UNIVERSAL WISER
PUBLISHER

Extended Local Convergence for High Order Schemes Under ω-Continuity Conditions

Gus Argyros[1], Michael Argyros[2], Ioannis K. Argyros[3*], Santhosh George[4]
[1]Department of Computing and Technology, Cameron University, Lawton, Oklahoma, USA
[2]Department of Computing and Technology, Cameron University, Lawton, Oklahoma, USA
[3]Department of Mathematical Sciences, Cameron University, Lawton, Oklahoma, USA
[4]Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka, India
E-mail: iargyros@cameron.edu

Abstract: There is a plethora of schemes of the same convergence order for generating a sequence approximating a solution of an equation involving Banach space operators. But the set of convergence criteria is not the same in general. This makes the comparison between them challenging and only numerically. Moreover, the convergence is established using Taylor series and by assuming the existence of high order derivatives that do not even appear on these schemes. Furthermore, no computable error estimates, uniqueness for the solution results or a ball of convergence is given. We address all these problems by using only the first derivative that actually appears on these schemes and under the same set of convergence conditions. Our technique is so general that it can be used to extend the applicability of other schemes along the same lines.

Keywords: sixth convergence order, ball of convergence, convergence criteria
MSC: 65H05, 65J15, 49M15

1. Introduction

There is a plethora of high convergence order iterative schemes (IS) producing a sequence approaching a solution x_* of equation

$$F(x) = 0. \tag{1}$$

Here $F : D \subset X \rightarrow Y$ with X, Y denoting Banach space; D a nonempty, open convex set and operator F continuously differentiable according to Fréchet. One needs to use IS, since the preferred closed form of x_* is obtainable in special situations. In particular, when $X = Y = \mathbb{R}^i$ (i a natural number) Chun and Neta [1] presented a unified way of dealing with the local convergence of the three step IS given for $x_0 \in D$ by

$$\begin{aligned} y_n &= x_n - \alpha F'(x_n)^{-1}F(x_n) \\ y_n &= x_n - \varphi_1(x_n, y_n)F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= z_n - \varphi_2(x_n, y_n)F'(x_n)^{-1}F(x_n), \end{aligned} \tag{2}$$

where $\alpha \in \mathbb{R}$ or $\alpha \in \mathbb{C}$, and $\varphi_1 : D \times D \rightarrow L(X, Y)$, $\varphi_2 : D \times D \rightarrow L(X, Y)$ with $L(X, Y)$ being the space of continuous linear operators mapping X into Y.

The weights α, φ_1 and φ_2 were chosen as

$$\alpha = \frac{2}{3}, \tag{3}$$

DOI: https://doi.org/10.37256/cm.152020709

$\varphi_1(x_n, y_n) = \alpha_1 I + \alpha_2 u_n + \alpha_3 v_n + \alpha_4 u_n^2 + \alpha_5 v_n^2 + \alpha_6 v_n^3$, $\varphi_2(x_n, y_n) = \beta_1 I + \beta_2 u_n + \beta_3 v_n + \beta_4 u_n^2 + \beta_5 v_n^2$ provided that $u_n = F'(y_n)^{-1} F(x_n)$ and $v_n = F'(x_n)^{-1} F'(y_n)$ and α_j, β_j, $j = 1, 2, \dots, 6$ are scalars. The sixth order of convergence was established in [1] provided that the derivatives exist up to the seventh order

$$\begin{aligned} \alpha_1 &= -\frac{1}{2} + 3\alpha_4 + 3\alpha_5 + 8\alpha_6, \\ \alpha_2 &= -\frac{9}{8} - 3\alpha_4 - \alpha_5 - 6\alpha_6, \\ \alpha_3 &= -\frac{3}{8} - \alpha_4 - 3\alpha_5 - 6\alpha_6, \\ \beta_1 &= -\frac{1}{2} - 2\beta_3 + \beta_4 - 3\beta_5 \end{aligned} \tag{4}$$

and $\beta_2 = \frac{3}{2} + \beta_3 - 2\beta_4 + 2\beta_5$, where α_4, α_5, α_6, β_3, β_4, β_5 are free scalars. But these derivatives do not appear on (2) and also limit applicability.

For example: Let $X = Y = \mathbb{R}$ and $D = [-\frac{1}{2}, \frac{3}{2}]$. Define f on D by

$$f(t) = \begin{cases} t^3 \log t^2 + t^5 - t^4 & if\ t \neq 0 \\ 0 & if\ t = 0. \end{cases}$$

Then, we have $x_* = 1$,

$$f'(t) = 3t^2 \log t^2 + 5t^4 - 4t^3 + 2t^2,$$

$$f''(t) = 6t \log t^2 + 20t^3 - 12t^2 + 10t,$$

and

$$f'''(t) = 6 \log t^2 + 60t^2 - 24t + 22.$$

Obviously $f'''(t)$ is not bounded on D. So, the convergence of schemes (2) and (3) are not guaranteed to converge by the analysis in [1].

Other concerns include the facts that no computable estimates on $\| x_n - x_* \|$ or uniqueness of x_* results are given either.

Table 1. Weight functions

α	$\varphi_1(x_n, y_n)$	$\varphi_2(x_n, y_n)$
1	$2[F'(x_n) + F'(y_n)]^{-1} F(x_n)$	u_n
γ	$(1 + \frac{1}{2\gamma})l - \frac{1}{2\gamma} v_n$	$(1 + \frac{1}{\gamma})l - \frac{1}{\gamma} v_n$
$\frac{2}{3}$	$-\frac{1}{2} l + \frac{9}{8} u_n + \frac{3}{8} v_n$	$\frac{11}{8} l - \frac{9}{4} u_n + \frac{15}{8} u_n^2$
$\frac{2}{3}$	$\frac{5}{8} l + \frac{3}{8} u_n^2$	$\frac{11}{8} l - \frac{9}{4} u_n + \frac{15}{8} u_n^2$
$\frac{2}{3}$	$\frac{23}{8} l - 3v_n + \frac{9}{8} v_n^2$	$\frac{5}{2} l - \frac{3}{2} v_n$
$\frac{2}{3}$	$l + \frac{21}{8} v_n - \frac{9}{2} v_n^2 + \frac{15}{8} v_n^3$	$3l - \frac{5}{2} v_n + \frac{1}{2} v_n^2$

Hence, there is a need to address these matters using conditions on F' which only appears on (2), and on the more general setting of Banach space valued operators. Moreover, we do not specialize α, φ_1, φ_2 to satisfy (3) or (4). This way we include all other specializations of (2) listed in the Table 1.

The idea presented is general enough, so it can be utilized for the extension of other schemes [1-41].

2. Convergence

The local convergence of scheme (2) requires some scalar functions and parameters. Set $M = [0, \infty)$.

Suppose that there exist function $\omega_0 : M \to M$ continuous and nondecreasing such that equation

$$\omega_0(t) - 1 = 0 \tag{5}$$

has a minimal positive solution ρ_0. Set $M_0 = [0, \rho_0)$.

Suppose there exit functions $\omega : M_0 \to M$, $\omega_1 : M_0 \to M$ continuous and nondecreasing such that for

$$f_1(t) = \frac{\int_0^1 \omega((1-\theta)t)d\theta + |1-\alpha| \int_0^1 \omega_1(\theta t)dt}{1-\omega_0(t)}$$

and equation

$$\bar{f}_1(t) = f_1(t) - 1 \tag{6}$$

has a minimal solution $\rho_1 \in (0, \rho_0)$. Consider a continuous and nondecreasing function $\psi_1 : M_0 \to M$.

Suppose that for functions

$$f_2(t) = \frac{\int_0^1 \omega((1-\theta)t)d\theta + \psi_1(t)\int_0^1 \omega_1(\theta t)d\theta}{1-\omega_0(t)}$$

$$\bar{f}_2(t) = f_2(t) - 1,$$

equation

$$\bar{f}_2(t) = 0 \tag{7}$$

has a minimal solution $\rho_2 \in (0, \rho_0)$.

Suppose that equation

$$\omega_2(f_2(t)t) - 1 = 0 \tag{8}$$

has a minimal solution $\bar{\rho}_2 \in (0, \rho_0)$. Set $M_1 = [0, \bar{\rho}_2)$.

Suppose that for all $t \in M_1$

$$f_3(t) = \left[\frac{\int_0^1 \omega((1-\theta)f_2(t)t)d\theta}{1-\omega_0(f_2(t)t)} + \frac{(\omega_0(t)+\omega_0(f_2(t)t))\int_0^1 \omega_1(\theta f_2(t)t)d\theta}{(1-\omega_0(t))(1-\omega_0(f_2(t)t))}\right.$$

$$\left. + \frac{\psi_2(t)\int_0^1 \omega_1(\theta t)d\theta}{1-\omega_0(t)}\right] f_2(t),$$

$$\bar{f}_3(t) = f_3(t) - 1,$$

equation

$$\bar{f}_3(t) = 0 \tag{9}$$

has a minimal solution $\rho_3 \in (0, \bar{\rho}_2)$, where $\psi_2 : M_1 \to M$ is a continuous and nondecreasing function. We shall show that

$$\rho = \min\{\rho_j\},\ j = 1, 2, 3 \tag{10}$$

is a radius of convergence for scheme (2). Notice that by these definitions for all $t \in [0, \rho)$

$$0 \le \omega_0(t) < 1 \tag{11}$$

$$0 \le \omega_0(f_1(t)t) < 1 \tag{12}$$

and

$$0 \le f_j(t) < 1. \tag{13}$$

The notations $U(u, \lambda)$, $\bar{U}(u, \lambda)$ are used for the open and closed balls in X with center $u \in X$ and of radius $\lambda > 0$.

Next, the local convergence of scheme (2) is provided based on conditions (Γ):

(Γ_1) There exists a simple solution $x_* \in D$ of equation $F(x) = 0$.

(Γ_2) There exists a continuous and nondecreasing function $\omega_0 : M \to M$ such that for all $x \in D$

$$\| F'(x_*)^{-1}(F'(x) - F'(x_*))\| \le \omega_0(\| x - x_*\|).$$

Set $U_0 = D \cap U(x_*, \rho_0)$.

(Γ_3) There exist continuous and nondecreasing functions $\omega : M_0 \to M$, $\omega_1 : M_0 \to M$, $\psi_1 : M_0 \to M$, $\psi_2 : M_0 \to M$ such that for each $x, y \in U_0$

$$\| F'(x_*)^{-1}(F'(y) - F'(x))\| \le \omega(\| y - x\|),$$

$$\| F'(x_*)^{-1} F'(x)\| \le \omega_1(\| x - x_*\|),$$

$$\| I - \varphi_1(x, y)\| \le \psi_1(\| x - x_*\|)$$

and

$$\| I - \varphi_2(x, y)\| \le \psi_2(\| x - x_*\|)$$

for $y = x - \alpha F'(x)^{-1}F(x)$.

(Γ_4) $\bar{U}(x_*, \rho) \subseteq D$ and

(Γ_5) There exists $\rho_* \ge \rho$ such that

$$\int_0^1 \omega_0(\theta\rho_*)d\theta < 1.$$

Set $U_1 = D \cap \bar{U}(x_*, \rho_**)$.

Theorem 2.1 Suppose the conditions (Γ) are satisfied. Then, starting from $x_0 \in U(x_*, \rho) - \{x_*\}$, sequence $\{x_n\}$ generated by scheme (2) is well defined in $U(x_*, \rho)$, remains in $U(x_*, \rho)$ for each $n = 0, 1, 2,...$ and $\lim_{n\to\infty} x_n = x_*$.

Moreover, the following items hold for $e_n = \| x_n - x_*\|$

$$\| y_n - x_*\| \le f_1(e_n)e_n \le e_n < \rho, \tag{14}$$

$$\| z_n - x_* \| \le f_2(e_n)e_n \le e_n, \tag{15}$$

and

$$e_{n+1} \le f_3(e_n)e_n \le e_n, \tag{16}$$

where functions f_j are given previously and ρ is defined in (10). Furthermore, the limit point x_* is the only solution of equation $F(x) = 0$ in the set U_1 given in (Γ_5).

Proof. We shall show using mathematical induction on k that first iterates exist for all k; remain in $U(x_*, \rho)$; converge to x_* and secondly the solution is unique on the set U_1 given in (Γ_5).

Let $u \in U(x_*, \rho)$. It then follows by (Γ_1), (Γ_2), (10), and (11) that

$$\| F'(x_*)^{-1}(F'(u) - F'(x_*))\| \le \omega_0(\| u - x_*\|) \le \omega_0(\rho) < 1, \tag{17}$$

so $F'(u)^{-1}$ exists and

$$\| F'(u)^{-1} F'(x_*)\| \le \frac{1}{1 - \omega_0(\| u - x_*\|)} \tag{18}$$

holds by the lemma on invertible operators due to Banach [25]. If one sets set $u = x_0$, then y_0, z_0, x_1 exist by scheme (2), if $n = 0$. We can write by the first substep of scheme (2), (10), (13) (for $m = 1$), (Γ_1), (Γ_3) and (17) for $u = x_0$

$$\begin{aligned}
\| y_0 - x_*\| &\le \| x_0 - x_* - F'(x_0)^{-1}F(x_0) + (1-\alpha)F'(x_0)^{-1}F'(x_0)\| \\
&\le \| F'(x_0)^{-1}F'(x_*)\| \times \| \textstyle\int_0^1 F'(x_*)^{-1}(F'(x_* + \theta(x_0 - x_*)) - F'(x_0))(x_0 - x_*)d\theta\| \\
&\qquad + |1 - \alpha| \| F'(x_0)^{-1}F'(x_*)\| \ \| F'(x_*)^{-1}F'(x_0)\| \\
&\le f_1(e_0)e_0 \le e_0 < \rho,
\end{aligned} \tag{19}$$

showing $y_0 \in U(x_*, \rho)$ and (14) for $n = 0$. Moreover, by (10), (13) (for $m = 2, 3$), we obtain in turn

$$\begin{aligned}
\|z_0 - x_*\| &\le \|(x_0 - x_* - F'(x_0)^{-1}F(x_0)) \\
&\qquad + (I - \varphi_1(x_0, y_0)F'(x_0)^{-1}F(x_0))\|]e_0 \\
&\le f_2(e_0)e_0 \le e_0,
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
e_1 &\le \|(z_0 - x_* - F'(z_0)^{-1}F(z_0)) \\
&\quad + (F'(z_0)^{-1} - F'(x_0)^{-1})F(z_0) + (I - \varphi_2(x_0, y_0))F'(x_0)^{-1}F(z_0))\| \\
&\quad \left[\frac{\int_0^1 \omega((1-\theta)\| z_0 - x_*\|)d\theta}{1 - \omega_0(\| z_0 - x_*\|)} + \frac{(\omega_0(\| z_0 - x_*\|) + \omega_0(e_0))\int_0^1 \omega_1(\theta \| z_0 - x_*\|)d\theta}{(1 - \omega_0(\| z_0 - x_*\|))(1 - \omega_0(e_0))}\right. \\
&\quad \left. + \frac{\psi_1(e_0)\int_0^1 \omega_1(\theta e_0)d\theta}{1 - \omega_0(e_0)}\right]\| z_0 - x_*\| \\
&\le f_3(e_0)e_0 \le e_0,
\end{aligned} \tag{21}$$

so z_0, $x_1 \in U(x_*, \rho)$ and (15), (16) hold for $n = 0$. Hence, the induction for (14)-(16) is shown for $n = 0$. Suppose these estimations hold for all $k = 0, 1, 2, ..., n$. Then, by simply switching x_0, y_0, x_1 by x_k, y_k, x_{k+1} in the preceding calculations, we conclude (14)-(16) hold for all n. Then, from

$$e_{k+1} \leq ce_k < \rho, \tag{22}$$

where $c = f_3(e_0) \in [0, 1)$, we have $\lim_{k\to\infty} x_k = x_*$, and $x_{k+1} \in U(x_*, \rho)$. Let $q \in D_1$ be such that $F(q) = 0$. Set $T = \int_0^1 F'(x_* + \theta(q - x_*))d\theta$. Then, by (Γ_2) and (Γ_5),

$$\begin{aligned} \| F'(x_*)^{-1}(T - F'(x_*))\| &\leq \|\int_0^1 F'(x_*)^{-1}(F'(x_* + \theta(q - x_*)) - F'(x_*))d\theta\| \\ &\leq \int_0^1 \omega_0(\theta\| x_* - q\|)d\theta < 1, \end{aligned} \tag{23}$$

so T^{-1} exists, by the Banach lemma on invertible operators [25], and $x_* = q$ follows from $0 = F(x_*) - F(q) = T(x_* - q)$.

Remark 2.2 1. By (a2), and the estimate

$$\begin{aligned} \|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \\ &\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq 1 + w_0(\|x - x^*\|) \end{aligned}$$

second condition in (a3) can be dropped, and w_1 be defined as

$$w_1(t) = 1 + w_0(t) \text{ or } w_1(t) = 2. \tag{24}$$

Notice that, if $w_1(t) < 1 + w_0(t)$, then ρ can be larger (see Example 3.1).

2. The results obtained here can be used for operators G satisfying autonomous differential equations [3-10] of the form

$$F'(x) = T(F(x))$$

where T is a continuous operator. Then, since $F'(x^*) = T(F(x^*)) = T(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then, we can choose: $T(x) = x + 1$.

3. The local results obtained here can be used for projection schemes such as the Arnoldi's algorithm, the generalized minimum residual algorithm (GMRES), the generalized conjugate algorithm (GCA) for combined Newton/finite projection schemes and in connection to the mesh independence principle can be used to develop the cheapest and most efficient mesh refinement strategies [3-10, 17].

4. Let $w_0(t) = L_0 t$, and $w(t) = Lt$. The parameter $r_A = \frac{2}{2L_0 + L}$ was shown by us to be the convergence radius of Newton's algorithm [3]

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \text{ for each } n = 0, 1, 2, \cdots \tag{25}$$

under the conditions (a1)-(a3) (w_1 is not used). It follows that the convergence radius R of algorithm (2) cannot be larger than the convergence radius r_A of the second order Newton's algorithm (25). As already noted in [4] r_A is at least as large as the convergence ball given by Rheinboldt [20]

$$r_{TR} = \frac{2}{3L_1},$$

where L_1 is the Lipschitz constant on Ω, $L_0 \leq L_1$ and $L \leq L_1$. In particular, for $L_0 < L_1$ or $L < L_1$, we have that

$$r_{TR} < r_A$$

and

$$\frac{r_{TR}}{r_A} \to \frac{1}{3} \text{ as } \frac{L_0}{L_1} \to 0.$$

That is our convergence ball r_A is at most three times larger than Rheinboldt's. The same value for r_{TR} was given by Traub [24].

5. It is worth noticing that solver (2) is not changing, when we use the conditions (A) of Theorem 2.1 instead of the stronger conditions used in [12, 14]. Moreover, we can compute the computational order of convergence (COC) defined by

$$\xi = \ln\left(\frac{\| x_{n+1} - x^* \|}{\| x_n - x^* \|}\right) / \ln\left(\frac{\| x_n - x^* \|}{\| x_{n-1} - x^* \|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{\| x_{n+1} - x_n \|}{\| x_n - x_{n-1} \|}\right) / \ln\left(\frac{\| x_n - x_{n-1} \|}{\| x_{n-1} - x_{n-2} \|}\right).$$

This way we obtain in practice the order of convergence in a way that avoids the existence of the seventh Fréchet derivative for operator F.

3. Numerical examples

As in [1], we choose $\alpha_4 = \frac{63}{64}$, $\alpha_5 = 0$, $\alpha_6 = 0$, $\beta_3 = \frac{15}{8}$, $\beta_5 = 0$ and $\beta_4 = 0$.

Example 3.1 Let $B_1 = B_2 = D = \mathbb{R}$. Define $F(x) = \sin x$. Then, we get that $x_* = 0$, and by (Γ_2) and the first condition in (Γ_3)

$$\begin{aligned}
\|F'(x_*)^{-1}(F'(x) - F'(x_*))\| &= |1(\cos x - \cos x_*)| \\
&= |\cos x - \cos x_*| \\
&= |\cos v(x - x_*)| \\
&= |\cos v||x - x_*| \\
&\le |x - x_*| \ (v \in \mathbb{R}),
\end{aligned}$$

so $w_0(t) = w(t) = t$. Then, obtain the roots by solving (6), (7), (9) (knowing now w_0, w, w_1, ψ_1, ψ_2) using Mathematica. Then, we determine the radii using (10). The same is done in the rest of the examples. One can also see examples about how these scalar functions are found in [3-10, 21, 25]. Further, we have the following error estimates for Example 3.1.

$$\|y_5 - x_*\| = 0.0266 \le e_5 = 0.0802 < \rho = 0.195081,$$

$$\|z_5 - x_*\| = 0.0373 \le e_5 = 0.0802 < \rho = 0.195081,$$

and

$$\|x_6 - x_*\| = 0.0056 \le e_5 = 0.0802 < \rho = 0.195081.$$

Table 2. Radius for Example 3.1

Radius	$\omega_1(t) = 1$	$\omega_1(t) = 1 + \omega_0(t)$
r_1	0.44444	0.4
r_2	0.473532	0.437925
r_3	0.195081	0.187903

Example 3.2 Let $B_1 = B_2 = C[0, 1]$, be the space of continuous functions defined on $[0, 1]$ with the max norm. Let $D = \bar{U}(0, 1)$. Define function F on D by

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{26}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2 \xi(\theta)d\theta, \text{ for each } \xi \in D.$$

Then, we get that $x_* = 0$, $F'(x_*) = I$, $\omega_0(t) = \frac{15}{2}t$, $\omega(t) = 15t$ and $\omega_1(t) = 2$. This way, we have that

Table 3. Radius for Example 3.2

Radius	$\omega_1(t) = 2$	$\omega_1(t) = 1 + \omega_0(t)$
r_1	0.0296296	0.0533
r_2	0.231907	0.0903624
r_3	0.0126373	0.0284867

Example 3.3 Let $B_1 = B_2 = \mathbb{R}^3$, $D = U(0, 1)$, $x_* = (0, 0, 0)^T$, and define F on D by

$$F(x) = F(u_1, u_2, u_3) = (e^{u_1} - 1, \frac{e-1}{2}u_2^2 + u_2, u_3)^T. \tag{27}$$

For the points $u = (u_1, u_2, u_3)^T$, the Fréchet derivative is given by

$$F'(u) = \begin{pmatrix} e^{u_1} & 0 & 0 \\ 0 & (e-1)u_2 + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using the norm of the maximum of the rows and since $G'(x_*) = \text{diag}(1, 1, 1)$, we get by conditions (A) $\omega_0(t) = (e - 1)t$, $\omega(t) = \frac{1}{e^{e-1}}t$, and $\omega_1(t) = \frac{1}{e^{e-1}}$. Then we have

Example 3.4 Returning back to the motivational example at the introduction of this study, we have $\omega_0(t) = \omega(t) = 96.662907t$, $\omega_1(t) = 1.0631$. Then, we have

Table 4. Radius for Example 3.3

Radius	$\omega_1(t) = \frac{1}{e^{e-1}}$	$\omega_1(t) = 1 + \omega_0(t)$
r_1	0.073878	0.229929
r_2	0.911242	0.309664
r_3	0.0600865	0.115891

Table 5. Radius for Example 3.4

Radius	$\omega_1(t) = \frac{1}{e^{e-1}}$	$\omega_1(t) = 1 + \omega_0(t)$
r_1	0.00445282	0.00413809
r_2	0.0145963	0.00749138
r_3	0.002212	no solution

Further, we have the following error estimates for Example 3.4.

$$\|y_{10} - x_*\| = 0.0006 \leq e_{10} = 0.0008 < \rho = 0.002212,$$

$$\|z_{10} - x_*\| = 0.0006 \leq e_{10} = 0.0008 < \rho = 0.002212,$$

and

$$\|x_{11} - x_*\| = 0.0004 \leq e_{10} = 0.0008 < \rho = 0.002212.$$

References

[1] Chun C, Neta B. Developing high order methods for the solution of systems of nonlinear equations. *Applied Mathematics and Computation*. 2019; 342: 178-190.

[2] Amat S, Bermudez C, Hernandez, MA, Martinez E. On an efficient k-step iterative method for nonlinear equations. *Journal of Computational and Applied Mathematics*. 2016; 302: 258-271.

[3] Amat S, Argyros IK, Busquier S, Hernández MA. On two high-order families of frozen Newton-type methods. *Numerical Linear Algebra with Applications*. 2018; 25: 1-13.

[4] Argyros IK. *Computational Theory of Iterative Solvers*. 1st ed. New York, USA: Elsevier Publ. Co.; 2007.

[5] Argyros IK, Hilout S. Weaker conditions for the convergence of Newton's method. *Journal of Complexity*. 2012; 28: 364-387.

[6] Argyros IK, Magréñan AA. *A Contemporary Study of Iterative Methods*. New York: Elsevier (Academic Press); 2018.

[7] Argyros IK, Magréñan AA. *Iterative Methods and Their Dynamics with Applications*. New York, USA: CRC Press; 2017.

[8] Argyros IK. *Computational Theory of Iterative Methods*. New York: Elsevier Publ. Company; 2007.

[9] Argyros IK, George S. *Mathematical Modeling for the Solution of Equations and Systems of Equations with Applications. Volume-IV*. NY: Nova Publishes; 2020.

[10] Argyros IK, George S. On the complexity of extending the convergence region for Traub's method. *Journal of Complexity*. 2020; 56: 101423. Available from: https://doi.org/10.1016/j.jco.2019.101423.

[11] Cordero A, Torregrosa JR. Variants of Newton method using fifth-order quadrature formulas. *Applied Mathematics and Computation*. 2007; 190: 686-698.

[12] Cordero A, Hueso JL, Martínez E, Torregrosa JR. A modified Newton-Jarratt s composition. *Numerical Algorithms*. 2010; 55: 87-99.

[13] Cordero A, Hueso JL, Martínez E, Torregrosa JR. Increasing the convergence order of an iterative method for nonlinear systems. *Applied Mathematics Letters*. 2012; 25: 2369-2374.

[14] Darvishi MT, Barati A. A fourth-order method from quadrature formulae to solve systems of nonlinear equations. *Applied Mathematics and Computation*. 2007; 188(1): 257-261. Available from: DOI: 10.1016/j.amc.2006.09.115.

[15] Esmaeili H, Ahmad M. An efficient three step method to solve system of nonlinear equations. *Applied Mathematics and Computation*. 2015; 266: 1093-1101.

[16] Ezquerro JA, Hernández MA, Romero N, Velasco AI. On Steffensen's method on Banach spaces. *Journal of Computational and Applied Mathematics*. 2013; 249: 9-23.

[17] Artidiello S, Cordero A, Torregrosa JR, Vasileva MP. Design and multi-dimensional extension of iterative methods for solving nonlinear equations. *Applied Mathematics and Computation*. 2017; 293: 194-203.

[18] Grau-Sánchez M, Grau A, Noguera M. Ostrowski type methods for solving systems of nonlinear equations. *Applied Mathematics and Computation*. 2011; 218: 2377-2385.

[19] Homeier HHH. A modified Newton method with cubic convergence: the multivariable case. *Journal of Computational and Applied Mathematics*. 2004; 169: 161-169.

[20] Hueso JL, Martinez E, Teruel C. Convergence, efficiency and dynamics of new fourth and sixth order families of iterative methods for nonlinear systems. *Computational & Applied Mathematics*. 2015; 275: 412-420.

[21] Kelley CT. *Solving Nonlinear Equations with Newton's Method*. SIAM, Philadelphia; 2003.

[22] Nedzhibov GH. A family of multi-point iterative methods for solving systems of nonlinear equations. *Journal of Computational and Applied Mathematics*. 2008; 222(2): 244-250.

[23] Neta B. A sixth-order family of methods for nonlinear equations. *International Journal of Computer Mathematics*. 1979; 7:157-161.

[24] Torregrosa JR. Increasing the convergence order of an iterative method for nonlinear systems. *Applied Mathematics Letters*. 2012; 25: 2369-2374.

[25] Ostrowski AM. *Solution of Equations and Systems of Equations*. New York: Academic Press; 1966.

[26] Montazeri H, Soleymani F, Shateyi S, Motsa SS. On a new method for computing the numerical solution of systems of nonlinear equations. *Journal of Applied Mathematics*. 2012. Available from: https://doi.org/10.1155/2012/751975.
[27] Sharma JR, Arora H. Efficient Jarratt-like methods for solving systems of nonlinear equations. *Calcolo*. 2014; 51: 193-210.
[28] Petkovic MS, Neta B, Petkovic LD, Dzunic J. *Multipoint Methods for Solving Nonlinear Equations*. Elsevier; 2012.
[29] Rheinboldt WC. An adaptive continuation process for solving systems of nonlinear equations. *Mathematical Models and Numerical Solvers*. 1977; 3(1): 129-142.
[30] Semenov VS. The method of determining all real nonmultiple roots of systems of nonlinear equations. C*omputational Mathematics and Mathematical Physics*. 2007; 47: 1428-1434.
[31] Sauer T. *Numerical Analysis*. 2nd ed. London: Pearson; 2012.
[32] Sharma JR, Gupta P. An efficient fifth order method for solving systems of nonlinear equations. *Computers & Mathematics with Applications*. 2014; 67: 591-601.
[33] Traub JF. *Iterative Solvers for the Solution of Equations*. New York: AMS Chelsea Publishing; 1982.
[34] Wang X, Kou J, Li Y. Modified Jarratt method with sixth-order convergence. *Applied Mathematics Letters*. 2009; 22: 1798-1802.
[35] Weerakoon S, Fernando TGI. A variant of Newton's method with accelerated third-order convergence. *Applied Mathematics Letters*. 2000; 13(8): 87-93.
[36] Benbernou S, Gala S, Ragusa MA. On the regularity criteria for the 3D magnetohydrodynamic equations via two components in terms of BMO space. *Mathematical Methods in the Applied Sciences*. 2014; 37(15): 2320-2325.
[37] Chen XM. The Kaczmarz algorithm, row action methods, and statistical learning algorithms, Frames and harmonic analysis. *Contemporary Mathematics*. 2018; 706: 115-127.
[38] Ragusa MA, Scapellato A. Mixed Morrey spaces and their applications to partial differential equations. *Nonlinear Analysis-Theory Methods and Applications*. 2017; 151: 51-65.
[39] Ragusa MA, Tachikawa A. Regularity for minimizers for functionals of double phase with variable exponents. *Advances in Nonlinear Analysis*. 2020; 9: 710-728.
[40] Turkyilmazoglu M. A simple algorithm for high order Newton iteration for-mulae and some new variants. *Hacettepe Journal of Mathematics & Statistics*. 2020; 49(1): 425-438.
[41] Babajee DKR, Dauhoo MZ, Darvishi MT, Barati A. A note on the local convergence of iterative methods based on Adomian decomposition method and 3-node quadrature rule. *Applied Mathematics and Computation*. 2008; 200(1): 452-458.