Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

UNIVERSAL WISER PUBLISHER

Research Article

The Univalency of Analytic Functions Involving a Linear Operator Defined by the Mittag-Leffler Function and the Lambert Series

Jamal Salah

College of Applied and Health Sciences, A'Sharqiyah University, Ibra, 400, Sultanate of Oma
E-mail: damous73@yahoo.com

Received: 31 March 2025; **Revised:** 27 May 2025; **Accepted:** 23 June 2025

Abstract: This study introduces a novel linear operator incorporating both the Mittag-Leffler function and Lambert series to establish a new subclass of analytic functions denoted as $\mathscr{A}_{\mathscr{J}}(\xi, \rho, \psi)$. We first obtain sufficient conditions for functions to be members of $\mathscr{A}_{\mathscr{J}}(\xi, \rho, \psi)$. Consequently, we examine key properties including extreme points, coefficient constraints, and the radii of Univalency and starlikeness for functions in this subclass. In addition, by the means of Robin's inequalities, we provide some coefficients' lower bounds when applicable.

***Keywords*:** Univalent functions, starlike functions, convolution product, Lambert series, Mittag-Leffler function

MSC: 30C45, 30C50

1. Background and context

The single-parameter Mittag-Leffler function, denoted as $E_\alpha(z)$ for $\alpha \in \mathbb{C}$, with $\Re(\alpha) > 0$ (see [1] and [2]), is expressed by the power series:

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad z \in \mathbb{C}. \tag{1}$$

Wiman [3] expanded this concept to include a two-parameter version. For $\alpha, \beta \in \mathbb{C}$, with $\Re(\alpha), \Re(\beta) > 0$, the two-parameter function $E_{\alpha, \beta}(z)$ takes the form:

$$E_{\alpha, \beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad z \in \mathbb{C}. \tag{2}$$

Various mathematicians have explored different generalizations of this function (see [4]). Our focus centers on the generalization proposed by Salah and Darus [5]:

DOI: https://doi.org/10.37256/cm.6520256891

$$
{}_qF^{\theta,\,k}_{\alpha,\,\beta} = \sum_{n=0}^{\infty}\prod_{j=1}^{q}\frac{(\theta_j)_{k_j n}}{(\beta_j)_{\alpha_j n}}\cdot\frac{z^n}{n!}. \tag{3}
$$

In this formation, $(\theta)_v$ represents the Pochhammer symbol, defined as:

$$
(\theta)_v := \frac{\Gamma(\theta+v)}{\Gamma(\theta)} = \begin{cases} 1, & \text{if } v = 0,\ \theta\in\mathbb{C}\setminus\{0\},\\ \theta(\theta+1)\dots(\theta+n-1), & \text{if } v = n\in\mathscr{N},\ \theta\in\mathbb{C}, \end{cases} \tag{4}
$$

$$
(1)_n = n!, \quad n\in\mathscr{N}_0, \quad \mathscr{N}_0 = \mathscr{N}\cup\{0\}, \quad \mathscr{N} = \{1,\,2,\,3,\,\dots\}, \tag{5}
$$

and

$$
(q\in\mathscr{N},\ \ j=1,\,2,\,3,\,\dots q;\ \ \Re\{\theta_j,\,\beta_j\}>0,\ \text{ and } \Re(\alpha_j)>\max\{0,\,\Re(k_j)-1;\ \Re(k_j)\};\ \Re(k_j)>0). \tag{6}
$$

In number theory, the Lambert series proves valuable for certain problems due to its connection with significant arithmetic functions (see [6–9]), such as:

$$
\sum_{n=1}^{\infty}\sigma_0(n)x^n = \sum_{n=1}^{\infty}\frac{x^n}{1-x^n}, \tag{7}
$$

where $\sigma_0(n) = d(n)$ is the number of positive divisors of n.

$$
l(z) = \sum_{n=1}^{\infty}\sigma_\alpha(n)x^n = \sum_{n=1}^{\infty}\frac{n^\alpha x^n}{1-x^n}, \tag{8}
$$

where $\sigma_\alpha(n)$ is the higher-order sum of divisors function of n.

Our examination specifically focuses on the series expressed in the second equation. When $\alpha = 1$, we denote $\sigma_1(n) = \sigma(n)$, where $\sigma(n)$ represents the sum of divisors function that appears in elementary statements equivalent to the renowned Riemann Hypothesis.

We distinguish between the Lambert series discussed here and the Lambert W function, which emerges naturally in various scientific and engineering problem solutions [10].

In 1984, Robin [11] demonstrated that

$$
\sigma(n) < e^\gamma n\log\log n + \frac{0.6483n}{\log\log n}, \quad n\geq 3. \tag{9}
$$

Furthermore, he established that the Riemann hypothesis is equivalent to

$$\sigma(n) < e^{\gamma} n \log\log n, \quad n > 5040, \tag{10}$$

where $\gamma = 0.57721\cdots$, represents the Euler-Mascheroni constant.

This paper does not attempt to prove or disprove Robin's inequality or the Riemann hypothesis. For deeper exploration of these topics, readers are directed to references [12–17].

2. Fundamental concepts

Let $\mathscr{A}$ denote the class of analytic functions expressed as:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}, \tag{11}$$

and $\mathscr{S}$ denote the subclass of $\mathscr{A}$ comprising univalent (or one-to-one) functions on $\mathbb{D}$.

The significance of the coefficients in the power series formulation emerged during the early development of univalent function theory.

This research aims to establish a linear operator that defines a new subclass of analytic functions.

We begin by recalling the definition of the Hadamard product (convolution). For functions $f \in \mathscr{A}$ of the form given above and $g \in \mathscr{A}$ expressed as:

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad z \in \mathbb{D}, \tag{12}$$

the convolution $(*)$ of these functions is computed as:

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{D}. \tag{13}$$

We employ the Lambert series $\mathscr{L}(z)$, with coefficients given by the sum of divisors function $\sigma(n)$:

$$\mathscr{L}(z) = \sum_{n=1}^{\infty} \frac{n z^n}{1 - z^n} = \sum_{n=1}^{\infty} \sigma(n) z^n = z + \sum_{n=2}^{\infty} \sigma(n) z^n, \quad z \in \mathbb{D}. \tag{14}$$

Since ${}_qF_{\alpha,\beta}^{\theta,k}$ is not a member of class $\mathscr{A}$, we normalize it by introducing:

$${}_q\mathbb{F}_{\alpha,\beta}^{\theta,k} = \prod_{j=1}^{q} \frac{(\beta_j)_{\alpha_j}}{(\alpha_j)_{k_j}} \left({}_qF_{\alpha,\beta}^{\theta,k} - 1\right) = z + \sum_{n=2}^{\infty} \prod_{j=1}^{q} \frac{(\beta_j)_{\alpha_j}}{(\alpha_j)_{k_j}} \cdot \frac{(\theta_j)_{k_j n}}{(\beta_j)_{\alpha_j n}} \cdot \frac{z^n}{n!}. \tag{15}$$

We reference a linear operator recently explored by Salah [18]. For a function $f \in \mathscr{A}$ of the previously defined form, we define the linear operator $\mathscr{J}(\mathscr{L}, \mathbb{F})(z) : \mathscr{A} \longrightarrow \mathscr{A}$ as:

$$\mathscr{J}(\mathscr{L}, \mathbb{F})(z) := ({}_q\mathbb{F}^{\theta, k}_{\alpha, \beta} * \mathscr{L})(z) = z + \sum_{n=2}^{\infty} \prod_{j=1}^{q} \frac{(\beta_j)_{\alpha_j}}{(\alpha_j)_{k_j}} \cdot \frac{(\theta_j)_{k_j n}}{(\beta_j)_{\alpha_j n}} \cdot \frac{\sigma(n)}{n!} a_n z^n, \quad z \in \mathbb{D}. \tag{16}$$

Based on this operator, we propose the following definition:

A function $f \in \mathscr{A}$ of the form (11) is considered a member of class $\mathscr{A}_{\mathscr{J}}(\xi, \rho, \psi)$ if it satisfies:

$$\operatorname{Re}\left\{\xi \frac{\mathscr{J}(\mathscr{L}, \mathbb{F})(z)}{z} + \rho(\mathscr{J}(\mathscr{L}, \mathbb{F})(z))'\right\} > \psi, \quad z \in \mathbb{D},\ \xi, \rho > 0, \text{ and } 0 \le \psi \le \xi + \rho \le 1. \tag{17}$$

In this study, we apply the well-known approach in the theory of geometric functions, that is involving special functions to introduce new subclasses of univalent functions and hence evoking various properties such as duality, coefficients bounds, extreme values and subordination, see for example [19–24].

3. Charactering the extreme points of $\mathscr{A}_{\mathscr{J}}(\xi, \rho, \psi)$

Theorem 1 A function $f \in \mathscr{A}$ of the form (11) belongs to class $\mathscr{A}_{\mathscr{J}}(\xi, \rho, \psi)$ if and only if $f(z)$ can be expressed as:

$$f(z) = z + \int_{|x|=1} \left(\sum_{n=2}^{\infty} \prod_{j=1}^{q} \frac{(\alpha_j)_{k_j}(\beta_j)_{\alpha_j n}}{(\beta_j)_{\alpha_j}(\theta_j)_{k_j n}} \cdot \frac{n! x^{n-1}}{(\xi + \rho n)\sigma(n)} z^n\right) d\mu(x), \tag{18}$$

where $\mu(x)$ is the probability measure on the set $X = \{x : |x| = 1\}$. For fixed parameters ξ, ρ and ψ, the class $\mathscr{A}_{\mathscr{J}}(\xi, \rho, \psi)$ and the probability measures $\{\mu\}$ defined on X establish a one-to-one correspondence through this expression.

Proof. From condition (17), we know that $f(z) \in \mathscr{A}_{\mathscr{J}}(\xi, \rho, \psi)$ if and only if:

$$\frac{\xi\left(\frac{\mathscr{J}(\mathscr{L}, \mathbb{F})(z)}{z}\right) + \rho(\mathscr{J}(\mathscr{L}, \mathbb{F})(z))' - \psi}{\xi + \rho - \psi} \in \mathscr{P}, \tag{19}$$

where $\mathscr{P}$ represents the normalized class of analytic functions which have positive real part.

Utilizing Herglotz expressions of functions in $\mathscr{P}$:

$$\frac{\xi\left(\frac{\mathscr{J}(\mathscr{L}, \mathbb{F})(z)}{z}\right) + \rho(\mathscr{J}(\mathscr{L}, \mathbb{F})(z))' - \psi}{\xi + \rho - \psi} = \int_{|x|=1} \frac{1 + xz}{1 - xz} d\mu(x), \tag{20}$$

or equivalently:

$$\frac{\xi}{\rho}\left(\frac{\mathscr{J}(\mathscr{L}, \mathbb{F})(z)}{z}\right) + (\mathscr{J}(\mathscr{L}, \mathbb{F})(z))' = \frac{1}{\rho}\int_{|x|=1} \frac{\xi + \rho + (\xi + \rho - 2\psi)xz}{1 - xz} d\mu(x). \tag{21}$$

Therefore:

$$z^{-\frac{\xi}{\rho}} \int_0^z \left[\frac{\xi}{\rho}\left(\frac{\mathscr{J}(\mathscr{L}, \mathbb{F})(t)}{t}\right) + (\mathscr{J}(\mathscr{L}, \mathbb{F})(t))'\right] t^{\frac{\xi}{\rho}} dt = \frac{1}{\rho} \int_{|x|}$$

$$=1 \left[z^{-\frac{\xi}{\rho}} \int_0^z \frac{\xi+\rho+(\xi+\rho-2\psi)xt}{1-xt} t^{-\frac{\xi}{\rho}} dt\right] d\mu(x). \tag{22}$$

This expression equates to:

$$\mathscr{J}(\mathscr{L}, \mathbb{F})(z) = \frac{1}{\xi+\rho} \int_{|x|=1} \left[(2\psi-\xi-\rho)z + 2(\xi+\rho-\psi) \sum_{n=0}^{\infty} \frac{(\xi+\rho)x^n z^{n+1}}{(n+1)\rho+\xi}\right] d\mu(x), \tag{23}$$

that is:

$$\mathscr{J}(\mathscr{L}, \mathbb{F})(z) = z + \int_{|x|=1} \left(\sum_{n=2}^{\infty} \frac{2(\xi+\rho-\psi)}{\xi+n\rho} x^{n-1} z^n\right) d\mu(x). \tag{24}$$

This formulation is equivalent to the expression in Theorem 1. The reasoning process is reversible, proving the first part of the theorem. Additionally, since both probability measures $\{\mu\}$ and class $\mathscr{P}$, as well as class $\mathscr{P}$ and class $\mathscr{A}_{\mathscr{J}}(\xi, \rho, \psi)$, establish one-to-one correspondences, the second part of the theorem is validated. □

Corollary 1 The extreme points of class $\mathscr{A}_{\mathscr{J}}(\xi, \rho, \psi)$ are functions defined as:

$$f_x(z) = z + \sum_{n=2}^{\infty} \prod_{j=1}^{q} \frac{(\alpha_j)_{k_j} (\beta_j)_{\alpha_j n}}{(\beta_j)_{\alpha_j} (\theta_j)_{k_j n}} \cdot \frac{n! x^{n-1}}{(\xi+\rho n)\sigma(n)} z^n, \quad |x| = 1. \tag{25}$$

Proof. Using notation $f_x(z)$, the expression in Theorem 1 can be rewritten as:

$$f_\mu(z) = \int_{|x|=1} f_x(z) d\mu(x). \tag{26}$$

By Theorem 1, the map $\mu \to f_\mu$ is one-to-one, confirming the assertion. □

Corollary 2 For a function f defined by (6) that belongs to class $\mathscr{A}_{\mathscr{J}}(\xi, \rho, \psi)$:

$$|a_n| \le \prod_{j=1}^{q} \frac{(\alpha_j)_{k_j} (\beta_j)_{\alpha_j n}}{(\beta_j)_{\alpha_j} (\theta_j)_{k_j n}} \cdot \frac{n!}{(\xi+\rho n)\sigma(n)}, \quad n \ge 2. \tag{27}$$

This result is sharp.

Proof. The coefficient bounds reach their maximum at extreme points, thus following from Corollary 1. □

The following result is directly derived from Corollary 2:

Corollary 3 For a function f defined by (11) that belongs to class $\mathscr{A}_{\mathscr{J}}(\xi, \rho, \psi)$, when $|z| = r < 1$:

$$|f(z)| \leq r + \sum_{n=2}^{\infty} \prod_{j=1}^{q} \frac{(\alpha_j)_{k_j} (\beta_j)_{\alpha_j n}}{(\beta_j)_{\alpha_j} (\theta_j)_{k_j n}} \cdot \frac{n!}{(\xi + \rho n)\sigma(n)} r^n. \tag{28}$$

We now establish lower bounds for coefficients a_n using Robin's inequalities from (9) and (10), with the latter referred to as the Riemann hypothesis.

Corollary 4 For a function f defined by (11) in class $\mathscr{A}_{\mathscr{J}}(\xi, \rho, \psi)$, if:

$$|a_n| = \prod_{j=1}^{q} \frac{(\alpha_j)_{k_j} (\beta_j)_{\alpha_j n}}{(\beta_j)_{\alpha_j} (\theta_j)_{k_j n}} \cdot \frac{n!}{(\xi + \rho n)\sigma(n)}, \tag{29}$$

then:

$$|a_n| > \frac{1}{(\xi + \rho n)} \cdot \frac{(n-1)! \log \log n}{e^{\gamma} (\log \log n)^2 + 0.6483} \cdot \prod_{j=1}^{q} \frac{(\alpha_j)_{k_j} (\beta_j)_{\alpha_j n}}{(\beta_j)_{\alpha_j} (\theta_j)_{k_j n}}, \quad n \geq 3. \tag{30}$$

Proof. This follows directly from Corollary 1 and inequality (9). □

Corollary 5 For a function f defined by (11) in class $\mathscr{A}_{\mathscr{J}}(\xi, \rho, \psi)$, assuming the Riemann hypothesis holds true, and:

$$|a_n| = \prod_{j=1}^{q} \frac{(\alpha_j)_{k_j} (\beta_j)_{\alpha_j n}}{(\beta_j)_{\alpha_j} (\theta_j)_{k_j n}} \cdot \frac{n!}{(\xi + \rho n)\sigma(n)}, \tag{31}$$

then:

$$|a_n| > \frac{1}{(\xi + \rho n)} \cdot \frac{(n-1)!}{e^{\gamma} \log \log n} \cdot \prod_{j=1}^{q} \frac{(\alpha_j)_{k_j} (\beta_j)_{\alpha_j n}}{(\beta_j)_{\alpha_j} (\theta_j)_{k_j n}}, \quad n > 5040. \tag{32}$$

Proof. This follows from Corollary 1 and inequality (10). □

4. Examining univalency and starlikeness radii

Theorem 2 For a function f defined by (6) in class $\mathscr{A}_{\mathscr{J}}(\xi, \rho, \psi)$, $f(z)$ is univalent within $|z| < R(\xi, \rho, \psi)$, where:

$$R(\xi, \rho, \psi) = \inf_{n} \left\{ \prod_{j=1}^{q} \frac{(\beta_j)_{\alpha_j} (\theta_j)_{k_j n}}{(\alpha_j)_{k_j} (\beta_j)_{\alpha_j n}} \cdot \frac{(\xi + \rho n)\sigma(n)}{n \cdot n!} \right\}^{\frac{1}{n-1}}. \tag{33}$$

This result is sharp.

Proof. It is sufficient to demonstrate that:

$$|f'(z)-1|<1. \tag{34}$$

For the left side of this inequality:

$$\left|\sum_{n=2}^{\infty} na_n z^{n-1}\right| \leq \sum_{n=2}^{\infty} n|a_n||z|^{n-1}, \tag{35}$$

this expression is less than 1 when:

$$|z|^{n-1} < \prod_{j=1}^{q} \frac{(\beta_j)_{\alpha_j}(\theta_j)_{k_j n}}{(\alpha_j)_{k_j}(\beta_j)_{\alpha_j n}} \cdot \frac{(\xi+\rho n)\sigma(n)}{n \cdot n!}. \tag{36}$$

To establish that $R(\xi,\,\rho,\,\psi)$ is optimal, consider $f(z) \in \mathscr{A}$ defined as:

$$f(z) = z - \prod_{j=1}^{q} \frac{(\alpha_j)_{k_j}(\beta_j)_{\alpha_j n}}{(\beta_j)_{\alpha_j}(\theta_j)_{k_j n}} \cdot \frac{n!}{(\xi+\rho n)\sigma(n)} z^n. \tag{37}$$

If $\tau > R(\xi,\,\rho,\,\psi)$, then there exists $n \geq 2$ such that:

$$\left\{\prod_{j=1}^{q} \frac{(\beta_j)_{\alpha_j}(\theta_j)_{k_j n}}{(\alpha_j)_{k_j}(\beta_j)_{\alpha_j n}} \cdot \frac{(\xi+\rho n)\sigma(n)}{n \cdot n!}\right\}^{\frac{1}{n-1}} < \tau. \tag{38}$$

Since $f'(0)=1$ and

$$f'(\tau) = 1 - \prod_{j=1}^{q} \frac{(\alpha_j)_{k_j}(\beta_j)_{\alpha_j n}}{(\beta_j)_{\alpha_j}(\theta_j)_{k_j n}} \cdot \frac{n \cdot n!}{(\xi+\rho n)\sigma(n)} \tau^{n-1} < 0, \tag{39}$$

there exists $\tau_0 \in (0,\,\tau)$ where $f'(\tau_0)=0$, indicating $f(z)$ is not univalent within $|z|<\tau$, which completes the proof. □

Theorem 3 For a function f defined by (11) in class $\mathscr{A}_{\mathscr{J}}(\xi,\,\rho,\,\psi)$, $f(z)$ is starlike of order δ $(0 \leq \delta < 1)$ within $|z| < R_0$, where:

$$R_0 = \inf_n \left\{\prod_{j=1}^{q} \frac{(\beta_j)_{\alpha_j}(\theta_j)_{k_j n}}{(\alpha_j)_{k_j}(\beta_j)_{\alpha_j n}} \cdot \frac{(\xi+\rho n)(1-\delta)\sigma(n)}{2(n-\delta)n!}\right\}^{\frac{1}{n-1}}. \tag{40}$$

Proof. Using techniques similar to Theorem 2, this result is established by showing:

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \delta. \tag{41}$$

□

5. Conclusion

This research has established a new subclass $\mathscr{A}_{\mathscr{J}}(\xi, \rho, \psi)$ by integrating a generalized form of the Mittag-Leffler function with the Lambert series. We have characterized this subclass by determining its extreme points, deriving coefficient boundaries, and calculating the radii of Univalency and starlikeness. Where applicable, we have incorporated Robin's inequalities to derive additional results. This study can be extended to other subclasses of analytic functions such as bi-valent functions and p-valent functions. By the means of Robin's inequalities, coefficients' lower bound can also be provided.

Acknowledgement

The author expresses gratitude to those who assisted with this work and acknowledges the valuable input from peer reviewers. No funding was received for this research.

Conflict of interest

The author declares no competing financial interest.

References

[1] Rehman H, Darus M, Salah J. Coefficient properties involving the generalized k-Mittag-Leffler functions. *Transylvanian Journal of Mathematics and Mechanics*. 2017; 9: 155-164.

[2] Nisar K, Purohit S, Abouzaid M, Qurashi M. Generalized k-Mittag-Leffler function and its composition with pathway integral operators. *Journal of Nonlinear Science and Applications*. 2016; 9: 3519-3526.

[3] Wiman A. Über den fundamentalsatz in der teorie der funktionen $E_\alpha(x)$ [On the fundamental theorem in the theory of functions $E_\alpha(x)$]. *Acta Mathematica*. 1905; 29: 191-201.

[4] Shukla A, Prajapti J. On a generalization of Mittag-Leffler function and its properties. *Journal of Mathematical Analysis and Applications*. 2007; 336: 797-811.

[5] Salah J, Darus M. A note on generalized Mittag-Leffler function and application. *Far East Journal of Mathematical Sciences*. 2011; 48: 33-46.

[6] Lóczi L. Guaranteed- and high-precision evaluation of the Lambert W function. *Applied Mathematics and Computation*. 2022; 433: 1-22.

[7] Schmidt M. A catalog of interesting and useful Lambert series identities. *arXiv:200402976*. 2020. Available from: https://arxiv.org/abs/2004.02976.

[8] Postnikov A. *Introduction to Analytical Number Theory*. Moscow; 1971.

[9] Apostol T. *Introduction to Analytic Number Theory*. Springer-Verlag; 1976.

[10] Kesisoglou I, Singh G, Nikolaou M. The Lambert function should be in the engineering mathematical toolbox. *Computers & Chemical Engineering*. 2021; 148: 107259.

[11] Robin G. Grande valeurs de la function somme des diviseurs et hypothese de Riemann [Large values of the divisor sum function and the Riemann hypothesis]. *Journal of Pure and Applied Mathematic*. 1984; 63: 187-213.

[12] Axler C. On Robin's inequality. *The Ramanujan Journal*. 2023; 61(3): 909-919. Available from: https://doi.org/10.1007/s11139-022-00683-0.

[13] Choie Y, Lichiardopol N, Moree P, Solé P. On Robin's criterion for the Riemann hypothesis. *Journal of Number Theory of Bordeaux*. 2007; 19: 357-372.

[14] Salah J, Rehman H, Al-Buwaiqi I. The non-trivial zeros of the Riemann zeta function through Taylor series expansion and incomplete gamma function. *Mathematical Statistics*. 2022; 10: 410-418.

[15] Salah J. Some remarks and propositions on Riemann hypothesis. *Mathematical Statistics*. 2021; 9: 159-165.

[16] Eswaran K. The pathway to the Riemann hypothesis. In: Reddy PN. (ed.) *Open Reviews of the Proof of The Riemann Hypothesis*. Sreenidhi Institute of Science and Technology; 2021. p.189.

[17] Bombieri E. *The Riemann Hypothesis-Official Problem Description*. Clay Mathematics Institute; 2000.

[18] Salah J. On uniformly starlike functions with respect to symmetrical points involving the Mittag-Leffler function and the Lambert series. *Symmetry*. 2024; 16(5): 580.

[19] Salah J. Properties of a linear operator involving Lambert series and Rabotnov function. *International Journal of Mathematics and Mathematical Sciences*. 2024; 2024(1): 3657721.

[20] Oluwayemi M, Fadipe-Joseph O. Results on a certain class of analytic functions associated with Chebyshev polynomial. *Scientific African*. 2023; 19: e01482.

[21] Oluwayemi MO, Olatunji SO, Ogunlade TO. On certain properties of a univalent function associated with beta function. In: *Abstract and Applied Analysis*. Wiley Online Library; 2022. p.8150057. Available from: https://doi.org/10.1155/2022/8150057.

[22] Lecko A, Madhumitha S, Ravichandran V. Ma-Minda starlikeness of univalent functions. *Results in Mathematics*. 2025; 80: 44.

[23] Giri M, Raghavendar K. Inclusion results on hypergeometric functions in a class of analytic functions associated with linear operators. *Contemporary Mathematics*. 2024; 5(2): 2315-2334.

[24] Raghavendar K, Swaminathan A. Integral transforms of functions to be in certain class defined by the combination of starlike and convex functions. *Computers & Mathematics with Applications*. 2012; 63(8): 1296-1304.