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Research Article

Dark and Singular Optical Solitons for Kundu-Eckhaus Equation with Differential Group Delay by the Generalized ϕ^6-Model Expansion

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Abstract: This paper recovers optical solitons solutions with polarization-mode dispersion in optical fibers. The governing model is the Kundu-Eckhaus equation. The adopted integration scheme is the generalized ϕ^6-model expansion. The intermediary functions are the Jacobi's elliptic functions and the Weierstrass' elliptic functions. The parameter constraints for the existence of such solitons are also presented.

Keywords: solitons, integrability

MSC: 78A60, 81V80

1. Introduction

Pulse splitting is a standard detrimental effect that takes place in the propagation of solitons across intercontinental distances. This effect stems from natural causes such as random variation of fiber diameter, bending and twisting of fibers as well as other forms of rough handling of optical fibers. Therefore, pulse splitting is something that we must live with. This paper addresses this effect of differential group delay with Kundu-Eckhaus equation. Although this model was first established in Fluid Dynamics, it is commonly used to study pulse propagation dynamics through optical fibers for transcontinental and transoceanic distances because this model is eerily similar to the standard nonlinear Schrödinger's equation which is the standard model to address pulse propagation dynamics [1–10].

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The study in this paper focuses on the retrieval of solitons in birefringent fibers for the model by the aid of the generalized ϕ^6-model expansion approach. The soliton solutions are recovered through the intermediate Jacobi's elliptic function which approaches soliton solutions when the modulus of ellipticity approaches unity. The other form of intermediary functions that naturally emerged from the adopted integration scheme is the Weierstrass' elliptic functions which yield bright solitons under special parameter choice. The details are enumerated and exhibited in the rest of the paper.

2. Governing model

It's well known that Kundu-Eckhaus equation in polarization preserving fibers is written as:

$$iu_t + au_{xx} + b|u|^4 u + i\{\rho |u|^2 u_x + \mu u^2 u_x^*\} = 0, \tag{1}$$

where $u = u(x, t)$ is a complex-valued function that represents the wave profile, and $u^* = u^*(x, t)$ is a complex conjugate, while a, b, c, β are real-valued constants, $i = \sqrt{-1}$. The first term is the linear temporal evolution, while the second term represents the Chromatic Dispersion (CD). The physical significance of Eq. (1) is that it could describe the propagation of ultrashort femto-second pulses in an optical fiber. If $b = 0$ Eq. (1) reduces to the Kaup-Newell equation. If $\mu = 0$ and $b = 0$, Eq. (1) reduces to the Chen-Lee-Liu equation. If $\rho = 0$, Eq. (1) reduces to the Gerjikov-Ivanov equation. Eq. (1) is different from the Kundu-Eckhaus equation.

In birefringent fibers, Eq. (1) splits into two components, for the first time, as:

$$iq_t + a_1 q_{xx} + \left(f_1 |q|^4 + g_1 |q|^2 |r|^2 + h_1 |r|^4\right) q + i\left\{\left(b_1 |q|^2 + c_1 |r|^2\right) q_x + \left(d_1 q^2 + e_1 r^2\right) q_x^*\right\} = 0, \tag{2}$$

$$ir_t + a_2 r_{xx} + \left(f_2 |r|^4 + g_2 |r|^2 |q|^2 + h_2 |q|^4\right) r + i\left\{\left(b_2 |r|^2 + c_2 |q|^2\right) r_x + \left(d_2 r^2 + e_2 q^2\right) r_x^*\right\} = 0, \tag{3}$$

where $q(x, t)$ and $r(x, t)$ are complex-valued functions represent the wave profiles, while a_j, b_j, c_j, d_j, e_j, f_j, g_j, h_j and β, $(j = 1, 2)$ are real constants.

The objective of this paper is to solve Eqs. (2) and (3) using generalized ϕ^6-model expansion method.

3. Mathematical analysis

In order to solve the coupled system (2) and (3), we make the transformations:

$$q(x, t) = P_1(\eta)\exp[i\varphi_1(x, t], \tag{4}$$

$$r(x, t) = P_2(\eta)\exp[i\varphi_2(x, t], \quad i = \sqrt{-1}, \tag{5}$$

In Eqs. (4) and (5), the amplitude component of the solitons are shown as $P_\ell(\eta)$, $(\ell = 1, 2)$ and $\eta = x + \alpha t$. Here the phase components of the solitons are $\varphi_\ell(x, t) = kx + \lambda t + \theta$, where the parameters α, k, λ and θ sequentially correspond to the velocity of the soliton, frequency of the soliton, wave number of the soliton and phase constant.

Substituting (4) and (5) into Eqs. (2) and (3) and separating the real and imaginary parts, we deduce that the real parts are

$$a_1P_1^{''}-\left(\lambda+a_1k^2\right)P_1+\left[f_1P_1^5+g_1P_1^3P_2^2+h_1P_1P_2^4\right]$$

$$-k\left(b_1P_1^3+c_1P_1P_2^2\right)+k\left(d_1P_1^3+e_1P_1P_2^2\right)=0, \tag{6}$$

and

$$a_2P_2^{''}-\left(\lambda+a_2k^2\right)P_2+\left[f_2P_2^5+g_2P_2^3P_1^2+h_2P_2P_1^4\right]$$

$$-k\left(b_2P_2^3+c_2P_2P_1^2\right)+k\left(d_2P_2^3+e_2P_2P_1^2\right)=0, \tag{7}$$

while the imaginary parts are

$$(\alpha+2a_1k)P_1'+[b_1+d_1]P_1'P_1^2+[c_1+e_1]P_1'P_2^2=0, \tag{8}$$

and

$$(\alpha+2a_2k)P_2'+[b_2+d_2]P_2'P_2^2+[c_2+e_2]P_2'P_1^2=0. \tag{9}$$

Using the transformation $P_2=AP_1$, into Eqs. (6)-(9), where A is a nonzero constant, such that $A\neq 1$. Then applying the linearly independent principle on Eqs. (8) and (9), we get the soliton frequency:

$$k=\frac{-\alpha}{2a_j}, \qquad (j=1,2) \tag{10}$$

which leads to the relation

$$a_1=a_2,$$

along with the constrain conditions:

$$[b_1+d_1]+A^2[c_1+e_1]=0,$$

$$[b_2+d_2]A^2+[c_2+e_2]=0. \tag{11}$$

The two Equations (6) and (7) are equivalent under the constraint conditions:

$$\frac{a_1}{a_2}=\frac{\lambda+a_1k^2}{\lambda+a_2k^2}=\frac{-\left(b_1+c_1A^2\right)+\left(d_1+e_1A^2\right)}{-\left(b_2A^2+c_2\right)+\left(d_2A^2+e_2\right)}=\frac{h_1A^4+f_1+g_1A^2}{h_2+f_2A^4+g_2A^2}.$$

When the condition $a_1=a_2$ is imposed, the expression simplifies to:

$$\Leftrightarrow \frac{-\left(b_1+c_1A^2\right)+\left(d_1+e_1A^2\right)}{-\left(b_2A^2+c_2\right)+\left(d_2A^2+e_2\right)}=\frac{h_1A^4+f_1+g_1A^2}{h_2+f_2A^4+g_2A^2}=1,$$

which lead to the relations

$$f_1=h_2+(f_2-h_1)A^4+(g_2-g_1)A^2,$$

$$b_1=c_2+(b_2-c_1)A^2-\left[(d_2-e_1)A^2+e_2-d_1\right]. \tag{12}$$

Now, Eq. (6) is rewritten as

$$P_1^{''}+\Delta_1P_1+\Delta_2P_1^3+\Delta_3P_1^5=0, \tag{13}$$

where

$$\Delta_1=\frac{-\left(\lambda+a_1k^2\right)}{a_1},$$

$$\Delta_2=\frac{k}{a_1}\left[-\left(b_1+c_1A^2\right)+\left(d_1+e_1A^2\right)\right],$$

$$\Delta_3=\frac{1}{a_1}\left[f_1+g_1A^2+h_1A^4\right], \tag{14}$$

provided that $a_1\neq 0$.

Balancing the terms P_1'' and P_1^5 in Eq. (13), we get the balance number $N=\frac{1}{2}$. To reduce Eq. (13) to the desired form, we take into consideration the transformation

$$P_1(\eta)=\psi^{\frac{1}{2}}(\eta), \tag{15}$$

where $\psi(\eta)$ is a new function of η such that $\psi(\eta)>0$, and then Eq. (13) changes to the new equation

$$2\psi''\psi-\left(\psi'\right)^2+4\Delta_1\psi^2+4\Delta_2\psi^3+4\Delta_3\psi^4=0. \tag{16}$$

Balancing the terms $\psi''\psi$ and ψ^4 in Eq. (16) yields $N = 1$. The task now is to solve Eq. (16) by implementing the algorithm as detailed in the subsequent section.

4. Generalized ϕ^6-model expansion

The generalized ϕ^6 model expansion method is used in theoretical physics to extend the capabilities of the simpler ϕ^4 theory, particularly in the study of phase transitions, nonlinear dynamics, and topological defects. While the ϕ^4 model typically describes second-order (continuous) phase transitions, the inclusion of a sixth-order term in the potential enables the ϕ^6 model to capture both first-order and second-order transitions, as well as tricritical points where these behaviors meet. This makes the ϕ^6 model especially valuable in systems exhibiting metastability, multistability, or more complex energy landscapes. It also supports more soliton and kink solutions, which are important in studying interfaces and defect structures in field theory and condensed matter systems. Moreover, it provides a useful framework for renormalization group analyses by accommodating more complex fixed points and critical phenomena. However, these advantages come with increased mathematical complexity, a higher number of parameters to constrain, and potential non-renormalizability in higher-dimensional field theories. As a result, while the ϕ^6 model offers greater descriptive power and flexibility, it also demands more sophisticated analytical and numerical tools and careful interpretation.

The generalized ϕ^6 model expansion method is applicable in a wide range of physical contexts, particularly where systems exhibit rich and complex phase behavior. It is commonly employed in condensed matter physics, quantum field theory, and cosmology to describe phenomena such as first-order and second-order phase transitions, tricritical points, and topological defects. Its ability to account for multiple minima in the potential makes it especially suitable for modeling systems with metastable or multistable states, including structural phase transitions, magnetic systems, and interfaces in nonlinear media. In quantum field theory, the ϕ^6 model is used to investigate spontaneous symmetry breaking, vacuum structure, and nonperturbative soliton solutions. Additionally, in cosmology, it helps model early-universe scenarios, such as inflationary potentials with nontrivial features.

Despite its broad applicability, the generalized ϕ^6 model also has notable limitations. One major constraint is its increasing mathematical complexity, which can make exact analytical solutions intractable and require heavy reliance on numerical methods. Furthermore, the model involves more parameters than lower-order expansions, necessitating careful tuning or empirical input to produce physically meaningful results. Another limitation is its potential non-renormalizability in higher dimensions (typically beyond 3 + 1 spacetime dimensions), which restricts its use in high-energy quantum field theories where renormalizability is essential. Moreover, the model's rich potential structure can complicate physical interpretations, especially in systems with closely spaced energy minima or where phase coexistence plays a critical role. In summary, while the ϕ^6 model provides a powerful and versatile framework for exploring complex physical behavior, its use must be balanced against practical considerations related to tractability, interpretability, and theoretical consistency.

Based on the generalized ϕ^6-model expansion method, Eq. (16) has the formal solution:

$$\psi(\eta) = \rho_0 + \rho_1\phi(\eta) + \rho_2\phi^2(\eta), \tag{17}$$

where ρ_0, ρ_1, ρ_2 are constants to be determined later, such that $\rho_2 \neq 0$, while $\phi(\eta)$ satisfies the following nonlinear ODEs:

$$\phi'^2(\eta) = h_{01} + h_{21}\phi^2(\eta) + h_{41}\phi^4(\eta) + h_{61}\phi^6(\eta),$$

$$\phi''(\eta) = h_{21}\phi(\eta) + 2h_{41}\phi^3(\eta) + 3h_{61}\phi^5(\eta), \tag{18}$$

where h_{j1} $(j = 0, 2, 4, 6)$ are real constants, such that $h_{61} \neq 0$.

It is well known that Eqs. (18) have the solution:

$$\phi(\eta) = \frac{U(\eta)}{\sqrt{fU^2(\eta)+g}}, \tag{19}$$

where $\left(fU^2(\eta)+g\right) > 0$ and $U(\eta)$ is the solution of the Jacobian elliptic equation:

$$U'^2 = l_0 + l_2U^2(\eta) + l_4U^4(\eta), \tag{20}$$

and l_j $(j = 0,\ 2,\ 4)$ are constants to be determined later, while f and g are given by the new forms

$$f = \frac{l_2 - h_{21}}{3h_{01}}, \qquad g = \frac{l_0}{h_{01}}, \tag{21}$$

under the constraint conditions:

$$27h_{61}h_{01}^2 + (l_2 - h_{21})\left[9l_0l_4 - (l_2 - h_{21})(2l_2 + h_{21})\right] = 0,$$

$$3h_{01}h_{41} - \left(3l_0l_4 - \left(l_2^2 - h_{21}^2\right)\right) = 0. \tag{22}$$

It is well-known that Eq. (20) has many generalized Jacobian and Weierstrass elliptic functions solutions, as well as hyperbolic functions solutions.

Substituting (17) along with (18) into Eq. (16), collecting the coefficients of each power of $\phi^i(\eta)$, $(i = 0, \ldots, 8)$ to zero, we get a set of algebraic equations. Solving these algebraic equations with the aid of Maple, we get the result:

Result:

$$\rho_0 = 0,\ \ \rho_1 = 0,\ \ \rho_2 = \rho_2,\ \ h_{01} = h_{01},\ \ \Delta_1 = -h_{21},\ \ \Delta_2 = \frac{-2h_{41}}{\rho_2},\ \ \Delta_3 = \frac{-3h_{61}}{\rho_2^2}. \tag{23}$$

From (17)-(20), Eq. (16) has the formal solutions:

$$\psi(\eta) = \rho_2\phi^2(\eta) = \frac{3\rho_2h_{01}U^2(\eta)}{(l_2 - h_{21})U^2(\eta) + 3l_0}, \tag{24}$$

where $U(\eta)$ is given before.

Equations (4), (5), (14) and (15), under the constraints (11) and (12), Eqs. (2) and (3) has many new generalized Jacobi elliptic functions and other solutions as follows:

(1) If $l_0 = \dfrac{w^4m^2(m^2-1)}{l_4}$, $w^2 = \dfrac{l_2}{2m^2-1}$, $l_4 < 0$, $0 < m < 1$, then $U(\eta) = \dfrac{wm}{\sqrt{-l_4}}\mathrm{cn}(w\eta)$, and we get

$$q(x,t)=\exp\left\{i\left[\frac{-\alpha}{2a_1}x+\lambda t+\theta\right]\right\}$$

$$\times\left[\frac{3\rho_2 h_{01}\,\mathrm{cn}^2\left(\sqrt{\frac{l_2}{2m^2-1}}\,[x+\alpha t]\,,\,m\right)}{(l_2-h_{21})\,\mathrm{cn}^2\left(\sqrt{\frac{l_2}{2m^2-1}}\,[x+\alpha t]\,,\,m\right)-\frac{3l_2\left(m^2-1\right)}{2m^2-1}}\right]^{\frac{1}{2}}, \tag{25}$$

and

$$r(x,t)=Aq(x,t), \tag{26}$$

under the constraint conditions:

$$(l_2-h_{21})\left[\frac{9l_2^2m^2(m^2-1)}{(2m^2-1)^2}-(l_2-h_{21})\,(2l_2+h_{21})\right]+27h_{61}h_{01}^2=0,$$

$$3h_{01}h_{41}-\frac{3l_2^2m^2(m^2-1)}{(2m^2-1)^2}+(l_2^2-h_{21}^2)=0. \tag{27}$$

(2) If $l_0=\frac{w^4m^2}{l_4}$, $w^2=-\frac{l_2}{1+m^2}$, $l_2<0$, $l_4>0$, $0<m<1$, then $U(\eta)=\frac{wm}{\sqrt{l_4}}\mathrm{sn}\,(w\eta)$ or $U(\eta)=\frac{wm}{\sqrt{l_4}}\mathrm{cd}\,(w\eta)$, and we get

$$q(x,t)=\exp\left\{i\left[\frac{-\alpha}{2a_1}x+\lambda t+\theta\right]\right\}$$

$$\times\left[\frac{3\rho_2 h_{01}\,\mathrm{sn}^2\left(\sqrt{\frac{-l_2}{1+m^2}}\,[x+\alpha t]\,,\,m\right)}{(l_2-h_{21})\,\mathrm{sn}^2\left(\sqrt{\frac{-l_2}{1+m^2}}\,[x+\alpha t]\,,\,m\right)-\frac{3l_2}{1+m^2}}\right]^{\frac{1}{2}}, \tag{28}$$

and

$$r(x,t)=Aq(x,t), \tag{29}$$

or

$$q(x,t)=\exp\left\{i\left[\frac{-\alpha}{2a_1}x+\lambda t+\theta\right]\right\}$$

$$\times\left[\frac{3\rho_2 h_{01}\,\mathrm{cd}^2\left(\sqrt{\frac{-l_2}{1+m^2}}\,[x+\alpha t]\,,\,m\right)}{(l_2-h_{21})\,\mathrm{cd}^2\left(\sqrt{\frac{-l_2}{1+m^2}}\,[x+\alpha t]\,,\,m\right)-\frac{3l_2}{1+m^2}}\right]^{\frac{1}{2}},\tag{30}$$

and

$$r(x,t)=Aq(x,t),\tag{31}$$

under the constraint conditions:

$$(l_2-h_{21})\left[\frac{9l_2^2m^2}{(1+m^2)^2}-(l_2-h_{21})(2l_2+h_{21})\right]+27h_{61}h_{01}^2=0,$$

$$3h_{01}h_{41}-\frac{3l_2^2m^2}{(1+m^2)^2}+\left(l_2^2-h_{21}^2\right)=0.\tag{32}$$

In particular, if $m\to 1$, then we have the dark soliton solutions of Eqs. (2) and (3) in the form:

$$q(x,t)=\exp\left\{i\left[\frac{-\alpha}{2a_1}x+\lambda t+\theta\right]\right\}$$

$$\times\left[\frac{3\rho_2 h_{01}\tanh^2\left(\sqrt{\frac{-l_2}{2}}\,[x+\alpha t]\right)}{(l_2-h_{21})\tanh^2\left(\sqrt{\frac{-l_2}{2}}\,[x+\alpha t]\right)-\frac{3l_2}{2}}\right]^{\frac{1}{2}},\tag{33}$$

and

$$r(x,t)=Aq(x,t),\tag{34}$$

under the constraint conditions:

$$(l_2-h_{21})\left(\frac{1}{2}l_2+h_{21}\right)^2+27h_{61}h_{01}^2=0,$$

$$3h_{01}h_{41}+\frac{1}{4}l_2^2-h_{21}^2=0. \tag{35}$$

(3) If $l_0=\dfrac{w^4(1-m^2)}{l_4}$, $w^2=\dfrac{l_2}{2-m^2}$, $l_4<0$, $0<m<1$, then $U(\eta)=\dfrac{w}{\sqrt{-l_4}}\mathrm{dn}(w\eta)$, and we have

$$q(x,\,t)=\exp\left\{i\left[\frac{-\alpha}{2a_1}x+\lambda t+\theta\right]\right\}$$

$$\times\left[\frac{3\rho_2h_{01}\mathrm{dn}^2\left(\sqrt{\dfrac{l_2}{2-m^2}}\,[x+\alpha t]\,,\,m\right)}{(l_2-h_{21})\,\mathrm{dn}^2\left(\sqrt{\dfrac{l_2}{2-m^2}}\,[x+\alpha t]\,,\,m\right)-\dfrac{3l_2\left(1-m^2\right)}{2-m^2}}\right]^{\frac{1}{2}}, \tag{36}$$

and

$$r(x,\,t)=Aq(x,\,t), \tag{37}$$

under the constraint conditions:

$$(l_2-h_{21})\left[\frac{9l_2^2(1-m^2)}{(2-m^2)^2}-(l_2-h_{21})\,(2l_2+h_{21})\right]+27h_{61}h_{01}^2=0,$$

$$3h_{01}h_{41}-\frac{3l_2^2(1-m^2)}{(2-m^2)^2}+\left(l_2^2-h_{21}^2\right)=0. \tag{38}$$

(4) If $l_0=\dfrac{w^4(1-m^2)}{l_4}$, $w^2=\dfrac{l_2}{2-m^2}$, $l_4>0$, $0<m<1$, then $U(\eta)=\dfrac{w}{\sqrt{l_4}}\mathrm{cs}(w\eta)$, and we have

$$q(x,\,t)=\exp\left\{i\left[\frac{-\alpha}{2a_1}x+\lambda t+\theta\right]\right\}$$

$$\times\left[\frac{3\rho_2h_{01}\mathrm{cs}^2\left(\sqrt{\dfrac{l_2}{2-m^2}}\,[x+\alpha t]\,,\,m\right)}{(l_2-h_{21})\,\mathrm{cs}^2\left(\sqrt{\dfrac{l_2}{2-m^2}}\,[x+\alpha t]\,,\,m\right)+\dfrac{3l_2\left(1-m^2\right)}{2-m^2}}\right]^{\frac{1}{2}}, \tag{39}$$

and

$$r(x,\, t) = Aq(x,\, t), \tag{40}$$

under the same constraint conditions (38).

(5) If $l_0 = \dfrac{w^4 m^2 (m^2-1)}{l_4}$, $w^2 = \dfrac{l_2}{2m^2-1}$, $l_4 > 0$, $0 < m < 1$, then $U(\eta) = \dfrac{w}{\sqrt{l_4}} \mathrm{ds}\,(w\eta)$, and we have

$$q(x,\, t) = \exp\left\{ i \left[\frac{-\alpha}{2a_1} x + \lambda t + \theta \right] \right\}$$

$$\times \left[\frac{3\rho_2 h_{01} \mathrm{ds}^2 \left(\sqrt{\dfrac{l_2}{2m^2-1}} \left[x + \alpha t \right],\, m \right)}{(l_2 - h_{21})\, \mathrm{ds}^2 \left(\sqrt{\dfrac{l_2}{2m^2-1}} \left[x + \alpha t \right],\, m \right) + \dfrac{3 l_2 m^2 \left(m^2 - 1 \right)}{(2m^2-1)}} \right]^{\frac{1}{2}}, \tag{41}$$

and

$$r(x,\, t) = Aq(x,\, t), \tag{42}$$

under the same constraint conditions (27).

(6) If $l_0 = \dfrac{w^4 m^2 (m^2-1)}{l_4}$, $w^2 = \dfrac{l_2}{2m^2-1}$, $l_4 > 0$, $0 < m < 1$, then $U(\eta) = \dfrac{w\sqrt{1-m^2}}{\sqrt{l_4}} \mathrm{nc}\,(w\eta)$, and we have

$$q(x,\, t) = \exp\left\{ i \left[\frac{-\alpha}{2a_1} x + \lambda t + \theta \right] \right\}$$

$$\times \left[\frac{3\rho_2 h_{01} \mathrm{nc}^2 \left(\sqrt{\dfrac{l_2}{2m^2-1}} \left[x + \alpha t \right],\, m \right)}{(l_2 - h_{21})\, \mathrm{nc}^2 \left(\sqrt{\dfrac{l_2}{2m^2-1}} \left[x + \alpha t \right],\, m \right) - \dfrac{3 l_2 m^2}{2m^2-1}} \right]^{\frac{1}{2}}, \tag{43}$$

and

$$r(x,\, t) = Aq(x,\, t), \tag{44}$$

under the same constraint conditions (27).

(7) If $l_0 = \dfrac{w^4 (1-m^2)}{l_4}$, $w^2 = \dfrac{l_2}{2-m^2}$, $l_4 < 0$, $0 < m < 1$, then $U(\eta) = \dfrac{w\sqrt{1-m^2}}{\sqrt{-l_4}} \mathrm{nd}\,(w\eta)$, and we have

$$q(x,t)=\exp\left\{i\left[\frac{-\alpha}{2a_1}x+\lambda t+\theta\right]\right\}$$

$$\times\left[\frac{3\rho_2 h_{01}\,\mathrm{nd}^2\left(\sqrt{\frac{l_2}{2-m^2}}\,[x+\alpha t],\,m\right)}{(l_2-h_{21})\,\mathrm{nd}^2\left(\sqrt{\frac{l_2}{2-m^2}}\,[x+\alpha t],\,m\right)-\frac{3l_2}{2-m^2}}\right]^{\frac{1}{2}}, \tag{45}$$

and

$$r(x,t)=Aq(x,t), \tag{46}$$

under the same constraint conditions (38).

(8) If $l_0=\frac{w^4(1-m^2)}{l_4}$, $w^2=\frac{l_2}{2-m^2}$, $l_4>0$, $0<m<1$, then $U(\eta)=\frac{w\sqrt{1-m^2}}{\sqrt{l_4}}\mathrm{sc}\,(w\eta)$, and we have

$$q(x,t)=\exp\left\{i\left[\frac{-\alpha}{2a_1}x+\lambda t+\theta\right]\right\}$$

$$\times\left[\frac{3\rho_2 h_{01}\,\mathrm{sc}^2\left(\sqrt{\frac{l_2}{2-m^2}}\,[x+\alpha t],\,m\right)}{(l_2-h_{21})\,\mathrm{sc}^2\left(\sqrt{\frac{l_2}{2-m^2}}\,[x+\alpha t],\,m\right)+\frac{3l_2}{2-m^2}}\right]^{\frac{1}{2}}, \tag{47}$$

and

$$r(x,t)=Aq(x,t), \tag{48}$$

under the same constraint conditions (38).

(9) If $l_0=\frac{w^4m^2}{l_4}$, $w^2=-\frac{l_2}{1+m^2}$, $l_2<0$, $l_4>0$, $0<m<1$, then $U(\eta)=\frac{w}{\sqrt{l_4}}\mathrm{dc}\,(w\eta)$ or $U(\eta)=\frac{w}{\sqrt{l_4}}\mathrm{ns}\,(w\eta)$, and we get

$$q(x,t)=\exp\left\{i\left[\frac{-\alpha}{2a_1}x+\lambda t+\theta\right]\right\}$$

$$\times\left[\frac{3\rho_2 h_{01}\,\mathrm{dc}^2\left(\sqrt{\frac{-l_2}{1+m^2}}\,[x+\alpha t],\,m\right)}{(l_2-h_{21})\,\mathrm{dc}^2\left(\sqrt{\frac{-l_2}{1+m^2}}\,[x+\alpha t],\,m\right)-\frac{3l_2m^2}{1+m^2}}\right]^{\frac{1}{2}}, \tag{49}$$

and

$$r(x,\, t) = Aq(x,\, t), \tag{50}$$

or

$$q(x,\, t) = \exp\left\{ i \left[\frac{-\alpha}{2a_1} x + \lambda t + \theta \right] \right\}$$

$$\times \left[\frac{3\rho_2 h_{01} \operatorname{ns}^2 \left(\sqrt{\frac{-l_2}{1+m^2}} [x + \alpha t],\, m \right)}{(l_2 - h_{21}) \operatorname{ns}^2 \left(\sqrt{\frac{-l_2}{1+m^2}} [x + \alpha t],\, m \right) - \frac{3 l_2 m^2}{1+m^2}} \right]^{\frac{1}{2}}, \tag{51}$$

and

$$r(x,\, t) = Aq(x,\, t), \tag{52}$$

under the same constraint conditions (32).

In particular if $m \to 1$, then we have the singular soliton solutions

$$q(x,\, t) = \exp\left\{ i \left[\frac{-\alpha}{2a_1} x + \lambda t + \theta \right] \right\}$$

$$\times \left[\frac{3\rho_2 h_{01} \coth^2 \left(\sqrt{\frac{-l_2}{2}} [x + \alpha t] \right)}{(l_2 - h_{21}) \coth^2 \left(\sqrt{\frac{-l_2}{2}} [x + \alpha t] \right) - \frac{3 l_2}{2}} \right]^{\frac{1}{2}}, \tag{53}$$

and

$$r(x,\, t) = Aq(x,\, t), \tag{54}$$

under the same constraint conditions (35).

(10) If $l_0 = \dfrac{w^4 m^2 (m^2 - 1)}{l_4}$, $w^2 = \dfrac{l_2}{2m^2 - 1}$, $l_4 < 0$, $0 < m < 1$, then $U(\eta) = \dfrac{wm\sqrt{1 - m^2}}{\sqrt{-l_4}} \operatorname{sd}(w\eta)$, and we have

$$q(x,t) = \exp\left\{ i\left[\frac{-\alpha}{2a_1}x + \lambda t + \theta\right]\right\}$$

$$\times\left[\frac{3\rho_2 h_{01}\,\mathrm{sd}^2\left(\sqrt{\frac{l_2}{2m^2-1}}\,[x+\alpha t],\, m\right)}{(l_2-h_{21})\,\mathrm{sd}^2\left(\sqrt{\frac{l_2}{2m^2-1}}\,[x+\alpha t],\, m\right)+\frac{3l_2}{2m^2-1}}\right]^{\frac{1}{2}}, \tag{55}$$

and

$$r(x,t) = Aq(x,t), \tag{56}$$

under the same constraint conditions (27).

(11) If $l_0 = \frac{w^4(1-m^2)^2}{16 l_4}$, $w^2 = \frac{2l_2}{1+m^2}$, $l_4 < 0$, $0 < m < 1$, then $U(\eta) = \frac{w}{2\sqrt{-l_4}}\left[m\mathrm{cn}(w\eta) \pm \mathrm{dn}(w\eta)\right]$, and we have

$$q(x,t) = \exp\left\{ i\left[\frac{-\alpha}{2a_1}x + \lambda t + \theta\right]\right\}$$

$$\times\left[\frac{3\rho_2 h_{01}\begin{bmatrix} m\mathrm{cn}\left(\sqrt{\frac{2l_2}{1+m^2}}\,[x+\alpha t],\, m\right) \\ \pm\mathrm{dn}\left(\sqrt{\frac{2l_2}{1+m^2}}\,[x+\alpha t],\, m\right)\end{bmatrix}^2}{(l_2-h_{21})\begin{bmatrix} m\mathrm{cn}\left(\sqrt{\frac{2l_2}{1+m^2}}\,[x+\alpha t],\, m\right) \\ \pm\mathrm{dn}\left(\sqrt{\frac{2l_2}{1+m^2}}\,[x+\alpha t],\, m\right)\end{bmatrix}^2 - \frac{3l_2\left(1-m^2\right)^2}{2\left(1+m^2\right)}}\right]^{\frac{1}{2}}, \tag{57}$$

and

$$r(x,t) = Aq(x,t), \tag{58}$$

under the constraint conditions:

$$(l_2-h_{21})\left[\frac{9l_2^2(1-m^2)^2}{4\left(1+m^2\right)^2} - (l_2-h_{21})(2l_2+h_{21})\right] + 27h_{61}h_{01}^2 = 0,$$

$$3h_{01}h_{41} - \frac{3l_2^2(1-m^2)^2}{4\left(1+m^2\right)^2} + \left(l_2^2 - h_{21}^2\right) = 0. \tag{59}$$

(12) If $l_0 = \dfrac{w^4(1-m^2)^2}{16l_4}$, $w^2 = \dfrac{2l_2}{1+m^2}$, $l_4 > 0$, $0 < m < 1$, then $U(\eta) = \dfrac{w\sqrt{1-m^2}}{2\sqrt{l_4}}\left[\mathrm{nc}(w\eta) \pm \mathrm{sc}(w\eta)\right]$ or $U(\eta) = \dfrac{w\sqrt{1-m^2}}{2\sqrt{l_4}}\left[\dfrac{\mathrm{cn}(w\eta)}{1 \pm \mathrm{sn}(w\eta)}\right]$, and we get

$$q(x,t) = \exp\left\{i\left[\frac{-\alpha}{2a_1}x + \lambda t + \theta\right]\right\}$$

$$\times\left[\frac{3\rho_2 h_{01}\begin{bmatrix}\mathrm{nc}\left(\sqrt{\dfrac{2l_2}{1+m^2}}[x+\alpha t],\, m\right)\\ \pm\mathrm{sc}\left(\sqrt{\dfrac{2l_2}{1+m^2}}[x+\alpha t],\, m\right)\end{bmatrix}^2}{(l_2 - h_{21})\begin{bmatrix}\mathrm{nc}\left(\sqrt{\dfrac{2l_2}{1+m^2}}[x+\alpha t],\, m\right)\\ \pm\mathrm{sc}\left(\sqrt{\dfrac{2l_2}{1+m^2}}[x+\alpha t],\, m\right)\end{bmatrix}^2 + \dfrac{3l_2(1-m^2)}{2(1+m^2)}}\right]^{\frac{1}{2}}, \tag{60}$$

and

$$r(x,t) = Aq(x,t), \tag{61}$$

or

$$q(x,t) = \exp\left\{i\left[\frac{-\alpha}{2a_1}x + \lambda t + \theta\right]\right\}$$

$$\times\left[\frac{3\rho_2 h_{01}\,\mathrm{cn}^2\left(\sqrt{\dfrac{2l_2}{1+m^2}}[x+\alpha t],\, m\right)}{\left(\begin{array}{l}(l_2 - h_{21})\,\mathrm{cn}^2\left(\sqrt{\dfrac{2l_2}{1+m^2}}[x+\alpha t],\, m\right)\\ +\dfrac{3l_2(1-m^2)}{2(1+m^2)}\left[1+\mathrm{sn}\left(\sqrt{\dfrac{2l_2}{1+m^2}}[x+\alpha t],\, m\right)\right]^2\end{array}\right)}\right]^{\frac{1}{2}}, \tag{62}$$

and

$$r(x,t) = Aq(x,t), \tag{63}$$

under the same constraint conditions (59).

(13) If $l_0 = \frac{w^4 m^4}{16 l_4}$, $w^2 = \frac{2l_2}{m^2-2}$, $l_4 > 0$, $0 < m < 1$, then $U(\eta) = \frac{w}{2\sqrt{l_4}}\left[\sqrt{1-m^2}\mathrm{nc}(w\eta) \pm \mathrm{dc}(w\eta)\right]$, and we have

$$q(x, t) = \exp\left\{i\left[\frac{-\alpha}{2a_1}x + \lambda t + \theta\right]\right\}$$

$$\times\left[\frac{3\rho_2 h_{01}\left[\begin{array}{c}\sqrt{1-m^2}\mathrm{nc}\left(\sqrt{\frac{2l_2}{m^2-2}}[x+\alpha t], m\right)\\ \pm\mathrm{dc}\left(\sqrt{\frac{2l_2}{m^2-2}}[x+\alpha t], m\right)\end{array}\right]^2}{(l_2 - h_{21})\left[\begin{array}{c}\sqrt{1-m^2}\mathrm{nc}\left(\sqrt{\frac{2l_2}{m^2-2}}[x+\alpha t], m\right)\\ \pm\mathrm{dc}\left(\sqrt{\frac{2l_2}{m^2-2}}[x+\alpha t], m\right)\end{array}\right]^2 + \frac{3l_2 m^4}{2(m^2-2)}}\right]^{\frac{1}{2}}, \tag{64}$$

and

$$r(x, t) = Aq(x, t), \tag{65}$$

under the constraint conditions:

$$(l_2 - h_{21})\left[\frac{9l_2^2 m^4}{4(m^2-2)^2} - (l_2 - h_{21})(2l_2 + h_{21})\right] + 27h_{61}h_{01}^2 = 0,$$

$$3h_{01}h_{41} - \frac{3l_2^2 m^4}{4(m^2-2)^2} + \left(l_2^2 - h_{21}^2\right) = 0. \tag{66}$$

(14) If $l_0 = \frac{w^4\left(1-m^2\right)^2}{16 l_4}$, $w^2 = \frac{2l_2}{1+m^2}$, $l_4 < 0$, $0 < m < 1$, then $U(\eta) = \frac{w\sqrt{1-m^2}}{2\sqrt{-l_4}}[m\mathrm{sd}(w\eta) \pm \mathrm{nd}(w\eta)]$, and we have

$$q(x,t)=\exp\left\{i\left[\frac{-\alpha}{2a_1}x+\lambda t+\theta\right]\right\}$$

$$\times\left[\frac{3\rho_2 h_{01}\left[\begin{array}{c} m\,\mathrm{sd}\left(\sqrt{\frac{2l_2}{1+m^2}}\left[x+\alpha t\right],\, m\right)\\ \pm\mathrm{nd}\left(\sqrt{\frac{2l_2}{1+m^2}}\left[x+\alpha t\right],\, m\right)\end{array}\right]^2}{\left(l_2-h_{21}\right)\left[\begin{array}{c} m\,\mathrm{sd}\left(\sqrt{\frac{2l_2}{1+m^2}}\left[x+\alpha t\right],\, m\right)\\ \pm\mathrm{nd}\left(\sqrt{\frac{2l_2}{1+m^2}}\left[x+\alpha t\right],\, m\right)\end{array}\right]^2-\frac{3l_2\left(1-m^2\right)}{2\left(1+m^2\right)}}\right]^{\frac{1}{2}}, \tag{67}$$

and

$$r(x,t)=Aq(x,t), \tag{68}$$

under the same constraint conditions (59).

Also, Eqs. (2) and (3) has the following Weierstrass elliptic functions solutions:

(15) $U(\eta)=\frac{3\wp'(\eta,\,g_2,\,g_3)}{\sqrt{l_4}\left[6\wp(\eta,\,g_2,\,g_3)+l_2\right]}$ or $U(\eta)=\frac{\sqrt{l_0}\left[6\wp(\eta,\,g_2,\,g_3)+l_2\right]}{3\wp'(\eta,\,g_2,\,g_3)}$, where $g_2=l_0l_4+\frac{l_2^2}{12}$, $g_3=\frac{l_2\left(36l_0l_4-l_2^2\right)}{216}$, we have

$$q(x,t)=\exp\left\{i\left[\frac{-\alpha}{2a_1}x+\lambda t+\theta\right]\right\}$$

$$\times\left[\frac{9\rho_2 h_{01}\left[\wp'\left(\left[x+\alpha t\right],\, l_0l_4+\frac{l_2^2}{12},\, \frac{l_2\left(36l_0l_4-l_2^2\right)}{216}\right)\right]^2}{\begin{array}{c}3\left(l_2-h_{21}\right)\left[\wp'\left(\left[x+\alpha t\right],\, l_0l_4+\frac{l_2^2}{12},\, \frac{l_2\left(36l_0l_4-l_2^2\right)}{216}\right)\right]^2\\ +l_0l_4\left[6\wp\left(\left[x+\alpha t\right],\, l_0l_4+\frac{l_2^2}{12},\, \frac{l_2\left(36l_0l_4-l_2^2\right)}{216}\right)+l_2\right]^2\end{array}}\right]^{\frac{1}{2}}, \tag{69}$$

and

$$r(x,t)=Aq(x,t), \tag{70}$$

or

$$q(x,t)=\exp\left\{i\left[\frac{-\alpha}{2a_1}x+\lambda t+\theta\right]\right\}$$

$$\times\left[\frac{3\rho_2 h_{01}\left[6\wp\left([x+\alpha t],\, l_0l_4+\frac{l_2^2}{12},\, \frac{l_2\left(36l_0l_4-l_2^2\right)}{216}\right)+l_2\right]^2}{\begin{array}{l}(l_2-h_{21})\left[6\wp\left([x+\alpha t],\, l_0l_4+\frac{l_2^2}{12},\, \frac{l_2\left(36l_0l_4-l_2^2\right)}{216}\right)+l_2\right]^2\\ +27\left[\wp'\left([x+\alpha t],\, l_0l_4+\frac{l_2^2}{12},\, \frac{l_2\left(36l_0l_4-l_2^2\right)}{216}\right)\right]^2\end{array}}\right]^{\frac{1}{2}}, \tag{71}$$

and

$$r(x,t)=Aq(x,t), \tag{72}$$

under the same constraint conditions

$$(l_2-h_{21})\left[9l_0l_4-(l_2-h_{21})(2l_2+h_{21})\right]+27h_{61}h_{01}^2=0,$$

$$3h_{01}h_{41}-3l_0l_4+\left(l_2^2-h_{21}^2\right)=0. \tag{73}$$

(16) $U(\eta)=\left[\frac{3\wp(\eta,\, g_2,\, g_3)-l_2}{3l_4}\right]^{\frac{1}{2}}$ or $U(\eta)=\left[\frac{3l_0}{3\wp(\eta,\, g_2,\, g_3)-l_2}\right]^{\frac{1}{2}}$, where $g_2=\frac{4}{3}\left(l_2^2-3l_0l_4\right)$, $g_3=\frac{4}{27}\left(9l_0l_2l_4-2l_2^3\right)$, we have

$$q(x,t)=\exp\left\{i\left[\frac{-\alpha}{2a_1}x+\lambda t+\theta\right]\right\}$$

$$\times\left[\frac{3\rho_2 h_{01}\left[3\wp\left([x+\alpha t],\, \frac{4}{3}\left(l_2^2-3l_0l_4\right),\, \frac{4}{27}\left(9l_0l_2l_4-2l_2^3\right)\right)-l_2\right]}{(l_2-h_{21})\left[3\wp\left([x+\alpha t],\, \frac{4}{3}\left(l_2^2-3l_0l_4\right),\, \frac{4}{27}\left(9l_0l_2l_4-2l_2^3\right)\right)-l_2\right]+9l_4l_0}\right]^{\frac{1}{2}}, \tag{74}$$

and

$$r(x,t)=Aq(x,t), \tag{75}$$

or

$$q(x,\,t)=\exp\left\{i\left[\frac{-\alpha}{2a_1}x+\lambda t+\theta\right]\right\}$$

$$\times\left[\frac{3\rho_2 h_{01}}{3\wp\left([x+\alpha t]\,,\,\frac{4}{3}\left(l_2^2-3l_0l_4\right),\,\frac{4}{27}\left(9l_0l_2l_4-2l_2^3\right)\right)-h_{21}}\right]^{\frac{1}{2}}, \tag{76}$$

and

$$r(x,\,t)=Aq(x,\,t), \tag{77}$$

under the same constraint conditions (73).

(17) $U(\eta)=\dfrac{3\wp'(\eta,\,g_2,\,g_3)}{\sqrt{l_4}\,[6\wp(\eta,\,g_2,\,g_3)+l_2]}$ or $U(\eta)=\dfrac{\sqrt{l_0}\,[6\wp(\eta,\,g_2,\,g_3)+l_2]}{3\wp'(\eta,\,g_2,\,g_3)}$, where $l_0=\dfrac{5l_2^2}{36l_4}g_2=\dfrac{2l_2^2}{9}$, $g_3=\dfrac{l_2^3}{54}$, we have

$$q(x,\,t)=\exp\left\{i\left[\frac{-\alpha}{2a_1}x+\lambda t+\theta\right]\right\}$$

$$\times\left[\frac{9\rho_2h_{01}\left[\wp'\left([x+\alpha t]\,,\,\frac{2l_2^2}{9},\,\frac{l_2^3}{54}\right)\right]^2}{\begin{array}{c}3\left(l_2-h_{21}\right)\left[\wp'\left([x+\alpha t]\,,\,\frac{2l_2^2}{9},\,\frac{l_2^3}{54}\right)\right]^2\\+\frac{5}{36}l_2^2\left[6\wp\left([x+\alpha t]\,,\,\frac{2l_2^2}{9},\,\frac{l_2^3}{54}\right)+l_2\right]^2\end{array}}\right]^{\frac{1}{2}}, \tag{78}$$

and

$$r(x,\,t)=Aq(x,\,t), \tag{79}$$

or

$$q(x,\, t) = \exp\left\{ i \left[\frac{-\alpha}{2a_1} x + \lambda t + \theta \right] \right\}$$

$$\times \left[\frac{3\rho_2 h_{01} \left[6\wp\left([x+\alpha t],\, \frac{2l_2^2}{9},\, \frac{l_2^3}{54} \right) + l_2 \right]^2}{(l_2 - h_{21}) \left[6\wp\left([x+\alpha t],\, \frac{2l_2^2}{9},\, \frac{l_2^3}{54} \right) + l_2 \right]^2 + 27 \left[\wp' \left([x+\alpha t],\, \frac{2l_2^2}{9},\, \frac{l_2^3}{54} \right) \right]^2} \right]^{\frac{1}{2}}, \tag{80}$$

and

$$r(x,\, t) = Aq(x,\, t), \tag{81}$$

under the constraint conditions:

$$(l_2 - h_{21}) \left[\frac{5l_2^2}{4} - (l_2 - h_{21})(2l_2 + h_{21}) \right] + 27 h_{61} h_{01}^2 = 0,$$

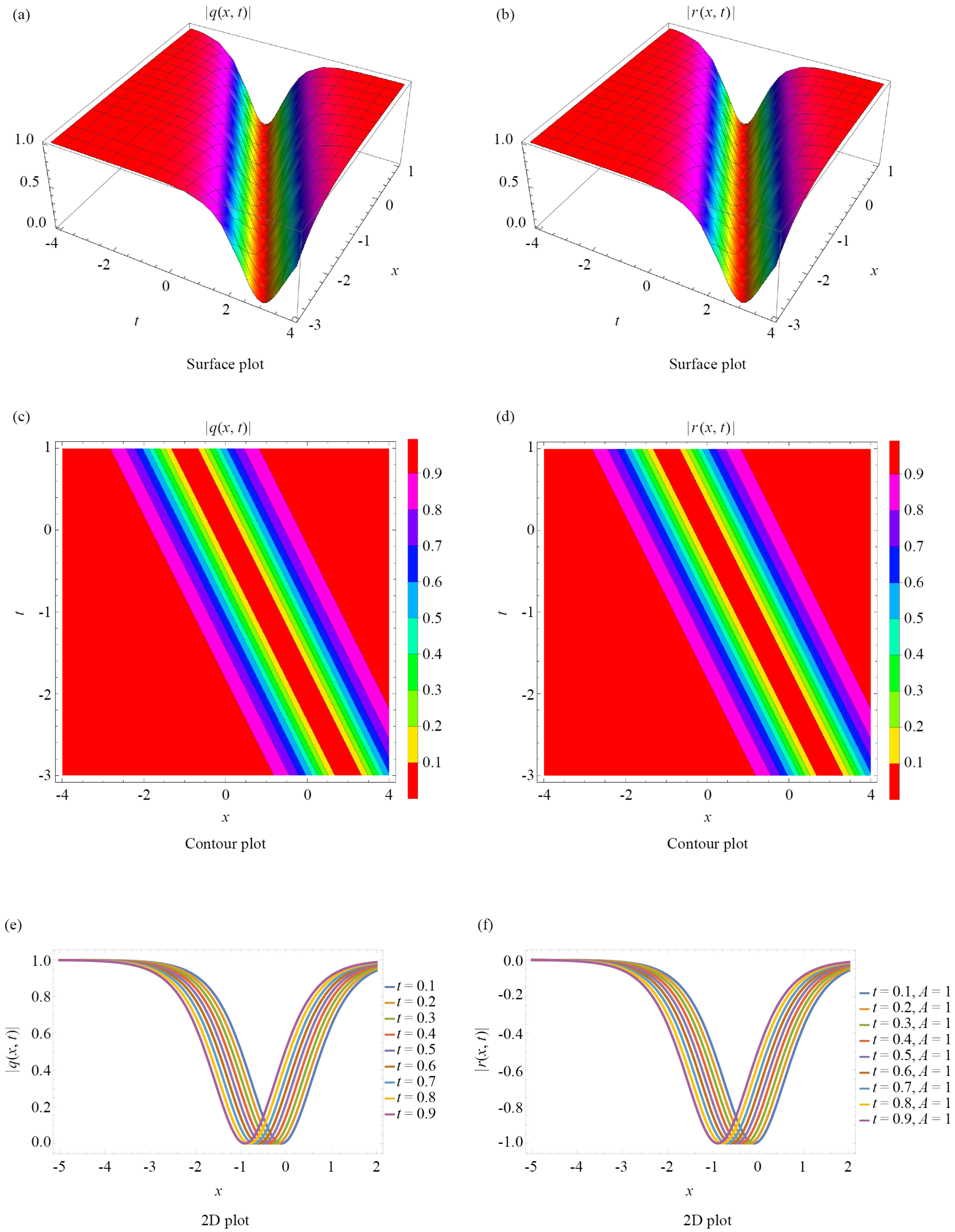
$$3h_{01} h_{41} + \frac{7l_2^2}{12} - h_{21}^2 = 0. \tag{82}$$

This is true provided $(l_2 - h_{21}) > 0$ and $\rho_2 h_{01} > 0$.

5. Results and discussion

Figure 1 presents the evolution of dark soliton solutions $q(x,\, t)$ and $r(x,\, t)$, which are described by the complex-valued solutions (33) and (34), respectively. The analysis is conducted by setting the time variable t at incremental values from 0.1 to 0.9 and considering different amplitude parameters $A = 1, 2, 3$, and 4. Additionally, the parameters are chosen as $\rho_2 = 1$, $l_2 = -1$, $h_{21} = -2$, $h_{01} = 1$, and $\alpha = 1$. These parameter settings are crucial in determining the nature of the soliton propagation and their characteristics under varying conditions.

Figures 1a and 1b illustrate the surface plots of the dark soliton solutions, depicting their propagation over the spatial and temporal domains. The three-dimensional representation highlights the intensity dips of the dark solitons, which remain stable as they evolve with time. The soliton structure is well-preserved, showcasing a characteristic intensity drop in the center of the profile, which is a hallmark of dark solitons. Additionally, the variation in amplitude parameters influences the depth and width of the soliton, demonstrating the impact of nonlinearity on soliton formation.


(a)
|q(x, t)|
Surface plot
(b)
|r(x, t)|
Surface plot
(c)
|q(x, t)|
Contour plot
(d)
|r(x, t)|
Contour plot
(e)
t = 0.1
t = 0.2
t = 0.3
t = 0.4
t = 0.5
t = 0.6
t = 0.7
t = 0.8
t = 0.9
2D plot
(f)
t = 0.1, A = 1
t = 0.2, A = 1
t = 0.3, A = 1
t = 0.4, A = 1
t = 0.5, A = 1
t = 0.6, A = 1
t = 0.7, A = 1
t = 0.8, A = 1
t = 0.9, A = 1
2D plot

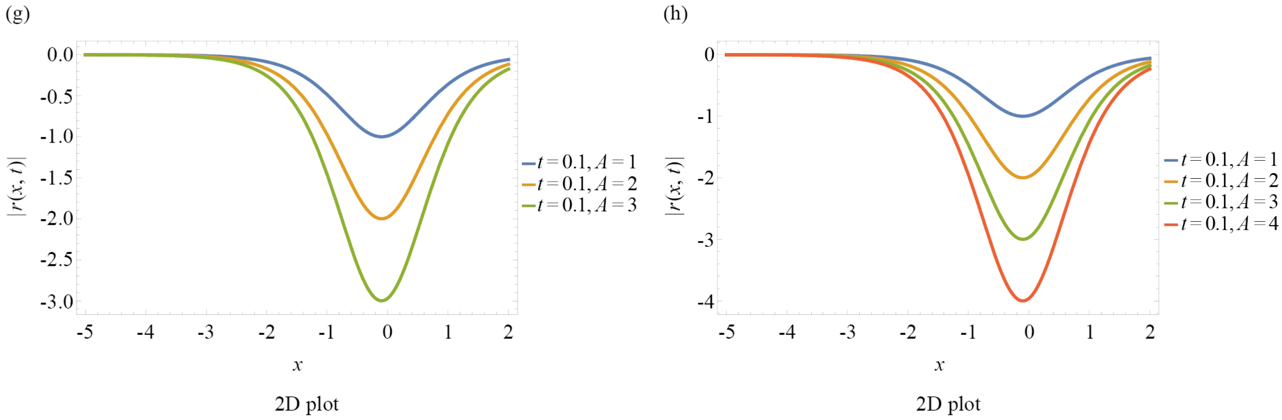


Figure 1. Exploring the features of dark solitons (33) and (34), particularly its magnitude

Figures 1c and 1d provide contour plots of the dark soliton solutions, emphasizing the density variations and spatial distribution of the solitons. These plots effectively highlight the soliton localization and periodicity in the transverse direction. The darker regions indicate lower intensity zones corresponding to the soliton core, while the lighter areas signify the surrounding continuous wave background. The contour plots confirm that the solitons maintain their shape and stability under the given parameter conditions, reinforcing their robustness against perturbations.

Figures 1e, 1f, 1g, and 1h present 2D plots illustrating the effect of different amplitude parameters $A = 1$, 2, 3, and 4 on the dark soliton solutions. As A increases, the soliton dips become more pronounced, indicating a stronger modulation depth. Higher amplitude values result in a more significant contrast between the soliton core and its background, leading to sharper transitions in the soliton profile. These plots confirm that the dark soliton solutions exhibit an increase in intensity variation with larger amplitude parameters, further validating the theoretical predictions of soliton dynamics in nonlinear optical systems.

Overall, Figure 1 provides a comprehensive visualization of dark soliton behavior under the influence of specific nonlinear parameters. The analysis demonstrates that dark solitons maintain their characteristic shape and stability over time, with their amplitude-dependent properties dictating the soliton depth and width. These findings contribute to the broader understanding of nonlinear wave propagation in optical fibers and other dispersive media, highlighting the critical role of parameter selection in soliton dynamics.

6. Conclusions

This paper derived the optical soliton solutions to the Kundu-Eckhaus equation with differential group delay by the aid of the generalized ϕ^6-model expansion approach. The intermediary functions are Jacobi's elliptic functions and Weierstrass' elliptic functions. The soiton solutions are classified. The results are interesting and can be applied in quantum optics if Kundu-Eckhaus equation is implemented in quantum optoelectronics. These results thus pave the way for further future studies [11–17]. They are the extension of the model to address dispersion-flattened fibers where the same mathematical principle will be implemented to retrieve the optical solitons. The results are going to be disseminated across the journals with time. This is just a tip of the iceberg!

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Conflict of interest

The authors claim that there is no conflict of interest.

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