, k}_{\alpha,\, \beta} * \mathscr{L}\right)(z) = z + \sum_{n=2}^{\infty} \prod_{j=1}^{q} \frac{(\beta_j)_{\alpha_j}}{(\alpha_j)_{k_j}} \frac{(\theta_j)_{k_j n}}{(\beta_j)_{\alpha_j n}} \cdot \frac{\sigma(n)}{n!} a_n z^n, \ z \in \mathbb{D}.$$

This linear operator leads us to our proposed definition:
A function $f \in \mathscr{A}$ of the form (6) is considered to be in the class $\mathscr{A}_{\mathscr{J}}(\xi, \rho, A, B)$ if it satisfies:

$$\left| \frac{(\mathscr{J}(\mathscr{L}, \mathbb{F})(z))' - 1}{(B + (A-B)(1-\xi)) - B(\mathscr{J}(\mathscr{L}, \mathbb{F})(z))'} \right| < \rho,$$

where

$$0 \le \xi < 1,\ 0 < \rho \le 1,\ -1 \le B < A \le 1 \text{ and } 0 < A \le 1.$$

We define the class: $\mathscr{T}_{\mathscr{J}}(\xi, \rho, A, B) = \mathscr{A}_{\mathscr{J}}(\xi, \rho, A, B) \cap \mathscr{T}$.

The main objective of this study is to derive the characteristic properties of the class $\mathscr{T}_{\mathscr{J}}(\xi, \rho, A, B)$. Consequently, we obtain the coefficients bounds and the extreme points. In addition, we discuss the distortion theorems. By utilizing the Robin's inequalities, we extend the results by conditionally providing lower bounds to the coefficients.

3. Characterization

In here, we obtain sufficient conditions for an analytic function to be a member of the class $\mathscr{T}_{\mathscr{J}}(\xi, \rho, A, B)$.

Theorem 1 A function $f \in \mathscr{A}$ of the form (7) is in class $\mathrm{T}_{\mathscr{J}}(\xi, \rho, A, B)$ if and only if:

$$\sum_{n=2}^{\infty} \prod_{j=1}^{q} \frac{(\beta_j)_{\alpha_j}}{(\alpha_j)_{k_j}} \frac{(\theta_j)_{k_j n}}{(\beta_j)_{\alpha_j n}} \cdot \frac{\sigma(n)}{(n-1)!} (1 - B\rho) a_n \le \rho(A-B)(1-\xi). \tag{11}$$

Proof. Assuming the condition holds true, and let $|z| = 1$, we have:

$$|\,(J(L, \mathbb{F})(z))' - 1 - \rho \left| (B + (A-B)(1-\xi)) - B(J(L, \mathbb{F})(z))' \right|$$

$$= \left| - \sum_{n=2}^{\infty} \prod_{j=1}^{q} \frac{(\beta_j)_{\alpha_j}}{(\alpha_j)_{k_j}} \frac{(\theta_j)_{k_j n}}{(\beta_j)_{\alpha_j n}} \cdot \frac{\sigma(n)}{(n-1)!} a_n z^{n-1} \right|$$

$$- \rho \left| (A-B)(1-\xi) + B \sum_{n=2}^{\infty} \prod_{j=1}^{q} \frac{(\beta_j)_{\alpha_j}}{(\alpha_j)_{k_j}} \frac{(\theta_j)_{k_j n}}{(\beta_j)_{\alpha_j n}} \cdot \frac{\sigma(n)}{(n-1)!} a_n z^{n-1} \right|$$

$$\le \sum_{n=2}^{\infty} \prod_{j=1}^{q} \frac{(\beta_j)_{\alpha_j}}{(\alpha_j)_{k_j}} \frac{(\theta_j)_{k_j n}}{(\beta_j)_{\alpha_{jn}}} \cdot \frac{\sigma(n)}{(n-1)!} (1 - \rho B) a_n - (A-B)(1-\xi) \le 0$$

by hypothesis.

Therefore $f(z) \in \mathscr{T}_{\mathscr{J}}(\xi,\ \rho,\ A,\ B)$.

Conversely, assuming $f(z) \in \mathscr{T}_{\mathscr{J}}(\xi,\ \rho,\ A,\ B)$, we have

$$\left|\frac{(J(L,\ \mathbb{F})(z))'-1}{(B+(A-B)(1-\xi))-B(J(L,\ \mathbb{F})(z))'}\right| = \frac{\left|\sum_{n=2}^{\infty}\prod_{j=1}^{q}\frac{(\beta_j)_{\alpha_j}}{(\alpha_j)_{k_j}}\frac{(\theta_j)_{k_jn}}{(\beta_j)_{\alpha_jn}}\cdot\frac{\sigma(n)}{(n-1)!}a_nz^{n-1}\right|}{\left|(A-B)(1-\xi)+B\sum_{n=2}^{\infty}\prod_{j=1}^{q}\frac{(\beta_j)_{\alpha_j}}{(\alpha_j)_{k_j}}\frac{(\theta_j)_{k_jn}}{(\beta_j)_{\alpha_jn}}\cdot\frac{\sigma(n)}{(n-1)!}a_nz^{n-1}\right|} < \rho.$$

Since $\mathrm{Re}(z) \leq z$ for all z, we have:

$$\mathrm{Re}\left(\frac{\sum_{n=2}^{\infty}\prod_{j=1}^{q}\frac{(\beta_j)_{\alpha_j}}{(\alpha_j)_{k_j}}\frac{(\theta_j)_{k_jn}}{(\beta_j)_{\alpha_jn}}\cdot\frac{\sigma(n)}{(n-1)!}a_nz^{n-1}}{(A-B)(1-\xi)+B\sum_{n=2}^{\infty}\prod_{j=1}^{q}\frac{(\beta_j)_{\alpha_j}}{(\alpha_j)_{k_j}}\frac{(\theta_j)_{k_jn}}{(\beta_j)_{\alpha_jn}}\cdot\frac{\sigma(n)}{(n-1)!}a_nz^{n-1}}\right) < \rho$$

selecting z on real axis, simplifying, and letting $z \to 1^{-}$ through real values, we obtain the desired result.

The assertion is sharp with the extremal function given by:

$$f(z) = z - \frac{\rho(A-B)(1-\xi)(n-1)!}{(1-B\rho)\sigma(n)}\prod_{j=1}^{q}\frac{(\alpha_j)_{k_j}(\beta_j)_{\alpha_{jn}}}{(\beta_j)_{\alpha_j}(\theta_j)_{k_{jn}}}z^n,\ \ n \geq 2.$$

Corollary 1 For a function f defined by (7) in class $\mathscr{T}_{\mathscr{J}}(\xi,\ \rho,\ A,\ B)$:

$$a_n \leq \frac{\rho(A-B)(1-\xi)(n-1)!}{(1-B\rho)\sigma(n)}\prod_{j=1}^{q}\frac{(\alpha_j)_{k_j}(\beta_j)_{\alpha_{jn}}}{(\beta_j)_{\alpha_j}(\theta_j)_{k_{jn}}}z^n,\ \ n \geq 2.$$

Corollary 2 If $f \in T_{\mathscr{J}}(\xi,\ \rho,\ A,\ B)$ and:

$$a_n = \frac{\rho(A-B)(1-\xi)(n-1)!}{(1-B\rho)\sigma(n)}\prod_{j=1}^{q}\frac{(\alpha_j)_{k_j}(\beta_j)_{\alpha_{jn}}}{(\beta_j)_{\alpha_j}(\theta_j)_{k_{jn}}}z^n,$$

then:

$$a_n > \frac{\rho(A-B)(1-\xi)(n-1)!\ \log\log n}{(1-B\rho)(e^{\gamma}(\log\log n)^2+0.6483)}.\prod_{j=1}^{q}\frac{(\alpha_j)_{k_j}(\beta_j)_{\alpha_{jn}}}{(\beta_j)_{\alpha_j}(\theta_j)_{k_{jn}}},\ \ n \geq 3.$$

Proof. This follows from Corollary 1 and inequality (4).

Corollary 3 Assuming the Riemann hypothesis holds true, if $f \in \mathscr{T}_{\mathscr{J}}(\xi,\ \rho,\ A,\ B)$ and:

$$a_n = \frac{\rho(A-B)(1-\xi)(n-1)!}{(1-B\rho)\sigma(n)} \prod_{j=1}^{q} \frac{(\alpha_j)_{k_j}(\beta_j)_{\alpha_j n}}{(\beta_j)_{\alpha_j}(\theta_j)_{k_j n}} z^n,$$

then:

$$a_n > \frac{\rho(A-B)(1-\xi)(n-1)!}{(1-B\rho)e^{\gamma}\log\log n} . \prod_{j=1}^{q} \frac{(\alpha_j)_{k_j}(\beta_j)_{\alpha_j n}}{(\beta_j)_{\alpha_j}(\theta_j)_{k_j n}}, \quad n > 5,040.$$

Proof. Derived from Corollary 1 and inequality (5).

4. Distortion theorem

In this section, we derive the upper and lower bounds of $|f(z)|$ provided that $f \in \mathscr{T}_{\mathscr{J}}(\xi,\ \rho,\ A,\ B)$.
Theorem 2 For a function $f \in \mathscr{T}_{\mathscr{J}}(\xi,\ \rho,\ A,\ B)$, the following inequalities hold:

$$|f(z)| \geq |z| - \frac{\rho(A-B)(1-\xi)}{3(1-B\rho)} \prod_{j=1}^{q} \frac{(\alpha_j)_{k_j}(\beta_j)_{2\alpha_j}}{(\beta_j)_{\alpha_j}(\theta_j)_{2k_j}} |z|^2, \tag{12}$$

$$|f(z)| \leq |z| + \frac{\rho(A-B)(1-\xi)}{3(1-B\rho)} \prod_{j=1}^{q} \frac{(\alpha_j)_{k_j}(\beta_j)_{2\alpha_j}}{(\beta_j)_{\alpha_j}(\theta_j)_{2k_j}} |z|^2. \tag{13}$$

Proof. For $f(z) \in \mathscr{T}_{\mathscr{J}}(\xi,\ \rho,\ A,\ B)$, by Theorem 1:

$$\prod_{j=1}^{q} \frac{(\beta_j)_{\alpha_j}}{(\alpha_j)_{k_j}} \frac{(\theta_j)_{2k_j}}{(\beta_j)_{2\alpha_j}} . \frac{\sigma(2)}{(2-1)!} (1-B\rho) \sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} \prod_{j=1}^{q} \frac{(\beta_j)_{\alpha_j}}{(\alpha_j)_{k_j}} \frac{(\theta_j)_{k_j n}}{(\beta_j)_{\alpha_j n}} . \frac{\sigma(n)}{(n-1)!} (1-B\rho)a_n \leq \rho(A-B)(1-\xi),$$

this yields:

$$\sum_{n=2}^{\infty} a_n \leq \frac{\rho(A-B)(1-\xi)}{\prod_{j=1}^{q} \frac{(\beta_j)_{\alpha_j}}{(\alpha_j)_{k_j}} \frac{(\theta_j)_{2k_j}}{(\beta_j)_{2\alpha_j}} . \frac{\sigma(2)}{(2-1)!} (1-B\rho)} = \frac{\rho(A-B)(1-\xi)}{3(1-B\rho)} \prod_{j=1}^{q} \frac{(\alpha_j)_{k_j}(\beta_j)_{2\alpha_j}}{(\beta_j)_{\alpha_j}(\theta_j)_{2k_j}}.$$

Therefore:

$$|f(z)| \geq |z| - |z|^2 \sum_{n=2}^{\infty} a_n \geq |z| - \frac{\rho(A-B)(1-\xi)}{3(1-B\rho)} \prod_{j=1}^{q} \frac{(\alpha_j)_{k_j}(\beta_j)_{2\alpha_j}}{(\beta_j)_{\alpha_j}(\theta_j)_{2k_j}} |z|^2,$$

and:

$$|f(z)| \leq |z| + |z|^2 \sum_{n=2}^{\infty} a_n \leq |z| + \frac{\rho(A-B)(1-\xi)}{3(1-B\rho)} \prod_{j=1}^{q} \frac{(\alpha_j)_{k_j} (\beta_j)_{2\alpha_j}}{(\beta_j)_{\alpha_j} (\theta_j)_{2k_j}} |z|^2.$$

Corollary 4 Under Theorem 2 conditions, $f(z)$ exists within a disk centred at the origin with radius:

$$r = 1 + \frac{\rho(A-B)(1-\xi)}{3(1-B\rho)} \prod_{j=1}^{q} \frac{(\alpha_j)_{k_j} (\beta_j)_{2\alpha_j}}{(\beta_j)_{\alpha_j} (\theta_j)_{2k_j}}.$$

Using Robin's inequalities (4) and (5), we derive additional constraints for assertions in Theorem 2:

Corollary 5 If $f \in \mathscr{T}_{\mathscr{J}}(\xi,\ \rho,\ A,\ B)$ and:

$$(1-B\rho)\left[3.\prod_{j=1}^{q} \frac{(\beta_j)_{\alpha_j}}{(\alpha_j)_{k_j}} \frac{(\theta_j)_{2k_j}}{(\beta_j)_{2\alpha_j}} + \sum_{n=3}^{\infty} \prod_{j=1}^{q} \frac{(\beta_j)_{\alpha_j}}{(\alpha_j)_{k_j}} \frac{(\theta_j)_{k_j n}}{(\beta_j)_{\alpha_j n}} . \frac{n}{(n-1)!} . \frac{\left[e^{\gamma} (\log\log n)^2 + 0.6483\right]}{\log\log n} a_n\right]$$

$$\leq \rho(A-B)(1-\xi),$$

Then Theorem 2's assertions remain valid.

Corollary 6 Assuming Riemann Hypothesis, if $f \in \mathscr{T}_{\mathscr{J}}(\xi,\ \rho,\ A,\ B)$ and:

$$\sum_{n=2}^{5,040} \prod_{j=1}^{q} \frac{(\beta_j)_{\alpha_j}}{(\alpha_j)_{k_j}} \frac{(\theta_j)_{k_j n}}{(\beta_j)_{\alpha_j n}} . \frac{\sigma(n)}{(n-1)!} (1-B\rho)a_n + \sum_{n=5,041}^{\infty} \prod_{j=1}^{q} \frac{(\beta_j)_{\alpha_j}}{(\alpha_j)_{k_j}} \frac{(\theta_j)_{k_j n}}{(\beta_j)_{\alpha_j n}} . \frac{e^{\gamma} n\ log\log n}{(n-1)!} (1-B\rho)a_n$$

$$\leq \rho(A-B)(1-\xi),$$

Then Theorem 2's assertions remain valid.

5. Conclusion

This study introduced the mathematical subclass $\mathscr{T}_{\mathscr{J}}(\xi,\ \rho,\ A,\ B)$ through a generalized Mittag-Leffler function combined with the Lambert series. We examined its characteristic properties and established the distortion theorem. In addition, we obtained the coefficients bounds and the extreme points of functions in the introduced subclass. We applied the two Robin's inequalities to provide lower bounds where applicable. Certainly, one can extend the study to various subclasses of analytic functions and evoke the lower bounds problems by involving the Robin's inequalities and assuming the Riemann Hypothesis.

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Conflict of interest

The author declares no conflict of interest related to this publication.

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