



Research Article

Solvency II, Undertaking Specific Parameters (USPs) Validation, Generalization and Criticism

Alexandros A. Zimbidis

Department of Statistics, Athens University of Economics and Business, Athens, Greece
E-mail: aaz@aueb.gr

Received: 7 November 2020; **Revised:** 18 January 2021; **Accepted:** 23 February 2021

Abstract: This paper provides a deeper actuarial insight in the mathematics and algorithms described in Delegate Act 35/2015 of Solvency II legislative framework with respect to the determination of Undertaking Specific Parameters (USPs). The numerical investigation is based on typical input signals used by control engineers in order to check the system response. It is finally revealed the close relationship between the USPs values and the values of the typical function of standard deviation. Finally, a generalization of the mathematical framework covering the special cases of the Pareto and Gamma distributions as inputs for the aggregate losses of an insurance company is provided.

Keywords: Solvency II, Standard Formula (SF), Undertaking Specific Parameters (USPs), Maximum likelihood, Control engineering, Spike-Ramp-Step signals

MSC: 91B30

1. Introduction

The new regulatory framework Solvency II [1], for all insurance companies across European Union member states, is already in force effectively from 1st of January 2016. Solvency II has adopted similar standards as Basel regulation that is in force into the banking industry and is divided into three pillars: The first pillar places all the quantitative requirements for the capital of an insurer while the second pillar deals with governance and risk management of insurers and finally the third pillar imposes the disclosure reports addressed to public and regulators.

In May 2017, all the insurance companies in Europe published their first financial reports according to the measures and guidelines imposed by Solvency II while three times more up to now in 2018, 19 and 20 have prepared such statements. These reports (encoded as SFCR-Solvency and Financial Conditions Reports) and in line with the previous measures and guidelines [2-4] were designed in order to reveal strengths and weaknesses of each insurance company. The majority of the insurers calculate the Solvency Capital Requirement (SCR) by using the Standard Formula (SF) approach all these years up today. The SF is actually a set of equations and algorithms that should be evaluated in order to calculate the amount of necessary SCR for each insurer.

Of course, the huge challenge and the great outstanding question is whether standard formula truly and adequately measures the risk within an insurance company. There is serious criticism for the SF which according to Scherer M,

et al. [5] “lacks sound economic and mathematical reasoning even minimal requirements (such as monotonicity, no arbitrage, etc.) are violated” and so the SF does not appear to be a reliable benchmark for the construction or validation of an internal model. The creation and use of internal models [6-8] may be a trustworthy alternative solution to the SF but only a few companies (these are actually amongst the largest worldwide international insurers) till today have chosen to follow this path, as it requires much technical and scientific effort.

The regulator (European Insurance Occupational Pension Authority, EIOPA) and the relevant legislative framework offer the alternative solution, before proceeding with a full or partial interval model, for the measurement of the basic risk faced by an insurer (i.e. the underwriting risk) and that is: the adoption of special distinct parameters within the application of the standard formula. These parameters are encoded as USPs-Undertaking Specific Parameters and are calculated on the basis of the own experience of an insurance company within the preceding last five or more years. The relevant framework of mathematics of USPs is determined in Delegated Act (DA) 35/2015 [9-11] that appears in the Official Journal of the European Union.

Although the calculation of USPs is a quite important topic for the whole European insurance market as it is directly related to the capital requirements of each company, there is a relevant small number of research efforts in this area [12-13]. Furthermore, there are research efforts towards the evaluation of premium and reserve risks especially in non-life business [14-16] and aggregation of risks within the Solvency II framework [17-18]. Appropriate and accurate figures of capital requirements will secure a stable and viable market that is absolutely necessary for the current turbulent economic environment.

In this paper, we question the mathematics of USPs as appear in the DA 35/2015 while also investigate the efficiency of the proposed process towards the simple calculation of the measure of standard deviation. Additionally, we substitute the basic hypothesis that appears in the text of DA 35/2015 which states that claims incurred by an insurance company are log-normally distributed on an annual basis, with other typical distributions. Furthermore, we investigate the formulas of USPs using a control engineering approach testing the output response against different input signals (premium patterns) and finally obtain generalizations and approximations with respect to other typical distributions.

The different typical input signals: spike, step, ramp and cycle, test the sensitivity of the determination algorithm and reveal the relationship with the value of the respective standard deviation. So, having established this connection between the two results, an insurer has a simple yet reliable tool for the validation of the complex algorithm of USPs' final value.

The rest of this paper is organized as follows:

In Section 2, we shortly review the concept of underwriting risk, the calculations proposed by the SF for this risk and also the first (premium risk) method that appears in DA 35/2015 for the determination of the USPs values. In Section 3, we fully investigate the mathematical mechanism with respect to the development pattern of premiums of an insurance company and compare the numerical results with the numerical results of the typical values of standard deviation. In Section 4, we remove the basic hypothesis of the lognormal distribution for the claim experience and using an alternative option for a Pareto and Gamma distribution, we obtain the relevant formulas. In Section 5, we formulate the conclusions and some further remarks for the specific topic.

2. The underwriting risk-standard formula (SF)-USPs determination

In this section, we actually provide the general background of the problem and formulate the basis for our theoretical approach. We split the analysis into three directions:

The underwriting risk which deals with nature, definition and management options,

SF which deals with the numerical calculations of the volume of the underwriting risk and,

USPs determination which deals with the mathematical framework imposed by DA 35/2015 and aims to substitute certain parameters of the formula involved in the underwriting risk calculation.

2.1 Underwriting risk

The central component within the total risk inherited in the insurance business is the underwriting risk. Actually, that is the insurance risk. The other risks are market risk, catastrophe risk, health risk, operational risk etc.

The underwriting risk is directly related to the nature of the insurance business and can be split into two sub-risks: (a) the premium risk and (b) the reserve risk. In the next lines, we provide short definitions for them.

2.1.1 Premium risk

As quoted in IFRS 17 (the new International Financial Reporting Standard for insurance business), an insurance contract is a contract where an entity (insurer) “accepts significant insurance risk from another party (the policyholder) by agreeing to compensate the policyholder if a specified uncertain future event (the insured event) adversely affects the policyholder”.

This agreement is accompanied by an initial money transfer (premium) from the policyholder to the insurer. So, the insurer receives lots of premiums in exchange for the promise to cover uncertain future events. The premium risk is the risk borne by the insurer that the received total premiums will not be adequate to cover the total claims caused by the uncertain future events.

2.1.2 Reserve Risk

Furthermore, in the usual insurance business context, there is another risk involved, that is the reserve risk. It is quite common upon the occurrence of an uncertain event that the relevant amount of claim is not fixed and fully determinable. So, the insurer should produce estimations as regards the amount of money needed for the claim settlement. At this point, the reserve risk appears as the risk that the initially estimated amount of money (initial claim reserve) will not be adequate to cover the policyholder’s final appeal.

2.2 Standard Formula (SF) for underwriting risk

As stated before, the Solvency II framework suggests a typical mathematical framework for the calculation of all risks (if the insurer has not formed his own view) which is described as: Standard Formula (SF).

The SF approach with respect to underwriting risk recommends a relatively simple calculation. Actually, the required capital (C) for the coverage of the underwriting risk is obtained by applying a 3σ factor upon the total volume of premiums and reserves. The σ parameter corresponds to the well-known standard deviation symbol and the coefficient of number 3 comes from the basic characteristic of the normal distribution where its mass is almost spread in the range of left and right of the relevant mean value.

Of course, the value is different for each insurance line of business. Below, we provide the full table for sigmas as regards the premium and reserve risks for each line of business (as quoted in DA 35/2015). (Table 1)

Table 1. (Sigmas values (σ)) as quoted in DA 35/2015

	Segment(s)	Standard deviation for gross premium risk of the segment	Standard deviation for reserve risk of the segment
1	Motor vehicle liability insurance and proportional reinsurance	10.0%	9.0%
2	Other motor insurance and proportional reinsurance	8.0%	8.0%
3	Marine, aviation and transport insurance and proportional reinsurance	15.0%	11.0%
4	Fire and other damage to property insurance and proportional reinsurance	8.0%	10.0%
5	General liability insurance and proportional reinsurance	14.0%	11.0%
6	Credit and suretyship insurance and proportional reinsurance	12.0%	19.0%
7	Legal expenses insurance and proportional reinsurance	7.0%	12.0%
8	Assistance insurance and proportional reinsurance	9.0%	20.0%
9	Miscellaneous financial loss insurance and proportional reinsurance	13.0%	20.0%
10	Non-proportional casualty reinsurance	17.0%	20.0%
11	Non-proportional marine, aviation and transport reinsurance	17.0%	20.0%
12	Non-proportional property reinsurance	17.0%	20.0%

2.3 USPs determination procedure

The framework imposed by DA 35/2015 describes the mechanism in order to replace the figures in Table 1 and find the specific parameters for each insurance company. First of all, the framework demands that all the input data should fulfill certain criteria of sufficiency, completeness and accuracy.

Additionally, the framework requires a minimum number of five years as the basis for the relevant calculations incorporating credibility factors where they are escalated from the lower credibility factor of 34% up to the higher one of 100%. The relevant table is provided below.

Table 2. (Credibility factors) as quoted in DA 35/2015

Time length (years)	5	6	7	8	9	10	11	12	13	14	≥ 15
Factor c (LoB 1, 5, 6)	34%	43%	51%	59%	67%	74%	81%	87%	92%	96%	100%
Factor c (LoB 2-4, 7-12)	34%	51%	67%	81%	92%	100%	100%	100%	100%	100%	100%

Then, the framework presents three mathematical methods for the USPs determination or the calculation (and substitution) of sigma values in Table 1.

In this paper, we investigate only the first basic method which is applicable both for the premium and the reserve risk. We quote here the basic assumption underlying the method as appears in DA 35/2015.

i. Expected aggregated losses in a particular segment and accident year are linearly proportional in premiums earned in a particular accident year, i.e. the company has a constant expected loss ratio (i.e. does not allow for premium changes),

ii. The variance of aggregated losses in a particular segment and accident year is quadratic in premiums earned in a particular accident year,

iii. Aggregated losses follow a log normal distribution,

iv. Maximum likelihood estimation is appropriate.

So, the new sigma value is obtained as a weighted average using the credibility factors of Table 2 according to the specific line of business (refer to [19] for justification).

$$\sigma_{(prem,s,USP)} = c\hat{\sigma}(\hat{\delta}, \hat{\gamma})\sqrt{\frac{T+1}{T-1}} + (1-c)\sigma_{(prem,s)} \quad (1)$$

where

s : the reporting line of business

c : the credibility factor referring to Table 2

δ : mixing parameter

γ : logarithmic variation coefficient

$\sigma_{(prem,s)}$: the standard deviation described in Table 1 for each line of business.

$$\hat{\sigma}(\hat{\gamma}, \hat{\delta}) = \exp \left\{ \hat{\gamma} + \frac{\frac{1}{2}T + \sum_{t=1}^T \pi_t(\hat{\gamma}, \hat{\delta}) \ln \left(\frac{Y_t}{X_t} \right)}{\sum_{t=1}^T \pi_t(\hat{\gamma}, \hat{\delta})} \right\} \quad (2)$$

where

X_t : represents the premiums earned within the year t

Y_t : represents the ultimate claims within the year t

$$\pi_t(\hat{\gamma}, \hat{\delta}) = \frac{1}{\ln \left[1 + \left[(1 - \hat{\delta}) \frac{\bar{X}}{X_t} + \hat{\delta} \right] e^{2\hat{\gamma}} \right]} \quad (3)$$

$$\bar{X} = \frac{1}{T} \sum_{t=1}^T X_t \quad (4)$$

(the arithmetic mean value of the premiums over the total time period of years).

The mixing parameter $\hat{\delta}$ and the logarithmic variation coefficient $\hat{\gamma}$ is obtained via the minimization of the following objective function (with respect to $\hat{\gamma}$ and $\hat{\delta}$), subject to the constraint that the mixing parameter $\hat{\delta}$ should not be less than zero or more than the unity i.e. $0 \leq \hat{\delta} \leq 1$.

$$\sum_{t=1}^T \pi_t(\hat{\gamma}, \hat{\delta}) \left[\ln \left(\frac{Y_t}{X_t} \right) + \frac{1}{2\pi_t(\hat{\gamma}, \hat{\delta})} + \hat{\gamma} - \ln(\hat{\sigma}(\hat{\gamma}, \hat{\delta})) \right]^2 - \sum_{t=1}^T \ln(\pi_t(\hat{\gamma}, \hat{\delta})) \quad (5)$$

3. Numerical investigation of the standard mathematical formula of USPs

In this section, we carry out a full numerical investigation with respect to standard mathematical formulas of USPs (appearing in DA 35/2015) as described at the end of previous section. We assume an insurance company and consider the Motor Vehicle Liability line of business. Furthermore, we simulate numerical values for aggregate losses from a family of lognormal distributions assuming a constant loss ratio of 90%. We should stress that the absolute value of loss ratio does not affect the whole analysis. Then, we consider a set of different patterns for the time development of the volume of premiums and calculate the sigma value of USPs against the typical value of the standard deviation of the loss ratio. We use a time period of 16 years (> 15 years) in order to obtain full credibility with respect to our data experience. The different patterns for the time development of the volume of premiums (in millions of €) are presented in Table 3.

All the patterns below are typical input signals (Table 3) used by control engineers to investigate the behavioral response of a mechanism. Figure 1 may facilitate our view on the four typical signals (constant, ramp, spike, step and cycle).

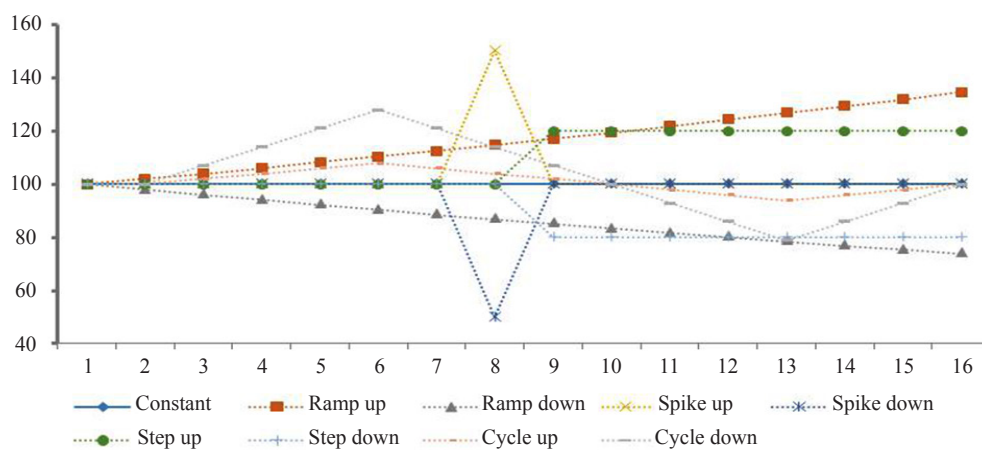


Figure 1. Geometric Patterns of the basic input signals (spike, step, ramp, cycle)

Table 3. Input signals

S/N	Name	Description of the patten	Numerical values of Premiums (P_1, P_2, P_3, \dots)
1	Constant	Constant volume of premium for the whole time period	100, 100, 100, ...
2.a	Low increase	Constant annual premium increase of 2%	100, 102, 104, ...
2.b	Medium increase	Constant annual premium increase of 7%	100, 107, 114, ...
2.c	High increase	Constant annual premium increase of 20%	100, 120, 144, ...
2.d	Low decrease	Constant annual premium decrease of 2%	100, 98, 96, ...
2.e	Medium decrease	Constant annual premium decrease of 7%	100, 93, 86, ...
2.f	High decrease	Constant annual premium decrease of 20%	100, 80, 64, ...
3.a	Up-High	Constant volume of premium apart a special year with a high increase	100, 100, ..., 100, 150, 100, ..., 100
3.b	Up-Low	Constant volume of premium apart a special year with a low increase	100, 100, ..., 100, 110, 100, ..., 100
3.c	Down-High	Constant volume of premium apart a special year with a high decrease	100, 100, ..., 100, 50, 100, ..., 100
3.d	Down-Low	Constant volume of premium apart a special year with a low decrease	100, 100, ..., 100, 90, 100, ..., 100
4.a	Up-Low	Constant volume of premium up to the middle and a steady low (2%) increase thereafter	100, 100, ..., 102, 102, 102, ..., 102
4.b	Up-Medium	Constant volume of premium up to the middle and a steady Medium (7%) increase thereafter	100, 100, ..., 100, 107, 107, ..., 107
4.c	Up-High	Constant volume of premium up to the middle and a steady high (20%) increase thereafter	100, 100, ..., 100, 120, 120, ..., 120
4.d	Up-Extreme	Constant volume of premium up to the middle and a steady extreme (50%) increase thereafter	100, 100, ..., 100, 150, 150, ..., 150
4.e	Down-Low	Constant volume of premium up to the middle and a steady low (2%) decrease thereafter	100, 100, ..., 100, 98, 98, ..., 98
4.f	Down-Medium	Constant volume of premium up to the middle and a steady medium (7%) decrease thereafter	100, 100, ...100, 93, 93, ..., 93
4.g	Down-High	Constant volume of premium up to the middle and a steady high (20%) decrease thereafter	100, 100, ...100, 80, 80, ..., 80
4.h	Down-Extreme	Constant volume of premium up to the middle and a steady extreme (50%) decrease thereafter	100, 100, ...100, 50, 50, ..., 50
5.a	Low-Long	Variable volume of premium exhibiting low (± 2 mil €/per year) oscillations within long (16-years) time periods	100, 102, 104, 106, 108, 106, 104, 102, 100, 98, 96, 94, 96, 98, 100
5.b	Medium-Long	Variable volume of premium exhibiting low (± 7 mil €/per year) oscillations within long (16-years) time periods	100, 107, 114, 121, 128, 121, 114, 107, 100, 93, 86, 79, 86, 93, 100
5.c	High-Long	Variable volume of premium exhibiting low (± 20 mil €/per year) oscillations within long (16-years) time periods	100, 120, 140, 160, 180, 160, 140, 120, 100, 80, 60, 40, 60, 80, 100
5.d	Low-Short	Variable volume of premium exhibiting low (± 2 mil €/per year) oscillations within short (8-years) time periods	100, 102, 104, 102, 100, 98, 96, 98, 100, 102, 104, 102, 100, 98, 96, 98, 100
5.e	Medium-Short	Variable volume of premium exhibiting low (± 7 mil €/per year) oscillations within short (8-years) time periods	100, 107, 114, 107, 100, 93, 86, 93, 100, 107, 114, 107, 100, 93, 86, 93, 100
5.f	High-Short	Variable volume of premium exhibiting low (± 20 mil €/per year) oscillations within short (8-years) time periods	100, 120, 140, 120, 100, 80, 60, 80, 100, 120, 140, 120, 100, 80, 60, 80, 100

We use a series of simulations for aggregate losses that exhibit a standard deviation value of 7.90% for the loss ratio. The results of our simulations are presented in Table 4 and should be compared with the value of 7.90% (standard deviation of loss ratio).

Furthermore, we enhance our analysis with respect to the difference of standard deviation and USPs calculated deviation value when we have a step pattern for the time development of the volume of premium. We ran the model for many different values of increasing or decreasing and we obtain the Table 5 and Figure 2.

Table 4. Results of USPs for different input signals

	Numbering		USPs calculated sigma value	Difference (%) of USPs from the standard deviation value (7.90%)
ramp	1	-	8.25%	4.4%
	2	a	8.25%	4.4%
	2	b	8.25%	4.4%
	2	c	8.25%	4.4%
	2	d	8.10%	2.5%
	2	e	7.65%	-3.2%
spike	2	f	6.96%	-11.9%
	3	a	8.25%	4.4%
	3	b	8.21%	3.9%
	3	c	8.06%	2.0%
steps	3	d	8.25%	4.4%
	4	a	8.25%	4.4%
	4	b	8.25%	4.4%
	4	c	8.25%	4.4%
	4	d	8.25%	4.4%
	4	e	8.23%	4.2%
	4	f	8.17%	3.4%
	4	g	8.00%	1.3%
cycles	4	h	7.43%	-5.9%
	5	a	8.21%	3.9%
	5	b	8.10%	2.5%
	5	c	7.86%	-0.5%
	5	d	8.25%	4.4%
	5	e	8.25%	4.4%
	5	f	8.25%	4.4%

Table 5. Results of USPs for the step signal

Numbering	Step increase (decrease)	USPs	Difference (%) of USPs from the standard deviation value (7.90%)
1	100%	8.25%	4.4%
2	90%	8.25%	4.4%
3	70%	8.25%	4.4%
4	70%	8.25%	4.4%
5	60%	8.25%	4.4%
6	50%	8.25%	4.4%
7	40%	8.25%	4.4%
8	30%	8.25%	4.4%
9	20%	8.25%	4.4%
10	10%	8.25%	4.4%
11	0%	8.25%	4.4%
12	-10%	8.14%	3.0%
13	-20%	8.00%	1.3%
14	-30%	7.84%	-0.8%
15	-40%	7.65%	-3.2%
16	-50%	7.43%	-5.9%
17	-60%	7.15%	-9.5%
18	-70%	6.80%	-13.9%
19	-80%	6.39%	-19.1%
20	-90%	5.87%	-25.7%
21	99%	5.22%	-33.9%

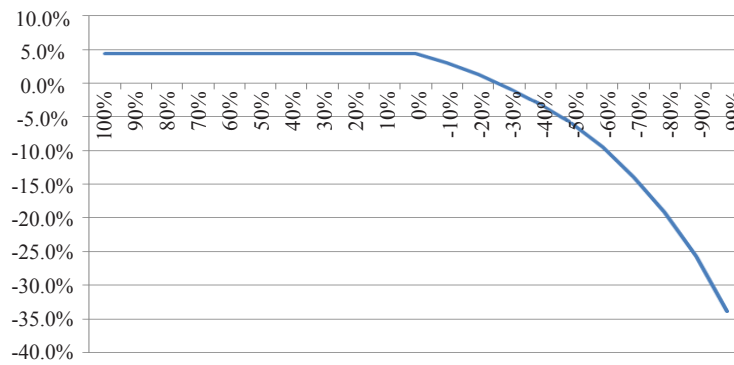


Figure 2. Results of USPs for the step signal

It is obvious from Table 4 and 5 while also from Figure 2 that:

- (a) Normally (typical patterns for the time development of premiums) the USPs sigma value marginally differs ($\pm 10\%$) from the typical value of the well-known standard deviation of the loss ratio.
- (b) Even for the extreme case of large step decrease the difference of sigma values (USPs and standard deviation) equals 33.9%.
- (c) The USPs mechanism pays no attention (no sensitivity) when we have increased patterns for the time development of premiums.

4. Calculation of USPs objective functions for three different distributions

In this section, we fully investigate the relevant mathematics of the formula of USPs. Firstly, we provide a typical mathematical proof for the formula that appears in DA 35/2015. Then, we relax the basic hypothesis for a lognormal distribution with respect to claim experience (firstly we assume Pareto and then a Gamma distribution) and produce other formulas very useful for the insurance companies. Before going into the specific cases for each claim distribution, we provide the overall framework of assumptions and notation.

Generally speaking, insurance companies receive premiums and payback claims to the policyholders. The time development of the premium pattern and the loss ratio are quite critical for each company. (Typically the loss ratio is defined as the ratio of ultimate claims over the earned premiums). We adapt the following notions:

X_t : premiums earned in year t

Y_t : ultimate claims in year t

Furthermore, we assume that claims expectation is linearly related to premiums over time while claims variation is both linearly and quadratic related to premiums (this is normally excused assuming a compound Poisson process for the claim experience). So,

$$E(Y_t) = \beta X_t \tag{6}$$

$$Var(Y_t) = \sigma^2 [(1 - \delta)\bar{X}X_t + \delta X_t^2] \tag{7}$$

where

σ^2 : represents the total variability of claim experience produced from premium variation (σ_1^2) and loss ratio variation (σ_2^2) i.e.

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 \tag{8}$$

δ : is the mixing parameter for the linear part and the quadratic part of the claim experience and is defined as

follows

$$\delta = \left(\frac{\sigma_2}{\sigma} \right)^2 \quad (9)$$

hence $\delta \in [0, 1]$.

T : is the total time period

$$\bar{X} = \frac{1}{T} \sum_{t=1}^T X_t \quad (10)$$

We proceed with the derivation of the USPs by plugging three different claim distributions as inputs in the relevant formulas.

4.1 The lognormal distribution

Assuming a lognormal distribution for the claim experience i.e.

$Y_t \sim LN(\mu, \psi)$ with density function

$$f(y) = \frac{1}{y\sqrt{2\pi\psi}} e^{-\frac{1}{2\psi}(\ln y - \mu)^2}, \quad \mu \in (-\infty, +\infty), \quad \psi > 0, \quad y > 0 \quad (11)$$

and using equations (6) and (7), we have the following equations with respect to claims expectation and variability.

$$E(Y_t) = e^{\mu + \frac{1}{2}\psi} = \beta X_t \quad (12)$$

$$Var(Y_t) = e^{2\mu + \psi} [e^\psi - 1] = \sigma^2 [(1 - \delta)\bar{X}X_t + \delta X_t^2] \quad (13)$$

So, we obtain a system of two equations with two unknown variables. We solve the system of equations (12) and (13) and obtain the solution for μ (the mean value) and ψ (the variability value). i.e.

$$\mu = \ln(\beta X_t) - \frac{1}{2\pi_t} \quad (14)$$

$$\psi = \frac{1}{\pi_t} \quad (15)$$

where

$$\pi_t^{-1} = \ln \left[1 + \frac{\sigma^2}{\beta^2} \left[(1 - \delta) \frac{\bar{X}}{X_t} + \delta \right] \right] \quad (16)$$

In order to facilitate our calculations, we use the following transformation

$$V_t = \ln(Y_t) - \mu \sim N(0, \psi) \quad (17)$$

or equivalently using equation (14)

$$V_i = \ln(Y_i) - \ln(\beta X_i) - \frac{1}{2\pi_i} \Rightarrow V_i = \ln\left(\frac{Y_i}{X_i}\right) - \ln(\beta) + \frac{1}{2\pi_i} \quad (18)$$

Since Y_i follows a lognormal distribution the V_i parameter follows a normal distribution with zero mean value. So the density function of the new variable V is

$$f_V(V) = \frac{1}{\sqrt{2\pi\psi}} e^{-\frac{1}{2\psi}V^2} \quad (19)$$

Now, we calculate the likelihood for the total period $[0, T]$ using equation (19)

$$L = \prod_{i=1}^T f_V(V_i) = \prod_{i=1}^T \left[\frac{1}{\sqrt{2\pi\psi}} e^{-\frac{1}{2\psi}V_i^2} \right] \quad (20)$$

We proceed with the maximization of the log-likelihood of equation (20) i.e.

$$l = \ln(L) = \ln \left[\frac{1}{(\sqrt{2\pi})^T} \left[\prod_{i=1}^T \pi_i \right]^{\frac{1}{2}} e^{-\frac{1}{2} \sum_{i=1}^T \pi_i V_i^2} \right]$$

We should then minimize the equivalent quantity

$$l' = \sum_{i=1}^T \pi_i V_i^2 - \sum_{i=1}^T \ln \pi_i \quad (21)$$

Substituting also

$$\frac{\sigma^2}{\beta^2} = e^{2\gamma} \Rightarrow \ln \sigma = \gamma + \ln \beta \quad (22)$$

into equation (21) we obtain

$$l'(\beta, \gamma, \delta) = \sum_{i=1}^T \pi_i(\gamma, \delta) \left[\ln\left(\frac{Y_i}{X_i}\right) + \frac{1}{2\pi_i(\gamma, \delta)} - \ln \beta \right]^2 - \sum_{i=1}^T \ln \pi_i(\gamma, \delta) \quad (23)$$

We freeze variables γ and δ and try to minimize $l'(\beta, \gamma, \delta)$ with respect to variable β . We differentiate $l'(\beta) = l'(\beta, \gamma, \delta)$ with respect to variable β and equating with zero in order to obtain potential minimum. Equivalently, we may assume that $l'(l\ln\beta) \equiv l'(\beta)$ and minimize with respect to $l\ln\beta$, so

$$\frac{dl'(\ln\beta)}{d\ln\beta} = 0 \Rightarrow \sum_{i=1}^T \pi_i(\gamma, \delta) \ln\left(\frac{Y_i}{X_i}\right) + \frac{T}{2} - \ln\beta \sum_{i=1}^T \pi_i(\gamma, \delta) = 0 \Rightarrow \ln \hat{\beta} = \frac{\frac{T}{2} + \sum_{i=1}^T \pi_i(\gamma, \delta) \ln\left(\frac{Y_i}{X_i}\right)}{\sum_{i=1}^T \pi_i(\gamma, \delta)} \quad (24)$$

Now using equation (22) and substituting in equation (23) given also equation (24), we obtain the formula described in DA 35/2015 as appears at the end of Section 2. We have to minimize (with numerical methods) the objective function.

$$\sum_{i=1}^T \pi_i(\hat{\gamma}, \hat{\delta}) \left[\ln\left(\frac{Y_i}{X_i}\right) + \frac{1}{2\pi_i(\hat{\gamma}, \hat{\delta})} + \hat{\gamma} - \ln(\hat{\sigma}(\hat{\gamma}, \hat{\delta})) \right]^2 - \sum_{i=1}^T \ln[\pi_i(\hat{\gamma}, \hat{\delta})] \quad (25)$$

4.2 The Pareto distribution

Assuming a Pareto distribution for the claim experience

$Y_i \sim Par(\mu, \psi)$ with density function

$$f(y) = \mu\psi^\mu (\psi + y)^{-(\mu+1)}, \quad \mu, \psi > 0, y > 0 \quad (26)$$

Where μ is the shape parameter while ψ is the scale parameter (refer to [20] for more details) and using equation (6) and (7), we have the following equation with respect to claims expectation and variability.

$$E(Y_i) = \frac{\psi}{\mu-1} = \beta X_i \quad (27)$$

$$Var(Y_i) = \frac{\mu\psi^2}{(\mu-1)^2(\mu-2)} \sigma^2 [(1-\delta)\bar{X}X_i + \delta X_i^2] \quad (28)$$

So, we obtain a system of two equations with two unknown variables. We solve the system of equations (27) & (28) and obtain the solution for μ and ψ parameters.

We solve equation (27) with respect to ψ .

$$\psi = \beta X_i (\mu - 1) \quad (29)$$

and plug this into equation (28) and obtain

$$\frac{\mu}{\mu-2} = \frac{\sigma^2}{\beta^2} \left[(1-\delta) \frac{\bar{X}}{X_i} + \delta \right] \quad (30)$$

Similarly with the lognormal distribution, we denote

$$\pi_i^* = \frac{\sigma^2}{\beta^2} \left[(1-\beta) \frac{\bar{X}}{X_i} + \delta \right] \quad (31)$$

Hence, we obtain the solution of the system (equation (27) and (28)) as

$$\mu = \frac{2\pi_i^*}{\pi_i^* - 1} \quad (32)$$

$$\psi = \beta X_i \left(\frac{\pi_i^* + 1}{\pi_i^* - 1} \right) \quad (33)$$

In order to facilitate our calculations, we shall use a certain transformation for the Pareto distribution. For this purpose we shall present the following lemma.

Lemma (6): Let a Y Pareto (μ, ψ) random variable (as defined in equation (26)). We define the transformed random variable $Z = g(Y)$ as

$$Z = \ln\left(\frac{\psi + Y}{\psi}\right) \quad (34)$$

Then the Z random variable is an exponentially distributed random variable with parameter μ with density function

$$f_z(z) = \mu e^{-\mu z} \quad (35)$$

Proof. The equation (34) is a 1-1 transformation, so we may obtain the inverse one i.e.

$$Y = g^{-1}(z) = \psi e^z - \psi \quad (36)$$

and the relevant Jacobian

$$\frac{dY}{dZ} = \psi e^z \quad (37)$$

Then using the standard transformation formula for the probability density function we obtain

$$\begin{aligned} f_z(z) &= f_Y(g^{-1}(z)) \left| \frac{dY}{dZ} \right| \\ f_z(z) &= \mu \psi^\mu (\psi e^z)^{-(\mu+1)} |\psi e^z| \end{aligned} \quad (38)$$

or equivalently

$$f_z(z) = \mu e^{-\mu z} \quad (39)$$

Now we use the following transformation

$$V_t = \ln\left(\frac{\psi + Y_t}{\psi}\right) \sim \exp(\mu) \quad (40)$$

or equivalently using equation (33)

$$\begin{aligned} V_t &= \ln(\psi + Y_t) - \ln(\psi) = \ln(\psi + Y_t) - \ln\left(\beta X_t \frac{\pi_t^* + 1}{\pi_t^* - 1}\right) \\ V_t &= \ln\left(\beta \frac{\pi_t^* + 1}{\pi_t^* - 1} + \frac{Y_t}{X_t}\right) - \ln\left(\frac{\pi_t^* + 1}{\pi_t^* - 1}\right) - \ln\beta \end{aligned} \quad (41)$$

Since V_t follows an exponential distribution, we derive the relevant density function

$$f_{V_t}(V_t) = \frac{2\pi_t^*}{\pi_t^* - 1} e^{-\frac{2\pi_t^*}{\pi_t^* - 1} V_t} \quad (42)$$

So the relevant likelihood for the total period $[0, T]$ is defined as

$$\begin{aligned} L &= \prod_{t=1}^T f_{V_t}(V_t) = \prod_{t=1}^T \left[\frac{2\pi_t^*}{\pi_t^* - 1} e^{-\frac{2\pi_t^*}{\pi_t^* - 1} V_t} \right] \\ &= 2^T \left[\prod_{t=1}^T \frac{\pi_t^*}{\pi_t^* - 1} \right] e^{-2 \sum_{t=1}^T \frac{\pi_t^*}{\pi_t^* - 1} V_t} \end{aligned} \quad (43)$$

We proceed with the maximization of the log-likelihood of equation (43)

$$l = \ln(L) = \ln(2^T) + \ln \prod_{t=1}^T \frac{\pi_t^*}{\pi_t^* - 1} + \ln \left(e^{-2 \sum_{t=1}^T \frac{\pi_t^*}{\pi_t^* - 1} V_t} \right)$$

we should then minimize the equivalent quantity

$$l' = 2 \sum_{t=1}^T \frac{\pi_t^*}{\pi_t^* - 1} V_t + \sum_{t=1}^T \ln \frac{\pi_t^* - 1}{\pi_t^*} \quad (44)$$

Substituting also

$$\frac{\sigma^2}{\beta^2} = e^{2\gamma} \Rightarrow \ln \sigma = \gamma + \ln \beta \quad (45)$$

into equation (44) we obtain

$$l'(\beta, \gamma, \delta) = 2 \sum_{t=1}^T \frac{\pi_t^*(\gamma, \delta)}{\pi_t^*(\gamma, \delta) - 1} \left[\ln \left(\beta \frac{\pi_t^*(\gamma, \delta) + 1}{\pi_t^*(\gamma, \delta) - 1} + \frac{Y_t}{X_t} \right) - \ln \left(\frac{\pi_t^*(\gamma, \delta) + 1}{\pi_t^*(\gamma, \delta) - 1} \right) - \ln \beta \right] + \sum_{t=1}^T \ln \frac{\pi_t^*(\gamma, \delta) - 1}{\pi_t^*(\gamma, \delta)}$$

or equivalently

$$l'(\beta, \gamma, \delta) = 2 \sum_{t=1}^T \frac{\pi_t^*(\gamma, \delta)}{\pi_t^*(\gamma, \delta) - 1} \ln \left(1 + \frac{\pi_t^*(\gamma, \delta) - 1}{\beta [\pi_t^*(\gamma, \delta) + 1]} \frac{Y_t}{X_t} \right) + \sum_{t=1}^T \ln \frac{\pi_t^*(\gamma, \delta) - 1}{\pi_t^*(\gamma, \delta)} \quad (46)$$

that we have to minimize it by numerical methods with respect to variables β , γ and δ under the constraints

$$\pi_t^*(\gamma, \delta) = e^{2\gamma} \left[(1 - \delta) \frac{\bar{X}}{X_t} + \delta \right] \quad \text{and} \quad \bar{X} = \frac{1}{T} \sum_{t=1}^T X_t$$

and then the σ value will be obtained from equation (45) i.e.

$$\hat{\sigma} = e^{\hat{\gamma} + \ln \hat{\beta}} \quad (47)$$

4.3 The Gamma distribution

Assuming a Gamma distribution for the claim experience

$Y_t \sim \text{Gamma}(\mu, \psi)$ with density function

$$f(y) = \frac{\psi^\mu}{\Gamma(\mu)} y^{\mu-1} e^{-\psi y}, \quad \mu, \psi > 0, y > 0 \quad (48)$$

and using equations (6) and (7) we have the following equations with respect to claims expectation and variability

$$E(Y_t) = \frac{\mu}{\psi} = \beta X_t \quad (49)$$

$$\text{Var}(Y_t) = \frac{\mu}{\psi^2} \sigma^2 [(1-\delta)\bar{X}X_t + \delta X_t^2] \quad (50)$$

So, we solve the system above and obtain

$$\psi = \frac{\beta}{\sigma^2 [(1-\delta)\bar{X} + \delta X_t]} \quad (51)$$

while

$$\mu = \frac{\beta X_t}{\frac{\sigma^2}{\beta} [(1-\delta)\bar{X} + \delta X_t]} = \frac{1}{\frac{\sigma^2}{\beta^2} \left[(1-\delta) \frac{\bar{X}}{X_t} + \delta \right]} \quad (52)$$

Hence, we derive

$$\mu = \pi_t^{*-1} \text{ and } \psi = \frac{1}{\beta X_t \pi_t^*} \quad (53)$$

where

$$\pi_t^* = \frac{\sigma^2}{\beta^2} \left[(1-\delta) \frac{\bar{X}}{X_t} + \delta \right] \quad (54)$$

Now, we shall proceed with the calculation of the maximum likelihood without using any transformation

$$L = \prod_{t=1}^T f_{Y_t}(y_t) = \prod_{t=1}^T \frac{\psi^\mu}{\Gamma(\mu)} y_t^{\mu-1} e^{-\psi y_t}$$

Substituting the values of μ and ψ from equations (53) and (54) we obtain

$$L = \prod_{t=1}^T \frac{1}{(\beta X_t \pi_t^*) \prod_{t=1}^{*-1} \Gamma(\pi_t^{*-1})} y_t^{\pi_t^{*-1}-1} e^{-\frac{y_t}{\beta X_t \pi_t^*}}$$

or equivalently

$$L = \left[\prod_{t=1}^T \frac{1}{\Gamma(\pi_t^{*-1}) \pi_t^* \sqrt{\beta X_t \pi_t^*}} \frac{\pi_t^* \sqrt{y_t}}{y_t} \right] e^{-\sum_{t=1}^T \frac{y_t}{\beta X_t \pi_t^*}}$$

then we maximize the logarithm of L .

$$l = \ln(L) = \sum_{t=1}^T \ln \frac{\pi_t^* \sqrt{\frac{Y_t}{X_t}}}{Y_t \Gamma(\pi_t^{*-1}) \pi_t^* \sqrt{\beta \pi_t^*}} - \sum_{t=1}^T \left(\frac{Y_t}{X_t} \right) \frac{1}{\beta \pi_t^*} \quad (55)$$

Substituting also

$$\frac{\sigma^2}{\beta^2} = e^{2\gamma} \Rightarrow \ln \sigma = \gamma + \ln \beta \quad (56)$$

into equation (55) and changing the signs we obtain

$$l'(\beta, \gamma, \delta) = \sum_{t=1}^T \frac{\frac{Y_t}{X_t}}{\beta \pi_t^*(\gamma, \delta)} \sum_{t=1}^T \left[\ln Y_t + \ln \Gamma(\pi_t^{*-1}) + \frac{\ln(\beta \pi_t^*)}{\pi_t^*(\gamma, \delta)} \right] - \sum_{t=1}^T \frac{\ln \left(\frac{Y_t}{X_t} \right)}{\pi_t^*(\gamma, \delta)} \quad (57)$$

that we have to minimize it by numerical methods with respect to variables β , γ and δ where

$$\pi_t^*(\gamma, \delta) = e^{2\gamma} \left[(1-\delta) \frac{\bar{X}}{X_t} + \delta \right] \quad \text{and} \quad \bar{X} = \frac{1}{T} \sum_{t=1}^T X_t$$

and then the σ value will be obtained from equation (56) i.e.

$$\hat{\sigma} = e^{\hat{\gamma} + \ln \hat{\beta}} \quad (58)$$

4.4 An approximation for the Gamma distribution

In this subsection, we provide an approximation for the minimization of the objective function under the Gamma case. If we assume a (more or less) constant pattern for the time development of premiums then we obtain that each annual year volume of premiums will equal to the mean value i.e.

$$X_t \cong \bar{X} \quad (59)$$

then equation (31) may be written as

$$\pi_t^* \cong \frac{\sigma^2}{\beta^2} [(1-\delta) * 1 + \delta] \quad (60)$$

Furthermore, if we assume a line of business (LoB) with marginal (or zero) profits, then β equals to unity (and consequently $\ln \beta \cong 0$).

Hence, the final approximation is the following

$$\pi_t^* = \sigma^2 \quad (61)$$

We obtain the approximation for the Gamma case and the respective function (57)

$$\begin{aligned} l'_G(\sigma^2) &= \sum_{t=1}^T \frac{Y_t}{X_t \sigma^2} + \sum_{t=1}^T \left[\ln Y_t + \ln \Gamma\left(\frac{1}{\sigma^2}\right) + \frac{\ln(\sigma^2)}{\sigma^2} \right] - \sum_{t=1}^T \frac{\ln\left(\frac{Y_t}{X_t}\right)}{\sigma^2} \\ &= \frac{1}{\sigma^2} \sum_{t=1}^T \left(\frac{Y_t}{X_t}\right) + \sum_{t=1}^T \ln(Y_t) + T \ln \Gamma\left(\frac{1}{\sigma^2}\right) + \frac{1}{\sigma^2} \sum_{t=1}^T \ln(\sigma^2) - \frac{1}{\sigma^2} \sum_{t=1}^T \ln Y_t \end{aligned}$$

$$\text{We substitute } 1 + \xi = \frac{1}{\sigma^2} \quad (62)$$

and use Stirling's approximation

$$\ln n! = \ln \Gamma(n+1) \cong n \ln n - n + O(\ln n) \quad (63)$$

So, we obtain

$$\begin{aligned} l'_G(\xi) &= (1 + \xi) \left\{ \sum_{t=1}^T \left[\left(\frac{Y_t}{X_t}\right) - \ln(1 + \xi) - \ln(Y_t) \right] \right\} + T \ln \Gamma(1 + \xi) \\ &= (1 + \xi) \sum_{t=1}^T \left[\frac{Y_t}{X_t} - \ln Y_t \right] - (1 + \xi) T \ln(1 + \xi) + T[\xi \ln - \xi] \end{aligned}$$

We differentiate the last expression with respect to parameter and obtain

$$\frac{dl'_G(\xi)}{d\xi} = \sum_{t=1}^T \left[\frac{Y_t}{X_t} - \ln(Y_t) \right] + T \ln \xi - T(1 + \ln(1 + \xi)) \quad (64)$$

Equating the last equation with zero, we obtain

$$\ln\left(\frac{1 + \xi}{\xi}\right) = \ln(1 + \xi) - \ln \xi = \frac{1}{T} \sum_{t=1}^T \left[\frac{Y_t}{X_t} - \ln(Y_t) \right] - 1$$

$$\text{Or equivalently } \frac{1 + \xi}{\xi} = e^{\frac{1}{T} \sum_{t=1}^T \left(\frac{Y_t}{X_t} - \ln Y_t\right) - 1} \quad (65)$$

And using equation (62), we obtain the estimation for the σ value

$$\hat{\sigma}_G = \sqrt{1 - e^{-\frac{1}{T} \sum_{t=1}^T \left(\frac{Y_t}{X_t} - \ln Y_t\right)}} \quad (66)$$

Again, we may ascertain that the above value of $\hat{\sigma}_G$ is the minimum by differentiating once more the first derivative of $l'_G(\xi)$ (see expression (64)) and obtain

$$\frac{dl_G''(\xi)}{d\xi^2} = \frac{T}{\xi} - \frac{T}{1+\xi} = \frac{T}{\xi(1+\xi)} > 0 \quad (67)$$

where the last expression is obviously positive based on the choice of ζ (see expression (65)). So, the certain value of ζ corresponds to the minimum.

5. Conclusions-further research

In this paper, we have fully reviewed the mathematics and algorithms involved in the Delegated Act 35/2015 imposed by European Regulatory Authorities concerning the first method for the USPs determination method of the insurance companies. Actually, we have performed a full numerical analysis based on typical premium patterns as input signals used by control engineers in order to identify the sensitivities of the mathematical formulas. The relevant examples revealed that there is a low sensitivity with respect to time development of premium patterns and in many cases the sensitivity is vanished at all.

Furthermore, we compared the value produced under the framework of USPs with the value of the standard deviation function of the loss ratios finding only marginal differences. For the special cases of constant or increasing premium patterns, the two values (USPs and standard deviation) are too close (the difference is calculated less than 5%, see Table (4)). Of course, such a small distance (of less than 5%) accompanied with all the other small differences as appear in Table (4) and taking also into account the too complicated mathematics needed for the calculation of USPs values, may raise a quite serious criticism as: “whether the certain algorithm of USPs determination process, as appears in DA 35/2015, provides advanced safety levels with respect to an insurance company’s risk profile”.

Additionally, we have further elaborate and reveal the mathematics behind the DA 35/2015 providing detailed solutions for two other claim distributions (apart from the lognormal): a) the Pareto and b) the Gamma case. For both cases, we have derived similar objective functions (as appears in DA 35/2015, for the lognormal distribution) that may only be solved using numerical optimization procedures. Finally, we have produced a closed approximation formula for obtaining the minimum values for the Gamma distribution. Again, this formula (see equation (66)) indicates that USPs values are closely related to the typical standard deviation value of the respective loss ratio.

So, all interesting parties as to the top management of the insurer, auditors and regulators have a simple and reliable tool for the validation of the result produced by the complex algorithm of USPs and that is the function of the standard deviation. Additionally, the derivation of USPs values using the input of Pareto and Gamma distribution enhances and empowers the view of the insurers towards the assessment and valuation of the risk inherent in insurance branches that lognormal distribution is not the ideal vehicle for the description of the claims incurred.

Further research may be directed towards the investigation of other distributions (apart from the lognormal, Pareto and Gamma) which appear as appropriate distributions for modeling risks in the insurance industry. It would be also quite useful to obtain analytical and accurate mathematical formulas along with some (if exist) approximation formulas that may help the risk mechanisms and the practical day-to-day operations of an insurance company.

References

- [1] European Parliament legislative resolution of 22 April 2009 on the amended proposal for a directive of the European Parliament and of the Council on the taking-up and pursuit of the business of Insurance and Reinsurance. 2009. Available from: <https://eur-lex.europa.eu/legal-content/EN/TXT/PDF/?uri=CELEX.32009L0138&from=en> [Accessed 21 January 2016].
- [2] CEIOPS. Quantitative impact study 5-QIS5-technical specifications. Brussels. 2010. Available from: http://ec.europa.eu/internal_market/insurance/docs/solvency/qis5/201007/technical_specifications_en.pdf [Accessed 21 January 2016].
- [3] EIOPA. The underlying assumptions in the standard formula for the Solvency Capital Requirement calculation. 2014. Available from: https://eiopa.europa.eu/Publications/Standards/EIOPA-14-322_Underlying_Assumptions.pdf [Accessed 21 January 2016].

- [4] EIOPA. Technical specifications for the preparatory phase (Part I and II). 2014. Available from: <https://eiopa.europa.eu/regulation-supervision/insurance/solvency-ii/solvency-ii-technical-specifications> [Accessed 21 January 2016].
- [5] Scherer M, Stahl G. The standard formula of Solvency II: a critical discussion. *European Actuarial Journal*. 2020. Available from: <https://doi.org/10.1007/s13385-020-00252-z>
- [6] Asadi S, Janabi Mazin AMAI. Measuring market and credit risk under Solvency II: Evaluation of the standard technique versus internal models for stock and bond markets. *European Actuarial Journal*. 2020; 10: 425-456. Available from: <https://doi.org/10.1007/s13385-020-00235-0>
- [7] Durán SP, Otero GL, Martorell CO, Merigó LJM. Backtesting an equity risk model under Solvency II. *Journal of Business Research*. 2018; 89: 216-222. Available from: <https://doi.org/10.1016/j.jbusres.2018.01.004>
- [8] Durán SP, Otero GL, Martorell CO, Gil-Lafuente AM. Property risk under solvency II: Effects of different unsmoothing techniques. *Technological and Economic Development of Economy*. 2019; 25(1): 1-19. Available from: <https://doi.org/10.3846/tede.2019.6213>
- [9] European Commission. Text of the delegated Act on Solvency II. 2014. Available from: <https://eur-lex.europa.eu/legal-content/EN/TXT/PDF/?uri=CELEX:32015R0035&from=EN> [Accessed 21 January 2016].
- [10] EIOPA. Guidelines on undertaking-specific parameters. 2015. Available from: https://eiopa.europa.eu/Publications/Guidelines/USP_Final_document_EN.pdf [Accessed 21 January 2016].
- [11] EIOPA. Calibration of the premium and reserve risk factors in the standard formula of solvency II. *Report of the Joint Working Group on Non-Life and Health NSLT Calibration*. 2011. Available from: https://eiopa.europa.eu/Publications/Reports/EIOPA-11-163-A-Report_JWG_on_NL_and_Health_non-SLT_Calibration.pdf [Accessed 21 January 2016].
- [12] Cerchiara RR, Demarco V. Undertaking specific parameters under solvency II: Reduction of capital requirement or not? *European Actuarial Journal*. 2016; 6: 351-376.
- [13] Gisler A. The insurance risk in the SST and in Solvency II: Modelling and parameter estimation. 2009. Available from: <http://dx.doi.org/10.2139/ssrn.2704364>
- [14] Frezal S. Solvency II is not risk-based-could it be? evidence from non-life calibrations. *North American Actuarial Journal*. 2018; 22(3): 365-379. Available from: DOI: 10.1080/10920277.2017.1421473
- [15] Cerchiara RR, Acri F. Estimating the volatility of non-life premium risk under solvency II: Discussion of danish fire insurance data. *Risks*. 2020; 8(3): 74. Available from: DOI: 10.3390/risks8030074
- [16] Fröhlich A, Weng A. Parameter uncertainty and reserve risk under Solvency II. *Insurance: Mathematics and Economics*. 2018; 81: 130-141. Available from: <https://doi.org/10.1016/j.insmatheco.2017.10.004>
- [17] Carlehed M. Practical aspects of the aggregation of two risks in the Solvency II standard formula. *European Actuarial Journal*. 2019; 9: 155-171. Available from: <https://doi.org/10.1007/s13385-019-00196-z>
- [18] Bølviken E, Guillen M. Risk aggregation in Solvency II through recursive log-normals. *Insurance: Mathematics and Economics*. 2017; 73: 20-26. Available from: <https://doi.org/10.1016/j.insmatheco.2016.12.006>
- [19] Bühlman H, Gisler A. *A Course in Credibility Theory and Its Applications*. Heidelberg: Springer-Verlag Berlin Heidelberg; 2005.
- [20] Klugman SA, Panjer HH, Willmot GE. *Loss Models: From Data to Decisions*. State of New Jersey: Wiley; 2008.