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Research Article

Shock Wave and Singular Solitary Wave Perturbation with Gardners Equation Having Dispersion Triplet

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Abstract: The current paper recovers the shock wave and solitary wave solutions to the well-known Korteweg-de Vries-modified Korteweg-de Vries (KdV-mKdV) equation with perturbation terms that is alternately known as the Gardners equation. The integration scheme adopted in this work to retrieve such solutions is the G'/G-expansion. The results appear with parameter constraints for the existence of such waves and these are also presented.

Keywords: shock wave, solitary wave, Gardners equation

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1. Introduction

The dynamics of shallow water waves remain a central and extensively explored domain within Fluid Dynamics due to their vast applicability in coastal engineering, oceanography, and nonlinear wave theory. A significant body of research has emerged over the decades, addressing various analytical and numerical aspects of shallow water wave equations [1–12]. These contributions include, but are not limited to, the derivation of exact solitary and shock wave solutions, the identification of infinitely many conserved quantities, and the development of robust numerical schemes for simulating wave propagation and interactions.

In particular, classical models such as the Korteweg-de Vries (KdV), modified KdV (mKdV), and Gardners equations have been at the forefront of modeling dispersive wave phenomena. Recent works have investigated the effects of

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perturbations and higher-order corrections to these equations to more accurately reflect real-world physical processes such as bottom friction, wind stress, variable topography, and higher-order nonlinearities [13–20]. These modified equations have been addressed using a variety of tools, including the semi-inverse variational principle [13], homotopy analysis methods [14], Lie symmetry approaches [15], and the Hirota bilinear method [16]. Moreover, soliton perturbation theory has been effectively applied to analyze the persistence and stability of solitary waves under weak perturbations, especially in identifying fixed-point solutions that allow solitary waves to retain their amplitude and profile [17].

The perturbed versions of the KdV and mKdV equations have received considerable attention in recent years, particularly in contexts involving nonlinear evolution equations with damping, external forcing, and stochastic effects [18–20]. In this regard, the Gardners equation-being a nonlinear combination of the KdV and mKdV equations-serves as an important model, especially when addressing both bright-type and kink-type solitary structures under perturbative settings.

The present study revisits the perturbed Gardners equation by incorporating multiple perturbation terms that are physically relevant in modeling dissipative and dispersive effects. The main goal is to derive analytical solutions that represent singular solitary waves and shock waves within this perturbed framework. To achieve this, we employ the well-established G'/G-expansion method, which has proven effective for extracting closed-form solutions in various nonlinear partial differential equations [21, 22]. The structure of the paper is as follows: the next section provides a detailed formulation of the model, followed by the application of the integration algorithm. The resulting analytical solutions are discussed comprehensively, and complementary numerical simulations are presented to validate and visualize the dynamical behavior of the obtained waveforms.

2. Governing model

The dimensionless form of the unperturbed version of Gardners equation that describes the shallow water wave dynamics is given as [13–20]

$$q_t + \left(a_1 q + a_5 q^2\right) q_x + a_2 q_{xxx} + a_3 q_{xxt} + a_4 q_{xtt} = 0, \tag{1}$$

where x and t are the spatial and temporal variables for the model and the dependent variable $q(x,\ t)$ is the wave amplitude. The three dispersion terms are a_j for $j = 2,\ 3,\ 4$. Here a_2 represents the spatial dispersion while a_3 and a_4 accounts for the versions of the spatio-temporal dispersion. The first term is the linear temporal evolution, while the coefficients of a_1 and a_5 are the nonlinear terms that account for the KdV and mKdV equation respectively. Thus Gardners equation is occasionally referred to as the KdV-mKdV equation. The current paper will be addressed in presence of perturbation terms and will retrieve the shock waves and solitary wave solutions to the model.

This portion of study is manifested for Gardners equation with perturbation terms. The solitary wave and shock wave solutions are herein recovered using generalized G'/G-expansion approach [21, 22], based on the traveling wave hypothesis. It is noteworthy that the current study addresses perturbed Gardners equation for the first time in the following manner:

$$\begin{aligned} & Q_t + (a_1 Q + a_5 Q^2) Q_x + a_2 Q_{xxx} + a_3 Q_{xxt} + a_4 Q_{xtt} \\ &= \theta Q_x Q_{xx} + \delta Q^m Q_x + \Lambda Q Q_{xxx} + \nu Q Q_x Q_{xx} + \xi Q_x Q_{xxx} + \psi Q_{xxxxx} + \kappa Q Q_{xxxxx}, \end{aligned} \tag{2}$$

with $Q = Q(x,\ t)$. The perturbation terms are from linear and nonlinear dispersive effect that includes higher order dispersions as well.

Utilizing $\tau = x - \omega t$, along with $Q(x, t) = F(\tau)$, we have recovered the following ordinary differential equation from the equation (2):

$$\left((-\nu F - \theta) F'' - \xi F''' + a_1 F + a_5 F^2 - \delta (F)^m - \omega\right) F' + \left(a_4 \omega^2 - \Lambda F - a_3 \omega + a_2\right) F''' + (-\kappa F - \psi) F''''' = 0. \quad (3)$$

Next, equation (3) has been analyzed by using the generalized G'/G -expansion method with $m = 1, 2$, for discovering new analytic solutions as described in following manner.

2.1 *Case-I: m = 1*

Firstly, we have considered $m = 1$ for (3) and then equation (3) is rewritten as follows:

$$\left((-\nu F - \theta) F'' - \xi F''' + a_1 F + a_5 F^2 - \delta F - \omega\right) F' + \left(a_4 \omega^2 - \Lambda F - a_3 \omega + a_2\right) F''' + (-\kappa F - \psi) F''''' = 0. \quad (4)$$

Equation (4) has been solved via the following solution structure, that is recovered by balancing the highest order derivative term and highly nonlinear terms homogeneously [21, 22]:

$$F(\tau) = \alpha_0 + \alpha_1 \left(\frac{G'(\tau)}{G(\tau)}\right) + \alpha_2 \left(\frac{G'(\tau)}{G(\tau)}\right)^2, \quad (5)$$

with $G(\tau)$ following the subsequent auxiliary equation:

$$G'' + \lambda G' + \mu G = 0, \quad (6)$$

where α_0, α_1, α_2 are parameters that are assessed during the computational process. We have furnished the following parameter settings for extracting the solutions to equation (4) using (5) into (4), along with (6), as stated below:

$$\begin{aligned}
&\alpha_1 = 2\left(\frac{\lambda}{2} + \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\right)\alpha_2, \quad \nu = -\frac{60\kappa}{\alpha_2}, \quad \xi = -10\kappa\sqrt{\lambda^2 - 4\mu}, \\
&a_5 = \frac{3}{7}\,\frac{-95\kappa\lambda^2\alpha_2 + 240\left(\lambda/2 + 1/2\sqrt{\lambda^2 - 4\mu}\right)\alpha_2\kappa\lambda + 140\kappa\mu\alpha_2 + 13\Lambda\alpha_2 - 240\kappa\alpha_0 + 4\theta\alpha_2}{{\alpha_2}^2}, \\
&a_1 = \frac{Y_1}{7{\alpha_2}^2}, \quad \omega = -\frac{Y_2}{14{\alpha_2}^2}, \quad a_2 = -\frac{Y_3}{1176{\alpha_2}^4} \\
&\psi = \frac{37\kappa\lambda^2\alpha_2}{168} + \frac{2}{7}\left(\frac{\lambda}{2} + \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\right)\alpha_2\kappa\lambda - \frac{7}{6}\kappa\mu\alpha_2 - \frac{\Lambda\alpha_2}{56} - \frac{2}{7}\kappa\alpha_0 - \frac{\theta\alpha_2}{84},
\end{aligned} \quad (7)$$

with

$$
\begin{aligned}
Y_1 = {} & 15\,\alpha_2{}^2\kappa\lambda^3\sqrt{\lambda^2-4\mu}+344\,\kappa\lambda^4\alpha_2{}^2+1380\,\alpha_2{}^2\kappa\lambda\,\mu\,\sqrt{\lambda^2-4\mu}-1282\,\kappa\lambda^2\mu\,\alpha_2{}^2+39\,\Lambda\,\alpha_2{}^2\lambda\,\sqrt{\lambda^2-4\mu} \\
& +4\,\Lambda\lambda^2\alpha_2{}^2-1440\,\alpha_2\kappa\lambda\,\alpha_0\sqrt{\lambda^2-4\mu}+12\,\alpha_2{}^2\lambda\,\theta\,\sqrt{\lambda^2-4\mu}-30\,\kappa\lambda^2\alpha_0\alpha_2+1064\,\kappa\mu^2\alpha_2{}^2-2\,\lambda^2\theta\,\alpha_2{}^2 \\
& +62\,\Lambda\mu\,\alpha_2{}^2-2760\,\kappa\mu\,\alpha_0\alpha_2+32\,\mu\,\theta\,\alpha_2{}^2-78\,\Lambda\,\alpha_0\alpha_2+7\,\delta\,\alpha_2{}^2+1440\,\kappa\,\alpha_0{}^2-24\,\theta\,\alpha_0\alpha_2, \\
Y_2 = {} & -2608\,\kappa\mu^2\alpha_0\alpha_2{}^2+4\,\alpha_2{}^2\lambda^2\theta\,\alpha_0+4680\,\kappa\mu\,\alpha_0{}^2\alpha_2-64\,\alpha_2{}^2\mu\,\theta\,\alpha_0-18\,\Lambda\lambda^2\mu\,\alpha_2{}^3-8\,\Lambda\lambda^2\alpha_0\alpha_2{}^2 \\
& +922\,\kappa\lambda^4\mu\,\alpha_2{}^3-928\,\kappa\lambda^4\alpha_0\alpha_2{}^2-2810\,\kappa\lambda^2\mu^2\alpha_2{}^3-12\,\lambda^2\mu\,\theta\,\alpha_2{}^3 \\
& -90\,\kappa\lambda^2\alpha_0{}^2\alpha_2-4680\,\alpha_2{}^2\kappa\lambda\,\mu\,\alpha_0\sqrt{\lambda^2-4\mu} \\
& -24\,\alpha_2{}^2\lambda\,\theta\,\alpha_0\sqrt{\lambda^2-4\mu}-202\,\alpha_2{}^3\kappa\lambda^3\mu\,\sqrt{\lambda^2-4\mu}+90\,\alpha_2{}^2\kappa\lambda^3\alpha_0\sqrt{\lambda^2-4\mu}+1304\,\alpha_2{}^3\kappa\lambda\,\mu^2\sqrt{\lambda^2-4\mu} \\
& +62\,\Lambda\,\alpha_2{}^3\lambda\,\mu\,\sqrt{\lambda^2-4\mu}+32\,\alpha_2{}^3\lambda\,\mu\,\theta\,\sqrt{\lambda^2-4\mu}-78\,\Lambda\,\alpha_2{}^2\lambda\,\alpha_0\sqrt{\lambda^2-4\mu}+2160\,\alpha_2\kappa\lambda\,\alpha_0{}^2\sqrt{\lambda^2-4\mu} \\
& +104\,\alpha_2{}^3\kappa\lambda^5\sqrt{\lambda^2-4\mu}+4\,\Lambda\,\alpha_2{}^3\lambda^3\sqrt{\lambda^2-4\mu}-2\,\alpha_2{}^3\lambda^3\theta\,\sqrt{\lambda^2-4\mu}-124\,\Lambda\mu\,\alpha_0\alpha_2{}^2-1440\,\alpha_0{}^3\kappa \\
& +3284\,\kappa\lambda^2\mu\,\alpha_0\alpha_2{}^2-1624\,\kappa\mu^3\alpha_2{}^3+8\,\mu^2\theta\,\alpha_2{}^3-7\,\kappa\lambda^6\alpha_2{}^3+24\,\alpha_2\theta\,\alpha_0{}^2+13\,\Lambda\lambda^4\alpha_2{}^3-58\,\Lambda\mu^2\alpha_2{}^3 \\
& +78\,\Lambda\,\alpha_0{}^2\alpha_2+4\,\lambda^4\theta\,\alpha_2{}^3, \\
Y_3 = {} & 77448\,\kappa\lambda^4\mu\,a_3\alpha_2{}^5-414720\,\kappa\,\theta\,a_4\alpha_0{}^5\alpha_2+219360\,\Lambda\lambda^2\mu^2\theta\,a_4\alpha_0\alpha_2{}^5+50824704\,\kappa^2\mu^5a_4\alpha_0\alpha_2{}^5 \\
& -155904\,\kappa\mu^5\theta\,a_4\alpha_2{}^6+5460\,\kappa\lambda^3\alpha_2{}^5\sqrt{\lambda^2-4\mu}+588\,\Lambda\lambda\,\alpha_2{}^5\sqrt{\lambda^2-4\mu}-1347840\,\Lambda\,\kappa\,a_4\alpha_0{}^5\alpha_2 \\
& -150960\,\Lambda^2\lambda^2\mu\,a_4\alpha_0{}^2\alpha_2{}^4-25728\,\lambda^2\mu^3\theta^2a_4\alpha_2{}^6-18432\,\mu\,\theta^2a_4\alpha_0{}^3\alpha_2{}^3+393120\,\kappa\mu\,a_3\alpha_0{}^2\alpha_2{}^3 \\
& -66336\,\Lambda\lambda^4\mu\,\theta\,a_4\alpha_0\alpha_2{}^5-588\,\kappa\lambda^6a_3\alpha_2{}^5-76992\,\kappa\lambda^8\theta\,a_4\alpha_0\alpha_2{}^5+1351392\,\kappa\lambda^4\mu^3\theta\,a_4\alpha_2{}^6 \\
& +3456\,\theta^2a_4\alpha_0{}^4\alpha_2{}^2+316873728\,\kappa^2\lambda^2\mu^4a_4\alpha_0\alpha_2{}^5+21424704\,\Lambda\,\kappa\lambda^2\mu^3a_4\alpha_0\alpha_2{}^5+4704\,\lambda^4\theta^2a_4\alpha_0{}^2\alpha_2{}^4 \\
& -962784\,\kappa\lambda^4\theta\,a_4\alpha_0{}^3\alpha_2{}^3+7918944\,\kappa^2\lambda^8a_4\alpha_0{}^2\alpha_2{}^4+8587872\,\Lambda\,\kappa\lambda^4\mu\,a_4\alpha_0{}^2\alpha_2{}^4-6400512\,\Lambda\,\kappa\lambda^4\mu^2a_4\alpha_0\alpha_2{}^5 \\
& +1440\,\Lambda\lambda^2\theta\,a_4\alpha_0{}^3\alpha_2{}^3+384\,\mu^4\theta^2a_4\alpha_2{}^6+120\,\lambda^8\theta^2a_4\alpha_2{}^6+15824256\,\kappa^2\mu^6a_4\alpha_2{}^6+996864\,\kappa\mu^4\theta\,a_4\alpha_0\alpha_2{}^5
\end{aligned}
$$

$$+2453760\,\kappa\mu\,\theta\,a_4\alpha_0{}^4\alpha_2{}^2+193056\,\Lambda\,\kappa\,\lambda^8\mu\,a_4\alpha_2{}^6+2093\,\kappa\,\lambda^4\alpha_2{}^5+12441600\,\kappa^2 a_4\alpha_0{}^6-1176\,\Lambda\,\alpha_0\alpha_2{}^4$$

$$-2268\,\Lambda\,\mu\,\alpha_2{}^5+861\,\Lambda\,\lambda^2\alpha_2{}^5-118402560\,\kappa^2\mu^3 a_4\alpha_0{}^3\alpha_2{}^3+1555200\,\kappa^2\lambda^2 a_4\alpha_0{}^5\alpha_2-7488\,\Lambda^2\lambda^2 a_4\alpha_0{}^3\alpha_2{}^3$$

$$-2832\,\kappa\,\lambda^{10}\theta\,a_4\alpha_2{}^6+1038144\,\Lambda\,\kappa\,\lambda^6\mu\,a_4\alpha_0\alpha_2{}^5+86304\,\Lambda^2\mu^3 a_4\alpha_0\alpha_2{}^5-50393856\,\kappa^2\mu^4 a_4\alpha_0{}^2\alpha_2{}^4$$

$$+6523200\,\Lambda\,\kappa\,\mu\,a_4\alpha_0{}^4\alpha_2{}^2-919688064\,\kappa^2\lambda^2\mu^3 a_4\alpha_0{}^2\alpha_2{}^4-2304\,\Lambda\,\lambda^6\mu\,\theta\,a_4\alpha_2{}^6+4056\,\Lambda^2\lambda^4\mu^2 a_4\alpha_2{}^6$$

$$-10192\,\kappa\,\mu^2\alpha_2{}^5+182\,\lambda^2\theta\,\alpha_2{}^5-728\,\mu\,\theta\,\alpha_2{}^5+528\,\Lambda\,\lambda^8\theta\,a_4\alpha_2{}^6+264480\,\Lambda^2\lambda^2\mu^2 a_4\alpha_0\alpha_2{}^5$$

$$+65190\,\kappa^2\lambda^{12}a_4\alpha_2{}^6-237120\,\Lambda\,\kappa\,\lambda^8 a_4\alpha_0\alpha_2{}^5$$

$$-10920\,\kappa\,\lambda^2\alpha_0\alpha_2{}^4-4483584\,\kappa\,\mu^2\theta\,a_4\alpha_0{}^3\alpha_2{}^3-136416\,\kappa\,\mu^3 a_3\alpha_2{}^5+336\,\lambda^4\theta\,a_3\alpha_2{}^5$$

$$+3900\,\Lambda\,\kappa\,\lambda^{10}a_4\alpha_2{}^6-10416\,\Lambda\,\mu\,a_3\alpha_0\alpha_2{}^4+44078040\,\kappa^2\lambda^4 a_4\alpha_0{}^4\alpha_2{}^2+43680\,\kappa\,\mu\,\alpha_0\alpha_2{}^4-5824\,\kappa\,\lambda^2\mu\,\alpha_2{}^5$$

$$-8402688\,\Lambda\,\kappa\,\mu^2 a_4\alpha_0{}^3\alpha_2{}^3-59226864\,\kappa^2\lambda^6\mu\,a_4\alpha_0{}^2\alpha_2{}^4-556320\,\kappa\,\lambda^6\mu^2\theta\,a_4\alpha_2{}^6+8217144\,\kappa^2\lambda^8\mu^2 a_4\alpha_2{}^6$$

$$+275856\,\kappa\,\lambda^2\mu\,a_3\alpha_0\alpha_2{}^4-1008\,\lambda^2\mu\,\theta\,a_3\alpha_2{}^5-672\,\Lambda\,\lambda^2 a_3\alpha_0\alpha_2{}^4-15168\,\lambda^4\mu\,\theta^2 a_4\alpha_0\alpha_2{}^{56}$$

$$+10464\,\lambda^4\mu^2\theta^2 a_4\alpha_2+11461632\,\kappa\,\lambda^2\mu^3\theta\,a_4\alpha_0\alpha_2{}^5$$

$$+11156544\,\Lambda\,\kappa\,\lambda^2\mu\,a_4\alpha_0{}^3\alpha_2{}^3-12848256\,\kappa\,\lambda^2\mu^2\theta\,a_4\alpha_0{}^2\alpha_2{}^4-4992\,\Lambda^2\lambda^6 a_4\alpha_0\alpha_2{}^5$$

$$-245749728\,\kappa^2\lambda^4\mu^3 a_4\alpha_0\alpha_2{}^5+3935232\,\kappa\,\lambda^2\mu\,\theta\,a_4\alpha_0{}^3\alpha_2{}^3+674400\,\kappa\,\lambda^6\mu\,\theta\,a_4\alpha_0\alpha_2{}^5+6552\,\Lambda\,a_3\alpha_0{}^2\alpha_2{}^3$$

$$+104833248\,\kappa^2\lambda^6\mu^2 a_4\alpha_0\alpha_2{}^5+1130304\,\Lambda\,\kappa\,\mu^5 a_4\alpha_2{}^6-79728\,\Lambda^2\lambda^2\mu^3 a_4\alpha_2{}^6+4231680\,\Lambda\,\kappa\,\mu^4 a_4\alpha_0\alpha_2{}^5$$

$$-1512\,\Lambda\,\lambda^2\mu\,a_3\alpha_2{}^5+1110\,\Lambda^2\lambda^8 a_4\alpha_2{}^6+2016\,\theta\,a_3\alpha_0{}^2\alpha_2{}^3-120960\,\kappa\,a_3\alpha_0{}^3\alpha_2{}^2+672\,\mu^2\theta\,a_3\alpha_2{}^5$$

$$+13951296\,\kappa^2\lambda^2\mu^5 a_4\alpha_2{}^6+32640\,\Lambda\,\mu^3\theta\,a_4\alpha_0\alpha_2{}^5-88608\,\Lambda\,\lambda^2\mu^3\theta\,a_4\alpha_2{}^6$$

$$-17051328\,\kappa^2\lambda^8\mu\,a_4\alpha_0\alpha_2{}^5-116064\,\Lambda^2\mu\,a_4\alpha_0{}^3\alpha_2{}^3-6144\,\mu^3\theta^2 a_4\alpha_0\alpha_2{}^5+3861792\,\kappa\,\lambda^4\mu\,\theta\,a_4\alpha_0{}^2\alpha_2{}^4$$

$$+29568\,\Lambda\,\lambda^4\theta\,a_4\alpha_0{}^2\alpha_2{}^4+190272\,\kappa^2\lambda^{10}a_4\alpha_0\alpha_2{}^5-1574160\,\Lambda\,\kappa\,\lambda^2\mu^4 a_4\alpha_2{}^6-991056\,\Lambda\,\kappa\,\lambda^6\mu^2 a_4\alpha_2{}^6$$

$$-80870400\,\kappa^2\mu\,a_4\alpha_0{}^5\alpha_2+100032\,\kappa\,\lambda^8\mu\,\theta\,a_4\alpha_2{}^6+1984512\,\kappa\,\mu^3\theta\,a_4\alpha_0{}^2\alpha_2{}^4-1440\,\lambda^6\mu\,\theta^2 a_4\alpha_2{}^6$$

$$-173776320\,\kappa^2\lambda^2\mu\,a_4\alpha_0{}^4\alpha_2{}^2-41603184\,\kappa^2\lambda^6\mu^3a_4\alpha_2{}^6+86016\,\Lambda\,\mu^2\theta\,a_4\alpha_0{}^2\alpha_2{}^4+22464\,\Lambda\,\theta\,a_4\alpha_0{}^4\alpha_2{}^2$$

$$+46464\,\lambda^2\mu^2\theta^2a_4\alpha_0\alpha_2{}^5-77952\,\kappa\lambda^4a_3\alpha_0\alpha_2{}^4-896640\,\Lambda\,\kappa\,\mu^3a_4\alpha_0{}^2\alpha_2{}^4+87936\,\Lambda\,\kappa\lambda^6a_4\alpha_0{}^2\alpha_2{}^4$$

$$+176480640\,\kappa^2\mu^2a_4\alpha_0{}^4\alpha_2{}^2-2038848\,\kappa\lambda^2\mu^4\theta\,a_4\alpha_2{}^6-128640\,\kappa\lambda^6\theta\,a_4\alpha_0{}^2\alpha_2{}^4+960\,\Lambda\lambda^6\theta\,a_4\alpha_0\alpha_2{}^5$$

$$+24672\,\Lambda\lambda^4\mu^2\theta\,a_4\alpha_2{}^6+1032864\,\Lambda\,\kappa\lambda^4\mu^3a_4\alpha_2{}^6-60672\,\Lambda^2\lambda^4\mu\,a_4\alpha_0\alpha_2{}^5-95040\,\kappa\lambda^2\theta\,a_4\alpha_0{}^4\alpha_2{}^2$$

$$-202232160\,\kappa^2\lambda^4\mu\,a_4\alpha_0{}^3\alpha_2{}^3-95616\,\Lambda\mu\,\theta\,a_4\alpha_0{}^3\alpha_2{}^3-5568\,\Lambda\mu^4\theta\,a_4\alpha_2{}^6+768\,\lambda^6\theta^2a_4\alpha_0\alpha_2{}^5$$

$$-8736\,\alpha_2{}^6\kappa^2\lambda^{11}a_4\sqrt{\lambda^2-4\mu}+624\,\Lambda^2\alpha_2{}^6\lambda^7a_4\sqrt{\lambda^2-4\mu}-96\,\alpha_2{}^6\lambda^7\theta^2a_4\sqrt{\lambda^2-4\mu}$$

$$+8736\,\alpha_2{}^5\kappa\lambda^5a_3\sqrt{\lambda^2-4\mu}+336\,\Lambda\,\alpha_2{}^5\lambda^3a_3\sqrt{\lambda^2-4\mu}-168\,\alpha_2{}^5\lambda^3\theta\,a_3\sqrt{\lambda^2-4\mu}$$

$$-21840\,\kappa\lambda\,\mu\,\alpha_2{}^5\sqrt{\lambda^2-4\mu}+49056\,\Lambda^2\lambda^4a_4\alpha_0{}^2\alpha_2{}^4-236040\,\kappa\lambda^2\mu^2a_3\alpha_2{}^5$$

$$+721025280\,\kappa^2\lambda^2\mu^2a_4\alpha_0{}^3\alpha_2{}^3+336\,\lambda^2\theta\,a_3\alpha_0\alpha_2{}^4-3106368\,\Lambda\,\kappa\lambda^4a_4\alpha_0{}^3\alpha_2{}^3$$

$$-106080\,\Lambda\lambda^2\mu\,\theta\,a_4\alpha_0{}^2\alpha_2{}^4+52254744\,\kappa^2\lambda^4\mu^4a_4\alpha_2{}^6+54000\,\Lambda\,\kappa\lambda^2a_4\alpha_0{}^4\alpha_2{}^2-32164704\,\Lambda\,\kappa\lambda^2\mu^2a_4\alpha_0{}^2\alpha_2{}^4$$

$$+26880\,\mu^2\theta^2a_4\alpha_0{}^2\alpha_2{}^4+1152\,\lambda^2\theta^2a_4\alpha_0{}^3\alpha_2{}^3+354939984\,\kappa^2\lambda^4\mu^2a_4\alpha_0{}^2\alpha_2{}^4+1092\,\Lambda\lambda^4a_3\alpha_2{}^5$$

$$-4872\,\Lambda\mu^2a_3\alpha_2{}^5+36504\,\Lambda^2a_4\alpha_0{}^4\alpha_2{}^2-216\,\Lambda^2\lambda^6\mu\,a_4\alpha_2{}^6+20184\,\Lambda^2\mu^4a_4\alpha_2{}^6$$

$$+37968\,\Lambda^2\mu^2a_4\alpha_0{}^2\alpha_2{}^4+3456000\,\kappa^2\lambda^6a_4\alpha_0{}^3\alpha_2{}^3-4523040\,\kappa\lambda^4\mu^2\theta\,a_4\alpha_0\alpha_2{}^5-219072\,\kappa\mu^2a_3\alpha_0\alpha_2{}^4$$

$$-7560\,\kappa\lambda^2a_3\alpha_0{}^2\alpha_2{}^3-589128\,\kappa^2\lambda^{10}\mu\,a_4\alpha_2{}^6-5376\,\mu\,\theta\,a_3\alpha_0\alpha_2{}^4$$

$$-20352\,\lambda^2\mu\,\theta^2a_4\alpha_0{}^2\alpha_2{}^4+1167624\,\alpha_2{}^6\kappa^2\lambda^9\mu\,a_4\sqrt{\lambda^2-4\mu}+15888\,\Lambda\,\alpha_2{}^6\kappa\lambda^9a_4\sqrt{\lambda^2-4\mu}$$

$$-1165704\,\alpha_2{}^5\kappa^2\lambda^9a_4\alpha_0\sqrt{\lambda^2-4\mu}-5851344\,\alpha_2{}^6\kappa^2\lambda^7\mu^2a_4\sqrt{\lambda^2-4\mu}+5160\,\alpha_2{}^6\kappa\lambda^9\theta\,a_4\sqrt{\lambda^2-4\mu}$$

$$+19212144\,\alpha_2{}^6\kappa^2\lambda^5\mu^3a_4\sqrt{\lambda^2-4\mu}-120\,\Lambda\,\alpha_2{}^6\lambda^7\theta\,a_4\sqrt{\lambda^2-4\mu}-1296000\,\alpha_2{}^4\kappa^2\lambda^7a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu}$$

$$-40034304\,\alpha_2{}^6\kappa^2\lambda^3\mu^4a_4\sqrt{\lambda^2-4\mu}+8808\,\Lambda^2\alpha_2{}^6\lambda^5\mu\,a_4\sqrt{\lambda^2-4\mu}-25412352\,\alpha_2{}^6\kappa^2\lambda\,\mu^5a_4\sqrt{\lambda^2-4\mu}$$

$$+\,1824\,\alpha_2{}^6\lambda^5\mu\,\theta^2 a_4\sqrt{\lambda^2-4\mu}-12552\,\Lambda^2\alpha_2{}^5\lambda^5 a_4\alpha_0\sqrt{\lambda^2-4\mu}-16176\,\Lambda^2\alpha_2{}^6\lambda^3\mu^2 a_4\sqrt{\lambda^2-4\mu}$$

$$-\,25948080\,\alpha_2{}^3\kappa^2\lambda^5 a_4\alpha_0{}^3\sqrt{\lambda^2-4\mu}-1248\,\alpha_2{}^5\lambda^5\theta^2 a_4\alpha_0\sqrt{\lambda^2-4\mu}-4800\,\alpha_2{}^6\lambda^3\mu^2\theta^2 a_4\sqrt{\lambda^2-4\mu}$$

$$-\,43152\,\Lambda^2\alpha_2{}^6\lambda\,\mu^3 a_4\sqrt{\lambda^2-4\mu}+3072\,\alpha_2{}^6\lambda\,\mu^3\theta^2 a_4\sqrt{\lambda^2-4\mu}+11232\,\Lambda^2\alpha_2{}^4\lambda^3 a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu}$$

$$-\,3888000\,\alpha_2{}^2\kappa^2\lambda^3 a_4\alpha_0{}^4\sqrt{\lambda^2-4\mu}-1728\,\alpha_2{}^4\lambda^3\theta^2 a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu}-16968\,\alpha_2{}^5\kappa\,\lambda^3\mu\,a_3\sqrt{\lambda^2-4\mu}$$

$$-\,73008\,\Lambda^2\alpha_2{}^3\lambda\,a_4\alpha_0{}^3\sqrt{\lambda^2-4\mu}-37324800\,\alpha_2\kappa^2\lambda\,a_4\alpha_0{}^5\sqrt{\lambda^2-4\mu}+7560\,\alpha_2{}^4\kappa\,\lambda^3 a_3\alpha_0\sqrt{\lambda^2-4\mu}$$

$$+\,109536\,\alpha_2{}^5\kappa\,\lambda\,\mu^2 a_3\sqrt{\lambda^2-4\mu}-6912\,\alpha_2{}^3\lambda\,\theta^2 a_4\alpha_0{}^3\sqrt{\lambda^2-4\mu}+5208\,\Lambda\,\alpha_2{}^5\lambda\,\mu\,a_3\sqrt{\lambda^2-4\mu}$$

$$+\,2688\,\alpha_2{}^5\lambda\,\mu\,\theta\,a_3\sqrt{\lambda^2-4\mu}-6552\,\Lambda\,\alpha_2{}^4\lambda\,a_3\alpha_0\sqrt{\lambda^2-4\mu}+181440\,\alpha_2{}^3\kappa\,\lambda\,a_3\alpha_0{}^2\sqrt{\lambda^2-4\mu}$$

$$-\,2016\,\alpha_2{}^4\lambda\,\theta\,a_3\alpha_0\sqrt{\lambda^2-4\mu}-2280672\,\Lambda\,\alpha_2{}^5\kappa\,\lambda^5\mu\,a_4\alpha_0\sqrt{\lambda^2-4\mu}-1027872\,\alpha_2{}^5\kappa\,\lambda^5\mu\,\theta\,a_4\alpha_0\sqrt{\lambda^2-4\mu}$$

$$+\,6071904\,\Lambda\,\alpha_2{}^5\kappa\,\lambda^3\mu^2 a_4\alpha_0\sqrt{\lambda^2-4\mu}+3033216\,\alpha_2{}^5\kappa\,\lambda^3\mu^2\theta\,a_4\alpha_0\sqrt{\lambda^2-4\mu}$$

$$-\,3256416\,\Lambda\,\alpha_2{}^4\kappa\,\lambda^3\mu\,a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu}+896640\,\Lambda\,\alpha_2{}^5\kappa\,\lambda\,\mu^3 a_4\alpha_0\sqrt{\lambda^2-4\mu}$$

$$+\,16224\,\Lambda\,\alpha_2{}^5\lambda^3\mu\,\theta\,a_4\alpha_0\sqrt{\lambda^2-4\mu}-1755648\,\alpha_2{}^4\kappa\,\lambda^3\mu\,\theta\,a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu}$$

$$-\,1984512\,\alpha_2{}^5\kappa\,\lambda\,\mu^3\theta\,a_4\alpha_0\sqrt{\lambda^2-4\mu}+12604032\,\Lambda\,\alpha_2{}^4\kappa\,\lambda\,\mu^2 a_4\alpha_0{}^2\sqrt{\lambda^2}$$

$$-\,4\,\mu-86016\,\Lambda\,\alpha_2{}^5\lambda\,\mu^2\theta\,a_4\alpha_0\sqrt{\lambda^2-4\mu}+6725376\,\alpha_2{}^4\kappa\,\lambda\,\mu^2\theta\,a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu}$$

$$-\,13046400\,\Lambda\,\alpha_2{}^3\kappa\,\lambda\,\mu\,a_4\alpha_0{}^3\sqrt{\lambda^2-4\mu}-4907520\,\alpha_2{}^3\kappa\,\lambda\,\mu\,\theta\,a_4\alpha_0{}^3\sqrt{\lambda^2-4\mu}$$

$$-\,272833920\,\alpha_2{}^4\kappa^2\lambda^3\mu^2 a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu}+407376\,\alpha_2{}^4\kappa\,\lambda^5\theta\,a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu}$$

$$-\,498432\,\alpha_2{}^6\kappa\,\lambda\,\mu^4\theta\,a_4\sqrt{\lambda^2-4\mu}+4944\,\Lambda^2\alpha_2{}^5\lambda^3\mu\,a_4\alpha_0\sqrt{\lambda^2-4\mu}$$

$$-\,16320\,\Lambda\,\alpha_2{}^6\lambda\,\mu^3\theta\,a_4\sqrt{\lambda^2-4\mu}+98720640\,\alpha_2{}^3\kappa^2\lambda^3\mu\,a_4\alpha_0{}^3\sqrt{\lambda^2-4\mu}$$

$$+\,177603840\,\alpha_2{}^4\kappa^2\lambda\,\mu^3 a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu}+6528\,\alpha_2{}^5\lambda^3\mu\,\theta^2 a_4\alpha_0\sqrt{\lambda^2-4\mu}$$

$$- 37968\,\Lambda^2 {\alpha_2}^5 \lambda\, \mu^2 a_4 \alpha_0 \sqrt{\lambda^2-4\,\mu} - 108000\,\Lambda\, {\alpha_2}^3 \kappa\, \lambda^3 a_4 {\alpha_0}^3 \sqrt{\lambda^2-4\,\mu}$$

$$- 2160\,\Lambda\, {\alpha_2}^4 \lambda^3 \theta\, a_4 {\alpha_0}^2 \sqrt{\lambda^2-4\,\mu} - 352961280\, {\alpha_2}^3 \kappa^2 \lambda\, \mu^2 a_4 {\alpha_0}^3 \sqrt{\lambda^2-4\,\mu}$$

$$+ 190080\, {\alpha_2}^3 \kappa\, \lambda^3 \theta\, a_4 {\alpha_0}^3 \sqrt{\lambda^2-4\,\mu} - 26880\, {\alpha_2}^5 \lambda\, \mu^2 \theta^2 a_4 \alpha_0 \sqrt{\lambda^2-4\,\mu}$$

$$+ 174096\,\Lambda^2 {\alpha_2}^4 \lambda\, \mu\, a_4 {\alpha_0}^2 \sqrt{\lambda^2-4\,\mu} + 202176000\, {\alpha_2}^2 \kappa^2 \lambda\, \mu\, a_4 {\alpha_0}^4 \sqrt{\lambda^2-4\,\mu}$$

$$+ 27648\, {\alpha_2}^4 \lambda\, \mu\, \theta^2 a_4 {\alpha_0}^2 \sqrt{\lambda^2-4\,\mu} + 3369600\,\Lambda\, {\alpha_2}^2 \kappa\, \lambda\, a_4 {\alpha_0}^4 \sqrt{\lambda^2-4\,\mu} - 44928\,\Lambda\, {\alpha_2}^3 \lambda\, \theta\, a_4 {\alpha_0}^3 \sqrt{\lambda^2-4\,\mu}$$

$$+ 1036800\, {\alpha_2}^2 \kappa\, \lambda\, \theta\, a_4 {\alpha_0}^4 \sqrt{\lambda^2-4\,\mu} - 393120\, {\alpha_2}^4 \kappa\, \lambda\, \mu\, a_3 \alpha_0 \sqrt{\lambda^2-4\,\mu}$$

$$- 14928\,\Lambda\, {\alpha_2}^6 \kappa\, \lambda^7 \mu\, a_4 \sqrt{\lambda^2-4\,\mu} + 7736784\, {\alpha_2}^5 \kappa^2 \lambda^7 \mu\, a_4 \alpha_0 \sqrt{\lambda^2-4\,\mu} - 49488\, {\alpha_2}^6 \kappa\, \lambda^7 \mu\, \theta\, a_4 \sqrt{\lambda^2-4\,\mu}$$

$$- 33936\,\Lambda\, {\alpha_2}^5 \kappa\, \lambda^7 a_4 \alpha_0 \sqrt{\lambda^2-4\,\mu} + 725760\,\Lambda\, {\alpha_2}^6 \kappa\, \lambda^5 \mu^2 a_4 \sqrt{\lambda^2-4\,\mu}$$

$$- 80550864\, {\alpha_2}^5 \kappa^2 \lambda^5 \mu^2 a_4 \alpha_0 \sqrt{\lambda^2-4\,\mu} + 33600\, {\alpha_2}^5 \kappa\, \lambda^7 \theta\, a_4 \alpha_0 \sqrt{\lambda^2-4\,\mu}$$

$$+ 523152\, {\alpha_2}^6 \kappa\, \lambda^5 \mu^2 \theta\, a_4 \sqrt{\lambda^2-4\,\mu} - 2309664\,\Lambda\, {\alpha_2}^6 \kappa\, \lambda^3 \mu^3 a_4 \sqrt{\lambda^2-4\,\mu}$$

$$+ 7824\,\Lambda\, {\alpha_2}^6 \lambda^5 \mu\, \theta\, a_4 \sqrt{\lambda^2-4\,\mu} + 85620240\, {\alpha_2}^4 \kappa^2 \lambda^5 \mu\, a_4 {\alpha_0}^2 \sqrt{\lambda^2-4\,\mu}$$

$$+ 213765504\, {\alpha_2}^5 \kappa^2 \lambda^3 \mu^3 a_4 \alpha_0 \sqrt{\lambda^2-4\,\mu} - 1247232\, {\alpha_2}^6 \kappa\, \lambda^3 \mu^3 \theta\, a_4 \sqrt{\lambda^2-4\,\mu}$$

$$+ 1289952\,\Lambda\, {\alpha_2}^4 \kappa\, \lambda^5 a_4 {\alpha_0}^2 \sqrt{\lambda^2-4\,\mu} - 2115840\,\Lambda\, {\alpha_2}^6 \kappa\, \lambda\, \mu^4 a_4 \sqrt{\lambda^2-4\,\mu}$$

$$- 7104\,\Lambda\, {\alpha_2}^5 \lambda^5 \theta\, a_4 \alpha_0 \sqrt{\lambda^2-4\,\mu} - 14064\,\Lambda\, {\alpha_2}^6 \lambda^3 \mu^2 \theta\, a_4 \sqrt{\lambda^2-4\,\mu} + 143424\,\Lambda\, {\alpha_2}^4 \lambda\, \mu\, \theta\, a_4 {\alpha_0}^2 \sqrt{\lambda^2-4\,\mu}$$

$$+ 50393856\, {\alpha_2}^5 \kappa^2 \lambda\, \mu^4 a_4 \alpha_0 \sqrt{\lambda^2-4\,\mu}.$$

Its worth to mention that all the obtained parameter values are laced with δ, α_0, α_2, a_3, a_4, θ, κ, Λ as free parameters.

Thus, using (7) and (5) in conjunction with $\tau = -\omega t + x$, $F(\tau) = Q(x,\ t)$, along with the solution of equation (6), we have found the following solution structures for equation (2):

Concerning the instance $\lambda^2 - 4\mu > 0$,

$$Q(x,\ t)=\alpha_0+\left(\frac{\sqrt{\lambda^2-4\mu}\left(w_1\sinh\left(\frac{1}{2}(x-\omega t)\sqrt{\lambda^2-4\mu}\right)+w_2\cosh\left(\frac{1}{2}(x-\omega t)\sqrt{\lambda^2-4\mu}\right)\right)}{2w_2\sinh\left(\frac{1}{2}(x-\omega t)\sqrt{\lambda^2-4\mu}\right)+2w_1\cosh\left(\frac{1}{2}(x-\omega t)\sqrt{\lambda^2-4\mu}\right)}-\frac{\lambda}{2}\right)^2\alpha_2$$

$$+\left(\frac{2\sqrt{\lambda^2-4\mu}\left(w_1\sinh\left(\frac{1}{2}(x-\omega t)\sqrt{\lambda^2-4\mu}\right)+w_2\cosh\left(\frac{1}{2}(x-\omega t)\sqrt{\lambda^2-4\mu}\right)\right)}{2w_2\sinh\left(\frac{1}{2}(x-\omega t)\sqrt{\lambda^2-4\mu}\right)+2w_1\cosh\left(\frac{1}{2}(x-\omega t)\sqrt{\lambda^2-4\mu}\right)}-\lambda\right) \tag{8}$$

$$\left(\frac{\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}\right)\alpha_2,$$

equipped with $w_2,\ w_1$ as free parameters.

Case-1 Significantly, we have observed that considering $w_1=0$ in (8), results into singular solitary waves as:

$$Q(x,\ t)=\alpha_0+\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\coth\left(\frac{1}{2}(-\omega t+x)\sqrt{\lambda^2-4\mu}\right)-\frac{\lambda}{2}\right)^2\alpha_2$$
$$+\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\coth\left(\frac{1}{2}(-\omega t+x)\sqrt{\lambda^2-4\mu}\right)-\frac{\lambda}{2}\right)\left(\lambda+\sqrt{\lambda^2-4\mu}\right)\alpha_2. \tag{9}$$

Case-2 Significantly, we have observed that considering $w_2=0$ in (8), leads to shock wave solutions:

$$Q(x,\ t)=\alpha_0+\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}(-\omega t+x)\sqrt{\lambda^2-4\mu}\right)-\frac{\lambda}{2}\right)^2\alpha_2$$
$$+\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}(-\omega t+x)\sqrt{\lambda^2-4\mu}\right)-\frac{\lambda}{2}\right)\left(\lambda+\sqrt{\lambda^2-4\mu}\right)\alpha_2. \tag{10}$$

2.2 *Case-II: m = 2*

Taking $m=2$ into consideration for equation (3), leads to following equation for exploring solutions of perturbed Gardners equation (2):

$$\left((-\nu F-\theta)F''-\xi F'''+a_1F+a_5F^2-\delta(F)^2-\omega\right)F'+\left(a_4\omega^2-\Lambda F-a_3\omega+a_2\right)F'''+(-\kappa F-\psi)F'''''=0. \tag{11}$$

The following solution structure for equation (11) is produced as a result of homogeneous balancing between the highest order linear and most nonlinear derivative-containing terms:

$$F(\tau) = \alpha_0 + \alpha_1 \left(\frac{G'(\tau)}{G(\tau)}\right) + \alpha_2 \left(\frac{G'(\tau)}{G(\tau)}\right)^2, \tag{12}$$

with $\alpha_i,\ i = 0,\ 1,\ 2$ are constants, to be determined in the mean process of computation and $G(\tau)$ is following the auxiliary equation (6).

Then after, with the help of (12) and (6) in equation (11), we got an opportunity to find following values for involved parameters:

$$\alpha_1 = \left(\lambda + \sqrt{\lambda^2 - 4\mu}\right)\alpha_2,\ \ \nu = -\frac{60\kappa}{\alpha_2},\ \ \omega = -\frac{1}{14}\frac{Y_4}{{\alpha_2}^2},$$

$$a_5 = \frac{1}{7}\,\frac{-285\,\kappa\lambda^2\alpha_2 + 720\left(\lambda/2 + 1/2\sqrt{\lambda^2 - 4\mu}\right)\alpha_2\kappa\lambda + 7\,\delta\,{\alpha_2}^2 + 420\,\kappa\mu\,\alpha_2 + 39\,\Lambda\,\alpha_2 - 720\,\kappa\,\alpha_0 + 12\,\theta\,\alpha_2}{{\alpha_2}^2},$$

$$\psi = \frac{37\,\kappa\lambda^2\alpha_2}{168} + \frac{1}{7}\left(\lambda + \sqrt{\lambda^2 - 4\mu}\right)\alpha_2\kappa\lambda - \frac{7}{6}\kappa\mu\,\alpha_2 - \frac{\Lambda\alpha_2}{56} - \frac{2}{7}\kappa\,\alpha_0 - \frac{\alpha_2\theta}{84},$$

$$\xi = -10\,\kappa\sqrt{\lambda^2 - 4\mu},\ \ a_1 = \frac{Y_5}{7{\alpha_2}^2},\ \ a_2 = -\frac{Y_6}{1176{\alpha_2}^4}, \tag{13}$$

with

$$Y_4 = -64\,{\alpha_2}^2\mu\,\theta\,\alpha_0 - 8\,\Lambda\lambda^2\alpha_0{\alpha_2}^2 - 124\,\Lambda\mu\,\alpha_0\,{\alpha_2}^2 - 18\,\Lambda\lambda^2\mu\,{\alpha_2}^3 + 4\,{\alpha_2}^2\lambda^2\theta\,\alpha_0 + 4680\,\kappa\mu\,{\alpha_0}^2\alpha_2$$

$$-90\,\kappa\lambda^2{\alpha_0}^2\alpha_2 - 2608\,\kappa\mu^2\alpha_0{\alpha_2}^2 - 928\,\kappa\lambda^4\alpha_0{\alpha_2}^2 - 2810\,\kappa\lambda^2\mu^2{\alpha_2}^3 - 12\,\lambda^2\mu\,\theta\,{\alpha_2}^3 + 922\,\kappa\lambda^4\mu\,{\alpha_2}^3$$

$$+4\,\Lambda{\alpha_2}^3\lambda^3\sqrt{\lambda^2 - 4\mu} - 2\,{\alpha_2}^3\lambda^3\theta\sqrt{\lambda^2 - 4\mu} + 104\,{\alpha_2}^3\kappa\lambda^5\sqrt{\lambda^2 - 4\mu} - 1440\,{\alpha_0}^3\kappa$$

$$+13\,\Lambda\,\lambda^4{\alpha_2}^3 + 4\,\lambda^4\theta\,{\alpha_2}^3 - 7\,\kappa\lambda^6{\alpha_2}^3 + 8\,\mu^2\theta\,{\alpha_2}^3 - 1624\,\kappa\mu^3{\alpha_2}^3$$

$$+24\,\alpha_2\theta\,{\alpha_0}^2 - 58\,\Lambda\mu^2{\alpha_2}^3 + 78\,\Lambda{\alpha_0}^2\alpha_2 + 3284\,\kappa\lambda^2\mu\,\alpha_0{\alpha_2}^2 - 202\,{\alpha_2}^3\kappa\lambda^3\mu\sqrt{\lambda^2 - 4\mu}$$

$$+90\,{\alpha_2}^2\kappa\lambda^3\alpha_0\sqrt{\lambda^2 - 4\mu} + 1304\,{\alpha_2}^3\kappa\lambda\,\mu^2\sqrt{\lambda^2 - 4\mu} + 62\,\Lambda{\alpha_2}^3\lambda\,\mu\sqrt{\lambda^2 - 4\mu}$$

$$+32\,{\alpha_2}^3\lambda\,\mu\,\theta\sqrt{\lambda^2 - 4\mu} - 78\,\Lambda{\alpha_2}^2\lambda\,\alpha_0\sqrt{\lambda^2 - 4\mu}$$

$$+2160\,\alpha_2\kappa\lambda\,{\alpha_0}^2\sqrt{\lambda^2 - 4\mu} - 24\,{\alpha_2}^2\lambda\,\theta\,\alpha_0\sqrt{\lambda^2 - 4\mu} - 4680\,{\alpha_2}^2\kappa\lambda\,\mu\,\alpha_0\sqrt{\lambda^2 - 4\mu},$$

$$
\begin{aligned}
Y_5 = {} & 15\,\alpha_2{}^2\kappa\lambda^3\sqrt{\lambda^2-4\mu}+344\,\kappa\lambda^4\alpha_2{}^2+1380\,\alpha_2{}^2\kappa\lambda\,\mu\,\sqrt{\lambda^2-4\mu}-1282\,\kappa\,\lambda^2\mu\,\alpha_2{}^2+39\,\Lambda\,\alpha_2{}^2\lambda\,\sqrt{\lambda^2-4\mu}\\
& +4\,\Lambda\lambda^2\alpha_2{}^2-1440\,\alpha_2\kappa\lambda\,\alpha_0\sqrt{\lambda^2-4\mu}+12\,\alpha_2{}^2\lambda\,\theta\,\sqrt{\lambda^2-4\mu}-30\,\kappa\lambda^2\alpha_0\alpha_2+1064\,\kappa\mu^2\alpha_2{}^2-2\,\lambda^2\theta\,\alpha_2{}^2\\
& +62\,\Lambda\mu\,\alpha_2{}^2-2760\,\kappa\mu\,\alpha_0\alpha_2+32\,\mu\,\theta\,\alpha_2{}^2-78\,\Lambda\,\alpha_0\alpha_2+1440\,\kappa\,\alpha_0{}^2-24\,\theta\,\alpha_0\alpha_2,\\
Y_6 = {} & -6144\,\mu^3\theta^2a_4\alpha_0\alpha_2{}^5-118402560\,\kappa^2\mu^3a_4\alpha_0{}^3\alpha_2{}^3+44078040\,\kappa^2\lambda^4a_4\alpha_0{}^4\,\alpha_2{}^2-50393856\,\kappa^2\mu^4a_4\alpha_0{}^2\alpha_2{}^4\\
& -672\,\Lambda\lambda^2a_3\alpha_0\,\alpha_2{}^4+20184\,\Lambda^2\mu^4a_4\alpha_2{}^6+7918944\,\kappa^2\lambda^8a_4\alpha_0{}^2\alpha_2{}^4+52254744\,\kappa^2\lambda^4\mu^4a_4\alpha_2{}^6\\
& +13951296\,\kappa^2\lambda^2\mu^5a_4\alpha_2{}^6-1440\,\lambda^6\mu\,\theta^2a_4\alpha_2{}^6+4704\,\lambda^4\theta^2a_4\alpha_0{}^2\alpha_2{}^4+22464\,\Lambda\,\theta\,a_4\alpha_0{}^4\alpha_2{}^2\\
& -1347840\,\Lambda\,\kappa\,a_4\alpha_0{}^5\alpha_2+861\,\Lambda\lambda^2\alpha_2{}^5-2268\,\Lambda\mu\,\alpha_2{}^5-1176\,\Lambda\,\alpha_0\alpha_2{}^4+12441600\,\kappa^2a_4\alpha_0{}^6-5824\,\kappa\lambda^2\mu\,\alpha_2{}^5\\
& +43680\,\kappa\mu\,\alpha_0\alpha_2{}^4-10920\,\kappa\lambda^2\alpha_0\alpha_2{}^4+1152\,\lambda^2\theta^2a_4\alpha_0{}^3\alpha_2{}^3+176480640\,\kappa^2\mu^2a_4\alpha_0{}^4\alpha_2{}^2\\
& +86304\,\Lambda^2\mu^3a_4\alpha_0\alpha_2{}^5+36504\,\Lambda^2a_4\alpha_0{}^4\alpha_2{}^2+1092\,\Lambda\lambda^4a_3\alpha_2{}^5\\
& -7560\,\kappa\lambda^2a_3\alpha_0{}^2\alpha_2{}^3+336\,\lambda^2\theta\,a_3\alpha_0\alpha_2{}^4-236040\,\kappa\,\lambda^2\mu^2a_3\alpha_2{}^5-120960\,\kappa\,a_3\alpha_0{}^3\alpha_2{}^2\\
& -136416\,\kappa\mu^3a_3\,\alpha_2{}^5+672\,\mu^2\theta\,a_3\alpha_2{}^5+336\,\lambda^4\theta\,a_3\alpha_2{}^5+3456\,\theta^2a_4\alpha_0{}^4\alpha_2{}^2-588\,\kappa\lambda^6a_3\alpha_2{}^5-1008\,\lambda^2\mu\,\theta\,a_3\alpha_2{}^5\\
& +15824256\,\kappa^2\mu^6a_4\alpha_2{}^6+384\,\mu^4\theta^2a_4\alpha_2{}^6+65190\,\kappa^2\lambda^{12}a_4\alpha_2{}^6+120\,\lambda^8\theta^2a_4\alpha_2{}^6+1130304\,\Lambda\,\kappa\mu^5a_4\,\alpha_2{}^6\\
& +1110\,\Lambda^2\lambda^8a_4\alpha_2{}^6+2016\,\theta\,a_3\alpha_0{}^2\alpha_2{}^3+182\,\lambda^2\theta\,\alpha_2{}^5+2093\,\kappa\lambda^4\,\alpha_2{}^5-728\,\mu\,\theta\,\alpha_2{}^5-10192\,\kappa\mu^2\alpha_2{}^5\\
& -4872\,\Lambda\mu^2a_3\alpha_2{}^5-1512\,\Lambda\lambda^2\mu\,a_3\alpha_2{}^5+528\,\Lambda\lambda^8\theta\,a_4\alpha_2{}^6+6552\,\Lambda\,a_3\alpha_0{}^2\alpha_2{}^3\\
& -10416\,\Lambda\mu\,a_3\alpha_0\alpha_2{}^4-77952\,\kappa\lambda^4a_3\alpha_0\alpha_2{}^4-5376\,\mu\,\theta\,a_3\alpha_0\alpha_2{}^4\\
& +393120\,\kappa\mu\,a_3\alpha_0{}^2\alpha_2{}^3-5568\,\Lambda\mu^4\theta\,a_4\,\alpha_2{}^6-216\,\Lambda^2\lambda^6\mu\,a_4\alpha_2{}^6\\
& -4992\,\Lambda^2\lambda^6a_4\alpha_0\alpha_2{}^5+4056\,\Lambda^2\lambda^4\mu^2a_4\alpha_2{}^6-7488\,\Lambda^2\lambda^2a_4\alpha_0{}^3\,\alpha_2{}^3\\
& +3900\,\Lambda\,\kappa\lambda^{10}a_4\alpha_2{}^6-589128\,\kappa^2\lambda^{10}\mu\,a_4\alpha_2{}^6+190272\,\kappa^2\lambda^{10}a_4\alpha_0\alpha_2{}^5
\end{aligned}
$$

$$+8217144\,\kappa^2\lambda^8\mu^2 a_4\alpha_2{}^6-2832\,\kappa\lambda^{10}\theta\, a_4\alpha_2{}^6-41603184\,\kappa^2\lambda^6\mu^3 a_4\alpha_2{}^6$$

$$+1555200\,\kappa^2\lambda^2 a_4\alpha_0{}^5\alpha_2+50824704\,\kappa^2\mu^5 a_4\alpha_0\alpha_2{}^5-155904\,\kappa\mu^5\theta\, a_4\alpha_2{}^6+768\,\lambda^6\theta^2 a_4\alpha_0\alpha_2{}^5$$

$$+10464\,\lambda^4\mu^2\theta^2 a_4\alpha_2{}^6+49056\,\Lambda^2\lambda^4 a_4\alpha_0{}^2\alpha_2{}^4-25728\,\lambda^2\mu^3\theta^2 a_4\alpha_2{}^6-414720\,\kappa\theta\, a_4\alpha_0{}^5\alpha_2$$

$$-219072\,\kappa\mu^2 a_3\alpha_0\alpha_2{}^4+26880\,\mu^2\theta^2 a_4\alpha_0{}^2\alpha_2{}^4-80870400\,\kappa^2\mu\, a_4\alpha_0{}^5\alpha_2+3456000\,\kappa^2\lambda^6 a_4\alpha_0{}^3\alpha_2{}^3$$

$$+37968\,\Lambda^2\mu^2 a_4\alpha_0{}^2\alpha_2{}^4-79728\,\Lambda^2\lambda^2\mu^3 a_4\alpha_2{}^6-18432\,\mu\,\theta^2 a_4\alpha_0{}^3\alpha_2{}^3-116064\,\Lambda^2\mu\, a_4\alpha_0{}^3\alpha_2{}^3$$

$$+77448\,\kappa\lambda^4\mu\, a_3\alpha_2{}^5-1296000\,\alpha_2{}^4\kappa^2\lambda^7 a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu}-40034304\,\alpha_2{}^6\kappa^2\lambda^3\mu^4 a_4\sqrt{\lambda^2-4\mu}$$

$$+8808\,\Lambda^2\alpha_2{}^6\lambda^5\mu\, a_4\sqrt{\lambda^2-4\mu}-25412352\,\alpha_2{}^6\kappa^2\lambda\,\mu^5 a_4\sqrt{\lambda^2-4\mu}+1824\,\alpha_2{}^6\lambda^5\mu\,\theta^2 a_4\sqrt{\lambda^2-4\mu}$$

$$-12552\,\Lambda^2\alpha_2{}^5\lambda^5 a_4\alpha_0\sqrt{\lambda^2-4\mu}-16176\,\Lambda^2\alpha_2{}^6\lambda^3\mu^2 a_4\sqrt{\lambda^2-4\mu}-25948080\,\alpha_2{}^3\kappa^2\lambda^5 a_4\alpha_0{}^3\sqrt{\lambda^2-4\mu}$$

$$-1248\,\alpha_2{}^5\lambda^5\theta^2 a_4\alpha_0\sqrt{\lambda^2-4\mu}-4800\,\alpha_2{}^6\lambda^3\mu^2\theta^2 a_4\sqrt{\lambda^2-4\mu}-43152\,\Lambda^2\alpha_2{}^6\lambda\,\mu^3 a_4\sqrt{\lambda^2-4\mu}$$

$$+3072\,\alpha_2{}^6\lambda\,\mu^3\theta^2 a_4\sqrt{\lambda^2-4\mu}+11232\,\Lambda^2\alpha_2{}^4\lambda^3 a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu}-3888000\,\alpha_2{}^2\kappa^2\lambda^3 a_4\alpha_0{}^4\sqrt{\lambda^2-4\mu}$$

$$-1728\,\alpha_2{}^4\lambda^3\theta^2 a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu}-16968\,\alpha_2{}^5\kappa\lambda^3\mu\, a_3\sqrt{\lambda^2-4\mu}-73008\,\Lambda^2\alpha_2{}^3\lambda\, a_4\alpha_0{}^3\sqrt{\lambda^2-4\mu}$$

$$-37324800\,\alpha_2\kappa^2\lambda\, a_4\alpha_0{}^5\sqrt{\lambda^2-4\mu}+7560\,\alpha_2{}^4\kappa\lambda^3 a_3\alpha_0\sqrt{\lambda^2-4\mu}+109536\,\alpha_2{}^5\kappa\lambda\,\mu^2 a_3\sqrt{\lambda^2-4\mu}$$

$$-6912\,\alpha_2{}^3\lambda\,\theta^2 a_4\alpha_0{}^3\sqrt{\lambda^2-4\mu}+5208\,\Lambda\alpha_2{}^5\lambda\,\mu\, a_3\sqrt{\lambda^2-4\mu}+2688\,\alpha_2{}^5\lambda\,\mu\,\theta\, a_3\sqrt{\lambda^2-4\mu}$$

$$-6552\,\Lambda\alpha_2{}^4\lambda\, a_3\alpha_0\sqrt{\lambda^2-4\mu}+181440\,\alpha_2{}^3\kappa\lambda\, a_3\alpha_0{}^2\sqrt{\lambda^2-4\mu}-2016\,\alpha_2{}^4\lambda\,\theta\, a_3\alpha_0\sqrt{\lambda^2-4\mu}$$

$$+1167624\,\alpha_2{}^6\kappa^2\lambda^9\mu\, a_4\sqrt{\lambda^2-4\mu}+15888\,\Lambda\alpha_2{}^6\kappa\lambda^9 a_4\sqrt{\lambda^2-4\mu}-1165704\,\alpha_2{}^5\kappa^2\lambda^9 a_4\alpha_0\sqrt{\lambda^2-4\mu}$$

$$-5851344\,\alpha_2{}^6\kappa^2\lambda^7\mu^2 a_4\sqrt{\lambda^2-4\mu}+5160\,\alpha_2{}^6\kappa\lambda^9\theta\, a_4\sqrt{\lambda^2-4\mu}+19212144\,\alpha_2{}^6\kappa^2\lambda^5\mu^3 a_4\sqrt{\lambda^2-4\mu}$$

$$-5851344\,\alpha_2{}^6\kappa^2\lambda^7\mu^2 a_4\sqrt{\lambda^2-4\mu}+5160\,\alpha_2{}^6\kappa\lambda^9\theta\, a_4\sqrt{\lambda^2-4\mu}+19212144\,\alpha_2{}^6\kappa^2\lambda^5\mu^3 a_4\sqrt{\lambda^2-4\mu}$$

$$- 120\Lambda\alpha_2{}^6\lambda^7\theta\, a_4\sqrt{\lambda^2-4\mu} + 3033216\,\alpha_2{}^5\kappa\lambda^3\mu^2\theta\, a_4\alpha_0\sqrt{\lambda^2-4\mu} - 3256416\Lambda\ \alpha_2{}^4\kappa\lambda^3\mu\, a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu}$$

$$+ 896640\Lambda\alpha_2{}^5\kappa\lambda\mu^3 a_4\alpha_0\sqrt{\lambda^2-4\mu} + 16224\Lambda\alpha_2{}^5\lambda^3\mu\,\theta\, a_4\alpha_0\sqrt{\lambda^2-4\mu}$$

$$- 1755648\,\alpha_2{}^4\kappa\lambda^3\mu\,\theta\, a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu} - 1984512\,\alpha_2{}^5\kappa\lambda\mu^3\theta\, a_4\alpha_0\sqrt{\lambda^2}$$

$$-4\mu + 12604032\Lambda\alpha_2{}^4\kappa\lambda\mu^2 a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu} - 86016\Lambda\alpha_2{}^5\lambda\mu^2\theta\, a_4\alpha_0\sqrt{\lambda^2-4\mu}$$

$$+ 6725376\,\alpha_2{}^4\kappa\lambda\mu^2\theta\, a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu} - 13046400\Lambda\ \alpha_2{}^3\kappa\lambda\mu\, a_4\alpha_0{}^3\sqrt{\lambda^2-4\mu}$$

$$+ 143424\Lambda\alpha_2{}^4\lambda\mu\,\theta\, a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu} - 4907520\,\alpha_2{}^3\kappa\lambda\mu\,\theta\, a_4\ \alpha_0{}^3\sqrt{\lambda^2}$$

$$-4\mu - 2280672\Lambda\ \alpha_2{}^5\kappa\lambda^5\mu\, a_4\alpha_0\sqrt{\lambda^2-4\mu} - 1027872\,\alpha_2{}^5\kappa\lambda^5\mu\,\theta\, a_4\alpha_0\sqrt{\lambda^2-4\mu}$$

$$+ 6071904\Lambda\alpha_2{}^5\kappa\lambda^3\mu^2 a_4\alpha_0\sqrt{\lambda^2-4\mu} + 5460\,\kappa\lambda^3\alpha_2{}^5\sqrt{\lambda^2-4\mu} + 588\Lambda\lambda\,\alpha_2{}^5\sqrt{\lambda^2-4\mu}$$

$$+ 27648\,\alpha_2{}^4\lambda\mu\,\theta^2 a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu} + 3369600\Lambda\alpha_2{}^2\kappa\lambda\, a_4\alpha_0{}^4\sqrt{\lambda^2-4\mu} - 44928\Lambda\alpha_2{}^3\lambda\,\theta\, a_4\alpha_0{}^3\sqrt{\lambda^2-4\mu}$$

$$+ 1036800\,\alpha_2{}^2\kappa\lambda\,\theta\, a_4\alpha_0{}^4\sqrt{\lambda^2-4\mu} - 393120\,\alpha_2{}^4\kappa\lambda\mu\, a_3\alpha_0\sqrt{\lambda^2-4\mu} - 14928\Lambda\alpha_2{}^6\kappa\lambda^7\mu\, a_4\sqrt{\lambda^2-4\mu}$$

$$+ 7736784\,\alpha_2{}^5\kappa^2\lambda^7\mu\, a_4\alpha_0\sqrt{\lambda^2-4\mu} - 49488\,\alpha_2{}^6\kappa\lambda^7\mu\,\theta\, a_4\sqrt{\lambda^2-4\mu} - 33936\Lambda\alpha_2{}^5\kappa\lambda^7 a_4\alpha_0\sqrt{\lambda^2-4\mu}$$

$$+ 725760\Lambda\alpha_2{}^6\kappa\lambda^5\mu^2 a_4\sqrt{\lambda^2-4\mu} - 80550864\,\alpha_2{}^5\kappa^2\ \lambda^5\mu^2 a_4\alpha_0\sqrt{\lambda^2-4\mu} + 33600\,\alpha_2{}^5\kappa\lambda^7\theta\, a_4\alpha_0\sqrt{\lambda^2-4\mu}$$

$$+ 523152\,\alpha_2{}^6\kappa\ \lambda^5\mu^2\theta\, a_4\sqrt{\lambda^2-4\mu} - 2309664\Lambda\alpha_2{}^6\kappa\lambda^3\mu^3 a_4\sqrt{\lambda^2-4\mu} + 7824\Lambda\alpha_2{}^6\ \lambda^5\mu\,\theta\, a_4\sqrt{\lambda^2-4\mu}$$

$$+ 85620240\,\alpha_2{}^4\kappa^2\lambda^5\mu\, a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu} + 213765504\,\alpha_2{}^5\kappa^2\lambda^3\mu^3 a_4\alpha_0\sqrt{\lambda^2-4\mu}$$

$$- 1247232\,\alpha_2{}^6\kappa\lambda^3\mu^3\theta\, a_4\sqrt{\lambda^2-4\mu} + 1289952\Lambda\alpha_2{}^4\kappa\lambda^5 a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu}$$

$$- 2115840\Lambda\alpha_2{}^6\kappa\lambda\mu^4 a_4\sqrt{\lambda^2-4\mu} - 7104\Lambda\alpha_2{}^5\ \lambda^5\theta\, a_4\alpha_0\sqrt{\lambda^2-4\mu} - 14064\Lambda\alpha_2{}^6\lambda^3\mu^2\theta\, a_4\sqrt{\lambda^2-4\mu}$$

$$- 272833920\,\alpha_2{}^4\kappa^2\lambda^3\mu^2 a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu} + 50393856\,\alpha_2{}^5\kappa^2\lambda\mu^4 a_4\alpha_0\sqrt{\lambda^2-4\mu}$$

$$+407376\,\alpha_2{}^4\kappa\lambda^5\theta\,a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu}-498432\,\alpha_2{}^6\kappa\lambda\,\mu^4\theta\,a_4\sqrt{\lambda^2-4\mu}+4944\,\Lambda^2\alpha_2{}^5\lambda^3\mu\,a_4\alpha_0\sqrt{\lambda^2-4\mu}$$

$$-16320\,\Lambda\,\alpha_2{}^6\lambda\,\mu^3\theta\,a_4\sqrt{\lambda^2-4\mu}+98720640\,\alpha_2{}^3\kappa^2\lambda^3\mu\,a_4\alpha_0{}^3\sqrt{\lambda^2-4\mu}$$

$$+177603840\,\alpha_2{}^4\kappa^2\lambda\,\mu^3a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu}+6528\,\alpha_2{}^5\lambda^3\mu\,\theta^2a_4\alpha_0\sqrt{\lambda^2}$$

$$-4\mu-37968\,\Lambda^2\alpha_2{}^5\lambda\,\mu^2a_4\alpha_0\sqrt{\lambda^2-4\mu}-108000\,\Lambda\,\alpha_2{}^3\kappa\lambda^3a_4\alpha_0{}^3\sqrt{\lambda^2-4\mu}$$

$$-2160\,\Lambda\,\alpha_2{}^4\lambda^3\theta\,a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu}-352961280\,\alpha_2{}^3\kappa^2\lambda\,\mu^2a_4\alpha_0{}^3\sqrt{\lambda^2-4\mu}$$

$$+190080\,\alpha_2{}^3\kappa\,\lambda^3\theta\,a_4\alpha_0{}^3\sqrt{\lambda^2-4\mu}-26880\,\alpha_2{}^5\lambda\,\mu^2\theta^2a_4\alpha_0\sqrt{\lambda^2-4\mu}+174096\,\Lambda^2\alpha_2{}^4\lambda\,\mu\,a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu}$$

$$+202176000\,\alpha_2{}^2\kappa^2\lambda\,\mu\,a_4\alpha_0{}^4\sqrt{\lambda^2-4\mu}-8736\,\alpha_2{}^6\kappa^2\lambda^{11}a_4\sqrt{\lambda^2}$$

$$-4\mu+624\,\Lambda^2\alpha_2{}^6\lambda^7a_4\sqrt{\lambda^2-4\mu}-96\,\alpha_2{}^6\lambda^7\theta^2a_4\sqrt{\lambda^2-4\mu}+8736\,\alpha_2{}^5\kappa\lambda^5a_3\sqrt{\lambda^2-4\mu}$$

$$+336\,\Lambda\,\alpha_2{}^5\,\lambda^3a_3\sqrt{\lambda^2-4\mu}-168\,\alpha_2{}^5\lambda^3\theta\,a_3\sqrt{\lambda^2-4\mu}-21840\,\kappa\lambda\,\mu\,\alpha_2{}^5\sqrt{\lambda^2-4\mu}-20352\,\lambda^2\mu\,\theta^2a_4\alpha_0{}^2\alpha_2{}^4$$

$$+1984512\,\kappa\mu^3\theta\,a_4\alpha_0{}^2\alpha_2{}^4-173776320\,\kappa^2\lambda^2\mu\,a_4\,\alpha_0{}^4\alpha_2{}^2+46464\,\lambda^2\mu^2\,\theta^2a_4\alpha_0\alpha_2{}^5+996864\,\kappa\mu^4\theta\,a_4\alpha_0\alpha_2{}^5$$

$$-962784\,\kappa\,\lambda^4\theta\,a_4\alpha_0{}^3\alpha_2{}^3+721025280\,\kappa^2\lambda^2\mu^2a_4\alpha_0{}^3\alpha_2{}^3-15168\,\lambda^4\mu\,\theta^2a_4\alpha_0\alpha_2{}^5$$

$$-919688064\,\kappa^2\lambda^2\mu^3a_4\alpha_0{}^2\alpha_2{}^4-202232160\,\kappa^2\,\lambda^4\mu\,a_4\alpha_0{}^3\alpha_2{}^3$$

$$-2038848\,\kappa\lambda^2\mu^4\theta\,a_4\alpha_2{}^6-128640\,\kappa\lambda^6\theta\,a_4\alpha_0{}^2\alpha_2{}^4+316873728\,\kappa^2\lambda^2\mu^4a_4\alpha_0\,\alpha_2{}^5$$

$$+354939984\,\kappa^2\lambda^4\mu^2a_4\alpha_0{}^2\alpha_2{}^4+1351392\,\kappa\lambda^4\,\mu^3\theta\,a_4\alpha_2{}^6-245749728\,\kappa^2\,\lambda^4\mu^3a_4\alpha_0\alpha_2{}^5$$

$$-59226864\,\kappa^2\lambda^6\mu\,a_4\alpha_0{}^2\alpha_2{}^4-556320\,\kappa\lambda^6\mu^2\theta\,a_4\alpha_2{}^6-76992\,\kappa\lambda^8\theta\,a_4\alpha_0\alpha_2{}^5$$

$$+104833248\,\kappa^2\lambda^6\mu^2a_4\alpha_0\alpha_2{}^5+100032\,\kappa\lambda^8\mu\,\theta\,a_4\alpha_2{}^6$$

$$+275856\,\kappa\lambda^2\mu\,a_3\alpha_0\alpha_2{}^4+2453760\,\kappa\mu\,\theta\,a_4\alpha_0{}^4\alpha_2{}^2-4483584\,\kappa\mu^2\theta\,a_4\,\alpha_0{}^3\alpha_2{}^3-95040\,\kappa\lambda^2\theta\,a_4\alpha_0{}^4\alpha_2{}^2$$

$$+960\,\Lambda\lambda^6\theta\,a_4\alpha_0\alpha_2{}^5-1574160\,\Lambda\,\kappa\lambda^2\mu^4a_4\alpha_2{}^6+87936\,\Lambda\,\kappa\lambda^6a_4\alpha_0{}^2\alpha_2{}^4-2304\,\Lambda\lambda^6\mu\,\theta\,a_4\alpha_2{}^6$$

$$+1032864\Lambda\kappa\lambda^4\mu^3 a_4\alpha_2{}^6-991056\Lambda\kappa\lambda^6\mu^2 a_4\alpha_2{}^6-237120\Lambda\kappa\lambda^8 a_4\alpha_0\alpha_2{}^5-95616\Lambda\mu\theta a_4\alpha_0{}^3\alpha_2{}^3$$

$$+6523200\Lambda\kappa\mu a_4\alpha_0{}^4\alpha_2{}^2+86016\Lambda\mu^2\theta a_4\alpha_0{}^2\alpha_2{}^4+1440\Lambda\lambda^2\theta a_4\alpha_0{}^3\alpha_2{}^3-8402688\Lambda\kappa\mu^2 a_4\alpha_0{}^3\alpha_2{}^3$$

$$-17051328\kappa^2\lambda^8\mu a_4\alpha_0\alpha_2{}^5+3935232\kappa\lambda^2\mu\theta a_4\alpha_0{}^3\alpha_2{}^3-12848256\kappa\lambda^2\mu^2\theta a_4\alpha_0{}^2\alpha_2{}^4$$

$$+11461632\kappa\lambda^2\mu^3\theta a_4\alpha_0\alpha_2{}^5+3861792\kappa\lambda^4\mu\theta a_4\alpha_0{}^2\alpha_2{}^4-4523040\kappa\lambda^4\mu^2\theta a_4\alpha_0\alpha_2{}^5$$

$$+674400\kappa\lambda^6\mu\theta a_4\alpha_0\alpha_2{}^5-106080\Lambda\lambda^2\mu\theta a_4\alpha_0{}^2\alpha_2{}^4+11156544\Lambda\kappa\lambda^2\mu a_4\alpha_0{}^3\alpha_2{}^3$$

$$+219360\Lambda\lambda^2\mu^2\theta a_4\alpha_0\alpha_2{}^5-32164704\Lambda\kappa\lambda^2\mu^2 a_4\alpha_0{}^2\alpha_2{}^4-66336\Lambda\lambda^4\mu\theta a_4\alpha_0\alpha_2{}^5$$

$$+21424704\Lambda\kappa\lambda^2\mu^3 a_4\alpha_0\alpha_2{}^5+8587872\Lambda\kappa\lambda^4\mu a_4\alpha_0{}^2\alpha_2{}^4-6400512\Lambda\kappa\lambda^4\mu^2 a_4\alpha_0\alpha_2{}^5$$

$$+1038144\Lambda\kappa\lambda^6\mu a_4\alpha_0\alpha_2{}^5+193056\Lambda\kappa\lambda^8\mu a_4\alpha_2{}^6+54000\Lambda\kappa\lambda^2 a_4\alpha_0{}^4\alpha_2{}^2+32640\Lambda\mu^3\theta a_4\alpha_0\alpha_2{}^5$$

$$-896640\Lambda\kappa\mu^3 a_4\alpha_0{}^2\alpha_2{}^4-150960\Lambda^2\lambda^2\mu a_4\alpha_0{}^2\alpha_2{}^4+29568\Lambda\lambda^4\theta a_4\alpha_0{}^2\alpha_2{}^4+4231680\Lambda\kappa\mu^4 a_4\alpha_0\alpha_2{}^5$$

$$-3106368\Lambda\kappa\lambda^4 a_4\alpha_0{}^3\alpha_2{}^3+264480\Lambda^2\lambda^2\mu^2 a_4\alpha_0\alpha_2{}^5-88608\Lambda\lambda^2\mu^3\theta a_4\alpha_2{}^6-60672\Lambda^2\lambda^4\mu a_4\alpha_0\alpha_2{}^5$$

$$+24672\Lambda\lambda^4\mu^2\theta a_4\alpha_2{}^6,$$

equipped with $a_3,\ a_4,\ \alpha_0,\ \alpha_2,\ \Lambda,\ \delta,\ \kappa,\ \theta$ as free parameters.

Using solution of equation (6) along with parameter values (13) and (12), we may write the subsequent solution for equation (2), by reverting back to original variables $x,\ t$:

In relation to instance $\lambda^2-4\mu>0$,

$$Q(x,\ t)=\left(\frac{\sqrt{\lambda^2-4\mu}\left(w_1\sinh\left(\frac{1}{2}(-\omega t+x)\sqrt{\lambda^2-4\mu}\right)+w_2\cosh\left(\frac{1}{2}(-\omega t+x)\sqrt{\lambda^2-4\mu}\right)\right)}{2w_2\sinh\left(\frac{1}{2}(-\omega t+x)\sqrt{\lambda^2-4\mu}\right)+2w_1\cosh\left(\frac{1}{2}(-\omega t+x)\sqrt{\lambda^2-4\mu}\right)}-\frac{\lambda}{2}\right)^2\alpha_2+\alpha_0$$

$$+\left(\frac{\sqrt{\lambda^2-4\mu}\left(w_1\sinh\left(\frac{1}{2}(-\omega t+x)\sqrt{\lambda^2-4\mu}\right)+w_2\cosh\left(\frac{1}{2}(-\omega t+x)\sqrt{\lambda^2-4\mu}\right)\right)}{2w_2\sinh\left(\frac{1}{2}(-\omega t+x)\sqrt{\lambda^2-4\mu}\right)+2w_1\cosh\left(\frac{1}{2}(-\omega t+x)\sqrt{\lambda^2-4\mu}\right)}-\frac{\lambda}{2}\right)$$

$$\left(\lambda+\sqrt{\lambda^2-4\mu}\right)\alpha_2, \tag{14}$$

with w_1, w_2 as arbitrary parameters.

The solution (14) results into shock waves for $w_1 \neq 0$, $w_2 = 0$, while for $w_2 \neq 0$, $w_1 = 0$, obtains singular solitary waves.

3. Results and discussion

Figures 1 and 2 illustrate the characteristics of singular solitary waves and shock waves, respectively, by utilizing the solutions provided in equations (9) and (10). These figures allow for a comprehensive analysis of wave dynamics by examining their surface plots, contour plots, and 2D projections at a fixed time value of $t = 1$. The parameters governing the wave structures are set to specific values: $\lambda = 3$, $\mu = 1$, $\alpha_0 = 1$, $\alpha_2 = 1$, $\kappa = 1$, $\theta = 1$, and $\Lambda = 1$.

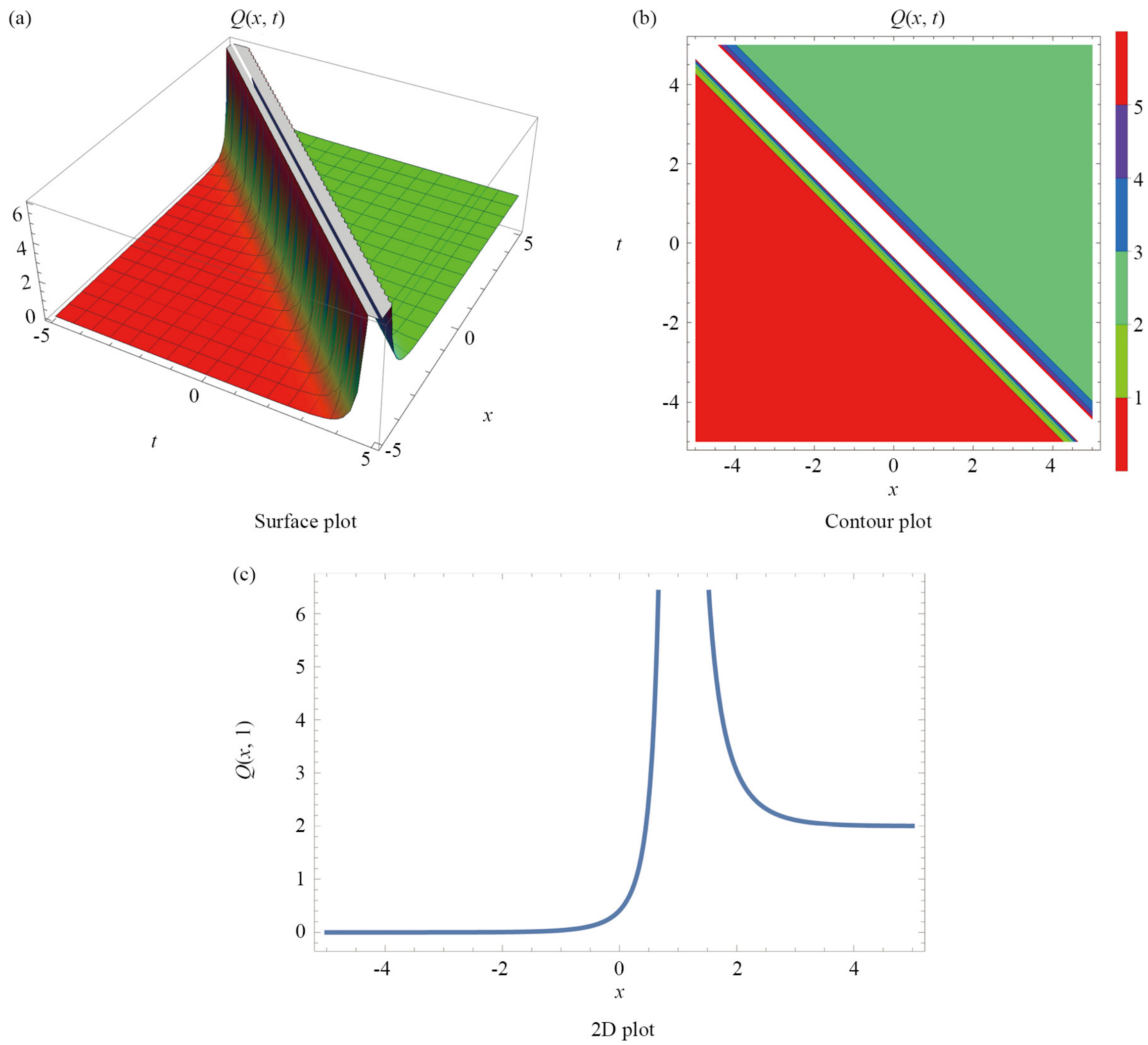

Figure 1. Exploring the features of a singular solitary wave

Figure 1 presents the singular solitary wave described by the solution (9). This wave exhibits distinct properties characterized by a singularity, where the function diverges. Singular solitary waves are significant in various physical systems where infinite energy localization or abrupt transitions occur, such as in plasma physics, optical fibers, and fluid dynamics. In Figure 1a, the surface plot provides a three-dimensional representation of the singular solitary wave over the spatial and temporal domain. The wave exhibits a steep gradient near its singularity, showing how the amplitude increases significantly in this region. The singularity is visually evident as the surface appears to approach infinite values, differentiating it from regular smooth solitary waves. This steep increase in amplitude suggests that singular solitary waves may be associated with extreme energy localization, which can impact wave propagation and stability. In Figure 1b, the contour plot further emphasizes the singular nature of the wave by showing closely packed contour lines near the singularity. The variation in intensity and spacing of the contours highlight the rapid changes in wave amplitude. The density of contour lines in the singular region indicates the abrupt variation in wave amplitude, which is a signature feature of singular solitary waves. The contrast between the high-amplitude and low-amplitude regions suggests a sharp energy concentration, a characteristic often seen in nonlinear wave interactions. In Figure 1c, the 2D representation of the singular solitary wave clearly shows the divergence behavior. Unlike regular solitary waves, the singular solitary wave does not settle into a stable peak; instead, it rises steeply near its singular point, confirming its unique structure. This behavior implies that singular solitary waves cannot be treated with standard perturbation methods used for smooth waves, as the singularity introduces complications in numerical modeling and analytical treatments.

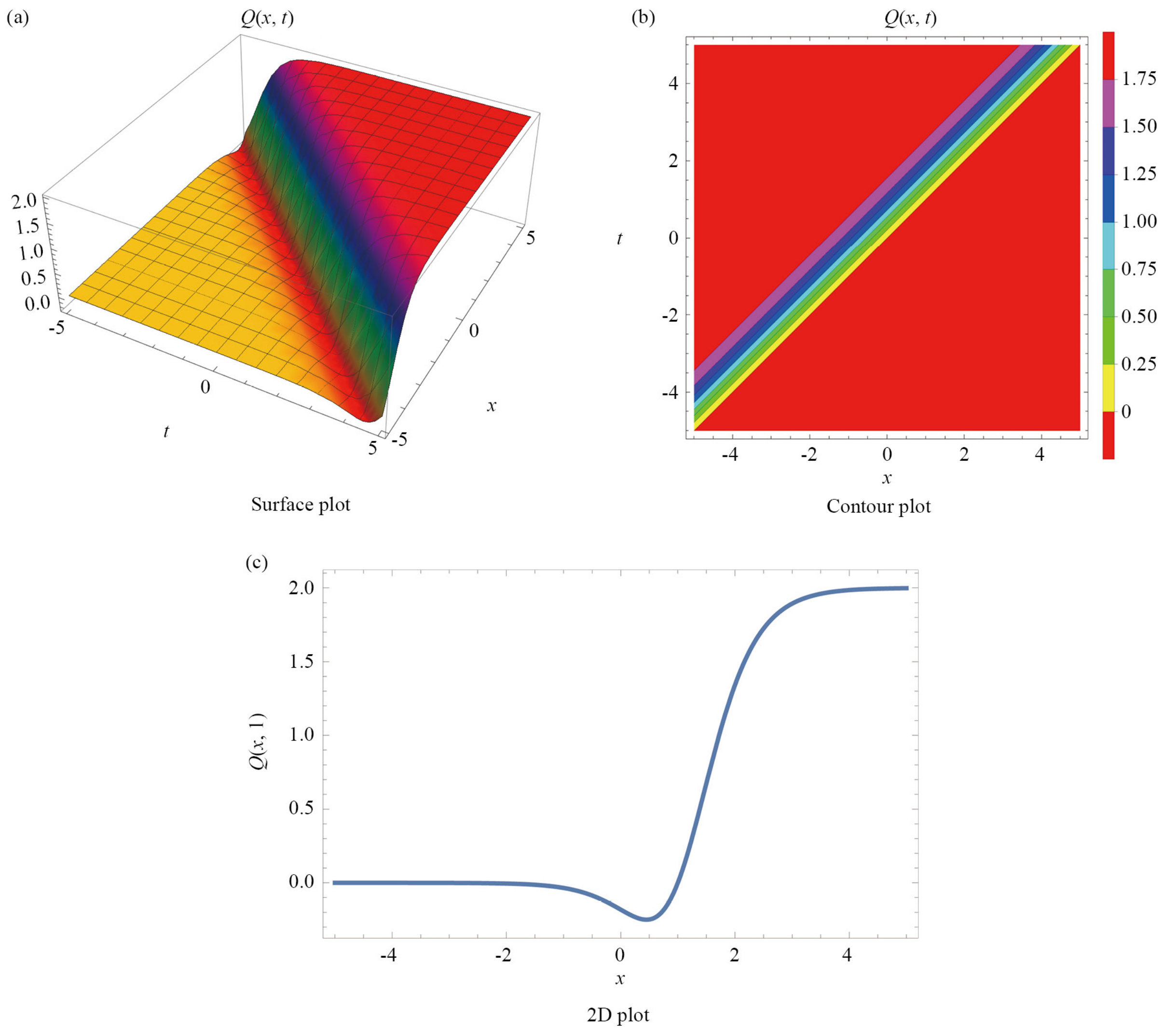

Figure 2. Exploring the features of a shock wave

Figure 2 represents the shock wave solution (10), which is characterized by a smooth yet abrupt transition between two distinct states. Unlike the singular solitary wave, the shock wave maintains finite amplitude at all points, making it a fundamental solution in wave propagation problems. Shock waves arise in various physical systems, including fluid dynamics, where discontinuous transitions in system properties occur. In Figure 2a, the surface plot of the shock wave reveals a smooth transition between two asymptotic states. Unlike the singular solitary wave, the shock wave exhibits a stable wavefront that moves through the medium without singular behavior. The wave amplitude changes sharply but remains bounded, ensuring a more predictable and physically realizable behavior compared to singular solitary waves. The surface plot shows a characteristic sigmoidal transition, where the wave smoothly evolves from one plateau to another, marking the key distinction between shock waves and solitary waves. In Figure 2b, the contour plot illustrates the structure of the shock wave transition. The contours are evenly spaced near the extremes and become denser in the transition region, which visually confirms the steep but finite gradient of the wave. The sharp gradient in the middle region signifies the presence of a wavefront, which is a fundamental aspect of shock wave behavior. The contour plot captures the essence of wave propagation in shock dynamics, where the wave maintains its form while propagating. In Figure 2c, the 2D plot of the shock wave demonstrates the sigmoidal profile typical of hyperbolic tangent functions. It clearly indicates the presence of a well-defined wavefront with a smooth but rapid shift in amplitude, marking the key characteristic of shock waves. Unlike singular solitary waves, shock waves exhibit a controlled and finite transition, making them important in applications involving signal transmission, energy transport, and aerodynamics.

The distinction between the singular solitary wave and the shock wave is evident from Figures 1 and 2. The singular solitary wave, given by the hyperbolic cotangent function, exhibits an unbounded growth near singular points, whereas the shock wave, described by the hyperbolic tangent function, smoothly transitions between two steady states. The presence of singularity in Figure 1 demonstrates the extreme behavior of singular solitary waves, which can have implications in physical systems where energy localization and wave-breaking phenomena occur. In contrast, the shock wave in Figure 2 represents a common structure in nonlinear wave dynamics, where rapid but continuous transitions are observed. A key difference between these two types of waves is in their physical applicability. Singular solitary waves, due to their infinite peaks, are often idealized representations that require careful handling in physical models. They may be associated with scenarios such as rogue waves in oceanography or extreme electromagnetic pulses in optics. On the other hand, shock waves have a more direct physical interpretation, as they represent realistic transitions in compressible fluids, supersonic flows, and even biological wave propagation in nerve signals. Furthermore, the stability and propagation characteristics of these waves differ significantly. Singular solitary waves tend to exhibit instability due to their unbounded nature, requiring special mathematical techniques to analyze their evolution and interactions. Shock waves, however, remain stable under various perturbations, making them more applicable in engineering and scientific applications where predictable wave behavior is essential.

In summary, the figures successfully illustrate the fundamental differences between these two types of nonlinear waves. The singular solitary wave is characterized by its singularity and steep gradient, while the shock wave exhibits a continuous but abrupt transition between states. These properties play crucial roles in various applications, from fluid dynamics to nonlinear optics, where wave behavior significantly impacts system dynamics. The analysis of these wave structures provides deeper insights into the mathematical modeling of nonlinear wave phenomena, emphasizing the need for specialized techniques to handle different types of wave solutions [23, 24].

4. Conclusions

This paper addressed the shallow water wave dynamics perturbation with Gardners equation. The model was successfully addressed by the G'/G-expansion approach. The scheme recovered shock waves and singular solitary wave solutions that could potentially model rogue waves. The results are indeed promising since one can, in future take a look at double-layered shallow water wave dynamics that is also modeled by the same Gardners equation and make an attempt to unravel the hidden form of waves that are supported by that model. The recovery of conservation laws and the application of soliton perturbation theory for the two-layered flow are yet to be addressed. Such studies are all under way. They are going to be recovered and disseminated all across the board.

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Conflict of interest

The authors claim that there is no conflict of interest.

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