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Research Article

Entropy Production and Thermodynamic Dynamics in Active and Passive Brownian Systems Driven by Time-Dependent Forces and Temperatures

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Abstract: In this work, we examine the impact of time-varying temperature and force on the thermodynamic features of active Brownian motor that moves with velocity v_0 against the force as well as passive Brownian motor. By deriving analytical expressions for entropy production and entropy extraction rates, we extend the existing theoretical frameworks by considering a force or temperature that varies exponentially, linearly, and quadratically. By studying the system analytically, we investigate how thermal relaxation, steady-state conditions, and nonlinear dissipation effects are affected over time. We find that the total entropy depends only on temperature and viscous friction if the Brownian particle moves freely, while the entropy production and dissipation rates are strongly influenced by the external force. When a Brownian particle is exposed to periodic forcing, entropy production exhibits oscillatory behavior with monotonic decay, whereas periodic impulsive forces induce discrete spikes followed by relaxation, reflecting intermittent energy injection and dissipation. On the other hand, nonperiodic impulsive forces lead to abrupt entropy surges, followed by gradual stabilization, ensuring long-term equilibration. At the stall force, when $f = \gamma v_0$, all thermodynamic rates-including the entropy production and extraction rates vanishes. We believe that our results have broad implications for the optimization of molecular motors, nanoscale transport, and pulse-driven systems. It also provides insights into the design of bio-inspired nanomachines, thermodynamically controlled microfluidic devices, and artificial nanorobots.

***Keywords*:** entropy production, Brownian systems, time-dependent driving forces

MSC: 60H10, 82C31

1. Introduction

Active matter systems, including molecular motors, motile bacteria, and catalytically driven colloids, operate intrinsically out of thermodynamic equilibrium by continuously consuming energy to sustain directed motion [1–4]. These systems violate the detailed balance and generate entropy, giving rise to persistent non-equilibrium fluxes and finite thermodynamic costs. Understanding how such systems convert energy into motion and how their performance metrics, such as entropy production, dissipation, and efficiency, are affected by external conditions remains a central pursuit in nonequilibrium statistical mechanics.

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The Fokker-Planck equation provides a foundational framework for describing the stochastic dynamics of Brownian and active systems under thermal and mechanical driving. Within this formalism, entropy production and energy dissipation can be quantified in terms of drift and diffusion processes, and their evolution analyzed under steady or driven regimes. Considerable progress has been made in understanding systems under isothermal and spatially varying environments, where Thermodynamic Uncertainty Relations (TURs), fluctuation theorems, and time-reversal symmetries constrain the system behavior [5–14].

Extensive studies have employed both discrete-state master equations [15–20] and continuous Fokker-Planck formulations [21–25] to analyze entropy production in classical [26, 27] and quantum systems [28–30] under static or spatially heterogeneous conditions. These investigations have illuminated the role of energy barriers, temperature gradients, and system asymmetry in determining steady-state currents, thermodynamic cost, and control precision [31].

Yet, a crucial aspect remains insufficiently addressed: in realistic physical and biological settings, external driving forces and thermal environments often vary explicitly in time. Intracellular motors, for instance, operate in fluctuating chemical landscapes where ATP concentrations and load forces evolve dynamically. Similarly, synthetic microswimmers and colloidal particles are frequently subjected to temporally modulated optical, thermal, or electrical control in laboratory experiments.

In such contexts, time-dependent force and temperature profiles are not mere theoretical abstractions but reflect experimentally realizable protocols. Exponential decay of temperature arises naturally during thermal relaxation, for example, following laser-induced heating, or in systems exchanging heat with a finite reservoir. Linear ramps are routinely applied in experimental setups involving electrophoretic forces, magnetic tweezers, and calorimetric cycles. Periodic and impulsive protocols are similarly standard in microrheology and molecular motor assays. As such, exponential, linear, and periodic time dependencies serve as realistic and analytically tractable models of experimentally relevant nonstationary driving.

Despite their relevance, the thermodynamics of systems subject to explicitly time-dependent driving, particularly in the context of entropy production, extraction, and dissipation, remains largely unexplored. Previous studies have seldom addressed how such protocols alter the nonequilibrium response and how they affect the fundamental thermodynamic limits. This work aims to address this gap by systematically analyzing Brownian and active systems under time-dependent thermal and mechanical control, thereby extending nonequilibrium thermodynamics into dynamic, experimentally grounded regimes.

The rest of the paper is organized as follows. In Section II, we introduce the stochastic model and derive expressions for the system's free energy under nonstationary dynamics. In Section III, we analyze the thermodynamic quantities under time-dependent temperature protocols. In Section IV, we study systems driven by temporally varying external forces. In Section V, we extend the analysis to time-dependent viscous friction. In Section VI, we discuss the implications for optimizing energy conversion in small-molecule motors. Finally, Section VII presents concluding remarks and a synthesis of the key results.

2. The model and derivation of thermodynamic relations

Prior to investigating the thermodynamic response under time-dependent forcing and thermal environments, we establish the foundational relations that govern entropy production and energy dissipation in active and passive Brownian systems.

The stochastic evolution of the system is described by the Fokker-Planck equation

$$\frac{\partial P(x,\,t)}{\partial t} = -\frac{\partial}{\partial x}\left[\frac{A(x,\,t)P(x,\,t)}{\gamma(t)}\right] + \frac{\partial^2}{\partial x^2}\left[\frac{D(t)P(x,\,t)}{\gamma(t)}\right], \tag{1}$$

where $P(x,\,t)$ denotes the probability density, $A(x,\,t)$ is the drift coefficient, and $\gamma(t)$ is the time-dependent viscous friction. For an active Brownian motor, the drift is given by $A(x,\,t) = f - \gamma v_0$, where f denotes the external load and v_0 is the self-propulsion velocity. In contrast, for a passive motor, $A(x,\,t) = f$. The diffusion coefficient is governed by the Einstein relation

$$D(t) = \frac{k_B T(t)}{\gamma(t)}, \tag{2}$$

with $T(t)$ denoting the instantaneous temperature and k_B the Boltzmann constant. Throughout, the spatial coordinate x is taken to be one-dimensional and subject to periodic boundary conditions, i.e., $P(x+L,\,t) = P(x,\,t)$ and $J(x+L,\,t) = J(x,\,t)$, where L is the domain length.

The associated probability current is defined as

$$J(x,\,t) = \frac{A(x,\,t)P(x,\,t) - D(t)\dfrac{\partial P(x,\,t)}{\partial x}}{\gamma(t)}. \tag{3}$$

The system's Shannon entropy is expressed as

$$S(t) = -\int_0^L P(x,\,t)\ln P(x,\,t)\,dx, \tag{4}$$

and its temporal evolution satisfies the entropy balance equation

$$\frac{dS(t)}{dt} = \dot{e}_p - \dot{h}_d, \tag{5}$$

where the instantaneous entropy production rate $\dot{e}_p$ and entropy extraction rate $\dot{h}_d$ are given respectively by

$$\dot{e}_p = \int_0^L \frac{\gamma(t)J^2(x,\,t)}{P(x,\,t)T(t)}\,dx, \tag{6}$$

$$\dot{h}_d = \int_0^L J(x,\,t)\frac{U'(x)}{T(t)}\,dx. \tag{7}$$

The entropy production rate $\dot{e}_p(t)$ quantifies the irreversible generation of entropy within the system due to nonconservative forces, stochastic fluctuations, and dissipation. The entropy extraction rate $\dot{h}_d(t)$, on the other hand, measures the net entropy flux transferred to the thermal environment. These time derivatives emerge naturally from the differential form of the second law and are now introduced explicitly prior to Eq. (9) for clarity.

To prevent conceptual ambiguity, it is important to distinguish between several entropy-related quantities. Function $S(t)$ quantifies the statistical uncertainty in the system's microstate distribution. The rate $\dot{E}_p(t)$ characterizes the irreversible production of entropy due to system-bath coupling. In contrast, the product $S(t)T(t)$ possesses units of energy and arises in equilibrium thermodynamic identities but does not carry a direct interpretation as entropy flow or production.

The time-integrated counterparts of the entropy flow and production rates are

$$\Delta h_d(t) = \int_{t_0}^{t} \dot{h}_d(t')\,dt', \tag{8}$$

$$\Delta e_p(t) = \int_{t_0}^{t} \dot{e}_p(t')\,dt', \tag{9}$$

$$\Delta S(t) = \Delta e_p(t) - \Delta h_d(t). \tag{10}$$

An alternative but thermodynamically equivalent expression for the entropy production is

$$\dot{E}_p = \int_0^L \frac{\gamma(t) J^2(x,\,t)}{P(x,\,t)}\,dx, \tag{11}$$

and the corresponding entropy dissipation (or extracted heat) rate is

$$\dot{H}_d = \int_0^L J(x,\,t) U'(x)\,dx. \tag{12}$$

The entropy balance law can then be expressed compactly as

$$\frac{dS^T(t)}{dt} = \dot{E}_p - \dot{H}_d, \tag{13}$$

and integrated to obtain

$$\Delta S^T(t) = \Delta E_p(t) - \Delta H_d(t). \tag{14}$$

The internal energy change rate is governed by the first law of thermodynamics

$$\dot{E}_{\text{in}} = -\dot{H}_d - \dot{W}, \text{ where } \dot{W} = \int_0^L J(x,\,t) f\,dx, \tag{15}$$

and the free energy dissipation rate is given by

$$\dot{F} = \dot{E}_{\text{in}} - \dot{S}^T = \dot{E}_{\text{in}} - \dot{E}_p + \dot{H}_d. \tag{16}$$

In the quasistatic limit, where $f = \gamma v_0$, both $\dot{E}_p$ and $\dot{H}_d$ vanish, indicating a reversible transformation. In contrast, for systems operating far from equilibrium, $\dot{E}_p > 0$, reflecting irreversible entropy production. For non-isothermal steady states, the balance $\dot{E}_p = \dot{H}_d$ ensures that the free energy change equals the internal energy change, i.e., $\Delta F(t) = \Delta U(t)$.

Finally, it is clarified that Eqs. (17) and (18) do not explicitly contain $T(t)$ because they are derived under the assumption of a constant temperature $T(t) = T_0$. This simplification has now been explicitly stated. The boundary conditions used to derive Eq. (16), from which Eqs. (17) and (18) follow, are periodic and consistent with the rest of the analysis.

2.1 *Exponentially decreasing temperature in time*

To explore the effect of thermal arrangements, we introduce exponentially and linearly decreasing thermal arrangements. For an exponential decay in temperature, the temperature $T(t)$ is given by

$$T(t) = T_{\text{st}} + (T_0 - T_{\text{st}})e^{-\beta t}, \tag{17}$$

where T_0 is the initial temperature and T_{st} is the steady-state temperature. The parameter β denotes the decaying constant. The variance is expressed as $\sigma^2(t) = \frac{2}{\gamma}\left[T_{\text{st}}t + \frac{(T_0 - T_{\text{st}})}{\beta}\left(1 - e^{-\beta t}\right)\right]$. For exponentially decreasing case, the entropy is given as

$$S(t) = \frac{1}{2}\ln\left(4\pi e\frac{2}{\gamma}\left[T_{\text{st}}t + \frac{(T_0 - T_{\text{st}})}{\beta}(1 - e^{-\beta t})\right]\right), \tag{18}$$

Please note that the expression for entropy is independent of load f' if only free boundary conditions are imposed. The change in entropy is given by

$$\Delta S(t) = S(t) - S(t_0) \tag{19}$$

$$= \frac{1}{2}\ln\left(\frac{T_{\text{st}}t + \frac{T_0 - T_{\text{st}}}{\beta}(1 - e^{-\beta t})}{T_{\text{st}}t_0 + \frac{T_0 - T_{\text{st}}}{\beta}(1 - e^{-\beta t_0})}\right).$$

The above equation indicates that the entropy increases with time.

The entropy evolution, $S(t)$, is analyzed for the exponential decreasing temperature case for various values of the decay parameter β ($\beta = 0.1,\ 0.3,\ 0.5,\ 0.7,\ 0.9,\ 1.1,\ 1.3,\ 1.5$) while maintaining the same initial and steady-state temperatures of $T_0 = 10.0$ and $T_{\text{st}} = 1.0$ (see Figure 1).

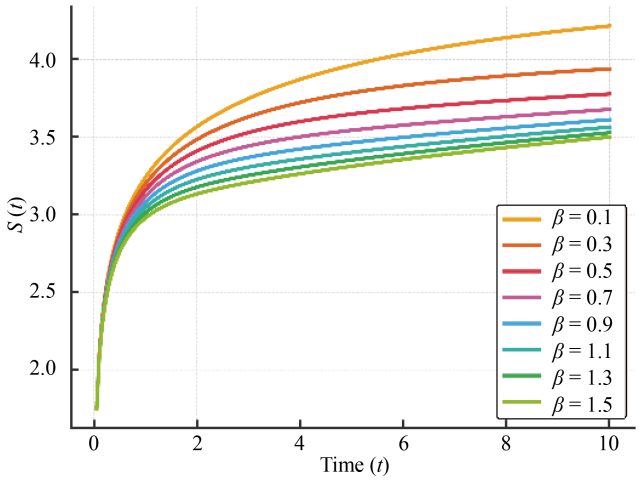

Figure 1. (Color online) Entropy evolution $S(t)$ for exponential decay model as a function of time for different values of β ($\beta = 0.1,\ 0.3,\ 0.5,\ 0.7,\ 0.9,\ 1.1,\ 1.3,\ 1.5$), with $T_0 = 10$ (initial temperature) and $T_{\text{st}} = 1$ (steady-state temperature). v_0

2.2 *Linearly decreasing temperature in time*

On the contrary, for linearly decreasing thermal arrangements

$$T(t) = T_0 - \alpha t \tag{20}$$

which is valid for the time interval $t \leq \dfrac{T_0 - T_{\text{st}}}{\alpha}$. We calculate the variance as $\sigma^2(t) = \dfrac{2}{\gamma}\left[T_0 t - \dfrac{\alpha t^2}{2}\right]$. We then calculate the entropy

$$S(t) = \frac{1}{2}\ln\left(4\pi e \frac{2}{\gamma}\left(T_0 t - \frac{\alpha t^2}{2}\right)\right), \tag{21}$$

As one can see that the entropy depends only on the temperature and viscous friction. The steady-state entropy can be written as

$$S = \frac{1}{2}\ln\left(4\pi e \frac{(T_0 - T_{\text{st}})^2}{\gamma\alpha}\right). \tag{22}$$

The change in entropy is given by

$$\Delta S(t) = \frac{1}{2}\ln\left(\frac{T_0 t - \dfrac{\alpha t^2}{2}}{T_0 t_0 - \dfrac{\alpha t_0^2}{2}}\right),\ t \leq \frac{T_0 - T_{\text{st}}}{\alpha}. \tag{23}$$

The entropy evolution, $S(t)$, is analyzed for the linear temperature decay model (see Figure 2), $S(t)$ is examined for different values of the parameter α ($\alpha = 0.1,\ 0.3,\ 0.5,\ 0.7,\ 0.9,\ 1.1,\ 1.3,\ 1.5$), with an initial temperature of $T_0 = 10.0$ and a steady-state temperature of $T_{\text{st}} = 1$.

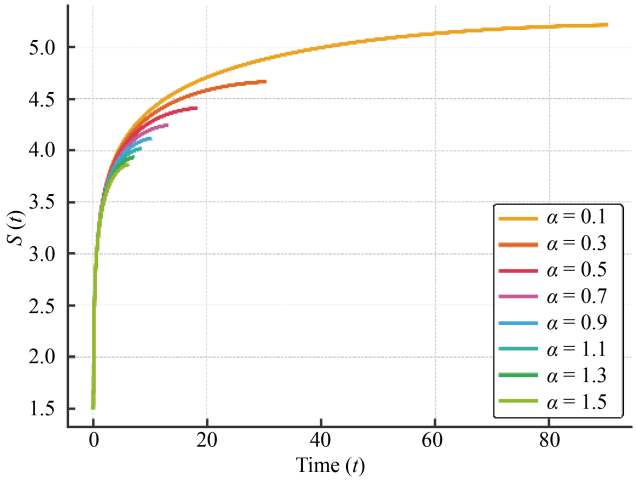

Figure 2. (Color online) Entropy evolution $S(t)$ as a function of time (linear temperature decay model) for different values of α ($\alpha = 0.1,\ 0.3,\ 0.5,\ 0.7,\ 0.9,\ 1.1,\ 1.3,\ 1.5$), with $T_0 = 10$ (initial temperature) and $T_{\text{st}} = 1$ (steady-state temperature)

Regardless of any thermal arrangements, the current density can be evaluated via

$$J(x,\ t) = P(x,\ t)v,\ v = \frac{f - \gamma v_0}{\gamma}, \tag{24}$$

After some algebra, for all thermal profiles, we derive the expressions for entropy production rate

$$\dot{e}_p = \frac{(f - \gamma v_0)^2}{\gamma T(t)} + \frac{T(t)}{2\sigma^2(t)} \tag{25}$$

and extraction rate

$$\dot{h}_p = \frac{(f - \gamma v_0)^2}{\gamma T(t)}. \tag{26}$$

Figure 3 illustrates the behavior of ΔE_p as a function of time for different values of f ($f = 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1.0$) under a linearly decreasing temperature profile. Similarly, Figure 4 presents the evolution of ΔH_d over time for various values of f. Both figures demonstrate that the entropy production or extraction rates increase monotonically with both force and time. In the absence of an applied force, the entropy production remains zero.

Let us now derive some of the key thermodynamic quantities for the linearly decreasing cases. The term related to the heat extraction rate can be writen as

$$\dot{H}_d = \frac{(f - \gamma v_0)^2}{\gamma}. \tag{27}$$

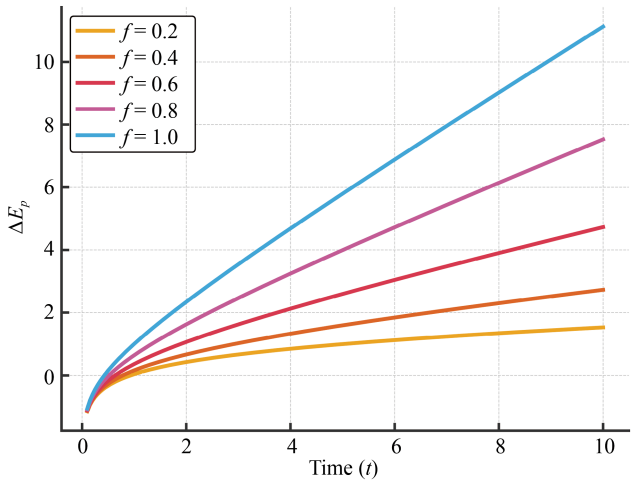

Figure 3. (Color online) The figure presents the temporal evolution of ΔE_p (in the absence of v_0)for various values of the force parameter f (f = 0.2, 0.4, 0.6, 0.8, 1.0) under a linearly decreasing temperature regime, with an initial temperature of $T_0 = 10$, a steady-state temperature of $T_{st} = 1$, and a decay parameter $\alpha = 0.1$. The result of this work also depicts that f enhances energy dissipation, showing the significant role of external forcing in the thermodynamic behavior of the system

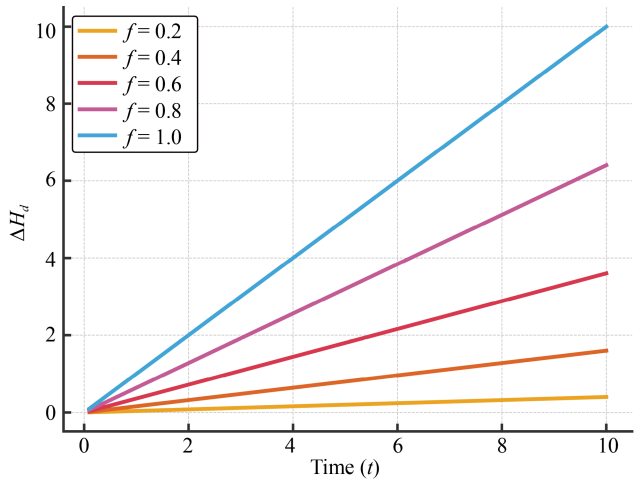

Figure 4. (Color online) The figure illustrates the variation of ΔH_d as a function of time (in the absence of v_0) for the same set of f values under identical thermal conditions. The change in the heat dissipation depict that as time increases, this thermodynamic relations increases linearly

The term related to the entropy production rate is given by:

$$\dot{E}_p = \frac{(f - \gamma v_0)^2}{\gamma} + \frac{(T_0 - \alpha t)^2}{\gamma\left(4\left(T_0 t - \frac{\alpha t^2}{2}\right)\right)}. \tag{28}$$

The time changes in entropy production and heat dissipation over time are

$$\Delta H_d = \frac{(f-\gamma v_0)^2 t}{\gamma} \tag{29}$$

and

$$\Delta E_p = \frac{(f-\gamma v_0)^2}{\gamma} t + \frac{1}{2} \ln \left(\frac{T_0 t - \frac{\alpha t^2}{2}}{T_0 t_0 - \frac{\alpha t_0^2}{2}} \right), \tag{30}$$

which is in agreement with $\Delta S = \Delta E_p - \Delta H_d$. The free energy

$$\Delta F(t) = -\Delta S(t) \tag{31}$$

as expected. Clearly, in the quasistatic limit, when the external force approaches the stall condition $f \to \gamma v_0$, the net velocity of the particle becomes zero. In this regime, all thermodynamic rates-including entropy production and extraction rates vanish, as shown in the equations above.

Here, we want to emphasize that entropy plays a crucial role in understanding these systems, as it quantifies energy dissipation, irreversibility, and the approach to thermal equilibrium. Brownian ratchets and stochastic engines, which rely on thermal noise to generate directed motion, exhibit reduced performance as the temperature drops, limiting the available thermal fluctuations necessary for movement. Active matter systems, such as bacterial colonies, artificial microswimmers, and cytoskeletal assemblies, rely on energy input to sustain motion; however, under decreasing temperatures, their activity diminishes, affecting their ability to self-organize and perform functions. By studying how systems behave under different temperature decay patterns, researchers can develop energy-efficient molecular machines, refine biomedical technologies, enhance industrial cooling strategies, and optimize quantum and nanoscale devices.

3. Analysis of brownian motion under a force that depends on time

In this section, we show that time-varying forces play a significant role in nonequilibrium thermodynamics by dictating transport efficiency, energy dissipation, and entropy production. We show that when the magnitude of these forces increases, the entropy production and extraction rates increase. We want to stress that most biological systems such as molecular motors, Brownian motors, active matter, and self-propelled particles either generate time-varying forces or operate under time-varying forces.

3.1 *Exponentially decreasing force in time*

Let us now consider a force that exponentially decays in time and approaches a constant value at a steady state:

$$F(t) = f_{st} - \gamma v_0 + (F_0 - f_{st}) e^{-\lambda t} \tag{32}$$

where F_0 denotes the initial force amplitude, f_{st} denotes the steady-state force before adjusting for stall, γv_0 is the stall force, and λ represents the decay rate.

Considering a constant temperature and the adjusted exponentially decreasing force, we rewrite the Fokker-Planck equation as

$$\frac{\partial P(x,\,t)}{\partial t} = \frac{\partial}{\partial x}\left(\frac{f_{st} - \gamma v_0 + (F_0 - f_{st})e^{-\lambda t}}{\gamma} P(x,\,t)\right) + \frac{T}{\gamma}\frac{\partial^2 P(x,\,t)}{\partial x^2}. \tag{33}$$

Note that this force can either be an external force applied to the system or a self-generated force arising from the active matter.

After some algebra, we also find the probability distribution

$$P(x,\,t) = \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{\left(x - \frac{F_0}{\gamma\lambda}(1 - e^{-\lambda t}) - \frac{(f_{st} - \gamma v_0)t}{\gamma}\right)^2}{4Dt}\right), \tag{34}$$

Using the above probability distribution, the entropy for the given system is simplified to

$$S(t) = \frac{1}{2}\ln\left(\frac{4\pi e T t}{\gamma L^2}\right) \tag{35}$$

Exploiting Eq. (37), one can see that as time progresses, the entropy increases, showing that the system irreversibility increases with time. Since the external energy of the system is zero, via Eq. (14), it is evident that:

$$\Delta F(t) = -\Delta S(t). \tag{36}$$

The change in the free energy decreases with time.

The entropy production rate is given by

$$\dot{e}_p = \frac{1}{2Tt} + \frac{(F_0 - f_{st})^2}{\gamma T} e^{-2\lambda t} + \frac{(f_{st} - \gamma v_0)^2}{\gamma T} \tag{37}$$

At steady state ($t \to \infty$), the entropy production rate becomes

$$\dot{e}_p = \frac{(f_{st} - \gamma v_0)^2}{\gamma T} \tag{38}$$

These equations depict that at small time t the entropy production rate is considerably high, and as time progresses, it decreases and stabilizes at a steady value dictated by the adjusted steady-state force. As long as a nonzero net force is imposed or operates in finite time, the system continuously dissipates energy.

The entropy extraction rate is calculated as

$$J(x,\,t) = -\frac{T}{\gamma}\frac{\partial P}{\partial x} + \frac{f_{st} - \gamma v_0 + (F_0 - f_{st})e^{-\lambda t}}{\gamma}P \tag{39}$$

$$\dot{h}_d = \int \left(J\frac{F(t)}{T} \right) dx \tag{40}$$

where the probability current is, as given above.

After some algebra, we obtain

$$\dot{h}_d = \frac{(F_0 - f_{st})^2}{\gamma T}e^{-2\lambda t} + \frac{(f_{st} - \gamma v_0)^2}{\gamma T} \tag{41}$$

At steady state ($t \to \infty$), the entropy production and extraction rates converge to

$$\dot{e}_p = \dot{h}_d = \frac{(f_{st} - \gamma v_0)^2}{\gamma T} \tag{42}$$

This result indicates that, at long times, the system reaches a balance where the entropy production rate matches the entropy extraction rate.

The velocity of the particle moving in this force field is given as

$$v = \frac{f_{st} - \gamma v_0 + (F_0 - f_{st})e^{-\lambda t}}{\gamma} \tag{43}$$

and at the steady state, one obtains $v = \dfrac{f_{st} - \gamma v_0}{\gamma}$.

All these results indicate that while the entropy $S(t)$ remains independent of the applied force, the entropy production and extraction rates significantly depend on the shifted force $f - \gamma v_0$. As the force increases beyond the stall point, the system exhibits greater irreversibility, reflecting higher dissipation and entropy generation. At the stall force, when $f = \gamma v_0$, or equivalently when $f - \gamma v_0 = 0$, all thermodynamic rates including entropy production and extraction vanish.

3.2 *Brownian motion in a periodic force*

Next, let us consider a force that varies periodically:

$$F(t) = F_0 \cos(\omega t + \phi) - \gamma v_0 \tag{44}$$

Here, F_0, ω, and ϕ denote the force amplitude, angular frequency, and phase shift, respectively.

We write the corresponding Fokker-Planck equation as

$$\frac{\partial P(x,\,t)}{\partial t} = \frac{\partial}{\partial x}\left(\frac{F_0 \cos(\omega t + \phi) - \gamma v_0}{\gamma} P(x,\,t) \right) + \frac{T}{\gamma}\frac{\partial^2 P(x,\,t)}{\partial x^2} \tag{45}$$

and one can simplify this equation to

$$\frac{\partial P}{\partial t} = -\frac{F_0 \cos(\omega t + \phi) - \gamma v_0}{\gamma} \frac{\partial P}{\partial x} + \frac{T}{\gamma} \frac{\partial^2 P}{\partial x^2} \tag{46}$$

The probability distribution for a given system is given by

$$P(x,\, t) = \frac{1}{\sqrt{4\pi D t}} \exp\left(- \frac{\left(x - \frac{F_0}{\gamma \omega} \sin(\omega t + \phi) - v_0 t \right)^2}{4 D t} \right) \tag{47}$$

After some algebra, one finds the entropy as

$$S(t) = \frac{1}{2} \ln\left(\frac{4\pi e T t}{\gamma L^2} \right) \tag{48}$$

which is independent of the force.

The entropy production rate simplifies to

$$\dot{e}_p = \frac{1}{2Tt} + \frac{(F_0 \cos(\omega t + \phi) - \gamma v_0)^2}{\gamma T} \tag{49}$$

The first term, $\frac{1}{2Tt}$, represents a dissipative contribution that decays over time, while the second term, $\frac{(F_0 \cos(\omega t + \phi) - \gamma v_0)^2}{\gamma T}$, accounts for entropy fluctuations due to periodic driving. Although the entropy fluctuates periodically, it decreases over time.

The time-averaged entropy production rate is also given by

$$\langle \dot{e}_p \rangle = \frac{1}{2\pi} \int_0^{2\pi} \frac{(F_0 \cos(\theta) - \gamma v_0)^2}{\gamma T} d\theta = \frac{F_0^2 + \gamma^2 v_0^2}{2\gamma T} \tag{50}$$

The heat dissipation rate is calculated as

$$\dot{h}_d = \frac{(F_0 \cos(\omega t + \phi) - \gamma v_0)^2}{\gamma T} \tag{51}$$

At steady state, the time-averaged entropy production rate and heat dissipation rate are equal:

$$\langle \dot{e}_p \rangle = \langle \dot{h}_d \rangle = \frac{F_0^2 + \gamma^2 v_0^2}{2\gamma T} \tag{52}$$

This result indicates that at a steady state, the system reaches a balance where the entropy production rate balances the entropy extraction rate.

The velocity:

$$v = \frac{F_0 \cos(\omega t + \phi) - \gamma v_0}{\gamma} \tag{53}$$

oscillates with the same frequency ω, and the time-averaged velocity is zero. For instance, if a charge is exposed to a time-varying electric field, the particle experiences a time-varying force. As a result, the particle exhibits oscillatory velocity.

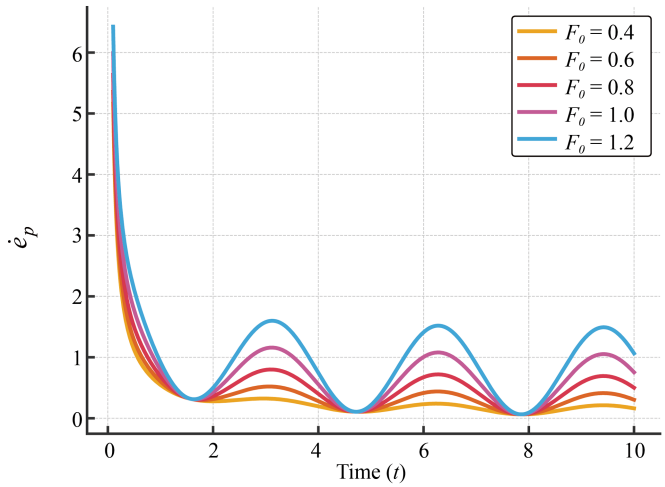

Figure 5. (Color online) The figure depicts the temporal evolution of the entropy production rate, $\dot{e}_p$, for different values of the force parameter F_0 ($F_0 = 0.4,\ 0.6,\ 0.8,\ 1.0,\ 1.2$), with fixed parameters $v_0 = 0$, $\omega = 10.0$, $T = 1.0$, $\gamma = 1$, and $\varphi = 0$. The results show that $\dot{e}_p$ exhibits oscillatory behavior while decreasing over time. Notably, larger values of F_0 lead to a considerable increase in the amplitude of the oscillations, indicating a stronger influence of the applied force on the entropy dynamics. Despite these fluctuations, the entropy production rate declines on average at a nearly constant rate, underscoring the interplay between external forcing and dissipative thermodynamic processes

In Figure 5, we present the time evolution of the entropy production rate, $\dot{e}_p$ (in the absence of v_0), for different values of the external force amplitude, F_0 ($F_0 = 0.4,\ 0.6,\ 0.8,\ 1.0,\ 1.2$). The results reveal that the entropy production rate exhibits oscillatory behavior while simultaneously decreasing over time. Notably, for larger values of F_0, the amplitude of these oscillations becomes significantly more pronounced, indicating a stronger influence of the applied force on the system's thermodynamic response. Despite these oscillations, the entropy production rate exhibits an overall monotonic decline, with an average decay that follows a nearly constant rate.

As discussed previously, periodic forces may play a crucial role in active matter, molecular motors, and thermodynamic heat engines. For instance, molecular motors receive periodic energy input from ATP hydrolysis to generate mechanical work, as well as to overcome thermal fluctuations and molecular friction.

3.3 *Analysis of brownian motion under a linearly increasing force*

In this section, we consider a linearly increasing force, which can be found in real systems such as molecular motors, Brownian ratchets, and active matter. Even in active matter, self-propelled particles experience gradually increasing forces due to controlled chemical gradients or time-dependent propulsion. In Brownian ratchets, we can apply external fields, such as optical or magnetic forces, that increase over time to drive directed motion. This force

$$F(t) = kt - \gamma v_0 \tag{54}$$

can also be used as a probe to study a given system. Here, k is a constant.

Using the Fokker-Planck equation:

$$\frac{\partial P(x,\, t)}{\partial t} = \frac{\partial}{\partial x}\left(\frac{kt - \gamma v_0}{\gamma} P(x,\, t)\right) + \frac{T}{\gamma}\frac{\partial^2 P(x,\, t)}{\partial x^2}, \tag{55}$$

one can solve the probability distribution:

$$P(x,\, t) = \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{\left(x - \frac{k}{2\gamma}t^2 + v_0 t\right)^2}{4Dt}\right), \tag{56}$$

The entropy is given as

$$S(t) = \frac{1}{2}\ln\left(\frac{4\pi e T t}{\gamma L^2}\right) \tag{57}$$

As one can see, the entropy logarithmically increases with time and temperature.

For the linearly increasing force case, the velocity is

$$v = \frac{kt - \gamma v_0}{\gamma} \tag{58}$$

linearly increases in time.

After some algebra, the entropy production rate is given by:

$$\dot{e}_p = \frac{1}{2Tt} + \frac{(kt - \gamma v_0)^2}{\gamma T} + \frac{k^2}{\gamma T} \tag{59}$$

The entropy production rate comprises three distinct contributions: a transient decay term, time-dependent growth term, and steady-state component. The initial decay indicates that entropy production is initially high, but diminishes as the system transitions from an out-of-equilibrium state to a driven steady state. The quadratic growth term in time reflects the increasing contribution of periodic forcing, signifying the cumulative effects of sustained energy injection into the system.

On the other hand, the heat dissipation rate is calculated as:

$$\dot{h}_d = \frac{(kt - \gamma v_0)^2}{\gamma T} + \frac{k^2}{\gamma T} \tag{60}$$

3.4 *Brownian motion moving in a periodic impulse force*

Let us now consider a Brownian particle that walks in a periodic impulsive force characterized by sudden bursts applied at regular intervals. The system is driven out of equilibrium when such a force is applied, and this in turn creates distinct entropy production patterns influenced by impulse frequency, magnitude, and duration. This force considerably affects the dynamics of the system because unlike the continuous force, the impulse causes rapid changes in energy and dynamics. In stochastic thermodynamics and nonlinear dynamics, this force

$$F(t) = F_0 \sum_{n=0}^{\infty} \delta(t - nT') - \gamma v_0 \sum_{n=0}^{\infty} \delta(t - nT') \tag{61}$$

regulates motion, enhances energy transfer, and sustains non-equilibrium steady states, making them essential for understanding driven systems under discrete perturbations. Here, the magnitude of the impulse force F_0 and the period of the applied force T' dictate the dynamics. The delta function $\delta(t - nT')$ represents the instantaneous impulse at discrete times, $t = nT'$.

After writing the Fokker-Planck equation:

$$\frac{\partial P(x,\, t)}{\partial t} = \frac{\partial}{\partial x}\left(\frac{(F_0 - \gamma v_0)\sum_{n=0}^{\infty}\delta(t - nT')}{\gamma} P(x,\, t)\right) + \frac{T}{\gamma}\frac{\partial^2 P(x,\, t)}{\partial x^2}, \tag{62}$$

we calculate the probability distribution as

$$P(x,\, t) = \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{(x - nv'T')^2}{4Dt}\right), \tag{63}$$

where $v' = \dfrac{F_0 - \gamma v_0}{\gamma}$ and $D = \dfrac{T}{\gamma}$ denote the effective velocity and diffusion coefficient, respectively.

The entropy:

$$S(t) = \frac{1}{2}\ln\left(\frac{4\pi e T t}{\gamma L^2}\right) \tag{64}$$

increases with time and temperature.

On the other hand, the velocity is

$$v = \frac{(F_0 - \gamma v_0)\sum_{n=0}^{\infty}\delta(t - nT')}{\gamma} \tag{65}$$

will have sudden jumps at each impulse, followed by exponential decay, as shown in Figure 6a. The time average of the velocity is given as

$$\langle v \rangle = \frac{1}{T'}\int_0^{T'} v(t),\, dt = \frac{F_0 - \gamma v_0}{\gamma^2 T'}(1 - e^{-\gamma T'}). \tag{66}$$

The entropy production rate is given as

$$\dot{e}_p = \frac{1}{2Tt} + \frac{(F_0 - \gamma v_0)^2}{\gamma T} \sum_{n=0}^{\infty} \delta(t - nT'). \tag{67}$$

It can be seen that the entropy production rate is dictated by the periodic impulsive force. Consequently, it exhibits discrete spikes at regular intervals whenever an impulsive force is applied, and after the spikes, the rate relaxes to lower values. This oscillatory behavior originates from intermittent energy injection, which temporarily increases the entropy production before the system dissipates excess energy. The instantaneous nature of external forcing is amplified by the presence of Dirac delta functions in both entropy production and heat extraction. This indicates that the energy input occurs in short bursts rather than continuously. We show that even though the entropy production rate fluctuates over time, as time increases, it declines to a steady state. Regardless of the magnitude of the force, the rate decays over time.

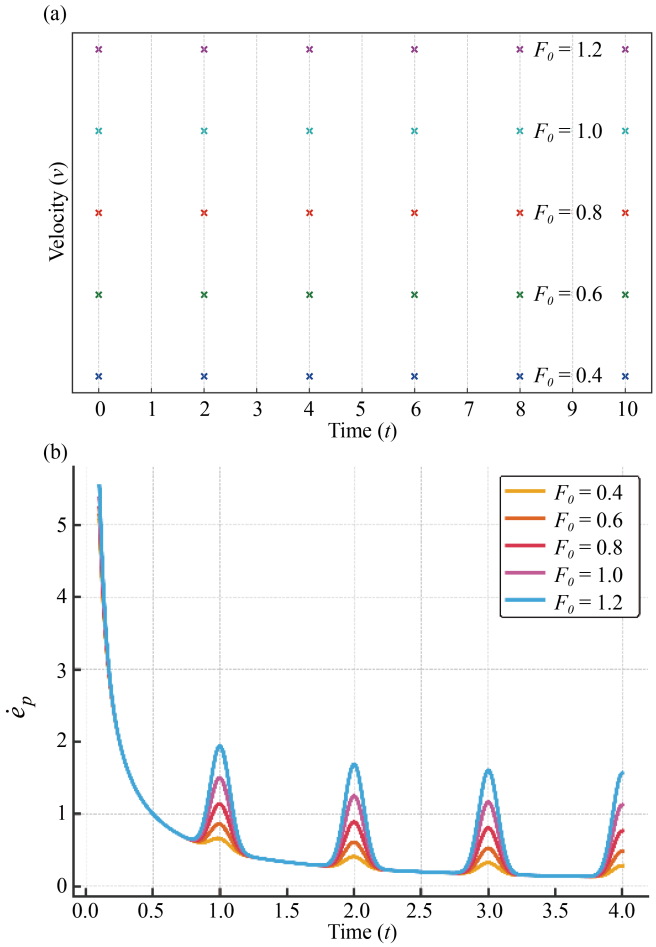

Figure 6. (a) Plot of velocity as a function of time in the absence of v_0. (b) The plot of $\dot{e}_p$ as a function of time t for different forces F_0 ($F_0 = 0.4,\ 0.6,\ 0.8,\ 1.0,\ 1.2$) and $v_0 = 0$. The results show that whenever a periodic impulse force is applied, the entropy production rate exhibits recurrent spikes, followed by relaxation. As the magnitude of F_0 increases, the amplitude of these oscillations becomes more pronounced. Despite these fluctuations, the entropy production rate decreases to its steady-state value, with an average decay that remains consistent regardless of the applied force

The entropy extraction rate is also calculated as

$$\dot{h}_d = \frac{(F_0 - \gamma v_0)^2}{\gamma T} \sum_{n=0}^{\infty} \delta(t - nT'). \tag{68}$$

At steady state, the entropy production and entropy extraction rates are equal:

$$\dot{e}_p = \dot{h}_d = \frac{(F_0 - \gamma v_0)^2}{\gamma T}. \tag{69}$$

This result indicates that, at long times, the system reaches a balance where the entropy produced matches the heat dissipated.

Figure 6a depicts $\dot{e}_p$ versus time for fixed F_0 ($F_0 = 0.4,\ 0.6,\ 0.8,\ 1.0,\ 1.2$) and $v_0 = 0$. The figure shows that in the presence of a periodic impulse force, the entropy production rate exhibits recurrent spikes. Each spike ultimately relaxes to a lower value. This periodic modulation arises due to the external forcing, which intermittently perturbs the system's entropy dynamics. Despite these oscillatory fluctuations, the overall entropy production rate decreases over time, following an average decay pattern.

3.5 *Brownian motion subjected to impulsive force*

Let us now explore the thermodynamic relations in the presence of impulsive force

$$F(t) = F_0 \sum_{n=0}^{\infty} \delta(t - t_0) - \gamma v_0 \sum_{n=0}^{\infty} \delta(t - t_0). \tag{70}$$

Fokker-Planck equation in terms of the applied force F at time t_0 is calculated as

$$\frac{\partial P(x,\, t)}{\partial t} = -\frac{F}{\gamma} \delta(t - t_0) \frac{\partial P}{\partial x} + \frac{T}{\gamma} \frac{\partial^2 P}{\partial x^2}. \tag{71}$$

After some algebra, the probability distribution reduces to

$$P(x,\, t) = \frac{1}{\sqrt{4\pi D t}} \exp\left(- \frac{(x - \frac{F}{\gamma} H(t - t_0))^2}{4Dt} \right) \tag{72}$$

where $H(t)$ is the Heaviside step function. Force F_0 is applied at time t_0.

The entropy:

$$S(t) = \frac{1}{2} \ln\left(\frac{4\pi e T t}{\gamma L^2} \right) \tag{73}$$

increases with time and temperature.

The velocity:

$$v = \frac{(F_0 - \gamma v_0)\sum_{n=0}^{\infty}\delta(t-t_0)}{\gamma} \tag{74}$$

will have sudden jumps.

The entropy production rate $\dot{e}_p$ is calculated as

$$\dot{e}_p = \frac{1}{2Tt} + \frac{(F_0 - \gamma v_0)^2}{\gamma T}\delta(t-t_0). \tag{75}$$

Similarly, the entropy extraction rate $\dot{h}_d$ is given by

$$\dot{h}_d = \frac{(F_0 - \gamma v_0)^2}{\gamma T}\delta(t-t_0). \tag{76}$$

At steady state ($t \to \infty$), the term $\frac{1}{2Tt}$ vanishes, and we obtain

$$\dot{e}_p = \dot{h}_d = \frac{(F_0 - \gamma v_0)^2}{\gamma T}\delta(t-t_0) \tag{77}$$

This confirms that the entropy production rate balances the entropy extraction rate. As can be clearly seen, the entropy production rate is dictated by nonperiodic impulsive forces. When a sudden impulsive force is applied, abrupt spikes in the entropy production rate are observed. The spikes are then followed by relaxation phases at the same time that the rate decreases in time to a steady state. Despite the presence of transient fluctuations, the system relaxes back.

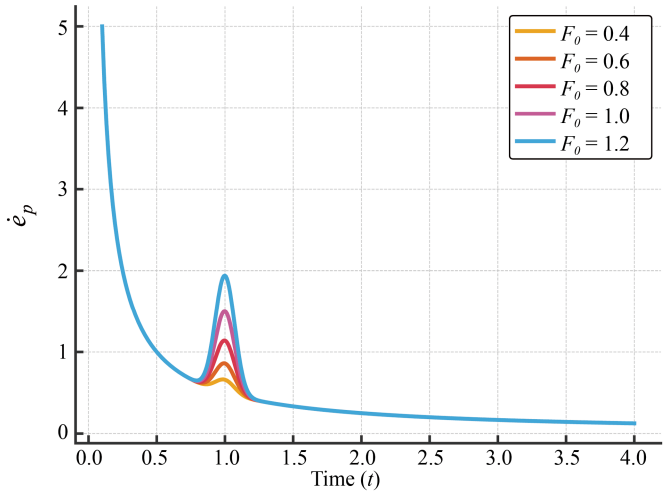

Figure 7. The entropy production rate, $\dot{e}_p$ as function of time t for fixed values of F_0 ($F_0 = 0.4,\ 0.6,\ 0.8,\ 1.0,\ 1.2$) and $v_0 = 0$ in the presence of an impulsive force. The results reveal that when a nonperiodic impulsive force is applied, the entropy production rate exhibits a sharp spike at a specific time. The system undergoes a relaxation period after this peak, during which the rate of entropy formation gradually decreases. As the magnitude of F_0 increases, the amplitude of the spike increases. In general, as time progresses, the entropy production or extraction rate decreases and saturates at a constant value

In Figure 7, we plot the entropy production rate (when $v_0 = 0$), $\dot{e}_p$ as a function of time t for fixed values of F_0 ($F_0 = 0.4,\ 0.6,\ 0.8,\ 1.0,\ 1.2$) in the presence of an impulsive force. The results reveal that when a nonperiodic impulsive force is applied, the entropy production rate exhibits a sharp spike at a specific time. After a pronounced peak, the system relaxes back. As the magnitude of F_0 increases, the amplitude of the spike increases. In general, as time progresses, the entropy production or extraction rate decreases and saturates at a constant value.

4. Time dependent viscous friction

Understanding time-dependent viscous friction along the reaction coordinate is essential for analyzing the thermodynamics of Brownian motors and molecular machines in nonequilibrium environments. Since viscosity modulates diffusion, motor velocity, and entropy production, its temporal variations significantly impact energy dissipation and transport efficiency. Increasing friction hinders motion, elevating dissipation and entropy production, whereas decreasing friction enhances efficiency and velocity. This study explores how time-dependent friction affects entropy and transport efficiency, offering insights into optimizing nanoscale motion and adaptive motor control.

Consider a Brownian particle that hops in a thermally uniform medium where the viscous friction is time dependent

$$\gamma = \frac{1}{g(1+t^z)} \tag{78}$$

and in this case the corresponding Fokker-Planck equation in overdamped medium is given as

$$\frac{\partial P(x,\ t)}{\partial t} = \frac{\partial}{\partial x}\left(\frac{f-\gamma v_0}{\gamma(t)}\right)P(x,\ t) + \frac{\partial}{\partial x}\left(\frac{T}{\gamma(t)}\frac{\partial P(x,\ t)}{\partial x}\right), \tag{79}$$

Imposing a periodic boundary condition $P(0,\ t) = P(L,\ t)$ and let us choose a Fourier cosine series

$$P(x,\ t) = \sum_{n=0}^{\infty} b_n(t)\cos\left(\frac{n\pi}{L}\left(x + \frac{f-\gamma v_0}{\gamma}\right)\right) \tag{80}$$

After some algebra we get the probability distribution as

$$P(x,\ t) = \sum_{n=0}^{\infty} \cos\left[\frac{n\pi}{L}\left(x + (f-\gamma v_0)\left(gt + \frac{gt^{z+1}}{z+1}\right)\right)\right]\zeta \tag{81}$$

where

$$\zeta = \exp\left(-\frac{(n\pi)^2 T\left(gt + \dfrac{gt^{z+1}}{z+1}\right)}{L^2}\right). \tag{82}$$

Here, f is the external load and T is the temperature of the medium.

The current is then given by

$$J(x,\,t) = -\left[\frac{(f-\gamma v_0)P(x,\,t)}{\gamma} + \frac{T}{\gamma}\frac{\partial P(x,\,t)}{\partial x}\right]. \tag{83}$$

As stated before $\dot{e}_p = \dot{h}_d + \dfrac{dS(t)}{dt}$ where

$$\frac{dS(t)}{dt} = -\int \frac{J}{P(x,\,t)}\frac{\partial P(x,\,t)}{\partial x}dx. \tag{84}$$

After some algebra, we write

$$\frac{dS(t)}{dt} = \int -J\frac{\sum n=0^{\infty}\dfrac{n\pi}{2}\cos\left[\dfrac{n\pi}{L}\left(x+(f-\gamma v_0)\left(gt+\dfrac{gt^{z+1}}{z+1}\right)\right)\right]\zeta}{\sum n=0^{\infty}\cos\left[\dfrac{n\pi}{L}\left(x+(f-\gamma v_0)\left(gt+\dfrac{gt^{z+1}}{z+1}\right)\right)\right]\zeta}dx. \tag{85}$$

For such a system we have

$$\dot{e}_p = -\int \frac{J^2}{P(x,\,t)Tg(1+t^z)}dx \tag{86}$$

and after some algebra we find

$$\dot{e}_p = \frac{(f-\gamma v_0)^2 g(1+t^z)}{T}. \tag{87}$$

The entropy extraction rate is given as

$$\dot{h}_d = -\int\left(\frac{J(f-\gamma v_0)}{T}\right)dx \tag{88}$$

and after some algebra one gets

$$\dot{h}_d = \frac{(f-\gamma v_0)^2 g(1+t^z)}{T}. \tag{89}$$

For $z<0$, within the limit $t\to\infty$, $\dot{e}_p = \dot{h}_d$. For the case where $z>0$, as t increases, $\dot{e}_p$ and $\dot{h}_d$ monotonously increase. On the other hand

$$\Delta h_d = \Delta e_p = \int_0^t \frac{(f-\gamma v_0)^2 g(1+t^z)}{T}dt = \frac{(f-\gamma v_0)^2 gt(t^z+z+1)}{T(z+1)}. \tag{90}$$

The heat dissipation rate is given by

$$\dot{H}_d = -\int (J(f-\gamma v_0))dx = (f-\gamma v_0)^2 g(1+t^z). \tag{91}$$

The term $\dot{E}_p$ is related to $\dot{e}_p$ and it is given by

$$\dot{E}_p = -\int \frac{J^2}{P(x,\, t)g(1+t^z)}dx = (f-\gamma v_0)^2 g(1+t^z) \tag{92}$$

while

$$\Delta H_d = \Delta E_p = \int_0^t (f-\gamma v_0)^2 g(1+t^z)dt = \frac{(f-\gamma v_0)^2 gt(t^z+z+1)}{(z+1)}. \tag{93}$$

On the other hand, the internal energy is given by

$$\dot{E}_{in} = \int JU's(x)dx = 0. \tag{94}$$

The total work done is then given by

$$\dot{W} = \int (J(f-\gamma v_0))dx. \tag{95}$$

The first law of thermodynamics can be written as

$$\dot{E}in = -\dot{H}_d(t) - \dot{W}. \tag{96}$$

5. Optimal driving force to minimize entropy production

Optimizing the driving force to minimize entropy production is essential for improving energy efficiency and reducing unnecessary dissipation in nonequilibrium systems. In many physical and biological processes, such as molecular motors, nanoscale transport, and active matter, external forces drive the system away from equilibrium, leading to energy loss in the form of heat. If the applied force is not carefully controlled, excessive dissipation occurs, reducing the efficiency of energy conversion and making the system less stable over time.

By designing an optimal force protocol that dynamically adjusts to changes in temperature, the system can operate closer to equilibrium, thereby reducing entropy production and minimizing energy waste. This optimization ensures that the work is extracted efficiently while keeping the dissipation as low as possible. It is particularly relevant in fields such as biophysics, where molecular machines rely on precise energy management, and nanotechnology, where minimizing heat generation is crucial for device performance. Understanding and implementing optimal driving strategies helps in designing energy-efficient processes, prolonging system stability, and improving the overall functionality of force-driven systems.

To minimize entropy production in a system with time-dependent temperature, an optimal driving force must be designed to reduce dissipation. Considering only the first term of the entropy production rate,

$$\dot{e}_p = \frac{f^2}{T(t)}. \tag{97}$$

It is evident that entropy production is directly proportional to the square of the applied force and inversely proportional to the instantaneous temperature. To achieve minimal entropy generation, the driving force $f(t)$ must be dynamically adjusted in response to temperature variations.

An optimal force protocol can be derived by enforcing a balance between the energy input and dissipation constraints. One effective approach is to define $f(t)$ as a function of temperature such that the ratio $f^2/T(t)$ remains as small as possible while maintaining system control. A natural choice is a power-law scaling,

$$f_{\text{opt}}(t) = F_0 \left(\frac{T(t)}{T_0} \right)^{\beta}, \tag{98}$$

where F_0 is the initial force magnitude, T_0 is the initial temperature, and β is an adjustable parameter that determines how the force compensates for thermal fluctuations. By selecting an appropriate β, one can tailor the protocol to minimize entropy production while ensuring that the system remains in a controlled nonequilibrium state.

Optimizing the driving force in this manner is crucial for improving the energy efficiency in stochastic thermodynamic systems, particularly in molecular machines, nanoscale engines, and active matter. Proper force modulation reduces unnecessary dissipation, extends system longevity, and enhances performance in thermodynamic control processes,

6. Summary and conclusion

In this work, we study the thermodynamics of Brownian motors that operate in media where both the driving force and thermal background vary dynamically in time. While previous studies have mainly focused on systems with isothermal or spatially varying forces or temperatures, real molecular machines, such as intracellular transport proteins, operate under temporally fluctuating forces and temperatures. These time-dependent protocols fundamentally alter the energy dissipation and entropy production, motivating a systematic theoretical investigation.

We analytically examined both active and passive Brownian motors subjected to exponentially, linearly, and quadratically varying thermal and force fields. The active motor is modeled as a particle self-propelling with velocity v_0, while the passive motor operates solely due to external driving and thermal asymmetries. Explicit expressions are derived for entropy, entropy production, and entropy extraction rates under each protocol.

Our results show that the total entropy depends solely on the temperature profile and viscous friction, provided that no boundary constraints are imposed. In contrast, entropy production and dissipation are directly modulated by the net driving force. In particular, we find that time-dependent periodic forces give rise to oscillatory entropy production superimposed on a monotonic decay, while periodic impulsive forces generate discrete entropy spikes. In contrast, nonperiodic impulses produce abrupt surges in entropy production, followed by relaxation toward a steady state. We also show that at stall force $f = \gamma v_0$, thermodynamic rates such as entropy production and extraction rates vanish.

In conclusion, in this work, we present an important model system that helps to understand the thermodynamic behavior of Brownian motors that operate under time-dependent forces and thermal arrangements. The analytical results derived herein give key insights into how nanoscale engines adapt to dynamic driving conditions. This, in turn, provides a theoretical foundation for the design and optimization of molecular machines capable of operating efficiently under fluctuating external stimuli.

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Data availability statement

This manuscript has no associated data, or the data will not be deposited. [Authors' comment: Since we presented an analytical work, we did not collect any data from simulations or experimental observations.]

Conflict of interests

The author declares that he has no competing interests.

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