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Research Article

Perturbation of Solitary Waves and Shock Waves with Dispersion Triplet

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Abstract: This paper recovers the solitary waves and shock waves from perturbed shallow water wave dynamics. The governing models considered are the KdV equation and the mKdV equation. The integration method is the G'/G-expansion method. The numerical simulations supplement the analytical results for such waves.

***Keywords*:** solitary waves, shock waves, G'/G-expansion

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1. Introduction

The dynamics of shallow water waves constitute a rich and captivating area of research within the broader field of Fluid Dynamics, due to their complex behaviors and relevance in both theoretical and applied contexts [1–10]. Numerous mathematical models have been developed over the decades to describe various aspects of shallow water wave propagation, dispersion, and nonlinearity. Among the most prominent are the Korteweg-de Vries (KdV) equation, the modified KdV (mKdV) equation, the Boussinesq equation, the Kawahara equation, the Gear-Grimshaw system, the Bona-Chen model, and the Zaeromaghaddam model, among others [11–18].

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These models can be broadly categorized based on the physical configurations they represent. The KdV, mKdV, Boussinesq, and Kawahara equations typically describe wave motion in single-layer fluid systems, such as water waves propagating along lakeshores or coastal regions. These models capture essential features such as dispersion, weak nonlinearity, and solitary wave formation. In contrast, the Gear-Grimshaw, Bona-Chen, and Zaeromaghaddam models address more complex configurations, particularly two-layered fluid systems, where interactions between different fluid layers with varying densities or viscosities are taken into account. Such models are crucial for understanding internal waves and stratified fluid flows in oceanography and atmospheric sciences.

The current paper focuses on the single-layer shallow water wave dynamics in the presence of various perturbation terms for the first time. These perturbations arise from physical effects such as topography, variable bottom boundaries, wind forcing, or dissipation, and they can be Hamiltonian (i.e., energy-conserving) or non-Hamiltonian (i.e., involving dissipation or gain). Specifically, we examine perturbed versions of the KdV and mKdV equations, which serve as canonical models for nonlinear wave evolution in shallow water settings [1, 3–5, 8, 9]

To analytically explore the solitary wave structures and shock-like solutions under the influence of these perturbations, we employ the well-established G'/G-expansion method [19, 20]. This method facilitates the systematic derivation of closed-form solutions by reducing the governing nonlinear partial differential equations to ordinary differential equations through a traveling wave transformation. The method's algebraic framework enables the construction of exact solutions, including solitons and shock waves, particularly when perturbations are treated as strong (i.e., non-negligible or dominant in the dynamics).

In the remainder of the paper, we first provide a concise but comprehensive introduction to the governing equations, followed by an in-depth discussion of their integrability properties. We then systematically derive exact solutions for the perturbed KdV and mKdV models using the G'/G-expansion method, highlighting the nature and behavior of the obtained solitary and shock wave structures under different perturbation regimes.

2. Governing model

The main structure of the model that is going to be addressed in this paper is given as

$$q_t + a_1 F(q) q_x + a_2 q_{xxx} + a_3 q_{xxt} + a_4 q_{xtt} = \theta q_x q_{xx} + \delta q^m q_x + \Lambda q q_{xxx} + \nu q q_x q_{xx} + \xi q_x q_{xxx} + \psi q_{xxxxx} + \kappa q q_{xxxxx}.$$

Here the first term is the temporal evolution of the wave while the functional F represents the nonlinearity structure. Thus for $F(q) = q$, one obtains KdV equation while for $F(q) = q^2$, the mKdV equation is obtained. This paper will therefore consider both cases along with the indicated perturbation terms. The dispersion terms are the coefficients of a_j for $j = 2, 3, 4$ which collectively constitute the dispersion triplet. From the perturbation terms, the coefficients of θ, δ, Λ are from nonlinear dispersions, while ψ is from higher order dispersion and κ means higher order nonlinear dispersion. Finally, ν and ξ are also from nonlinear higher order dispersions. This model will now be addressed in details for the KdV equation and the mKdV equation to recover the perturbed solitary and shock wave solutions. The details are exhibited and enlisted in the subsequent two subsections.

2.1 *Perturbed KdV equation*

This section is devoted to explore some different types of solutions for KdV equation with perturbation terms. It is remarkable to address that the perturbed KdV is addressed very first time in current study as follow:

$$Q_t + a_1 Q Q_x + a_2 Q_{xxx} + a_3 Q_{xxt} + a_4 Q_{xtt}$$

$$= \theta Q_x Q_{xx} + \delta Q^m q_x + \Lambda Q Q_{xxx} + \nu Q Q_x Q_{xx} + \xi Q_x Q_{xxx} + \psi Q_{xxxxx} + \kappa Q Q_{xxxxx}, \tag{1}$$

with $Q = Q(x, t)$.

Utilizing $\tau = -\omega t + x$, where ω is the speed of the wave, along with $Q(x, t) = F(\tau)$, we have recovered the following ordinary differential equation from the equation (1):

$$\left(\left(-\nu F - \theta\right) F'' - \xi F''' + a_1 F - \delta \left(F\right)^m - \omega\right) F' + \left(a_4 \omega^2 - \Lambda F - a_3 \omega + a_2\right) F'''$$

$$+ \left(-\kappa F - \psi\right) F''''' = 0. \tag{2}$$

Next, equation (2) has been examined with the help of $\left(\frac{G'}{G}\right)$-expansion method for $m = 1, 2$ to establish the new analytic solutions as in subsequent sections.

The $\left(\frac{G'}{G}\right)$-expansion method plays a significant role in the analytical treatment of Nonlinear Evolution Equations (NLEEs), particularly in soliton theory and mathematical physics [19, 20]. This method is widely regarded for its simplicity, computational efficiency, and ability to yield a broad class of exact solutions, including solitons, periodic waves, and rational-type solutions. One of the key strengths of the method lies in its algorithmic structure, which transforms a nonlinear Partial Differential Equation (PDE) into an Ordinary Differential Equation (ODE) via a wave transformation, followed by a systematic expansion in terms of $\left(\frac{G'}{G}\right)$, where G satisfies a second-order linear ODE. This approach enables the construction of closed-form solutions without requiring a priori assumptions about the form of the solution. The method is particularly advantageous for handling equations with polynomial-type nonlinearities and derivative-dependent terms. Its flexibility allows for generalization, such as the $\left(\frac{G'}{G}\right)^2$-expansion or the extended $\left(\frac{F'}{F}\right)$-expansion approaches, further enriching the solution space. In the context of the current study, the use of the $\left(\frac{G'}{G}\right)$-expansion method facilitates the derivation of various solution structures. The analytical tractability of this method makes it a valuable tool for exploring nonlinear wave phenomena in mathematical physics.

2.1.1 *Solutions to (2) for m = 1*

Firstly, we consider $m = 1$ for (2) and then equation (2) is rewritten as follows:

$$\left(\left(-\nu F - \theta\right) F'' - \xi F''' + a_1 F - \delta F - \omega\right) F' + \left(a_4 \omega^2 - \Lambda F - a_3 \omega + a_2\right) F'''$$

$$+ \left(-\kappa F - \psi\right) F''''' = 0. \tag{3}$$

Next equation (3) has been examined with the help of extended $\left(\frac{G'}{G}\right)$-expansion method for recovering the new analytic solutions as in subsequent portions of this paper: Speculating the homogeneous balance among highest order derivative term and highly non linear terms in equation (3) leads to the following solution structure for the equation (3) [19, 20]:

$$F(\tau)=\alpha_0+\alpha_1\left(\frac{G'(\tau)}{G(\tau)}\right)+\alpha_2\left(\frac{G'(\tau)}{G(\tau)}\right)^2, \tag{4}$$

with $G(\tau)$ following the auxiliary equation

$$G''+\lambda G'+\mu G=0, \tag{5}$$

It is worthy to notice that α_0, α_1, α_2 are parameters, to be evaluated in the mean process of calculations. Utilizing (4) into (3), along with (5), we furnish the subsequent parameter values for extracting the solutions to equation (3) as per mentioned details:

$$\begin{aligned}
&\Lambda=-\frac{3\left(-\kappa\lambda^2\alpha_2+8\left(\lambda+\sqrt{\lambda^2-4\mu}\right)\alpha_2\kappa\lambda-12\,\alpha_2\kappa\mu-16\,\kappa\alpha_0-16\,\psi\right)}{\alpha_2},\\
&\nu=-\frac{60\kappa}{\alpha_2},\quad \xi=-10\,\kappa\sqrt{\lambda^2-4\mu}\\
&\omega=-\frac{4\left(\lambda^2-4\mu\right)Z_1}{\alpha_2},\\
&\theta=\frac{2\left(7\,\kappa\lambda^2\alpha_2+24\left(\lambda+\sqrt{\lambda^2-4\mu}\right)\alpha_2\kappa\lambda-76\,\alpha_2\kappa\mu-48\,\kappa\alpha_0-78\,\psi\right)}{\alpha_2},\\
&a_1=-36\,\kappa\lambda^3\sqrt{\lambda^2-4\mu}-32\,\kappa\lambda^4+144\,\kappa\lambda\,\mu\sqrt{\lambda^2-4\mu}+184\,\kappa\lambda^2\mu+\frac{72\kappa\lambda^2\alpha_0}{\alpha_2}\\
&\qquad -224\,\kappa\mu^2-\frac{288\kappa\mu\,\alpha_0}{\alpha_2}+\frac{72\lambda^2\psi}{\alpha_2}+\delta-\frac{288\mu\,\psi}{\alpha_2},\\
&\alpha_1=\left(\lambda+\sqrt{\lambda^2-4\mu}\right)\alpha_2,\\
&a_2=-\frac{Z_2}{{\alpha_2}^2},
\end{aligned}\tag{6}$$

with

$$Z_1 = -4\sqrt{\lambda^2-4\mu}\alpha_2{}^2\kappa\lambda^3 - 4\kappa\alpha_2{}^2\lambda^4 - 2\sqrt{\lambda^2-4\mu}\alpha_2{}^2\kappa\lambda\mu + 6\kappa\lambda^2\mu\alpha_2{}^2 + 18\sqrt{\lambda^2-4\mu}\alpha_0\alpha_2\kappa\lambda$$

$$+8\kappa\lambda^2\alpha_0\alpha_2 + 22\alpha_2{}^2\kappa\mu^2 + 9\sqrt{\lambda^2-4\mu}\alpha_2\lambda\psi + 4\alpha_0\alpha_2\kappa\mu - 18\kappa\alpha_0{}^2 + 18\alpha_2\mu\psi - 18\psi\alpha_0,$$

$$Z_2 = 288\mu\psi a_3\alpha_0\alpha_2 - 72\lambda^2\psi a_3\alpha_0\alpha_2 + 288\kappa\mu a_3\alpha_0{}^2\alpha_2 + 72\lambda^2\mu\psi a_3\alpha_2{}^2 - 64\kappa\mu^2 a_3\alpha_0\alpha_2{}^2 - 72\kappa\lambda^2 a_3\alpha_0{}^2\alpha_2$$

$$-8\kappa\lambda^2\mu^2 a_3\alpha_2{}^3 + 32\kappa\lambda^4 a_3\alpha_0\alpha_2{}^2 + 88\kappa\lambda^4\mu a_3\alpha_2{}^3 - 114944\kappa^2\lambda^6\mu^2 a_4\alpha_0\alpha_2{}^3 + 35712\kappa^2\lambda^8\mu a_4\alpha_0\alpha_2{}^3$$

$$-165888\mu^3\psi^2 a_4\alpha_0\alpha_2 + 165888\kappa\mu^2\psi a_4\alpha_0{}^3 - 41472\lambda^2\mu\psi^2 a_4\alpha_0{}^2 - 124416\lambda^2\mu^3\psi^2 a_4\alpha_2{}^2$$

$$+10368\kappa\lambda^4\psi a_4\alpha_0{}^3 - 41472\kappa^2\lambda^2\mu a_4\alpha_0{}^4 - 36864\kappa^2\mu^3 a_4\alpha_0{}^3\alpha_2 + 67392\lambda^4\mu^2\psi^2 a_4\alpha_2{}^2$$

$$+202752\kappa\mu^5\psi a_4\alpha_2{}^3 - 15552\lambda^6\mu\psi^2 a_4\alpha_2{}^2 - 198656\kappa^2\mu^4 a_4\alpha_0{}^2\alpha_2{}^2 - 4608\kappa^2\lambda^6 a_4\alpha_0{}^3\alpha_2$$

$$+45056\kappa^2\mu^5 a_4\alpha_0\alpha_2{}^3 - 75200\kappa^2\lambda^4\mu^4 a_4\alpha_2{}^4 + 1536\kappa^2\lambda^2\mu^5 a_4\alpha_2{}^4 + 17280\kappa^2\lambda^8\mu^2 a_4\alpha_2{}^4$$

$$+4992\kappa^2\lambda^6\mu^3 a_4\alpha_2{}^4 + 8512\kappa^2\lambda^8 a_4\alpha_0{}^2\alpha_2{}^2 - 5632\kappa^2\lambda^{10}\mu a_4\alpha_2{}^4 + 27648\kappa\lambda^2\mu^2\psi a_4\alpha_0{}^2\alpha_2$$

$$-230400\kappa\lambda^2\mu^3\psi a_4\alpha_0\alpha_2{}^2 + 24192\kappa\lambda^4\mu\psi a_4\alpha_0{}^2\alpha_2 + 266112\kappa\lambda^4\mu^2\psi a_4\alpha_0\alpha_2{}^2$$

$$-79488\kappa\lambda^6\mu\psi a_4\alpha_0\alpha_2{}^2 - 5760\kappa\lambda^4\mu^3\psi a_4\alpha_2{}^3 + 153600\kappa^2\lambda^2\mu^4 a_4\alpha_0\alpha_2{}^3$$

$$+309120\kappa^2\lambda^4\mu^2 a_4\alpha_0{}^2\alpha_2{}^2 - 26496\kappa\lambda^6\mu^2\psi a_4\alpha_2{}^3 + 7488\kappa\lambda^8\psi a_4\alpha_0\alpha_2{}^2$$

$$+60160\kappa^2\lambda^4\mu^3 a_4\alpha_0\alpha_2{}^3 - 91264\kappa^2\lambda^6\mu a_4\alpha_0{}^2\alpha_2{}^2 + 10944\kappa\lambda^8\mu\psi a_4\alpha_2{}^3 - 1152\alpha_2{}^3\kappa\lambda^{10}\psi a_4$$

$$-3328\alpha_2{}^3\kappa^2\lambda^{10}a_4\alpha_0 + 512\alpha_2{}^4\kappa^2\lambda^{11}a_4\sqrt{\lambda^2-4\mu} - 16\alpha_2{}^3\kappa\lambda^5 a_3\sqrt{\lambda^2-4\mu} + 36\alpha_2{}^2\lambda^3\psi a_3\sqrt{\lambda^2-4\mu}$$

$$+16\alpha_2{}^3\kappa\lambda\mu\sqrt{\lambda^2-4\mu} + 48\alpha_2{}^2\kappa\lambda\alpha_0\sqrt{\lambda^2-4\mu} - 82944\kappa\lambda^2\mu\psi a_4\alpha_0{}^3 - 10368\lambda^4\mu\psi^2 a_4\alpha_0\alpha_2$$

$$+82944\lambda^2\mu^2\psi^2 a_4\alpha_0\alpha_2 - 112\kappa\lambda^2\mu a_3\alpha_0\alpha_2{}^2 - 288\mu^2\psi a_3\alpha_2{}^2 - 32\kappa\mu\alpha_0\alpha_2{}^2 + 48\kappa\lambda^2\mu\alpha_2{}^3$$

$$+32\kappa\lambda^2\alpha_0\alpha_2{}^2 + 123904\kappa^2\mu^6 a_4\alpha_2{}^4 + 4\mu\psi\alpha_2{}^2 - 48\kappa\alpha_0{}^2\alpha_2 + 11\lambda^2\psi\alpha_2{}^2 + 16\kappa\mu^2\alpha_2{}^3$$

$$-48\psi\alpha_0\alpha_2 - 165888\kappa\mu^4\psi a_4\alpha_0\alpha_2{}^2 - 24192\alpha_2{}^2\kappa\lambda^5\mu\psi a_4\alpha_0\sqrt{\lambda^2-4\mu}$$

$$-27648\alpha_2{}^2\kappa\lambda^3\mu^2\psi a_4\alpha_0\sqrt{\lambda^2-4\mu} + 124416\alpha_2\kappa\lambda^3\mu\psi a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu}$$

$$+202752\,\alpha_2{}^2\kappa\lambda\,\mu^3\psi\,a_4\alpha_0\sqrt{\lambda^2-4\mu}-248832\,\alpha_2\kappa\lambda\,\mu^2\psi\,a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu}$$

$$+512\,\alpha_2{}^4\kappa^2\lambda^{12}a_4-16\,\alpha_2{}^3\kappa\lambda^6a_3-16\,\alpha_2{}^3\kappa\lambda^3\sqrt{\lambda^2-4\mu}+24\,\alpha_2{}^2\lambda\,\psi\sqrt{\lambda^2-4\mu}$$

$$-16\,\alpha_2{}^3\kappa\lambda^4-4608\,\alpha_2{}^4\kappa^2\lambda^9\mu\,a_4\sqrt{\lambda^2-4\mu}-3328\,\alpha_2{}^3\kappa^2\lambda^9a_4\alpha_0\sqrt{\lambda^2-4\mu}$$

$$+9088\,\alpha_2{}^4\kappa^2\lambda^7\mu^2a_4\sqrt{\lambda^2-4\mu}+16000\,\alpha_2{}^4\kappa^2\lambda^5\mu^3a_4\sqrt{\lambda^2-4\mu}$$

$$-1152\,\alpha_2{}^3\kappa\lambda^9\psi\,a_4\sqrt{\lambda^2-4\mu}+6912\,\alpha_2{}^2\kappa^2\lambda^7a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu}-39936\,\alpha_2{}^4\kappa^2\lambda^3\mu^4a_4\sqrt{\lambda^2-4\mu}$$

$$-22528\,\alpha_2{}^4\kappa^2\lambda\,\mu^5a_4\sqrt{\lambda^2-4\mu}-10368\,\alpha_2\kappa^2\lambda^5a_4\alpha_0{}^3\sqrt{\lambda^2-4\mu}+5184\,\alpha_2{}^2\lambda^5\mu\,\psi^2a_4\sqrt{\lambda^2-4\mu}$$

$$-5184\,\alpha_2\lambda^5\psi^2a_4\alpha_0\sqrt{\lambda^2-4\mu}-41472\,\alpha_2{}^2\lambda^3\mu^2\psi^2a_4\sqrt{\lambda^2-4\mu}+56\,\alpha_2{}^3\kappa\lambda^3\mu\,a_3\sqrt{\lambda^2-4\mu}$$

$$+82944\,\alpha_2{}^2\lambda\,\mu^3\psi^2a_4\sqrt{\lambda^2-4\mu}+72\,\alpha_2{}^2\kappa\lambda^3a_3\alpha_0\sqrt{\lambda^2-4\mu}+32\,\alpha_2{}^3\kappa\lambda\,\mu^2a_3\sqrt{\lambda^2-4\mu}$$

$$-144\,\alpha_2{}^2\lambda\,\mu\,\psi\,a_3\sqrt{\lambda^2-4\mu}+29056\,\alpha_2{}^3\kappa^2\lambda^7\mu\,a_4\alpha_0\sqrt{\lambda^2-4\mu}-60288\,\alpha_2{}^3\kappa^2\lambda^5\mu^2a_4\alpha_0\sqrt{\lambda^2-4\mu}$$

$$+8640\,\alpha_2{}^3\kappa\lambda^7\mu\,\psi\,a_4\sqrt{\lambda^2-4\mu}-51840\,\alpha_2{}^2\kappa^2\lambda^5\mu\,a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu}$$

$$-60416\,\alpha_2{}^3\kappa^2\lambda^3\mu^3a_4\alpha_0\sqrt{\lambda^2-4\mu}+4608\,\alpha_2{}^2\kappa\lambda^7\psi\,a_4\alpha_0\sqrt{\lambda^2-4\mu}$$

$$-8640\,\alpha_2{}^3\kappa\lambda^5\mu^2\psi\,a_4\sqrt{\lambda^2-4\mu}+82944\,\alpha_2{}^2\kappa^2\lambda^3\mu^2a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu}$$

$$+198656\,\alpha_2{}^3\kappa^2\lambda\,\mu^4a_4\alpha_0\sqrt{\lambda^2-4\mu}-50688\,\alpha_2{}^3\kappa\lambda^3\mu^3\psi\,a_4\sqrt{\lambda^2-4\mu}$$

$$+82944\,\alpha_2\kappa^2\lambda^3\mu\,a_4\alpha_0{}^3\sqrt{\lambda^2-4\mu}+55296\,\alpha_2{}^2\kappa^2\lambda\,\mu^3a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu}$$

$$-15552\,\alpha_2\kappa\lambda^5\psi\,a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu}+82944\,\alpha_2{}^3\kappa\lambda\,\mu^4\psi\,a_4\sqrt{\lambda^2-4\mu}$$

$$-165888\,\alpha_2\kappa^2\lambda\,\mu^2a_4\alpha_0{}^3\sqrt{\lambda^2-4\mu}+41472\,\alpha_2\lambda^3\mu\,\psi^2a_4\alpha_0\sqrt{\lambda^2-4\mu}$$

$$-82944\,\alpha_2\lambda\,\mu^2\psi^2a_4\alpha_0\sqrt{\lambda^2-4\mu}-288\,\alpha_2{}^2\kappa\lambda\,\mu\,a_3\alpha_0\sqrt{\lambda^2-4\mu}$$

$$-55296\,\kappa^2\lambda^2\mu^2a_4\alpha_0{}^3\alpha_2-9216\,\kappa\lambda^2\mu^4\psi\,a_4\alpha_2{}^3-4608\,\kappa\lambda^6\psi\,a_4\alpha_0{}^2\alpha_2$$

$$-271360\,\kappa^2\lambda^2\mu^3 a_4\alpha_0{}^2\alpha_2{}^2+34560\,\kappa^2\lambda^4\mu\, a_4\alpha_0{}^3\alpha_2+82944\,\mu^2\psi^2 a_4\alpha_0-352\,\kappa\,\mu^3 a_3\alpha_2{}^3$$

$$+5184\,\lambda^4\psi^2 a_4\alpha_0{}^{22}+82944\,\mu^4\psi^2 a_4\alpha_2{}^2+1296\,\lambda^8\psi^2 a_4\alpha_2{}^2$$

$$+5184\,\kappa^2\lambda^4 a_4\alpha_0{}^4+82944\,\kappa^2\mu^2 a_4\alpha_0{}^4-202752\,\kappa\,\mu^3\psi\, a_4\alpha_0{}^2\alpha_2$$

It's worth to mention that all the obtained parameter values are laced with α_0, α_2, a_3, ψ, κ, δ as free parameters.

As a result, with the help of solution of equation (5) and using (6) and (4) along with $\tau=-\omega t+x$, $F(\tau)=q(x,\,t)$, we have obtained following solutions structures for equation (1):

Regrading the instance $\lambda^2-4\mu>0$, we have extracted following solution structure for equation (1):

$$Q(x,\,t)=\alpha_0+\left(\frac{\sqrt{\lambda^2-4\mu}\left(w_1\sinh\left(\frac{1}{2}(x-\omega t)\sqrt{\lambda^2-4\mu}\right)+w_2\cosh\left(\frac{1}{2}(x-\omega t)\sqrt{\lambda^2-4\mu}\right)\right)}{2w_2\sinh\left(\frac{1}{2}(x-\omega t)\sqrt{\lambda^2-4\mu}\right)+2w_1\cosh\left(\frac{1}{2}(x-\omega t)\sqrt{\lambda^2-4\mu}\right)}-\frac{\lambda}{2}\right)^2\alpha_2$$

$$+\left(\frac{2\sqrt{\lambda^2-4\mu}\left(w_1\sinh\left(\frac{1}{2}(x-\omega t)\sqrt{\lambda^2-4\mu}\right)+w_2\cosh\left(\frac{1}{2}(x-\omega t)\sqrt{\lambda^2-4\mu}\right)\right)}{2w_2\sinh\left(\frac{1}{2}(x-\omega t)\sqrt{\lambda^2-4\mu}\right)+2w_1\cosh\left(\frac{1}{2}(x-\omega t)\sqrt{\lambda^2-4\mu}\right)}-\lambda\right)$$

$$\left(\frac{\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}\right)\alpha_2, \tag{7}$$

equipped with w_2, w_1 as free parameters.

Case 1 Significantly, we have observed that considering $w_1=0$ in (7) results into singular solitary waves as:

$$Q(x,\,t)=\alpha_0+\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\coth\left(\frac{1}{2}(-\omega t+x)\sqrt{\lambda^2-4\mu}\right)-\frac{\lambda}{2}\right)^2\alpha_2$$

$$+\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\coth\left(\frac{1}{2}(-\omega t+x)\sqrt{\lambda^2-4\mu}\right)-\frac{\lambda}{2}\right)\left(\lambda+\sqrt{\lambda^2-4\mu}\right)\alpha_2. \tag{8}$$

Case 2 Significantly, we have observed that considering $w_2=0$ in (7) results into shock wave solutions:

$$Q(x,\,t)=\alpha_0+\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}(-\omega t+x)\sqrt{\lambda^2-4\mu}\right)-\frac{\lambda}{2}\right)^2\alpha_2$$

$$+\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}(-\omega t+x)\sqrt{\lambda^2-4\mu}\right)-\frac{\lambda}{2}\right)\left(\lambda+\sqrt{\lambda^2-4\mu}\right)\alpha_2. \tag{9}$$

2.1.2 *Solutions to (2) for m = 2*

Now considering ***m* = 2**, the equation (2), turns out subsequently as

$$\left((-\nu F-\theta)F''-\xi F'''+a_1F-\delta(F)^2-\omega\right)F'+\left(a_4\omega^2-\Lambda F-a_3\omega+a_2\right)F'''$$

$$+(-\kappa F-\psi)F''''' = 0. \tag{10}$$

Homogeneous balance among highest order linear and most nonlinear derivative containing terms in equation (10) is carried out, hence leading to generate following solution structure for equation (10):

$$F(\tau)=\alpha_0+\alpha_1\left(\frac{G'(\tau)}{G(\tau)}\right)+\alpha_2\left(\frac{G'(\tau)}{G(\tau)}\right)^2, \tag{11}$$

with α_i, $i=0,\,1,\,2$ are constants, to be determined in the mean process of calculations and $G(\tau)$ following the auxiliary equation (5). Using (11) in equation (10), along with (5), we have furnished following possibility to explore variety of solutions for equation (1):

$$\delta=-\frac{3}{7}\frac{-95\,\kappa\lambda^2\alpha_2+240\left(\lambda/2-1/2\sqrt{\lambda^2-4\mu}\right)\alpha_2\kappa\lambda+140\,\kappa\mu\,\alpha_2+13\Lambda\alpha_2-240\,\kappa\,\alpha_0+4\,\theta\,\alpha_2}{{\alpha_2}^2},$$

$$\nu=-\frac{60\kappa}{\alpha_2},\;\; \alpha_1=\lambda-\sqrt{\lambda^2-4\mu}$$

$$\omega=\frac{Z_3}{14{\alpha_2}^2},\;\; \xi=10\,\kappa\sqrt{\lambda^2-4\mu},$$

$$\psi=\frac{61\,\kappa\lambda^2\alpha_2}{168}-\frac{1}{7}\alpha_2\kappa\lambda\sqrt{\lambda^2-4\mu}-\frac{7}{6}\kappa\mu\,\alpha_2-\frac{\Lambda\alpha_2}{56}-\frac{2}{7}\kappa\,\alpha_0-\frac{\theta\,\alpha_2}{84},$$

$$a_1=-\frac{1}{7}\frac{Z_4}{{\alpha_2}^2},$$

$$a_2=-\frac{Z_5}{1176\,{\alpha_2}^4}, \tag{12}$$

with

$$Z_3 = 2608\,\kappa\,\mu^2\alpha_0\alpha_2{}^2 - 4\,\alpha_2{}^2\lambda^2\theta\,\alpha_0 - 4680\,\kappa\,\mu\,\alpha_0{}^2\alpha_2 + 64\,\alpha_2{}^2\mu\,\theta\,\alpha_0 + 18\,\Lambda\,\lambda^2\mu\,\alpha_2{}^3$$

$$+ 8\,\Lambda\,\lambda^2\alpha_0\alpha_2{}^2 + 124\,\Lambda\,\mu\,\alpha_0\alpha_2{}^2 - 922\,\kappa\,\lambda^4\mu\,\alpha_2{}^3 + 928\,\kappa\,\lambda^4\alpha_0\alpha_2{}^2 + 2810\,\kappa\,\lambda^2\mu^2\alpha_2{}^3$$

$$+ 12\,\lambda^2\mu\,\theta\,\alpha_2{}^3 + 90\,\kappa\,\lambda^2\alpha_0{}^2\alpha_2 - 4680\,\alpha_2{}^2\kappa\,\lambda\,\mu\,\alpha_0\sqrt{\lambda^2-4\mu} + 104\,\alpha_2{}^3\kappa\,\lambda^5\sqrt{\lambda^2-4\mu}$$

$$+ 4\,\Lambda\,\alpha_2{}^3\lambda^3\sqrt{\lambda^2-4\mu} - 2\,\alpha_2{}^3\lambda^3\theta\,\sqrt{\lambda^2-4\mu} + 2160\,\alpha_2\kappa\,\lambda\,\alpha_0{}^2\sqrt{\lambda^2-4\mu}$$

$$- 24\,\alpha_2{}^2\lambda\,\theta\,\alpha_0\sqrt{\lambda^2-4\mu} - 202\,\alpha_2{}^3\kappa\,\lambda^3\mu\,\sqrt{\lambda^2-4\mu} + 90\,\alpha_2{}^2\kappa\,\lambda^3\alpha_0\sqrt{\lambda^2-4\mu}$$

$$+ 1304\,\alpha_2{}^3\kappa\,\lambda\,\mu^2\sqrt{\lambda^2-4\mu} + 62\,\Lambda\,\alpha_2{}^3\lambda\,\mu\,\sqrt{\lambda^2-4\mu} + 32\,\alpha_2{}^3\lambda\,\mu\,\theta\,\sqrt{\lambda^2-4\mu}$$

$$- 78\,\Lambda\,\alpha_2{}^2\lambda\,\alpha_0\sqrt{\lambda^2-4\mu} + 1440\,\alpha_0{}^3\kappa - 3284\,\kappa\,\lambda^2\mu\,\alpha_0\alpha_2{}^2 + 58\,\Lambda\,\mu^2\alpha_2{}^3 - 13\,\Lambda\,\lambda^4\alpha_2{}^3$$

$$- 8\,\mu^2\theta\,\alpha_2{}^3 + 7\,\kappa\,\lambda^6\alpha_2{}^3 - 78\,\Lambda\,\alpha_0{}^2\alpha_2 + 1624\,\kappa\,\mu^3\alpha_2{}^3 - 4\,\lambda^4\theta\,\alpha_2{}^3 - 24\,\theta\,\alpha_2\alpha_0{}^2,$$

$$Z_4 = 15\,\alpha_2{}^2\kappa\,\lambda^3\sqrt{\lambda^2-4\mu} - 344\,\kappa\,\lambda^4\alpha_2{}^2 + 1380\,\alpha_2{}^2\kappa\,\lambda\,\mu\,\sqrt{\lambda^2-4\mu} + 1282\,\kappa\,\lambda^2\mu\,\alpha_2{}^2$$

$$+ 39\,\Lambda\,\alpha_2{}^2\lambda\,\sqrt{\lambda^2-4\mu} - 4\,\Lambda\,\lambda^2\alpha_2{}^2 - 1440\,\alpha_2\kappa\,\lambda\,\alpha_0\sqrt{\lambda^2-4\mu} + 12\,\alpha_2{}^2\lambda\,\theta\,\sqrt{\lambda^2-4\mu}$$

$$+ 30\,\kappa\,\lambda^2\alpha_0\alpha_2 - 1064\,\kappa\,\mu^2\alpha_2{}^2 + 2\,\alpha_2{}^2\lambda^2\theta - 62\,\Lambda\,\mu\,\alpha_2{}^2 + 2760\,\kappa\,\mu\,\alpha_0\alpha_2 - 32\,\alpha_2{}^2\mu\,\theta$$

$$+ 78\,\Lambda\,\alpha_0\alpha_2 - 1440\,\kappa\,\alpha_0{}^2 + 24\,\theta\,\alpha_0\alpha_2,$$

$$Z_5 = 6523200\,\Lambda\,\kappa\,\mu\,a_4\alpha_0{}^4\alpha_2{}^2 - 95616\,\Lambda\,\mu\,\theta\,a_4\alpha_0{}^3\alpha_2{}^3 + 190272\,\kappa^2\lambda^{10}a_4\alpha_0\alpha_2{}^5$$

$$+ 13951296\,\kappa^2\lambda^2\mu^5a_4\alpha_2{}^6 + 219360\,\Lambda\,\lambda^2\mu^2\theta\,a_4\alpha_0\alpha_2{}^5 - 991056\,\Lambda\,\kappa\,\lambda^6\mu^2a_4\alpha_2{}^6$$

$$- 962784\,\kappa\,\lambda^4\theta\,a_4\alpha_0{}^3\alpha_2{}^3 + 8217144\,\kappa^2\lambda^8\mu^2a_4\alpha_2{}^6 + 3900\,\Lambda\,\kappa\,\lambda^{10}a_4\alpha_2{}^6$$

$$+ 1032864\,\Lambda\,\kappa\,\lambda^4\mu^3a_4\alpha_2{}^6 + 10464\,\lambda^4\mu^2\theta^2a_4\alpha_2{}^6 + 528\,\Lambda\,\lambda^8\theta\,a_4\alpha_2{}^6 - 216\,\Lambda^2\lambda^6\mu\,a_4\alpha_2{}^6$$

$$+ 52254744\,\kappa^2\lambda^4\mu^4a_4\alpha_2{}^6 - 66336\,\Lambda\,\lambda^4\mu\,\theta\,a_4\alpha_0\alpha_2{}^5 + 86304\,\Lambda^2\mu^3a_4\alpha_0\alpha_2{}^5$$

$$- 4992\,\Lambda^2\lambda^6a_4\alpha_0\alpha_2{}^5 - 919688064\,\kappa^2\lambda^2\mu^3a_4\alpha_0{}^2\alpha_2{}^4 - 32164704\,\Lambda\,\kappa\,\lambda^2\mu^2a_4\alpha_0{}^2\alpha_2{}^4$$

$$+ 11461632\,\kappa\lambda^2\mu^3\theta\, a_4\alpha_0\alpha_2{}^5 - 15168\,\lambda^4\mu\,\theta^2 a_4\alpha_0\alpha_2{}^5 - 95040\,\kappa\lambda^2\theta\, a_4\alpha_0{}^4\alpha_2{}^2$$

$$- 59226864\,\kappa^2\lambda^6\mu\, a_4\alpha_0{}^2\alpha_2{}^4 - 1347840\,\Lambda\kappa\, a_4\alpha_0{}^5\alpha_2 + 7918944\,\kappa^2\lambda^8 a_4\alpha_0{}^2\alpha_2{}^4$$

$$- 7560\,\kappa\lambda^2 a_3\alpha_0{}^2\alpha_2{}^3 - 1574160\,\Lambda\kappa\lambda^2\mu^4 a_4\alpha_2{}^6 - 4483584\,\kappa\mu^2\theta\, a_4\alpha_0{}^3\alpha_2{}^3$$

$$+ 26880\,\mu^2\theta^2 a_4\alpha_0{}^2\alpha_2{}^4 - 77952\,\kappa\lambda^4 a_3\alpha_0\alpha_2{}^4 - 41603184\,\kappa^2\lambda^6\mu^3 a_4\alpha_2{}^6$$

$$+ 996864\,\kappa\mu^4\theta\, a_4\alpha_0\alpha_2{}^5 - 237120\,\Lambda\kappa\lambda^8 a_4\alpha_0\alpha_2{}^5 - 136416\,\kappa\mu^3 a_3\alpha_2{}^5$$

$$+ 316873728\,\kappa^2\lambda^2\mu^4 a_4\alpha_0\alpha_2{}^5 - 76992\,\kappa\lambda^8\theta\, a_4\alpha_0\alpha_2{}^5 + 77448\,\kappa\lambda^4\mu\, a_3\alpha_2{}^5$$

$$+ 4704\,\lambda^4\theta^2 a_4\alpha_0{}^2\alpha_2{}^4 + 50824704\,\kappa^2\mu^5 a_4\alpha_0\alpha_2{}^5 - 155904\,\kappa\mu^5\theta\, a_4\alpha_2{}^6$$

$$- 25728\,\lambda^2\mu^3\theta^2 a_4\alpha_2{}^6 + 3861792\,\kappa\lambda^4\mu\,\theta\, a_4\alpha_0{}^2\alpha_2{}^4 - 12848256\,\kappa\lambda^2\mu^2\theta\, a_4\alpha_0{}^2\alpha_2{}^4$$

$$+ 2453760\,\kappa\mu\,\theta\, a_4\alpha_0{}^4\alpha_2{}^2 - 1512\,\Lambda\lambda^2\mu\, a_3\alpha_2{}^5 + 22464\,\Lambda\theta\, a_4\alpha_0{}^4\alpha_2{}^2$$

$$+ 44078040\,\kappa^2\lambda^4 a_4\alpha_0{}^4\alpha_2{}^2 + 1130304\,\Lambda\kappa\mu^5 a_4\alpha_2{}^6 + 49056\,\Lambda^2\lambda^4 a_4\alpha_0{}^2\alpha_2{}^4$$

$$+ 3935232\,\kappa\lambda^2\mu\,\theta\, a_4\alpha_0{}^3\alpha_2{}^3 - 5568\,\Lambda\mu^4\theta\, a_4\alpha_2{}^6 - 7488\,\Lambda^2\lambda^2 a_4\alpha_0{}^3\alpha_2{}^3$$

$$+ 37968\,\Lambda^2\mu^2 a_4\alpha_0{}^2\alpha_2{}^4 - 116064\,\Lambda^2\mu\, a_4\alpha_0{}^3\alpha_2{}^3 - 219072\,\kappa\mu^2 a_3\alpha_0\alpha_2{}^4$$

$$+ 768\,\lambda^6\theta^2 a_4\alpha_0\alpha_2{}^5 + 3456000\,\kappa^2\lambda^6 a_4\alpha_0{}^3\alpha_2{}^3 - 1440\,\lambda^6\mu\,\theta^2 a_4\alpha_2{}^6$$

$$- 80870400\,\kappa^2\mu\, a_4\alpha_0{}^5\alpha_2 - 4523040\,\kappa\lambda^4\mu^2\theta\, a_4\alpha_0\alpha_2{}^5 + 8587872\,\Lambda\kappa\lambda^4\mu\, a_4\alpha_0{}^2\alpha_2{}^4$$

$$- 128640\,\kappa\lambda^6\theta\, a_4\alpha_0{}^2\alpha_2{}^4 - 2038848\,\kappa\lambda^2\mu^4\theta\, a_4\alpha_2{}^6 + 1038144\,\Lambda\kappa\lambda^6\mu\, a_4\alpha_0\alpha_2{}^5$$

$$- 2832\,\kappa\lambda^{10}\theta\, a_4\alpha_2{}^6 - 1008\,\lambda^2\mu\,\theta\, a_3\alpha_2{}^5 - 10416\,\Lambda\mu\, a_3\alpha_0\alpha_2{}^4 - 672\,\Lambda\lambda^2 a_3\alpha_0\alpha_2{}^4$$

$$- 245749728\,\kappa^2\lambda^4\mu^3 a_4\alpha_0\alpha_2{}^5 + 1984512\,\kappa\mu^3\theta\, a_4\alpha_0{}^2\alpha_2{}^4 + 1351392\,\kappa\lambda^4\mu^3\theta\, a_4\alpha_2{}^6$$

$$+ 354939984\,\kappa^2\lambda^4\mu^2 a_4\alpha_0{}^2\alpha_2{}^4 + 674400\,\kappa\lambda^6\mu\,\theta\, a_4\alpha_0\alpha_2{}^5 - 6400512\,\Lambda\kappa\lambda^4\mu^2 a_4\alpha_0\alpha_2{}^5$$

$$+ 11156544\,\Lambda\kappa\lambda^2\mu\, a_4\alpha_0{}^3\alpha_2{}^3 - 17051328\,\kappa^2\lambda^8\mu\, a_4\alpha_0\alpha_2{}^5 + 100032\,\kappa\lambda^8\mu\,\theta\, a_4\alpha_2{}^6$$

$$- 3106368\,\Lambda\,\kappa\,\lambda^4 a_4 {\alpha_0}^3 {\alpha_2}^3 - 202232160\,\kappa^2\lambda^4\mu\,a_4 {\alpha_0}^3 {\alpha_2}^3 - 896640\,\Lambda\,\kappa\,\mu^3 a_4 {\alpha_0}^2 {\alpha_2}^4$$

$$+ 36504\,\Lambda^2 a_4 {\alpha_0}^4 {\alpha_2}^2 - 106080\,\Lambda\,\lambda^2\mu\,\theta\,a_4 {\alpha_0}^2 {\alpha_2}^4 - 5376\,\mu\,\theta\,a_3 \alpha_0 {\alpha_2}^4$$

$$- 5824\,\kappa\,\lambda^2\mu\,{\alpha_2}^5 - 10920\,\kappa\,\lambda^2 \alpha_0 {\alpha_2}^4 - 50393856\,\kappa^2\mu^4 a_4 {\alpha_0}^2 {\alpha_2}^4 + 120\,\lambda^8\theta^2 a_4 {\alpha_2}^6$$

$$+ 43680\,\kappa\,\mu\,\alpha_0 {\alpha_2}^4 + 384\,\mu^4\theta^2 a_4 {\alpha_2}^6 + 15824256\,\kappa^2\mu^6 a_4 {\alpha_2}^6$$

$$- 118402560\,\kappa^2\mu^3 a_4 {\alpha_0}^3 {\alpha_2}^3 - 589128\,\kappa^2\lambda^{10}\mu\,a_4 {\alpha_2}^6 + 1555200\,\kappa^2\lambda^2 a_4 {\alpha_0}^5 \alpha_2$$

$$+ 336\,\lambda^2\theta\,a_3 \alpha_0 {\alpha_2}^4 + 29568\,\Lambda\,\lambda^4\theta\,a_4 {\alpha_0}^2 {\alpha_2}^4 + 1152\,\lambda^2\theta^2 a_4 {\alpha_0}^3 {\alpha_2}^3$$

$$+ 393120\,\kappa\,\mu\,a_3 {\alpha_0}^2 {\alpha_2}^3 - 236040\,\kappa\,\lambda^2\mu^2 a_3 {\alpha_2}^5 + 4056\,\Lambda^2\lambda^4\mu^2 a_4 {\alpha_2}^6$$

$$+ 3456\,\theta^2 a_4 {\alpha_0}^4 {\alpha_2}^2 - 6144\,\mu^3\theta^2 a_4 \alpha_0 {\alpha_2}^5 - 18432\,\mu\,\theta^2 a_4 {\alpha_0}^3 {\alpha_2}^3$$

$$- 1167624\,{\alpha_2}^6\kappa^2\lambda^9\mu\,a_4\sqrt{\lambda^2-4\mu} - 15888\,\Lambda\,{\alpha_2}^6\kappa\,\lambda^9 a_4\sqrt{\lambda^2-4\mu}$$

$$+ 1165704\,{\alpha_2}^5\kappa^2\lambda^9 a_4 \alpha_0\sqrt{\lambda^2-4\mu} + 5851344\,{\alpha_2}^6\kappa^2\lambda^7\mu^2 a_4\sqrt{\lambda^2-4\mu}$$

$$- 5160\,{\alpha_2}^6\kappa\,\lambda^9\theta\,a_4\sqrt{\lambda^2-4\mu} - 19212144\,{\alpha_2}^6\kappa^2\lambda^5\mu^3 a_4\sqrt{\lambda^2-4\mu}$$

$$+ 120\,\Lambda\,{\alpha_2}^6\lambda^7\theta\,a_4\sqrt{\lambda^2-4\mu} + 1296000\,{\alpha_2}^4\kappa^2\lambda^7 a_4 {\alpha_0}^2\sqrt{\lambda^2-4\mu}$$

$$+ 40034304\,{\alpha_2}^6\kappa^2\lambda^3\mu^4 a_4\sqrt{\lambda^2-4\mu} - 8808\,\Lambda^2{\alpha_2}^6\lambda^5\mu\,a_4\sqrt{\lambda^2-4\mu}$$

$$+ 25412352\,{\alpha_2}^6\kappa^2\lambda\,\mu^5 a_4\sqrt{\lambda^2-4\mu} - 1824\,{\alpha_2}^6\lambda^5\mu\,\theta^2 a_4\sqrt{\lambda^2-4\mu}$$

$$+ 12552\,\Lambda^2{\alpha_2}^5\lambda^5 a_4 \alpha_0\sqrt{\lambda^2-4\mu} + 16176\,\Lambda^2{\alpha_2}^6\lambda^3\mu^2 a_4\sqrt{\lambda^2-4\mu}$$

$$+ 25948080\,{\alpha_2}^3\kappa^2\lambda^5 a_4 {\alpha_0}^3\sqrt{\lambda^2-4\mu} + 1248\,{\alpha_2}^5\lambda^5\theta^2 a_4 \alpha_0\sqrt{\lambda^2-4\mu}$$

$$+ 4800\,{\alpha_2}^6\lambda^3\mu^2\theta^2 a_4\sqrt{\lambda^2-4\mu} + 43152\,\Lambda^2{\alpha_2}^6\lambda\,\mu^3 a_4\sqrt{\lambda^2-4\mu}$$

$$- 3072\,{\alpha_2}^6\lambda\,\mu^3\theta^2 a_4\sqrt{\lambda^2-4\mu} - 11232\,\Lambda^2{\alpha_2}^4\lambda^3 a_4 {\alpha_0}^2\sqrt{\lambda^2-4\mu}$$

$$+\,3888000\,{\alpha_2}^2\kappa^2\lambda^3a_4{\alpha_0}^4\sqrt{\lambda^2-4\mu}+1728\,{\alpha_2}^4\lambda^3\theta^2a_4{\alpha_0}^2\sqrt{\lambda^2-4\mu}$$

$$+\,16968\,{\alpha_2}^5\kappa\lambda^3\mu\,a_3\sqrt{\lambda^2-4\mu}+73008\,\Lambda^2{\alpha_2}^3\lambda\,a_4{\alpha_0}^3\sqrt{\lambda^2-4\mu}$$

$$+\,37324800\,\alpha_2\kappa^2\lambda\,a_4{\alpha_0}^5\sqrt{\lambda^2-4\mu}-7560\,{\alpha_2}^4\kappa\lambda^3a_3\alpha_0\sqrt{\lambda^2-4\mu}$$

$$-\,109536\,{\alpha_2}^5\kappa\lambda\,\mu^2a_3\sqrt{\lambda^2-4\mu}+6912\,{\alpha_2}^3\lambda\,\theta^2a_4{\alpha_0}^3\sqrt{\lambda^2-4\mu}$$

$$-\,5208\,\Lambda{\alpha_2}^5\lambda\,\mu\,a_3\sqrt{\lambda^2-4\mu}-2688\,{\alpha_2}^5\lambda\,\mu\,\theta\,a_3\sqrt{\lambda^2-4\mu}$$

$$+\,6552\,\Lambda{\alpha_2}^4\lambda\,a_3\alpha_0\sqrt{\lambda^2-4\mu}-181440\,{\alpha_2}^3\kappa\lambda\,a_3{\alpha_0}^2\sqrt{\lambda^2-4\mu}$$

$$+\,2016\,{\alpha_2}^4\lambda\,\theta\,a_3\alpha_0\sqrt{\lambda^2-4\mu}+32640\,\Lambda\mu^3\theta\,a_4\alpha_0{\alpha_2}^5+264480\,\Lambda^2\lambda^2\mu^2a_4\alpha_0{\alpha_2}^5$$

$$-\,177603840\,{\alpha_2}^4\kappa^2\lambda\,\mu^3a_4{\alpha_0}^2\sqrt{\lambda^2-4\mu}-6528\,{\alpha_2}^5\lambda^3\mu\,\theta^2a_4\alpha_0\sqrt{\lambda^2-4\mu}$$

$$+\,37968\,\Lambda^2{\alpha_2}^5\lambda\,\mu^2a_4\alpha_0\sqrt{\lambda^2-4\mu}+108000\,\Lambda{\alpha_2}^3\kappa\lambda^3a_4{\alpha_0}^3\sqrt{\lambda^2-4\mu}$$

$$+\,2160\,\Lambda{\alpha_2}^4\lambda^3\theta\,a_4{\alpha_0}^2\sqrt{\lambda^2-4\mu}+352961280\,{\alpha_2}^3\kappa^2\lambda\,\mu^2a_4{\alpha_0}^3\sqrt{\lambda^2-4\mu}$$

$$-\,190080\,{\alpha_2}^3\kappa\lambda^3\theta\,a_4{\alpha_0}^3\sqrt{\lambda^2-4\mu}+26880\,{\alpha_2}^5\lambda\,\mu^2\theta^2a_4\alpha_0\sqrt{\lambda^2-4\mu}$$

$$-\,174096\,\Lambda^2{\alpha_2}^4\lambda\,\mu\,a_4{\alpha_0}^2\sqrt{\lambda^2-4\mu}-202176000\,{\alpha_2}^2\kappa^2\lambda\,\mu\,a_4{\alpha_0}^4\sqrt{\lambda^2-4\mu}$$

$$-\,27648\,{\alpha_2}^4\lambda\,\mu\,\theta^2a_4{\alpha_0}^2\sqrt{\lambda^2-4\mu}-3369600\,\Lambda{\alpha_2}^2\kappa\lambda\,a_4{\alpha_0}^4\sqrt{\lambda^2-4\mu}$$

$$+\,44928\,\Lambda{\alpha_2}^3\lambda\,\theta\,a_4{\alpha_0}^3\sqrt{\lambda^2-4\mu}-1036800\,{\alpha_2}^2\kappa\lambda\,\theta\,a_4{\alpha_0}^4\sqrt{\lambda^2-4\mu}$$

$$+\,393120\,{\alpha_2}^4\kappa\lambda\,\mu\,a_3\alpha_0\sqrt{\lambda^2-4\mu}+87936\,\Lambda\kappa\lambda^6a_4{\alpha_0}^2{\alpha_2}^4$$

$$-\,88608\,\Lambda\lambda^2\mu^3\theta\,a_4{\alpha_2}^6-60672\,\Lambda^2\lambda^4\mu\,a_4\alpha_0{\alpha_2}^5+104833248\,\kappa^2\lambda^6\mu^2a_4\alpha_0{\alpha_2}^5$$

$$-\,414720\,\kappa\theta\,a_4{\alpha_0}^5\alpha_2+336\,\lambda^4\theta\,a_3{\alpha_2}^5+13046400\,\Lambda{\alpha_2}^3\kappa\lambda\,\mu\,a_4{\alpha_0}^3\sqrt{\lambda^2-4\mu}$$

$$-\,143424\,\Lambda{\alpha_2}^4\lambda\,\mu\,\theta\,a_4{\alpha_0}^2\sqrt{\lambda^2-4\mu}+4907520\,{\alpha_2}^3\kappa\lambda\,\mu\,\theta\,a_4{\alpha_0}^3\sqrt{\lambda^2-4\mu}$$

$$+2280672\,\Lambda\,{\alpha_2}^5\kappa\,\lambda^5\mu\,a_4\alpha_0\sqrt{\lambda^2-4\,\mu}+1027872\,{\alpha_2}^5\kappa\,\lambda^5\mu\,\theta\,a_4\alpha_0\sqrt{\lambda^2-4\,\mu}$$

$$-6071904\,\Lambda\,{\alpha_2}^5\kappa\,\lambda^3\mu^2a_4\alpha_0\sqrt{\lambda^2-4\,\mu}-3033216\,{\alpha_2}^5\kappa\,\lambda^3\mu^2\theta\,a_4\alpha_0\sqrt{\lambda^2-4\,\mu}$$

$$+3256416\,\Lambda\,{\alpha_2}^4\kappa\,\lambda^3\mu\,a_4{\alpha_0}^2\sqrt{\lambda^2-4\,\mu}-896640\,\Lambda\,{\alpha_2}^5\kappa\,\lambda\,\mu^3a_4\alpha_0\sqrt{\lambda^2-4\,\mu}$$

$$-16224\,\Lambda\,{\alpha_2}^5\lambda^3\mu\,\theta\,a_4\alpha_0\sqrt{\lambda^2-4\,\mu}+1755648\,{\alpha_2}^4\kappa\,\lambda^3\mu\,\theta\,a_4{\alpha_0}^2\sqrt{\lambda^2-4\,\mu}$$

$$+1984512\,{\alpha_2}^5\kappa\,\lambda\,\mu^3\theta\,a_4\alpha_0\sqrt{\lambda^2-4\,\mu}-12604032\,\Lambda\,{\alpha_2}^4\kappa\,\lambda\,\mu^2a_4{\alpha_0}^2\sqrt{\lambda^2-4\,\mu}$$

$$+86016\,\Lambda\,{\alpha_2}^5\lambda\,\mu^2\theta\,a_4\alpha_0\sqrt{\lambda^2-4\,\mu}-6725376\,{\alpha_2}^4\kappa\,\lambda\,\mu^2\theta\,a_4{\alpha_0}^2\sqrt{\lambda^2-4\,\mu}$$

$$-556320\,\kappa\,\lambda^6\mu^2\theta\,a_4{\alpha_2}^6+21424704\,\Lambda\,\kappa\,\lambda^2\mu^3a_4\alpha_0{\alpha_2}^5+960\,\Lambda\,\lambda^6\theta\,a_4\alpha_0{\alpha_2}^5$$

$$+24672\,\Lambda\,\lambda^4\mu^2\theta\,a_4{\alpha_2}^6+193056\,\Lambda\,\kappa\,\lambda^8\mu\,a_4{\alpha_2}^6-588\,\kappa\,\lambda^6a_3{\alpha_2}^5-4872\,\Lambda\,\mu^2a_3{\alpha_2}^5$$

$$+1110\,\Lambda^2\lambda^8a_4{\alpha_2}^6+8736\,{\alpha_2}^6\kappa^2\lambda^{11}a_4\sqrt{\lambda^2-4\,\mu}-624\,\Lambda^2{\alpha_2}^6\lambda^7a_4\sqrt{\lambda^2-4\,\mu}$$

$$+96\,{\alpha_2}^6\lambda^7\theta^2a_4\sqrt{\lambda^2-4\,\mu}-8736\,{\alpha_2}^5\kappa\,\lambda^5a_3\sqrt{\lambda^2-4\,\mu}-336\,\Lambda\,{\alpha_2}^5\lambda^3a_3\sqrt{\lambda^2-4\,\mu}$$

$$+168\,{\alpha_2}^5\lambda^3\theta\,a_3\sqrt{\lambda^2-4\,\mu}+21840\,\kappa\,\lambda\,\mu\,{\alpha_2}^5\sqrt{\lambda^2-4\,\mu}+14928\,\Lambda\,{\alpha_2}^6\kappa\,\lambda^7\mu\,a_4\sqrt{\lambda^2-4\,\mu}$$

$$-7736784\,{\alpha_2}^5\kappa^2\lambda^7\mu\,a_4\alpha_0\sqrt{\lambda^2-4\,\mu}+49488\,{\alpha_2}^6\kappa\,\lambda^7\mu\,\theta\,a_4\sqrt{\lambda^2-4\,\mu}$$

$$+33936\,\Lambda\,{\alpha_2}^5\kappa\,\lambda^7a_4\alpha_0\sqrt{\lambda^2-4\,\mu}-725760\,\Lambda\,{\alpha_2}^6\kappa\,\lambda^5\mu^2a_4\sqrt{\lambda^2-4\,\mu}$$

$$+80550864\,{\alpha_2}^5\kappa^2\lambda^5\mu^2a_4\alpha_0\sqrt{\lambda^2-4\,\mu}-33600\,{\alpha_2}^5\kappa\,\lambda^7\theta\,a_4\alpha_0\sqrt{\lambda^2-4\,\mu}$$

$$-523152\,{\alpha_2}^6\kappa\,\lambda^5\mu^2\theta\,a_4\sqrt{\lambda^2-4\,\mu}+2309664\,\Lambda\,{\alpha_2}^6\kappa\,\lambda^3\mu^3a_4\sqrt{\lambda^2-4\,\mu}$$

$$-7824\,\Lambda\,{\alpha_2}^6\lambda^5\mu\,\theta\,a_4\sqrt{\lambda^2-4\,\mu}-85620240\,{\alpha_2}^4\kappa^2\lambda^5\mu\,a_4{\alpha_0}^2\sqrt{\lambda^2-4\,\mu}$$

$$-213765504\,{\alpha_2}^5\kappa^2\lambda^3\mu^3a_4\alpha_0\sqrt{\lambda^2-4\,\mu}+1247232\,{\alpha_2}^6\kappa\,\lambda^3\mu^3\theta\,a_4\sqrt{\lambda^2-4\,\mu}$$

$$-1289952\,\Lambda\,{\alpha_2}^4\kappa\,\lambda^5a_4{\alpha_0}^2\sqrt{\lambda^2-4\,\mu}+2115840\,\Lambda\,{\alpha_2}^6\kappa\,\lambda\,\mu^4a_4\sqrt{\lambda^2-4\,\mu}$$

$$
\begin{aligned}
&+7104\,\Lambda\,\alpha_2{}^5\lambda^5\theta\,a_4\alpha_0\sqrt{\lambda^2-4\mu}+14064\,\Lambda\,\alpha_2{}^6\lambda^3\mu^2\theta\,a_4\sqrt{\lambda^2-4\mu}\\
&+272833920\,\alpha_2{}^4\kappa^2\lambda^3\mu^2a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu}-50393856\,\alpha_2{}^5\kappa^2\lambda\,\mu^4a_4\alpha_0\sqrt{\lambda^2-4\mu}\\
&-407376\,\alpha_2{}^4\kappa\,\lambda^5\theta\,a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu}+498432\,\alpha_2{}^6\kappa\,\lambda\,\mu^4\theta\,a_4\sqrt{\lambda^2-4\mu}\\
&-4944\,\Lambda^2\alpha_2{}^5\lambda^3\mu\,a_4\alpha_0\sqrt{\lambda^2-4\mu}+16320\,\Lambda\,\alpha_2{}^6\lambda\,\mu^3\theta\,a_4\sqrt{\lambda^2-4\mu}\\
&-98720640\,\alpha_2{}^3\kappa^2\lambda^3\mu\,a_4\alpha_0{}^3\sqrt{\lambda^2-4\mu}-588\,\Lambda\,\alpha_2{}^5\lambda\,\sqrt{\lambda^2-4\mu}-2304\,\Lambda\,\lambda^6\mu\,\theta\,a_4\alpha_2{}^6\\
&-150960\,\Lambda^2\lambda^2\mu\,a_4\alpha_0{}^2\alpha_2{}^4-8402688\,\Lambda\,\kappa\,\mu^2a_4\alpha_0{}^3\alpha_2{}^3+86016\,\Lambda\,\mu^2\theta\,a_4\alpha_0{}^2\alpha_2{}^4\\
&+54000\,\Lambda\,\kappa\,\lambda^2a_4\alpha_0{}^4\alpha_2{}^2+6552\,\Lambda\,a_3\alpha_0{}^2\alpha_2{}^3+1092\,\Lambda\,\lambda^4a_3\alpha_2{}^5\\
&-173776320\,\kappa^2\lambda^2\mu\,a_4\alpha_0{}^4\alpha_2{}^2+4231680\,\Lambda\,\kappa\,\mu^4a_4\alpha_0\alpha_2{}^5+20184\,\Lambda^2\mu^4a_4\alpha_2{}^6\\
&+46464\,\lambda^2\mu^2\theta^2a_4\alpha_0\alpha_2{}^5+275856\,\kappa\,\lambda^2\mu\,a_3\alpha_0\alpha_2{}^4-20352\,\lambda^2\mu\,\theta^2a_4\alpha_0{}^2\alpha_2{}^4\\
&+1440\,\Lambda\,\lambda^2\theta\,a_4\alpha_0{}^3\alpha_2{}^3+861\,\Lambda\,\lambda^2\alpha_2{}^5-728\,\mu\,\theta\,\alpha_2{}^5+12441600\,\kappa^2a_4\alpha_0{}^6-2268\,\Lambda\,\mu\,\alpha_2{}^5\\
&-1176\,\Lambda\,\alpha_0\alpha_2{}^4-10192\,\kappa\,\mu^2\alpha_2{}^5+182\,\lambda^2\theta\,\alpha_2{}^5+2093\,\kappa\,\lambda^4\alpha_2{}^5+65190\,\kappa^2\lambda^{12}a_4\alpha_2{}^6\\
&-79728\,\Lambda^2\lambda^2\mu^3a_4\alpha_2{}^6+176480640\,\kappa^2\mu^2a_4\alpha_0{}^4\alpha_2{}^2-5460\,\kappa\,\lambda^3\alpha_2{}^5\sqrt{\lambda^2-4\mu}\\
&+721025280\,\kappa^2\lambda^2\mu^2a_4\alpha_0{}^3\alpha_2{}^3-120960\,\kappa\,a_3\alpha_0{}^3\alpha_2{}^2+2016\,\theta\,a_3\alpha_0{}^2\alpha_2{}^3+672\,\mu^2\theta\,a_3\alpha_2{}^5,
\end{aligned}
$$

equipped with a_3, α_0, α_2, Λ, κ, θ as free parameters.

Using solution of equation (5) along with parameter values (12) and (11), we may write the subsequent solution for equation (1), by reverting back to original variables x, t:

$$Q(x,t)=\left(\frac{\sqrt{\lambda^2-4\mu}\left(w_1\sinh\left(\frac{1}{2}(-\omega t+x)\sqrt{\lambda^2-4\mu}\right)+w_2\cosh\left(\frac{1}{2}(-\omega t+x)\sqrt{\lambda^2-4\mu}\right)\right)}{2w_2\sinh\left(\frac{1}{2}(-\omega t+x)\sqrt{\lambda^2-4\mu}\right)+2w_1\cosh\left(\frac{1}{2}(-\omega t+x)\sqrt{\lambda^2-4\mu}\right)}-\frac{\lambda}{2}\right)^2\alpha_2+\alpha_0$$

$$+\left(\frac{\sqrt{\lambda^2-4\mu}\left(w_1\sinh\left(\frac{1}{2}(-\omega t+x)\sqrt{\lambda^2-4\mu}\right)+w_2\cosh\left(\frac{1}{2}(-\omega t+x)\sqrt{\lambda^2-4\mu}\right)\right)}{2w_2\sinh\left(\frac{1}{2}(-\omega t+x)\sqrt{\lambda^2-4\mu}\right)+2w_1\cosh\left(\frac{1}{2}(-\omega t+x)\sqrt{\lambda^2-4\mu}\right)}-\frac{\lambda}{2}\right)$$

$$\left(\lambda-\sqrt{\lambda^2-4\mu}\right)\alpha_2, \tag{13}$$

with w_1, w_2 as arbitrary parameters.

The solution (13) results into shock waves for $w_1 \neq 0$, $w_2 = 0$, while for $w_2 \neq 0$, $w_1 = 0$, obtains singular solitary waves.

2.2 *Perturbed mKdV equation*

The investigation of various exact solutions for modified KdV model with perturbation terms is the focus of this section. It is noteworthy that the current work addresses the perturbed modified KdV for the first time in the following manner:

$$Q_t+a_1Q^2Q_x+a_2Q_{xxx}+a_3Q_{xxt}+a_4Q_{xtt}$$

$$=\theta Q_xQ_{xx}+\delta Q^m q_x+\Lambda QQ_{xxx}+\nu QQ_xQ_{xx}+\xi Q_xQ_{xxx}+\psi Q_{xxxxx}+\kappa QQ_{xxxxx}, \tag{14}$$

with $Q=Q(x,t)$.

From the equation (14), we have extracted the following ordinary differential equation using $\tau=-\omega t+x$ and $Q(x,t)=F(\tau)$:

$$\left((-\nu F-\theta)F''-\xi F'''+a_1F^2-\delta\,(F)^m-\omega\right)F'+\left(a_4\omega^2-\Lambda F-a_3\omega+a_2\right)F'''$$

$$+(-\kappa F-\psi)F'''''=0. \tag{15}$$

To establish the new analytic solutions equation, (15), the $\left(\frac{G'}{G}\right)$-expansion approach has been implemented for $m=1, 2$ as shown in upcoming subsections.

2.2.1 *Solutions to (15) for m = 1*

After taking into account that $m=1$ for (15), the equation is rewritten as follows:

$$\left((-\nu F-\theta)F''-\xi F'''+a_1F^2-\delta F-\omega\right)F'+\left(a_4\omega^2-\Lambda F-a_3\omega+a_2\right)F'''$$

$$+(-\kappa F-\psi)F'''''=0. \tag{16}$$

The following solution structure for equation (16) is produced as a result of homogeneous balancing between the highest order linear and most nonlinear derivative-containing terms:

$$F(\tau)=\alpha_0+\alpha_1\left(\frac{G'(\tau)}{G(\tau)}\right)+\alpha_2\left(\frac{G'(\tau)}{G(\tau)}\right)^2, \tag{17}$$

with α_i, $i=0,\ 1,\ 2$ are constants, to be determined in complete computations and $G(\tau)$ following the auxiliary equation (5). We have provided the following opportunity to investigate a range of solutions for equation (14) by using (17) in equation (16) and (5):

$$\alpha_1=\left(\lambda-\sqrt{\lambda^2-4\mu}\right)\alpha_2,\ \xi=10\,\kappa\sqrt{\lambda^2-4\mu},\ \nu=-\frac{60\kappa}{\alpha_2}$$

$$a_1=\frac{3}{7}\,\frac{-120\,\alpha_2\kappa\lambda\sqrt{\lambda^2-4\mu}+25\,\kappa\lambda^2\alpha_2+140\,\kappa\mu\,\alpha_2+13\,\Lambda\alpha_2-240\,\kappa\alpha_0+4\,\theta\,\alpha_2}{\alpha_2{}^2},$$

$$\psi=\frac{61\,\kappa\lambda^2\alpha_2}{168}-\frac{1}{7}\,\alpha_2\kappa\lambda\sqrt{\lambda^2-4\mu}-\frac{7}{6}\,\kappa\mu\,\alpha_2-\frac{\Lambda\alpha_2}{56}-\frac{2}{7}\,\kappa\alpha_0-\frac{\theta\,\alpha_2}{84},$$

$$\delta=\frac{1}{7}\,\frac{Z_6}{\alpha_2{}^2},\ \ \omega=-\frac{1}{14}\,\frac{Z_7}{\alpha_2{}^2},$$

$$a_2=-\frac{Z_8}{1176\,\alpha_2{}^4}, \tag{18}$$

with

$$Z_6=15\,\alpha_2{}^2\kappa\lambda^3\sqrt{\lambda^2-4\mu}-344\,\kappa\lambda^4\alpha_2{}^2+1380\,\alpha_2{}^2\kappa\lambda\,\mu\sqrt{\lambda^2-4\mu}+1282\,\kappa\lambda^2\mu\,\alpha_2{}^2+39\,\Lambda\alpha_2{}^2\lambda\sqrt{\lambda^2-4\mu}$$

$$-4\,\Lambda\lambda^2\alpha_2{}^2-1440\,\alpha_2\kappa\lambda\,\alpha_0\sqrt{\lambda^2-4\mu}+12\,\alpha_2{}^2\lambda\,\theta\sqrt{\lambda^2-4\mu}+30\,\kappa\lambda^2\alpha_0\alpha_2-1064\,\kappa\mu^2\alpha_2{}^2+2\,\lambda^2\theta\,\alpha_2{}^2$$

$$-62\,\Lambda\mu\,\alpha_2{}^2+2760\,\kappa\mu\,\alpha_0\alpha_2-32\,\mu\,\theta\,\alpha_2{}^2+78\,\Lambda\alpha_0\alpha_2-1440\,\kappa\alpha_0{}^2+24\,\theta\,\alpha_0\alpha_2,$$

$$
\begin{aligned}
Z_7 = & -104\,\alpha_2{}^3\kappa\lambda^5\sqrt{\lambda^2-4\mu}-4\Lambda\alpha_2{}^3\lambda^3\sqrt{\lambda^2-4\mu}+2\,\alpha_2{}^3\lambda^3\theta\sqrt{\lambda^2-4\mu}+3284\,\kappa\lambda^2\mu\,\alpha_0\alpha_2{}^2\\
& -8\Lambda\lambda^2\alpha_0\alpha_2{}^2-124\Lambda\mu\,\alpha_0\alpha_2{}^2-18\Lambda\lambda^2\mu\,\alpha_2{}^3+922\,\kappa\lambda^4\mu\,\alpha_2{}^3-928\,\kappa\lambda^4\alpha_0\alpha_2{}^2\\
& -2810\,\kappa\lambda^2\mu^2\alpha_2{}^3-12\,\lambda^2\mu\,\theta\,\alpha_2{}^3-90\,\kappa\lambda^2\alpha_0{}^2\alpha_2-2608\,\kappa\mu^2\alpha_0\alpha_2{}^2+4\,\lambda^2\theta\,\alpha_0\alpha_2{}^2\\
& +4680\,\kappa\mu\,\alpha_0{}^2\alpha_2-64\,\mu\,\theta\,\alpha_0\alpha_2{}^2+4680\,\alpha_2{}^2\kappa\lambda\,\mu\,\alpha_0\sqrt{\lambda^2-4\mu}+202\,\alpha_2{}^3\kappa\lambda^3\mu\sqrt{\lambda^2-4\mu}\\
& -90\,\alpha_2{}^2\kappa\lambda^3\alpha_0\sqrt{\lambda^2-4\mu}-1304\,\alpha_2{}^3\kappa\lambda\,\mu^2\sqrt{\lambda^2-4\mu}-62\Lambda\alpha_2{}^3\lambda\,\mu\sqrt{\lambda^2-4\mu}\\
& -32\,\alpha_2{}^3\lambda\,\mu\,\theta\sqrt{\lambda^2-4\mu}+78\Lambda\alpha_2{}^2\lambda\,\alpha_0\sqrt{\lambda^2-4\mu}-2160\,\alpha_2\kappa\lambda\,\alpha_0{}^2\sqrt{\lambda^2-4\mu}\\
& +24\,\alpha_2{}^2\lambda\,\theta\,\alpha_0\sqrt{\lambda^2-4\mu}-1440\,\alpha_0{}^3\kappa-1624\,\kappa\mu^3\alpha_2{}^3-58\Lambda\mu^2\alpha_2{}^3+78\Lambda\alpha_0{}^2\alpha_2\\
& +13\Lambda\lambda^4\alpha_2{}^3-7\,\kappa\lambda^6\alpha_2{}^3+24\,\theta\,\alpha_0{}^2\alpha_2+8\,\mu^2\theta\,\alpha_2{}^3+4\,\lambda^4\theta\,\alpha_2{}^3,\\
Z_8 = & -6528\,\alpha_2{}^5\lambda^3\mu\,\theta^2a_4\alpha_0\sqrt{\lambda^2-4\mu}+37968\,\Lambda^2\alpha_2{}^5\lambda\,\mu^2a_4\alpha_0\sqrt{\lambda^2-4\mu}\\
& +108000\Lambda\alpha_2{}^3\kappa\lambda^3a_4\alpha_0{}^3\sqrt{\lambda^2-4\mu}+2160\Lambda\alpha_2{}^4\lambda^3\theta\,a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu}\\
& +352961280\,\alpha_2{}^3\kappa^2\lambda\,\mu^2a_4\alpha_0{}^3\sqrt{\lambda^2-4\mu}-190080\,\alpha_2{}^3\kappa\lambda^3\theta\,a_4\alpha_0{}^3\sqrt{\lambda^2-4\mu}\\
& +26880\,\alpha_2{}^5\lambda\,\mu^2\theta^2a_4\alpha_0\sqrt{\lambda^2-4\mu}-174096\,\Lambda^2\alpha_2{}^4\lambda\,\mu\,a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu}\\
& -202176000\,\alpha_2{}^2\kappa^2\lambda\,\mu\,a_4\alpha_0{}^4\sqrt{\lambda^2-4\mu}-27648\,\alpha_2{}^4\lambda\,\mu\,\theta^2a_4\alpha_0{}^2\sqrt{\lambda^2-4\mu}\\
& -3369600\Lambda\alpha_2{}^2\kappa\lambda\,a_4\alpha_0{}^4\sqrt{\lambda^2-4\mu}+44928\Lambda\alpha_2{}^3\lambda\,\theta\,a_4\alpha_0{}^3\sqrt{\lambda^2-4\mu}\\
& -1036800\,\alpha_2{}^2\kappa\lambda\,\theta\,a_4\alpha_0{}^4\sqrt{\lambda^2-4\mu}+393120\,\alpha_2{}^4\kappa\lambda\,\mu\,a_3\alpha_0\sqrt{\lambda^2-4\mu}\\
& +14928\Lambda\alpha_2{}^6\kappa\lambda^7\mu\,a_4\sqrt{\lambda^2-4\mu}-7736784\,\alpha_2{}^5\kappa^2\lambda^7\mu\,a_4\alpha_0\sqrt{\lambda^2-4\mu}\\
& +49488\,\alpha_2{}^6\kappa\lambda^7\mu\,\theta\,a_4\sqrt{\lambda^2-4\mu}+33936\Lambda\alpha_2{}^5\kappa\lambda^7a_4\alpha_0\sqrt{\lambda^2-4\mu}\\
& -725760\Lambda\alpha_2{}^6\kappa\lambda^5\mu^2a_4\sqrt{\lambda^2-4\mu}+80550864\,\alpha_2{}^5\kappa^2\lambda^5\mu^2a_4\alpha_0\sqrt{\lambda^2-4\mu}
\end{aligned}
$$

$$- 33600\,\alpha_2{}^5\kappa\,\lambda^7\theta\,a_4\alpha_0\sqrt{\lambda^2-4\,\mu} - 523152\,\alpha_2{}^6\kappa\,\lambda^5\mu^2\theta\,a_4\sqrt{\lambda^2-4\,\mu}$$

$$+ 2309664\,\Lambda\,\alpha_2{}^6\kappa\,\lambda^3\mu^3a_4\sqrt{\lambda^2-4\,\mu} - 7824\,\Lambda\,\alpha_2{}^6\lambda^5\mu\,\theta\,a_4\sqrt{\lambda^2-4\,\mu}$$

$$- 85620240\,\alpha_2{}^4\kappa^2\lambda^5\mu\,a_4\alpha_0{}^2\sqrt{\lambda^2-4\,\mu} - 213765504\,\alpha_2{}^5\kappa^2\lambda^3\mu^3a_4\alpha_0\sqrt{\lambda^2-4\,\mu}$$

$$+ 1247232\,\alpha_2{}^6\kappa\,\lambda^3\mu^3\theta\,a_4\sqrt{\lambda^2-4\,\mu} - 1289952\,\Lambda\,\alpha_2{}^4\kappa\,\lambda^5a_4\alpha_0{}^2\sqrt{\lambda^2-4\,\mu}$$

$$+ 2115840\,\Lambda\,\alpha_2{}^6\kappa\,\lambda\,\mu^4a_4\sqrt{\lambda^2-4\,\mu} + 7104\,\Lambda\,\alpha_2{}^5\lambda^5\theta\,a_4\alpha_0\sqrt{\lambda^2-4\,\mu}$$

$$+ 14064\,\Lambda\,\alpha_2{}^6\lambda^3\mu^2\theta\,a_4\sqrt{\lambda^2-4\,\mu} + 272833920\,\alpha_2{}^4\kappa^2\lambda^3\mu^2a_4\alpha_0{}^2\sqrt{\lambda^2-4\,\mu}$$

$$- 50393856\,\alpha_2{}^5\kappa^2\lambda\,\mu^4a_4\alpha_0\sqrt{\lambda^2-4\,\mu} - 407376\,\alpha_2{}^4\kappa\,\lambda^5\theta\,a_4\alpha_0{}^2\sqrt{\lambda^2-4\,\mu}$$

$$+ 498432\,\alpha_2{}^6\kappa\,\lambda\,\mu^4\theta\,a_4\sqrt{\lambda^2-4\,\mu} - 4944\,\Lambda^2\alpha_2{}^5\lambda^3\mu\,a_4\alpha_0\sqrt{\lambda^2-4\,\mu}$$

$$+ 16320\,\Lambda\,\alpha_2{}^6\lambda\,\mu^3\theta\,a_4\sqrt{\lambda^2-4\,\mu} - 98720640\,\alpha_2{}^3\kappa^2\lambda^3\mu\,a_4\alpha_0{}^3\sqrt{\lambda^2-4\,\mu}$$

$$- 177603840\,\alpha_2{}^4\kappa^2\lambda\,\mu^3a_4\alpha_0{}^2\sqrt{\lambda^2-4\,\mu} + 4800\,\alpha_2{}^6\lambda^3\mu^2\theta^2a_4\sqrt{\lambda^2-4\,\mu}$$

$$+ 43152\,\Lambda^2\alpha_2{}^6\lambda\,\mu^3a_4\sqrt{\lambda^2-4\,\mu} - 3072\,\alpha_2{}^6\lambda\,\mu^3\theta^2a_4\sqrt{\lambda^2-4\,\mu}$$

$$- 11232\,\Lambda^2\alpha_2{}^4\lambda^3a_4\alpha_0{}^2\sqrt{\lambda^2-4\,\mu} + 3888000\,\alpha_2{}^2\kappa^2\lambda^3a_4\alpha_0{}^4\sqrt{\lambda^2-4\,\mu}$$

$$+ 1728\,\alpha_2{}^4\lambda^3\theta^2a_4\alpha_0{}^2\sqrt{\lambda^2-4\,\mu} + 16968\,\alpha_2{}^5\kappa\,\lambda^3\mu\,a_3\sqrt{\lambda^2-4\,\mu}$$

$$+ 73008\,\Lambda^2\alpha_2{}^3\lambda\,a_4\alpha_0{}^3\sqrt{\lambda^2-4\,\mu} + 37324800\,\alpha_2\kappa^2\lambda\,a_4\alpha_0{}^5\sqrt{\lambda^2-4\,\mu}$$

$$- 7560\,\alpha_2{}^4\kappa\,\lambda^3a_3\alpha_0\sqrt{\lambda^2-4\,\mu} - 109536\,\alpha_2{}^5\kappa\,\lambda\,\mu^2a_3\sqrt{\lambda^2-4\,\mu}$$

$$+ 6912\,\alpha_2{}^3\lambda\,\theta^2a_4\alpha_0{}^3\sqrt{\lambda^2-4\,\mu} - 5208\,\Lambda\,\alpha_2{}^5\lambda\,\mu\,a_3\sqrt{\lambda^2-4\,\mu}$$

$$- 2688\,\alpha_2{}^5\lambda\,\mu\,\theta\,a_3\sqrt{\lambda^2-4\,\mu} + 6552\,\Lambda\,\alpha_2{}^4\lambda\,a_3\alpha_0\sqrt{\lambda^2-4\,\mu}$$

$$- 181440\,\alpha_2{}^3\kappa\,\lambda\,a_3\alpha_0{}^2\sqrt{\lambda^2-4\,\mu} + 2016\,\alpha_2{}^4\lambda\,\theta\,a_3\alpha_0\sqrt{\lambda^2-4\,\mu}$$

$$- 1167624\, {\alpha_2}^6 \kappa^2 \lambda^9 \mu\, a_4 \sqrt{\lambda^2 - 4\mu} - 15888\, \Lambda\, {\alpha_2}^6 \kappa\, \lambda^9 a_4 \sqrt{\lambda^2 - 4\mu}$$

$$+ 1165704\, {\alpha_2}^5 \kappa^2 \lambda^9 a_4 \alpha_0 \sqrt{\lambda^2 - 4\mu} + 5851344\, {\alpha_2}^6 \kappa^2 \lambda^7 \mu^2 a_4 \sqrt{\lambda^2 - 4\mu}$$

$$- 5160\, {\alpha_2}^6 \kappa\, \lambda^9 \theta\, a_4 \sqrt{\lambda^2 - 4\mu} - 19212144\, {\alpha_2}^6 \kappa^2 \lambda^5 \mu^3 a_4 \sqrt{\lambda^2 - 4\mu}$$

$$+ 120\, \Lambda\, {\alpha_2}^6 \lambda^7 \theta\, a_4 \sqrt{\lambda^2 - 4\mu} + 1296000\, {\alpha_2}^4 \kappa^2 \lambda^7 a_4 {\alpha_0}^2 \sqrt{\lambda^2 - 4\mu}$$

$$+ 40034304\, {\alpha_2}^6 \kappa^2 \lambda^3 \mu^4 a_4 \sqrt{\lambda^2 - 4\mu} - 8808\, \Lambda^2 {\alpha_2}^6 \lambda^5 \mu\, a_4 \sqrt{\lambda^2 - 4\mu}$$

$$+ 25412352\, {\alpha_2}^6 \kappa^2 \lambda\, \mu^5 a_4 \sqrt{\lambda^2 - 4\mu} - 1824\, {\alpha_2}^6 \lambda^5 \mu\, \theta^2 a_4 \sqrt{\lambda^2 - 4\mu}$$

$$+ 12552\, \Lambda^2 {\alpha_2}^5 \lambda^5 a_4 \alpha_0 \sqrt{\lambda^2 - 4\mu} + 16176\, \Lambda^2 {\alpha_2}^6 \lambda^3 \mu^2 a_4 \sqrt{\lambda^2 - 4\mu}$$

$$+ 25948080\, {\alpha_2}^3 \kappa^2 \lambda^5 a_4 {\alpha_0}^3 \sqrt{\lambda^2 - 4\mu} + 1248\, {\alpha_2}^5 \lambda^5 \theta^2 a_4 \alpha_0 \sqrt{\lambda^2 - 4\mu}$$

$$+ 2280672\, \Lambda\, {\alpha_2}^5 \kappa\, \lambda^5 \mu\, a_4 \alpha_0 \sqrt{\lambda^2 - 4\mu} + 1027872\, {\alpha_2}^5 \kappa\, \lambda^5 \mu\, \theta\, a_4 \alpha_0 \sqrt{\lambda^2 - 4\mu}$$

$$- 6071904\, \Lambda\, {\alpha_2}^5 \kappa\, \lambda^3 \mu^2 a_4 \alpha_0 \sqrt{\lambda^2 - 4\mu} - 3033216\, {\alpha_2}^5 \kappa\, \lambda^3 \mu^2 \theta\, a_4 \alpha_0 \sqrt{\lambda^2 - 4\mu}$$

$$+ 3256416\, \Lambda\, {\alpha_2}^4 \kappa\, \lambda^3 \mu\, a_4 {\alpha_0}^2 \sqrt{\lambda^2 - 4\mu} - 896640\, \Lambda\, {\alpha_2}^5 \kappa\, \lambda\, \mu^3 a_4 \alpha_0 \sqrt{\lambda^2 - 4\mu}$$

$$- 16224\, \Lambda\, {\alpha_2}^5 \lambda^3 \mu\, \theta\, a_4 \alpha_0 \sqrt{\lambda^2 - 4\mu} + 1755648\, {\alpha_2}^4 \kappa\, \lambda^3 \mu\, \theta\, a_4 {\alpha_0}^2 \sqrt{\lambda^2 - 4\mu}$$

$$+ 1984512\, {\alpha_2}^5 \kappa\, \lambda\, \mu^3 \theta\, a_4 \alpha_0 \sqrt{\lambda^2 - 4\mu} - 12604032\, \Lambda\, {\alpha_2}^4 \kappa\, \lambda\, \mu^2 a_4 {\alpha_0}^2 \sqrt{\lambda^2 - 4\mu}$$

$$+ 86016\, \Lambda\, {\alpha_2}^5 \lambda\, \mu^2 \theta\, a_4 \alpha_0 \sqrt{\lambda^2 - 4\mu} - 6725376\, {\alpha_2}^4 \kappa\, \lambda\, \mu^2 \theta\, a_4 {\alpha_0}^2 \sqrt{\lambda^2 - 4\mu}$$

$$+ 13046400\, \Lambda\, {\alpha_2}^3 \kappa\, \lambda\, \mu\, a_4 {\alpha_0}^3 \sqrt{\lambda^2 - 4\mu} - 143424\, \Lambda\, {\alpha_2}^4 \lambda\, \mu\, \theta\, a_4 {\alpha_0}^2 \sqrt{\lambda^2 - 4\mu}$$

$$+ 4907520\, {\alpha_2}^3 \kappa\, \lambda\, \mu\, \theta\, a_4 {\alpha_0}^3 \sqrt{\lambda^2 - 4\mu} + 8736\, {\alpha_2}^6 \kappa^2 \lambda^{11} a_4 \sqrt{\lambda^2 - 4\mu}$$

$$- 624\, \Lambda^2 {\alpha_2}^6 \lambda^7 a_4 \sqrt{\lambda^2 - 4\mu} + 96\, {\alpha_2}^6 \lambda^7 \theta^2 a_4 \sqrt{\lambda^2 - 4\mu} - 8736\, {\alpha_2}^5 \kappa\, \lambda^5 a_3 \sqrt{\lambda^2 - 4\mu}$$

$$- 336\, \Lambda\, {\alpha_2}^5 \lambda^3 a_3 \sqrt{\lambda^2 - 4\mu} + 168\, {\alpha_2}^5 \lambda^3 \theta\, a_3 \sqrt{\lambda^2 - 4\mu}$$

$$
\begin{aligned}
&+21840\,\kappa\lambda\,\mu\,\alpha_2{}^5\sqrt{\lambda^2-4\,\mu}-5460\,\kappa\lambda^3\alpha_2{}^5\sqrt{\lambda^2-4\,\mu}-588\,\Lambda\lambda\,\alpha_2{}^5\sqrt{\lambda^2-4\,\mu}\\
&-1176\,\Lambda\,\alpha_0\alpha_2{}^4+12441600\,\kappa^2 a_4\alpha_0{}^6-2268\,\Lambda\mu\,\alpha_2{}^5-728\,\mu\,\theta\,\alpha_2{}^5-5824\,\kappa\lambda^2\mu\,\alpha_2{}^5\\
&+1110\,\Lambda^2\lambda^8 a_4\alpha_2{}^6+20184\,\Lambda^2\mu^4 a_4\alpha_2{}^6+36504\,\Lambda^2 a_4\alpha_0{}^4\alpha_2{}^2+1092\,\Lambda\lambda^4 a_3\alpha_2{}^5\\
&-4872\,\Lambda\mu^2 a_3\alpha_2{}^5+6552\,\Lambda a_3\alpha_0{}^2\alpha_2{}^3-10920\,\kappa\lambda^2\alpha_0\alpha_2{}^4+65190\,\kappa^2\lambda^{12} a_4\alpha_2{}^6\\
&+120\,\lambda^8\theta^2 a_4\alpha_2{}^6+15824256\,\kappa^2\mu^6 a_4\alpha_2{}^6+384\,\mu^4\theta^2 a_4\alpha_2{}^6+3456\,\theta^2 a_4\alpha_0{}^4\alpha_2{}^2\\
&-588\,\kappa\lambda^6 a_3\alpha_2{}^5-10192\,\kappa\mu^2\alpha_2{}^5+861\,\Lambda\lambda^2\alpha_2{}^5+2093\,\kappa\lambda^4\alpha_2{}^5+182\,\lambda^2\theta\,\alpha_2{}^5\\
&+44078040\,\kappa^2\lambda^4 a_4\alpha_0{}^4\alpha_2{}^2-50393856\,\kappa^2\mu^4 a_4\alpha_0{}^2\alpha_2{}^4+4704\,\lambda^4\theta^2 a_4\alpha_0{}^2\alpha_2{}^4\\
&-118402560\,\kappa^2\mu^3 a_4\alpha_0{}^3\alpha_2{}^3-6144\,\mu^3\theta^2 a_4\alpha_0\alpha_2{}^5+1555200\,\kappa^2\lambda^2 a_4\alpha_0{}^5\alpha_2\\
&-79728\,\Lambda^2\lambda^2\mu^3 a_4\alpha_2{}^6+1130304\,\Lambda\kappa\mu^5 a_4\alpha_2{}^6+49056\,\Lambda^2\lambda^4 a_4\alpha_0{}^2\alpha_2{}^4\\
&-5568\,\Lambda\mu^4\theta\,a_4\alpha_2{}^6+86304\,\Lambda^2\mu^3 a_4\alpha_0\alpha_2{}^5-7488\,\Lambda^2\lambda^2 a_4\alpha_0{}^3\alpha_2{}^3\\
&-589128\,\kappa^2\lambda^{10}\mu\,a_4\alpha_2{}^6+190272\,\kappa^2\lambda^{10} a_4\alpha_0\alpha_2{}^5-414720\,\kappa\,\theta\,a_4\alpha_0{}^5\alpha_2\\
&+77448\,\kappa\lambda^4\mu\,a_3\alpha_2{}^5-77952\,\kappa\lambda^4 a_3\alpha_0\alpha_2{}^4-236040\,\kappa\lambda^2\mu^2 a_3\alpha_2{}^5\\
&-1008\,\lambda^2\mu\,\theta\,a_3\alpha_2{}^5-7560\,\kappa\lambda^2 a_3\alpha_0{}^2\alpha_2{}^3-219072\,\kappa\mu^2 a_3\alpha_0\alpha_2{}^4\\
&+336\,\lambda^2\theta\,a_3\alpha_0\alpha_2{}^4+393120\,\kappa\mu\,a_3\alpha_0{}^2\alpha_2{}^3-5376\,\mu\,\theta\,a_3\alpha_0\alpha_2{}^4+8217144\,\kappa^2\lambda^8\mu^2 a_4\alpha_2{}^6\\
&-2832\,\kappa\lambda^{10}\theta\,a_4\alpha_2{}^6-41603184\,\kappa^2\lambda^6\mu^3 a_4\alpha_2{}^6+7918944\,\kappa^2\lambda^8 a_4\alpha_0{}^2\alpha_2{}^4\\
&+52254744\,\kappa^2\lambda^4\mu^4 a_4\alpha_2{}^6+13951296\,\kappa^2\lambda^2\mu^5 a_4\alpha_2{}^6-1440\,\lambda^6\mu\,\theta^2 a_4\alpha_2{}^6\\
&+3456000\,\kappa^2\lambda^6 a_4\alpha_0{}^3\alpha_2{}^3+768\,\lambda^6\theta^2 a_4\alpha_0\alpha_2{}^5+10464\,\lambda^4\mu^2\theta^2 a_4\alpha_2{}^6\\
&+50824704\,\kappa^2\mu^5 a_4\alpha_0\alpha_2{}^5-155904\,\kappa\mu^5\theta\,a_4\alpha_2{}^6-25728\,\lambda^2\mu^3\theta^2 a_4\alpha_2{}^6\\
&+219360\,\Lambda\lambda^2\mu^2\theta\,a_4\alpha_0\alpha_2{}^5+11156544\,\Lambda\kappa\lambda^2\mu\,a_4\alpha_0{}^3\alpha_2{}^3-106080\,\Lambda\lambda^2\mu\,\theta\,a_4\alpha_0{}^2\alpha_2{}^4
\end{aligned}
$$

$$+674400\,\kappa\lambda^6\mu\,\theta\,a_4\alpha_0\alpha_2{}^5-4523040\,\kappa\lambda^4\mu^2\theta\,a_4\alpha_0\alpha_2{}^5+3861792\,\kappa\lambda^4\mu\,\theta\,a_4\alpha_0{}^2\alpha_2{}^4$$

$$+11461632\,\kappa\lambda^2\mu^3\theta\,a_4\alpha_0\alpha_2{}^5-12848256\,\kappa\lambda^2\mu^2\theta\,a_4\alpha_0{}^2\alpha_2{}^4+3935232\,\kappa\lambda^2\mu\,\theta\,a_4\alpha_0{}^3\alpha_2{}^3$$

$$+87936\,\Lambda\kappa\lambda^6a_4\alpha_0{}^2\alpha_2{}^4-1574160\,\Lambda\kappa\lambda^2\mu^4a_4\alpha_2{}^6+960\,\Lambda\lambda^6\theta\,a_4\alpha_0\alpha_2{}^5$$

$$+24672\,\Lambda\lambda^4\mu^2\theta\,a_4\alpha_2{}^6-60672\,\Lambda^2\lambda^4\mu\,a_4\alpha_0\alpha_2{}^5-88608\,\Lambda\lambda^2\mu^3\theta\,a_4\alpha_2{}^6$$

$$+264480\,\Lambda^2\lambda^2\mu^2a_4\alpha_0\alpha_2{}^5-3106368\,\Lambda\kappa\lambda^4a_4\alpha_0{}^3\alpha_2{}^3+4231680\,\Lambda\kappa\mu^4a_4\alpha_0\alpha_2{}^5$$

$$+29568\,\Lambda\lambda^4\theta\,a_4\alpha_0{}^2\alpha_2{}^4-150960\,\Lambda^2\lambda^2\mu\,a_4\alpha_0{}^2\alpha_2{}^4-896640\,\Lambda\kappa\mu^3a_4\alpha_0{}^2\alpha_2{}^4$$

$$+32640\,\Lambda\mu^3\theta\,a_4\alpha_0\alpha_2{}^5+193056\,\Lambda\kappa\lambda^8\mu\,a_4\alpha_2{}^6+54000\,\Lambda\kappa\lambda^2a_4\alpha_0{}^4\alpha_2{}^2$$

$$-8402688\,\Lambda\kappa\mu^2a_4\alpha_0{}^3\alpha_2{}^3+1440\,\Lambda\lambda^2\theta\,a_4\alpha_0{}^3\alpha_2{}^3+86016\,\Lambda\mu^2\theta\,a_4\alpha_0{}^2\alpha_2{}^4$$

$$+6523200\,\Lambda\kappa\mu\,a_4\alpha_0{}^4\alpha_2{}^2-95616\,\Lambda\mu\,\theta\,a_4\alpha_0{}^3\alpha_2{}^3-237120\,\Lambda\kappa\lambda^8a_4\alpha_0\alpha_2{}^5$$

$$-991056\,\Lambda\kappa\lambda^6\mu^2a_4\alpha_2{}^6+1032864\,\Lambda\kappa\lambda^4\mu^3a_4\alpha_2{}^6-2304\,\Lambda\lambda^6\mu\,\theta\,a_4\alpha_2{}^6$$

$$-95040\,\kappa\lambda^2\theta\,a_4\alpha_0{}^4\alpha_2{}^2-4483584\,\kappa\mu^2\theta\,a_4\alpha_0{}^3\alpha_2{}^3+2453760\,\kappa\mu\,\theta\,a_4\alpha_0{}^4\alpha_2{}^2$$

$$+275856\,\kappa\lambda^2\mu\,a_3\alpha_0\alpha_2{}^4-17051328\,\kappa^2\lambda^8\mu\,a_4\alpha_0\alpha_2{}^5+100032\,\kappa\lambda^8\mu\,\theta\,a_4\alpha_2{}^6$$

$$+104833248\,\kappa^2\lambda^6\mu^2a_4\alpha_0\alpha_2{}^5-76992\,\kappa\lambda^8\theta\,a_4\alpha_0\alpha_2{}^5-556320\,\kappa\lambda^6\mu^2\theta\,a_4\alpha_2{}^6$$

$$-59226864\,\kappa^2\lambda^6\mu\,a_4\alpha_0{}^2\alpha_2{}^4-245749728\,\kappa^2\lambda^4\mu^3a_4\alpha_0\alpha_2{}^5+1351392\,\kappa\lambda^4\mu^3\theta\,a_4\alpha_2{}^6$$

$$+354939984\,\kappa^2\lambda^4\mu^2a_4\alpha_0{}^2\alpha_2{}^4+316873728\,\kappa^2\lambda^2\mu^4a_4\alpha_0\alpha_2{}^5-128640\,\kappa\lambda^6\theta\,a_4\alpha_0{}^2\alpha_2{}^4$$

$$-2038848\,\kappa\lambda^2\mu^4\theta\,a_4\alpha_2{}^6-202232160\,\kappa^2\lambda^4\mu\,a_4\alpha_0{}^3\alpha_2{}^3-136416\,\kappa\mu^3a_3\alpha_2{}^5$$

$$+336\,\lambda^4\theta\,a_3\alpha_2{}^5-919688064\,\kappa^2\lambda^2\mu^3a_4\alpha_0{}^2\alpha_2{}^4-15168\,\lambda^4\mu\,\theta^2a_4\alpha_0\alpha_2{}^5$$

$$+721025280\,\kappa^2\lambda^2\mu^2a_4\alpha_0{}^3\alpha_2{}^3-962784\,\kappa\lambda^4\theta\,a_4\alpha_0{}^3\alpha_2{}^3+996864\,\kappa\mu^4\theta\,a_4\alpha_0\alpha_2{}^5$$

$$+46464\,\lambda^2\mu^2\theta^2a_4\alpha_0\alpha_2{}^5-173776320\,\kappa^2\lambda^2\mu\,a_4\alpha_0{}^4\alpha_2{}^2+1984512\,\kappa\mu^3\theta\,a_4\alpha_0{}^2\alpha_2{}^4$$

$$
\begin{aligned}
&-20352\,\lambda^2\mu\,\theta^2 a_4 {\alpha_0}^2 {\alpha_2}^4 + 672\,\mu^2\theta\, a_3 {\alpha_2}^5 + 1038144\,\Lambda\,\kappa\,\lambda^6\mu\, a_4\alpha_0 {\alpha_2}^5 \\
&-6400512\,\Lambda\,\kappa\,\lambda^4\mu^2 a_4\alpha_0 {\alpha_2}^5 + 8587872\,\Lambda\,\kappa\,\lambda^4\mu\, a_4 {\alpha_0}^2 {\alpha_2}^4 + 21424704\,\Lambda\,\kappa\,\lambda^2\mu^3 a_4\alpha_0 {\alpha_2}^5 \\
&-66336\,\Lambda\,\lambda^4\mu\,\theta\, a_4\alpha_0 {\alpha_2}^5 - 32164704\,\Lambda\,\kappa\,\lambda^2\mu^2 a_4 {\alpha_0}^2 {\alpha_2}^4 - 120960\,\kappa\, a_3 {\alpha_0}^3 {\alpha_2}^2 \\
&+2016\,\theta\, a_3 {\alpha_0}^2 {\alpha_2}^3 + 43680\,\kappa\,\mu\,\alpha_0 {\alpha_2}^4 + 37968\,\Lambda^2\mu^2 a_4 {\alpha_0}^2 {\alpha_2}^4 \\
&-116064\,\Lambda^2\mu\, a_4 {\alpha_0}^3 {\alpha_2}^3 + 176480640\,\kappa^2\mu^2 a_4 {\alpha_0}^4 {\alpha_2}^2 + 1152\,\lambda^2\theta^2 a_4 {\alpha_0}^3 {\alpha_2}^3 \\
&+26880\,\mu^2\theta^2 a_4 {\alpha_0}^2 {\alpha_2}^4 - 80870400\,\kappa^2\mu\, a_4 {\alpha_0}^5 \alpha_2 - 18432\,\mu\,\theta^2 a_4 {\alpha_0}^3 {\alpha_2}^3 \\
&+3900\,\Lambda\,\kappa\,\lambda^{10} a_4 {\alpha_2}^6 - 1347840\,\Lambda\,\kappa\, a_4 {\alpha_0}^5 \alpha_2 + 22464\,\Lambda\,\theta\, a_4 {\alpha_0}^4 {\alpha_2}^2 \\
&-1512\,\Lambda\,\lambda^2\mu\, a_3 {\alpha_2}^5 - 672\,\Lambda\,\lambda^2 a_3\alpha_0 {\alpha_2}^4 - 10416\,\Lambda\,\mu\, a_3\alpha_0 {\alpha_2}^4 \\
&+528\,\Lambda\,\lambda^8\theta\, a_4 {\alpha_2}^6 - 216\,\Lambda^2\lambda^6\mu\, a_4 {\alpha_2}^6 - 4992\,\Lambda^2\lambda^6 a_4\alpha_0 {\alpha_2}^5 + 4056\,\Lambda^2\lambda^4\mu^2 a_4 {\alpha_2}^6.
\end{aligned}
$$

It is important to note that α_0, α_2, a_3, Λ, κ, δ are used as free parameters in all of the obtained parameter values.

Then after, using (18) and (17) in conjunction with $\tau = -\omega t + x$, $F(\tau) = q(x,\, t)$ and the solution of equation (5), we have got the following solution structures for equation (14):

Regrading the instance $\lambda^2 - 4\mu > 0$, we have extracted following solution structure for equation (1):

$$
\begin{aligned}
Q(x,\, t) = \alpha_0 &+ \left(\frac{\sqrt{\lambda^2-4\mu}\left(w_1 \sinh\left(\frac{1}{2}(x-\omega t)\sqrt{\lambda^2-4\mu}\right) + w_2\cosh\left(\frac{1}{2}(x-\omega t)\sqrt{\lambda^2-4\mu}\right)\right)}{2w_2\sinh\left(\frac{1}{2}(x-\omega t)\sqrt{\lambda^2-4\mu}\right) + 2w_1\cosh\left(\frac{1}{2}(x-\omega t)\sqrt{\lambda^2-4\mu}\right)} - \frac{\lambda}{2} \right)^2 \alpha_2 \\
&+ \left(\frac{\sqrt{\lambda^2-4\mu}\left(w_1 \sinh\left(\frac{1}{2}(x-\omega t)\sqrt{\lambda^2-4\mu}\right) + w_2\cosh\left(\frac{1}{2}(x-\omega t)\sqrt{\lambda^2-4\mu}\right)\right)}{2w_2\sinh\left(\frac{1}{2}(x-\omega t)\sqrt{\lambda^2-4\mu}\right) + 2w_1\cosh\left(\frac{1}{2}(x-\omega t)\sqrt{\lambda^2-4\mu}\right)} - \frac{\lambda}{2} \right) \\
&\left(\lambda - \sqrt{\lambda^2-4\mu}\right)\alpha_2,
\end{aligned}
\tag{19}
$$

equipped with w_2, w_1 as free parameters.

The solution (19) recovers shock waves for $w_1 \neq 0$, $w_2 = 0$, while obtains singular solitary waves for $w_2 \neq 0$, $w_1 = 0$.

2.2.2 *Solutions to (15) for m = 2*

Taking into account $\boldsymbol{m} = \mathbf{2}$, the equation (15) now becomes as follows:

$$\Big((-\nu F-\theta)F''-\xi F'''+a_1F^2-\delta\,(F)^2-\omega\Big)F'+\left(a_4\omega^2-\Lambda F-a_3\omega+a_2\right)F'''$$

$$+(-\kappa F-\psi)F''''' = 0. \tag{20}$$

The solution structure for equation (20) is generated by achieving homogeneous equilibrium between the highest order linear and most nonlinear derivative containing terms:

$$F(\tau)=\alpha_0+\alpha_1\left(\frac{G'(\tau)}{G(\tau)}\right)+\alpha_2\left(\frac{G'(\tau)}{G(\tau)}\right)^2, \tag{21}$$

where $G(\tau)$ follows the auxiliary equation (5), and α_i, $i=0,1,2$ are constants that must be found in computation procedure. Together with (5), we have recovered the following parameter values for obtaining solutions for equation (14) by using (21) in equation (20):

$$\Lambda=\frac{1}{3}\kappa\left(\lambda^2-4\mu\right),\ \nu=-\frac{60\kappa}{\alpha_2},$$

$$\omega=-\frac{\alpha_2\left(23\kappa\lambda^6-276\kappa\lambda^4\mu+1104\kappa\lambda^2\mu^2-39\lambda^4\theta-1472\kappa\mu^3+312\lambda^2\mu\theta-624\mu^2\theta\right)}{252},$$

$$\psi=\frac{\alpha_2\left(32\kappa\lambda^2-128\kappa\mu-\theta\right)}{84},\ \xi=10\kappa\sqrt{\lambda^2-4\mu},$$

$$a_1=\frac{1}{7}\,\frac{148\kappa\lambda^2+7\delta\alpha_2-592\kappa\mu+12\theta}{\alpha_2},$$

$$\alpha_0=-\frac{7\lambda^2\alpha_2}{12}+\left(\frac{\lambda}{2}-\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\alpha_2\lambda+\frac{4}{3}\mu\,\alpha_2,$$

$$\alpha_1=\left(\lambda-\sqrt{\lambda^2-4\mu}\right)\alpha_2,$$

$$a_2=-\frac{\alpha_2\left(\lambda^2-4\mu\right)Z_9}{63504}, \tag{22}$$

with

$$Z_9 = 529\,\kappa^2\lambda^{10}a_4\alpha_2 - 10580\,\kappa^2\lambda^8\mu\,a_4\alpha_2 + 84640\,\kappa^2\lambda^6\mu^2a_4\alpha_2 - 1794\,\kappa\lambda^8\theta\,a_4\alpha_2$$

$$-338560\,\kappa^2\lambda^4\mu^3a_4\alpha_2 + 28704\,\kappa\lambda^6\mu\,\theta\,a_4\alpha_2 + 677120\,\kappa^2\lambda^2\mu^4a_4\alpha_2 - 172224\,\kappa\lambda^4\mu^2\theta\,a_4\alpha_2$$

$$+1521\,\lambda^6\theta^2a_4\alpha_2 - 541696\,\kappa^2\mu^5a_4\alpha_2 + 459264\,\kappa\lambda^2\mu^3\theta\,a_4\alpha_2 - 18252\,\lambda^4\mu\,\theta^2a_4\alpha_2$$

$$-459264\,\kappa\mu^4\theta\,a_4\alpha_2 + 73008\,\lambda^2\mu^2\theta^2a_4\alpha_2 - 97344\,\mu^3\theta^2a_4\alpha_2 + 5796\,\kappa\lambda^4a_3 - 46368\,\kappa\lambda^2\mu\,a_3$$

$$+92736\,\kappa\mu^2a_3 - 9828\,\lambda^2\theta\,a_3 + 179424\,\kappa\lambda^2 + 39312\,\mu\,\theta\,a_3 - 717696\,\kappa\mu + 9828\,\theta.$$

It is important to note that α_2, a_3, θ, κ, δ are all free parameters in all of the obtained parameter values.

Then after, (21) is updated using the solution of equation (5) and parameter values obtained in (22), followed by with reverting back to original variables x, t and we have got the following solution structures for equation (14):

Considering instance $\lambda^2 - 4\mu > 0$:

For equation (14), we have derived the solution structure as follows:

$$Q(x,\,t) = \left(\frac{\sqrt{\lambda^2-4\mu}\left(w_1\sinh\left(\frac{1}{2}(x-\omega t)\sqrt{\lambda^2-4\mu}\right)+w_2\cosh\left(\frac{1}{2}(x-\omega t)\sqrt{\lambda^2-4\mu}\right)\right)}{2w_2\sinh\left(\frac{1}{2}(x-\omega t)\sqrt{\lambda^2-4\mu}\right)+2w_1\cosh\left(\frac{1}{2}(x-\omega t)\sqrt{\lambda^2-4\mu}\right)}-\frac{\lambda}{2}\right)^2\alpha_2$$

$$+\left(\frac{\sqrt{\lambda^2-4\mu}\left(w_1\sinh\left(\frac{1}{2}(x-\omega t)\sqrt{\lambda^2-4\mu}\right)+w_2\cosh\left(\frac{1}{2}(x-\omega t)\sqrt{\lambda^2-4\mu}\right)\right)}{2w_2\sinh\left(\frac{1}{2}(x-\omega t)\sqrt{\lambda^2-4\mu}\right)+2w_1\cosh\left((x/2-1/2\,\omega t)\sqrt{\lambda^2-4\mu}\right)}-\frac{\lambda}{2}\right)$$

$$\left(\lambda-\sqrt{\lambda^2-4\mu}\right)\alpha_2-\frac{7\lambda^2\alpha_2}{12}+\left(\frac{\lambda}{2}-\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\alpha_2\lambda+\frac{4}{3}\mu\,\alpha_2, \tag{23}$$

accompanied with w_1, w_2 as free parameters.

The recovered solution (23) yields singular solitary waves for $w_2 \neq 0$, $w_1 = 0$ and it unfolds into shock waves for $w_1 \neq 0$, $w_2 = 0$.

3. Results and discussion

Figures 1 and 2 present a comprehensive visual analysis of two distinct wave structures: the singular solitary wave and the shock wave, respectively. These waves are governed by exact analytical solutions denoted as $Q(x,\,t)$ for the singular solitary wave, derived from solution (8), and $Q(x,\,t)$ for the shock wave, derived from solution (9). Both figures investigate the wave evolution at a fixed temporal point, $t = 1$, under a shared set of parameter values: $\alpha_0 = 1$, $\alpha_2 = 1$, $\lambda = 3$, $\mu = 1$, $\psi = 1$, and $\kappa = 1$.

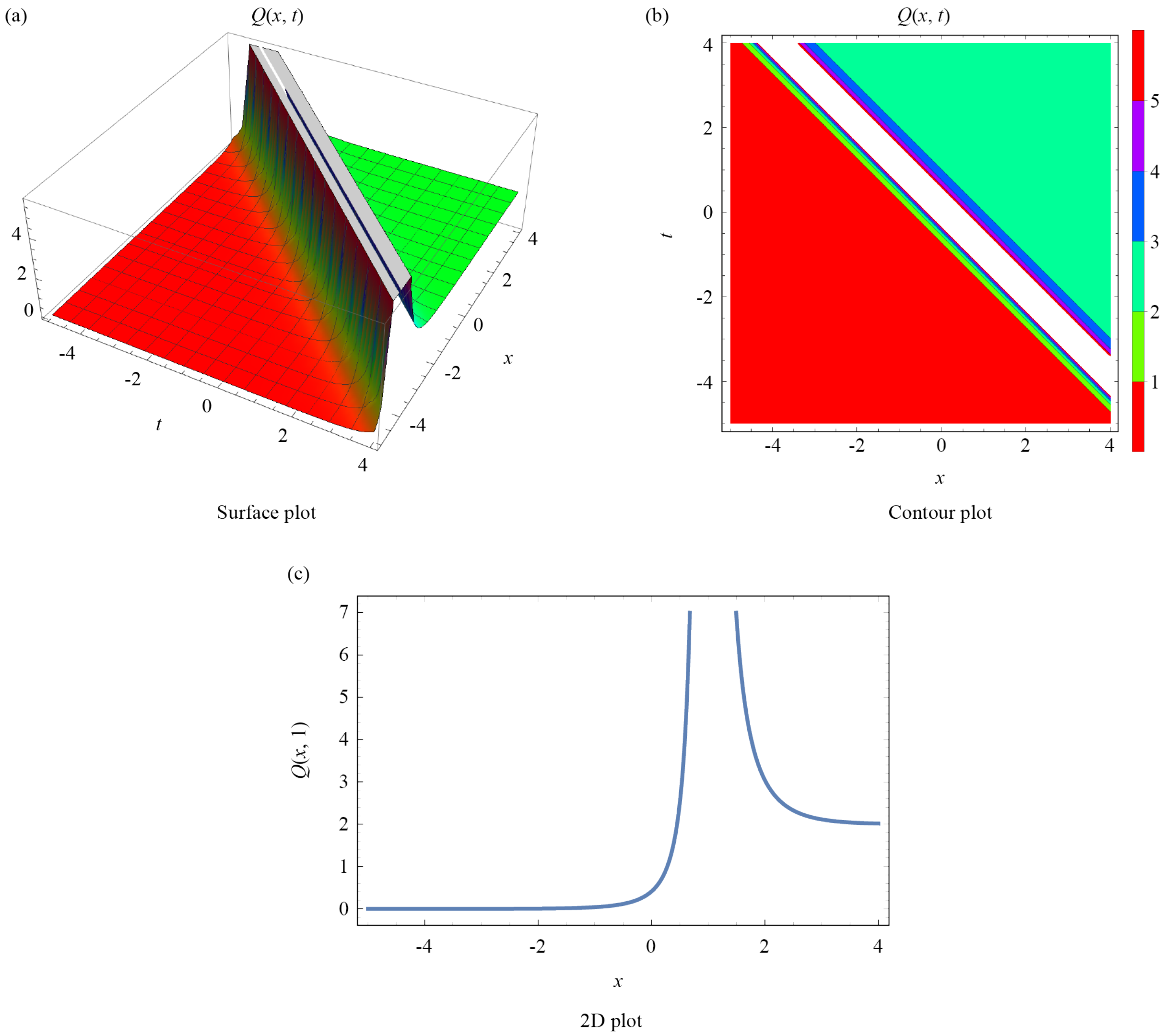

Figure 1. Analyzing the characteristics of a singular solitary wave

Figure 1 depicts the evolution of the singular solitary wave through three visual representations. Figure 1a shows the surface plot of $Q(x, t)$, where the wave exhibits a sharp, localized peak with an inherent vertical asymptote at the singularity point. This vertical spike confirms the non-regular nature of the wave, which diverges to infinity at a finite location in space-time, reflecting the classic behavior of singular solitons. Figure 1b, the contour plot, complements this interpretation by showing steep gradient transitions with tightly packed contour lines near the singular region. These contours emphasize the abrupt amplitude change and sharp localization typical of singular solitary waves. In Figure 1c, the 2D profile further clarifies the nature of the singularity, where the wave amplitude tends to infinity sharply at the singular point and decays symmetrically on either side, confirming the solitary yet singular behavior of the wave.

In contrast, Figure 2 illustrates the evolution of a shock wave governed by $Q(x, t)$. Figure 2a shows a smooth surface plot that transitions gradually from negative to positive amplitude, characteristic of a hyperbolic tangent profile. Unlike the singular solitary wave, the shock wave maintains a bounded and continuous structure without any asymptotic blow-up, highlighting its regular and non-singular nature. Figure 2b provides a contour plot that reveals a steep but finite gradient zone, marking the location of the shock front. The contour lines here are more evenly spaced compared to those of the singular wave, indicating a more gradual change in amplitude. Figure 2c, the 2D plot of the shock wave, distinctly captures the core feature of a shock wave: a sharp yet continuous transition zone that separates two asymptotic states.

This jump-like behavior, though steep, remains smooth and finite, distinguishing it from the infinite peak of the singular wave.

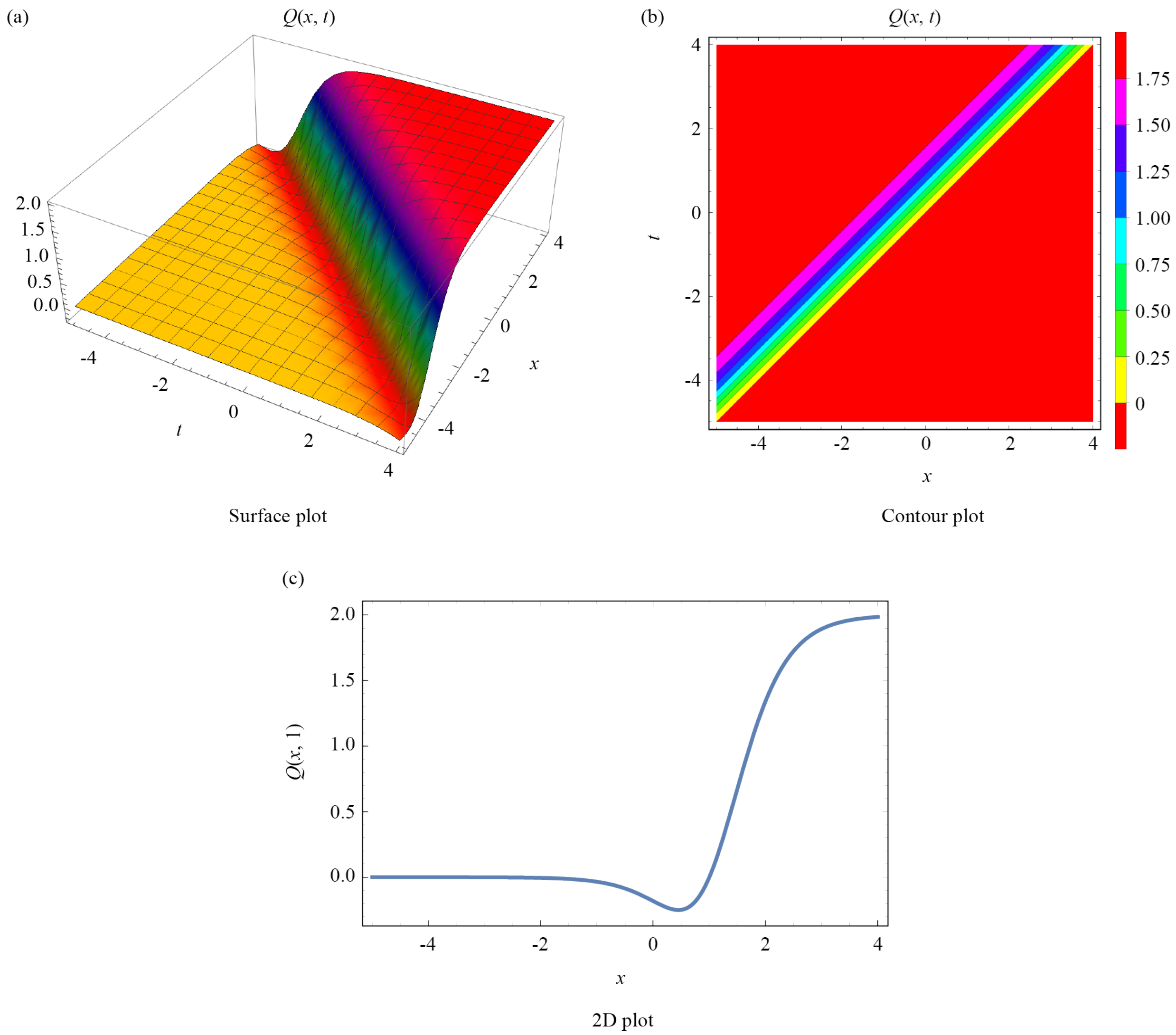

Figure 2. Analyzing the characteristics of a shock wave

Overall, Figures 1 and 2 provide a side-by-side comparison of singular and shock wave dynamics. The singular solitary wave, with its unbounded and asymmetric peak, represents an extreme localized energy concentration, potentially modeling physical scenarios involving singular perturbations or energy spikes. In contrast, the shock wave's bounded and monotonic transition reflects physical processes such as fluid shocks or nonlinear diffusion fronts. The consistent parameter setting across both figures ensures a fair comparison of the waveforms, thereby highlighting the intrinsic differences between singular solitary and shock wave structures in nonlinear dispersive media. As a result, these figures effectively visualize the distinct behaviors of two important nonlinear wave types, offering both mathematical clarity and physical interpretation for singular and shock wave solutions in nonlinear dynamical systems.

4. Conclusions

This paper recovered the solitary and shock waves for the perturbed KdV and mKdV equation. The integration algorithm that was adopted in the paper is G'/G-expansion scheme. The parameter constraints that naturally emerged from the integration scheme are also presented. The simulations of the two kinds of waves are also included. The results of the work do hold a lot of promise in this context. This scheme will be next applied to several forms of two-layered shallow water waves. This would give a wider perspective to the model. The results of such research activities are under way and once available they will be disseminated in various journals.

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Conflict of interest

The authors claim that there is no conflict of interest.

References

[1] Biswas A, Vega-Guzman J, Bansal A, Kara AH, Aphane M, Yildirim Y, et al. Solitary waves, shock waves and conservation laws with surface tension effect in Boussinesq equation. *Proceedings of the Estonian Academy of Sciences*. 2023; 72(1): 17-29. Available from: https://doi.org/10.3176/proc.2023.1.03.

[2] Antonova M, Biswas A. Adiabatic parameter dynamics of perturbed solitary waves. *Communications in Nonlinear Science and Numerical Simulation*. 2009; 14(3): 734-748. Available from: https://doi.org/10.1016/j.cnsns.2007.12.004.

[3] Clarkson PA, Kruskal MD. New similarity reductions of the Boussinesq equation. *Journal of Mathematical Physics*. 1989; 30: 2201-2213. Available from: https://doi.org/10.1063/1.528613.

[4] Lou SY, Ruan HY, Chen DF, Chen WZ. Similarity reductions of the KP equation by a direct method. *Journal of Physics A*. 1991; 24: 1455-1467. Available from: https://doi.org/10.1088/0305-4470/24/7/019.

[5] Cheng Y, Li YS. The constraint of the Kadomtsev-Petviashvili equation and its special solutions. *Physics Letters A*. 1991; 157: 22-26. Available from: https://doi.org/10.1016/0375-9601(91)90403-U.

[6] Tang XY, Lou SY. Extended multilinear variable separation approach and multivalued localized excitations for some (2 + 1)-dimensional integrable systems. *Journal of Mathematical Physics*. 2003; 64: 4000-4025. Available from: https://doi.org/10.1063/1.1598619.

[7] Wang M, Zhou Y, Li Z. Application of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics. *Physics Letters A*. 1996; 216(1-5): 67-75. Available from: https://doi.org/10.1016/0375-9601(96)00283-6.

[8] Tian SF. Lie symmetry analysis, conservation laws and solitary wave solutions to a fourth-order nonlinear generalized Boussinesq water wave equation. *Applied Mathematics Letters*. 2020; 100: 106056. Available from: https://doi.org/10.1016/j.aml.2019.106056.

[9] Wazwaz AM, Kaur L. New integrable Boussinesq equations of distinct dimensions with diverse variety of soliton solutions. *Nonlinear Dynamics*. 2029; 97: 83-94. Available from: https://doi.org/10.1007/s11071-019-04955-1.

[10] Jawad AJM, Biswas A, Yildirim Y, Alshomrani AS. A fresh perspective to the concatenation model in optical fibers with Kerr law of self-phase modulation. *Engineering Science and Technology*. 2024; 5(2): 195-208. Available from: https://doi.org/10.37256/est.5220244160.

[11] Rajan MM. Transition from bird to butterfly shaped nonautonomous soliton and soliton switching in erbium doped resonant fiber. *Physica Scripta*. 2020; 95(10): 105203. Available from: https://doi.org/10.1088/1402-4896/abb2df.

[12] Rajan MM. Unexpected behavior on nonlinear tunneling of chirped ultrashort soliton pulse in non-Kerr media with Raman effect. *Zeitschrift für Naturforschung A*. 2016; 71(8): 751-758. Available from: https://doi.org/10.1515/zna-2016-0187.

[13] Guo YS, Li W, Dong SH. Gaussian solitary solution for a class of logarithmic nonlinear Schrödinger equation in (1 + n) dimensions. *Results in Physics*. 2023; 44: 106187. Available from: https://doi.org/10.1016/j.rinp.2022.106187.

[14] López RC, Sun GH, Camacho-Nieto O, Yáñez-Márquez C, Dong SH. Analytical traveling-wave solutions to a generalized Gross-Pitaevskii equation with some new time and space varying nonlinearity coefficients and external fields. *Physics Letters A*. 2017; 381(35): 2978-2985. Available from: https://doi.org/10.1016/j.physleta.2017.07.012.

[15] Wang KJ, Liu XL, Wang WD, Li S, Zhu HW. Novel singular and non-singular complexiton, interaction wave and the complex multi-soliton solutions to the generalized nonlinear evolution equation. *Modern Physics Letters B*. 2025; 39(27): 2550135. Available from: https://doi.org/10.1142/S0217984925501350.

[16] Wang KJ, Zhu HW, Shi F, Liu XL, Wang GD, Li G. Lump wave, breather wave and other abundant wave solutions to the (2 + 1)-dimensional Sawada-Kotera-Kadomtsev-Petviashvili equation of fluid mechanics. *Pramana*. 2025; 99(1): 1-12. Available from: https://doi.org/10.1007/s12043-024-02884-2.

[17] Wang KJ, Zhu HW, Li S, Shi F, Li G, Liu XL. Bifurcation analysis, chaotic behaviors, variational principle, hamiltonian and diverse optical solitons of the fractional complex Ginzburg-Landau model. *International Journal of Theoretical Physics*. 2025; 64(5): 1-20. Available from: https://doi.org/10.1007/s10773-025-05977-9.

[18] Liang YH, Wang KJ. Diverse wave solutions to the new extended (2 + 1)-dimensional nonlinear evolution equation: Phase portrait, bifurcation and sensitivity analysis, chaotic pattern, variational principle and Hamiltonian. *International Journal of Geometric Methods in Modern Physics*. 2025. Available from: https://doi.org/10.1142/S0219887825501580.

[19] Mirzazadeh M, Eslami M, Biswas A. Soliton solutions of the generalized Klein-Gordon equation by using (G'/G)-expansion method. *Computational and Applied Mathematics*. 2014; 33: 831-839. Available from: https://doi.org/10.1007/s40314-013-0098-3.

[20] Zayed EME, Al-Joudi S. Applications of an extended (G'/G)-expansion method to find exact solutions of nonlinear PDEs in mathematical physics. *Mathematical Problems in Engineering*. 2010; 2010(1): 768573. Available from: https://doi.org/10.1155/2010/768573.