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Research Article

Maximization Problems for Second Order Elliptic Equations

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Abstract: We investigate the maxima in classes of rearrangements of some functionals associated with solutions to Dirichlet problems for second order elliptic equations.

***Keywords*:** rearrangements, second order elliptic equations, maximization problems

MSC: 35J25, 49K20, 49K30

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain, and let $g(x)$ and $f(x)$ be bounded non-negative functions. In the present paper, we are interested in optimal control problems that are related to the boundary value problem

$$\int_\Omega \left(a_{ij}(x)u_{x_i}\phi_{x_j} + g(x)u\phi\right) dx = \int_\Omega f(x)\phi(x)\, dx \quad \forall \phi \in H_0^1(\Omega), \tag{1}$$

where $A(x)$ is a $N \times N$ symmetric matrix satisfying, for some $0 < \lambda \le \Lambda$,

$$\lambda |\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^N, \ \forall x \in \Omega. \tag{2}$$

Here and in what follows, the summation convention from 1 to N over repeated index is in effect.

Let $g_0 \in L_+^\infty(\Omega)$, and let $\mathcal{G}$ be the class of rearrangements of g_0. For $g \in \mathcal{G}$, let u_g be the solution to problem (1) (with $f(x)$ fixed). We investigate the maximization problem

$$\sup_{g\in\mathcal{G}} \int_\Omega f(x)u_g(x)\, dx.$$

In case the operator is the Laplacian, similar problems are discussed in [1].

Next, we shall discuss the following problem. Let $f_0 \in L_+^\infty(\Omega)$ and let $\mathcal{F}$ be the class of rearrangements of f_0. For f

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$\in \mathcal{F}$ the energy integral associated with the solution $u = u_f$ to problem (1) is defined as $\int_\Omega f(x)u_f dx$. The maximization problem

$$\max_{f \in \mathcal{F}} \int_\Omega f(x)u_f(x)\,dx \tag{3}$$

has been widely investigated since the pioneering works [2-3]. It is well known that a maximizer $\hat{f}$ exists and that $\hat{f} = \varphi(u_{\hat{f}})$ for some non-decreasing function φ. This result is proved, usually, exploiting the continuity and the strict convexity of the functional "$\int_\Omega f(x)u_f(x)dx$ over $\overline{\mathcal{F}}$, the weak* closure of $\mathcal{F}$. In this paper, we introduce a new function $h_0 \in L^\infty_+(\Omega)$ and consider $\mathcal{H}$, the class of rearrangements of h_0. We investigate the maximization problem

$$\sup_{f \in \mathcal{F},\, h \in \mathcal{H}} \int_\Omega h(x)u_f(x)\,dx. \tag{4}$$

The investigation of this problem is easier than that in (3). However, we shall prove that when $\mathcal{H} = \mathcal{F}$ then a solution to problem (4) yields a solution to the well studied problem (3).

Optimization problems in classes of rearrangements started more than three decades ago with a paper by Alvino et al., see [1]. The differential equations considered in [1] were posed in *bounded* domains. In mid 1980's, Burton developed a comprehensive theory of rearrangements of functions, see [2-3], and applied his theory to prove existence of two or three dimensional non-compressible ideal flows having localized vortices while occupying an *unbounded* domain. Since then, several researchers have investigated maximization or minimization problems for the energy integral and for the first eigenvalue corresponding to suitable differential equations, see [4-5] and references therein. Recently, seems to be a new interest for this subject. A few years ago, Qiu et al. in [6] investigated a class of rearrangement optimization problems involving the p-Laplacian, and in [7] investigated optimization problems involving the fractional Laplacian. In 2018, the first and the second author of the present paper, together with A. Farjudian, investigated optimal harvesting strategy based on rearrangements [8]. In 2019, Kebede completed his doctoral dissertation "Optimization problems in classes of rearrangements" [9], and published the related papers [10-11]. In 2020, Amiri et al. discussed optimization problems related to a p-Laplacian equation on a multiply connected domain [12]. Again in 2020, Emamizadeh et al. investigated bang-bang and multiple valued optimal solutions of control problems related to quasi-linear elliptic equations [13]. In the same year, Anedda et al. investigated the Steiner symmetry of the minimizer (in classes of rearrangements) of a fractional eigenvalue problem [14].

These problems have physical interpretations. For example, they describe the vibration of a non-homogeneous membrane subject to a vertical force f. The solution u_f represents the displacement from the rest position. One may be interested to investigate the resilience of the membrane by maximizing the corresponding energy integral.

To investigate our problems, we shall use results from [2-3, 15].

2. Main results

We fix $f \in L^\infty_+(\Omega)$ such that the measure of the set $\{x \in \Omega : f(x) > 0\}$ is positive. For $g \in \overline{\mathcal{G}}$, define $u = u_g \in H^1_0(\Omega)$ such that

$$\int_\Omega \left(a_{ij}(x)u_{x_i}\phi_{x_j} + g(x)u\phi\right) dx = \int_\Omega f(x)\phi(x)\,dx \quad \forall \phi \in H^1_0(\Omega). \tag{5}$$

By the strong maximum principle (see [16]), we have $u_g(x) > 0$ in Ω. Putting $\phi = u$ in (5) and recalling that $g(x) \ge 0$, using (2) and Poincaré inequality, we find

$$\lambda \|\nabla u\|_2 \le C\|f\|_2, \tag{6}$$

with C independent of g.

We define

$$J(g) = \int_\Omega f(x) u_g(x)\, dx, \tag{7}$$

and investigate the optimization problem $\sup_{g \in \mathcal{G}} J(g)$.

The weak* continuity of $J(g)$ can be proved easily by using (6) and Rellich's Theorem. To get the convexity of $J(g)$, let us prove first the following result (which may have an own interest). Recall that f is non-negative and positive in a subset of positive measure.

Lemma 2.1 With $g \in L^\infty_+(\Omega)$, let $u \in H^1_0(\Omega)$ satisfy

$$\int_\Omega \left(a_{ij} u_{x_i} \varphi_{x_j} + gu\varphi\right) dx = \int_\Omega f\, \varphi\, dx \quad \forall \varphi \in H^1_0(\Omega). \tag{8}$$

With $h \in L^\infty_+(\Omega)$, let $v \in H^1_0(\Omega)$ satisfy

$$\int_\Omega \left(a_{ij} v_{x_i} \varphi_{x_j} + hv\varphi\right) dx = \int_\Omega f\, \varphi\, dx \quad \forall \varphi \in H^1_0(\Omega). \tag{9}$$

With $0 < t < 1$, let $w \in H^1_0(\Omega)$ satisfy

$$\int_\Omega \left(a_{ij} w_{x_i} \phi_{x_j} + (tg + (1-t)h) w\phi\right) dx = \int_\Omega f\, \phi\, dx \quad \forall \phi \in H^1_0(\Omega). \tag{10}$$

Then,

$$w(x) \le tu(x) + (1-t)v(x) \tag{11}$$

with equality if and only if $g = h$ almost everywhere in Ω.

Proof. Note that u, v and w are positive in Ω. If $\phi \in C^1_0(\Omega)$, we choose as test function $\varphi = t(\frac{v}{u})^{1-t} \phi$ in (8) to find

$$\int_\Omega \left[t\left(\frac{v}{u}\right)^{1-t} a_{ij} u_{x_i} \phi_{x_j} + t(1-t)\left(\frac{v}{u}\right)^{-t} a_{ij} u_{x_i} \left(\frac{v_{x_j}}{u} - \frac{v u_{x_j}}{u^2}\right)\phi + gut\left(\frac{v}{u}\right)^{1-t} \phi \right] dx = \int_\Omega ft\left(\frac{v}{u}\right)^{1-t} \phi\, dx. \tag{12}$$

Similarly, we choose as test function $\varphi = (1-t)(\frac{u}{v})^t \phi$ in (8) to find

$$\int_\Omega \left[(1-t)\left(\frac{u}{v}\right)^{t} a_{ij} v_{x_i} \phi_{x_j} + (1-t)t\left(\frac{u}{v}\right)^{t-1} a_{ij} v_{x_i} \left(\frac{u_{x_j}}{v} - \frac{u v_{x_j}}{v^2}\right)\phi + hv(1-t)\left(\frac{u}{v}\right)^{t} \phi \right] dx = \int_\Omega f(1-t)\left(\frac{u}{v}\right)^{t} \phi\, dx. \tag{13}$$

Since $a_{ij} = a_{ji}$, we claim that

$$\left(\frac{v}{u}\right)^{-t} a_{ij} u_{x_i} \left(\frac{v_{x_j}}{u} - \frac{v u_{x_j}}{u^2}\right) + \left(\frac{u}{v}\right)^{t-1} a_{ij} v_{x_i} \left(\frac{u_{x_j}}{v} - \frac{u v_{x_j}}{v^2}\right) = u^{t-1} v^{-t} \left[2a_{ij} u_{x_i} v_{x_j} - \frac{v}{u} a_{ij} u_{x_i} u_{x_j} - \frac{u}{v} a_{ij} v_{x_i} v_{x_j} \right] \le 0. \tag{14}$$

Indeed, we have

$$2a_{ij}u_{x_i}v_{x_j} - \frac{v}{u}a_{ij}u_{x_i}u_{x_j} - \frac{u}{v}a_{ij}v_{x_i}v_{x_j} = -a_{ij}\Big(\sqrt{\frac{v}{u}}u_{x_i} - \sqrt{\frac{u}{v}}v_{x_i}\Big)\Big(\sqrt{\frac{v}{u}}u_{x_j} - \sqrt{\frac{u}{v}}v_{x_j}\Big) \le 0. \tag{15}$$

We note that equality holds in (15) (and in (14)) if and only if

$$v\nabla u = u\nabla v.$$

If we add (12) and (13) with $\phi \ge 0$ and we take into account (14), we find

$$\int_\Omega \Big[t\Big(\frac{v}{u}\Big)^{1-t} a_{ij}u_{x_i}\phi_{x_j} + gut\Big(\frac{v}{u}\Big)^{1-t}\phi + (1-t)\Big(\frac{u}{v}\Big)^t a_{ij}v_{x_i}\phi_{x_j} + hv(1-t)\Big(\frac{u}{v}\Big)^t\phi\Big]dx \ge \int_\Omega \Big[t\Big(\frac{v}{u}\Big)^{1-t} + (1-t)\Big(\frac{u}{v}\Big)^t\Big]\phi f dx. \tag{16}$$

For $s > 0$, by Young inequality, we have

$$1 = s^{t(1-t)}s^{-t(1-t)} \le ts^{1-t} + (1-t)s^{-t}.$$

Therefore,

$$t\Big(\frac{v}{u}\Big)^{1-t} + (1-t)\Big(\frac{u}{v}\Big)^t \ge 1, \tag{17}$$

with equality sign if and only if $\frac{u}{v} = 1$. By (16) and (17), we find

$$\int_\Omega \Big[\Big(t\Big(\frac{v}{u}\Big)^{1-t} a_{ij}u_{x_i} + (1-t)\Big(\frac{u}{v}\Big)^t a_{ij}v_{x_i}\Big)\phi_{x_j} + \big(tg + (1-t)h\big)u^t v^{1-t}\phi\Big]dx \ge \int_\Omega f\phi dx. \tag{18}$$

Now, if we put $z = u^t v^{1-t}$, we have

$$z_{x_i} = t\Big(\frac{v}{u}\Big)^{1-t}u_{x_i} + (1-t)\Big(\frac{u}{v}\Big)^t v_{x_i},\ \ a_{ij}z_{x_i} = t\Big(\frac{v}{u}\Big)^{1-t}a_{ij}u_{x_i} + (1-t)\Big(\frac{u}{v}\Big)^t a_{ij}v_{x_i}.$$

Therefore, (18) yields

$$\int_\Omega \big[a_{ij}z_{x_i}\phi_{x_j} + \big(tg + (1-t)h\big)z\phi\big]dx \ge \int_\Omega f\phi dx.$$

Since $z = 0 = w$ on $\partial\Omega$, by (10) and the latter inequality, we get $w(x) \le z(x)$ in Ω. Finally, since $z = u^t v^{1-t} \le tu + (1-t)v$, inequality (11) follows.

We have proved that the map $g \to u_g$ is convex. Let us show that it is strictly convex. Indeed, if $w(x_0) = z(x_0)$ at some point $x_0 \in \Omega$ then, by the strong maximum principle (see [16]), $w(x) = z(x)$ in Ω. As a consequence, equality must hold in (15). Hence, $v\nabla u = u\nabla v$, which implies $u = cv$ in Ω for some $c > 0$.

If $w(x) = z(x)$, we must have equality also in (16) and (18). In particular, we must have

$$\int_\Omega \Big[t\Big(\frac{v}{u}\Big)^{1-t} + (1-t)\Big(\frac{u}{v}\Big)^t\Big]\phi f dx = \int_\Omega \phi f dx.$$

Since $\phi \in C_0^1(\Omega)$ is arbitrary, the latter equation implies

$$t\left(\frac{v}{u}\right)^{1-t} + (1-t)\left(\frac{u}{v}\right)^{t} = 1$$

for $x \in \{\operatorname{supp} f\}$. Hence $\dfrac{u(x)}{v(x)} = 1$ for $x \in \{\operatorname{supp} f\}$. Thus, $c = 1$ and $u(x) = v(x)$ in Ω. Finally, from the equations (16) and (17), we find $g = h$ almost everywhere in Ω, which yields the strict convexity of the map $g \rightarrow u_g$. The lemma is proved.

We are now in a position to prove the following result.

Theorem 2.2 Let $J(g)$ be defined as in (7), where u_g is the solution to (5).

There exists $\hat{g} \in \mathcal{G}$ such that $J(g) \leq J(\hat{g})$ for all $g \in \overline{\mathcal{G}}$. Furthermore, if $\hat{g} \in \mathcal{G}$ is any maximizer of $J(g)$ on $\overline{\mathcal{G}}$, there is a non-increasing function ψ such that $\hat{g} = \psi(u_{\hat{g}})$.

Proof. By Lemma 2.1, the functional $J(g)$ is strictly convex. Let us prove that $J(g)$ is weakly continuous and Gateaux differentiable.

By the variational characterization of the solution u_g to problem (5), we have

$$J(g) = \int_\Omega f u_g dx = \sup_{v \in H_0^1(\Omega)} \int_\Omega \left(2fv - a_{ij} v_{x_i} v_{x_j} - gv^2\right) dx.$$

For $h \in \overline{\mathcal{G}}$, we have

$$\begin{aligned} J(g) + \int_\Omega (g-h) u_g^2 dx &= \int_\Omega \left(2fu_g - a_{ij}(u_g)_{x_i}(u_g)_{x_j} - hu_g^2\right) dx \\ &\leq \int_\Omega \left(2fu_h - a_{ij}(u_h)_{x_i}(u_h)_{x_j} - hu_h^2\right) dx = J(h) \\ &= \int_\Omega (g-h) u_h^2 dx + \int_\Omega \left(2fu_h - a_{ij}(u_h)_{x_i}(u_h)_{x_j} - gu_h^2\right) dx \\ &\leq \int_\Omega (g-h) u_h^2 dx + J(g). \end{aligned}$$

Therefore,

$$\int_\Omega (g-h) u_g^2 dx \leq J(h) - J(g) \leq \int_\Omega (g-h) u_h^2 dx. \tag{19}$$

The continuity of $J(h)$ follows from (19).

Let us prove that $J(h)$ is Gateaux differentiable. Let $0 < t < 1$. If we replace h by $h_t = g + t(h - g)$ in (19), we find

$$\int_\Omega (g-h) u_g^2 dx \leq \frac{J(g + t(h-g)) - J(g)}{t} \leq \int_\Omega (g-h) u_{h_t}^2 dx.$$

As $t \rightarrow 0$, we have $h_t \rightarrow g$ in the norm of $L^\infty(\Omega)$. Hence, $u_{h_t} \rightarrow u_g$ in the norm of $L^2(\Omega)$ and

$$\lim_{t \to 0} \frac{J(g + t(h-g)) - J(g)}{t} = \int_\Omega (g-h) u_g^2 dx.$$

Therefore,

$$J'(g, h-g) = \int_\Omega (h-g)(-u_g^2)dx,$$

and $J'(g) = -u_g^2$. By Theorem 7 and Corollary 1 of [2], $J(g)$ attains a maximum value $\hat{g} \in \overline{\mathcal{G}}$. Moreover, if $\hat{g}$ is any maximizer then $\hat{g} = \varphi(-u_{\hat{g}}^2)$ almost everywhere for some non-decreasing function φ. Equivalently, we have $\hat{g} = \psi(u_{\hat{g}})$ for some non-increasing function ψ. The theorem is proved.

To investigate the second problem mentioned in the introduction, we recall the following result.

Lemma 2.3 Let $L : L^p(\Omega) \to \mathbb{R}$ be linear and weakly continuous. Then, there exists $\hat{f} \in \mathcal{F}$ such that

$$L(\hat{f}) = \sup_{\overline{\mathcal{F}}} L(f). \tag{20}$$

Proof. Although this result is known, we give here a short proof. Let $\{f_i\}$, $f_i \in \overline{\mathcal{F}}$ be a sequence such that

$$\lim_{i\to\infty} L(f_i) = \sup_{\overline{\mathcal{F}}} L(f).$$

Since $\overline{\mathcal{F}}$ is weakly compact, there is a subsequence (again denoted $\{f_i\}$) and an element $\hat{f} \in \overline{\mathcal{F}}$ such that

$$f_i \rightharpoonup \hat{f} \text{ (weakly).}$$

Since L is weakly continuous, we have

$$\lim_{i\to\infty} L(f_i) = L(\hat{f}).$$

We must show that $\hat{f} \in \mathcal{F}$. Denote

$$H = \{f \in \overline{\mathcal{F}} : L(f) = L(\hat{f})\}.$$

Let us prove that $H \cap \mathcal{F} \neq \emptyset$. Clearly, H is convex. Let us show that H is an extreme set in $\overline{\mathcal{F}}$. This means that if $f \in H$ and $f = \frac{f_1 + f_2}{2}$ with $f_1, f_2 \in \overline{\mathcal{F}}$, then necessarily $f_1, f_2 \in H$. To show this, we write

$$L(\hat{f}) = L(f) = \frac{L(f_1) + L(f_2)}{2} \le \frac{L(\hat{f}) + L(\hat{f})}{2} = L(\hat{f}).$$

Since equality must hold, we find

$$L(f_1) = L(\hat{f}) \text{ and } L(f_2) = L(\hat{f}), \text{ so } f_1, f_2 \in H.$$

On the other hand, since H is weakly compact and convex, we can apply the Krein-Milman theorem [17] to deduce that $\mathrm{ext}(H) \neq \emptyset$. Next, we show that

$$\mathrm{ext}(H) \subset \mathcal{F} = \mathrm{ext}(\overline{\mathcal{F}}).$$

To this end, let us assume the inclusion is false. Then, there exists $f \in \mathrm{ext}(H)$ which is not in $\mathcal{F}$. Therefore, we have $f = tf_1 + (1-t)f_2$ for some $f_1, f_2 \in \overline{\mathcal{F}}$ and some $t \in (0, 1)$. Since $\mathrm{ext}(H) \subset H$ and H is an extreme set, we deduce that f_1, $f_2 \in H$. But this contradicts $f \in \mathrm{ext}(H)$. We conclude that $\mathrm{ext}(H) \subset \mathcal{F}$. Furthermore, we have $H \cap \mathcal{F} \neq \emptyset$ as desired. The lemma is proved.

Theorem 2.4 Suppose f_0 and h_0 are bounded non-negative functions. Let $\mathcal{F}$ and $\mathcal{H}$ be the classes of rearrangements of f_0 and h_0 respectively. For $f \in \bar{\mathcal{F}}$, we denote with u_f the solution to (1). Similarly, for $h \in \bar{\mathcal{H}}$, we denote with u_h the solution to (1) with f replaced by h.

i) There exist $\hat{f} \in \mathcal{F}$ and $\hat{h} \in \mathcal{H}$ such that

$$\sup_{f \in \mathcal{F},\, h \in \mathcal{H}} \int_\Omega g(x) u_f(x) dx = \int_\Omega \hat{h}(x) u_{\hat{f}}(x) dx = \int_\Omega \hat{f}(x) u_{\hat{h}}(x) dx.$$

ii) If $\mathcal{F} = \mathcal{H}$ then $\hat{f} = \hat{h}$ almost everywhere in Ω.

Proof. Recall that $\bar{\mathcal{F}}$ and $\bar{\mathcal{H}}$ are convex and weak* sequentially compact (see [2-3]). For $k \in L^\infty_+(\Omega)$ define $u = u_k \in H^1_0(\Omega)$ such that

$$\int_\Omega \left(a_{ij} u_{x_i} \phi_{x_j} + gu\phi\right) dx = \int_\Omega k(x)\phi(x)\, dx \quad \forall \phi \in H^1_0(\Omega). \tag{21}$$

Putting $\phi = u$ in (21), recalling that $g(x) \geq 0$, using assumption (2) and Poincaré inequality, we find

$$\lambda \| \nabla u\|_{L^2(\Omega)} \leq C \| k\|_{L^2(\Omega)}, \tag{22}$$

with C independent of k.

If u_f is the solution of problem (1), let

$$\hat{I} = \sup_{f \in \mathcal{F},\, h \in \mathcal{H}} \int_\Omega h(x) u_f(x) dx.$$

Assume $\{f_i, h_i\}$ is a maximizing sequence for $\hat{I}$. Since the sequence $\{f_i\}$ is bounded in $L^\infty(\Omega)$ a subsequence (again denoted $\{f_i\}$) converges in the weak* topology to some $f \in \bar{\mathcal{F}}$. Similarly, a subsequence of $\{h_i\}$ (again denoted $\{h_i\}$) converges in the weak* topology to some $h \in \bar{\mathcal{G}}$. By (22) and Rellich's Theorem, the sequence $\{u_{f_i}\}$ converges strongly in $L^2(\Omega)$ to some $z \in H^1_0(\Omega)$. By using the equations for u_{f_i}, and u_f, one shows that $z = u_f$. Hence,

$$\hat{I} = \lim_{i \to \infty} \int_\Omega h_i(x) u_{f_i}(x) dx = \int_\Omega h(x) u_f(x) dx.$$

By Lemma 2.3, there is $\hat{h} \in \mathcal{H}$ such that

$$\int_\Omega h(x) u_f(x) dx \leq \int_\Omega \hat{h}(x) u_f(x) dx.$$

By using equation (1) and the corresponding equation with $f = \hat{h}$, it is easy to show that

$$\int_\Omega \hat{h}(x) u_f(x) dx = \int_\Omega f(x) u_{\hat{h}}(x) dx.$$

Hence, we have

$$\int_\Omega h(x) u_f(x) dx \leq \int_\Omega f(x) u_{\hat{h}}(x) dx.$$

Using Lemma 2.3 again, we find $\hat{f} \in \mathcal{F}$ such that

$$\int_\Omega f(x)u_{\hat{h}}(x)\,dx \le \int_\Omega \hat{f}(x)u_{\hat{h}}(x)dx.$$

Hence,

$$\hat{I} \le \int_\Omega \hat{f}(x)u_{\hat{h}}(x)dx \le \hat{I}.$$

It follows that

$$\hat{I} = \int_\Omega \hat{f}(x)u_{\hat{h}}(x)dx = \int_\Omega \hat{h}(x)u_{\hat{f}}(x)dx.$$

Part (i) is proved.

To discuss part (ii), we make some preparation. u_f and u_h satisfy

$$\int_\Omega \left(a_{ij}(u_f)_{x_i}\phi_{x_j} + gu_f\phi\right)dx = \int_\Omega f(x)\phi(x)\,dx \quad \forall\phi \in H_0^1(\Omega), \tag{23}$$

$$\int_\Omega \left(a_{ij}(u_h)_{x_i}\phi_{x_j} + gu_h\phi\right)dx = \int_\Omega h(x)\phi(x)\,dx \quad \forall\phi \in H_0^1(\Omega). \tag{24}$$

From (23) and (24) with $\phi = u_f - u_h$, we find

$$0 \le \int_\Omega \left(a_{ij}(u_f - u_h)_{x_i}(u_f - u_h)_{x_j} + g(u_f - u_h)^2\right)dx = \int_\Omega (f-h)(u_f - u_h)\,dx. \tag{25}$$

Since

$$\int_\Omega f(x)u_h(x)\,dx = \int_\Omega h(x)u_f(x)\,dx,$$

from (25) we find

$$2\int_\Omega f(x)u_h(x)\,dx \le \int_\Omega \left[f(x)u_f(x) + h(x)u_h(x)\right]dx.$$

It follows that

$$\int_\Omega f(x)u_h(x)\,dx \le \int_\Omega \frac{f(x)+h(x)}{2}\frac{u_f(x)+u_h(x)}{2}dx. \tag{26}$$

Consider now the case $\mathcal{F} = \mathcal{H}$. Then, if $f, h \in \mathcal{F}$, we have $\frac{f+h}{2} \in \overline{\mathcal{F}}$. If (f, h) is a maximizing pair, we find

$$\hat{I} = \int_\Omega f(x)u_h(x)dx \le \int_\Omega \frac{f+h}{2}\frac{u_f+u_h}{2}dx \text{ (by (26))}$$

$$= \int_\Omega \frac{f+h}{2}u_{\frac{f+h}{2}}dx \text{ (by the linearity of the equation)}$$

$$\le \hat{I} \text{ (by definition of } \hat{I}). \tag{27}$$

Since equality holds in (27), equality must hold in (26) and in (25), which implies $\|\nabla(u_f - u_h)\|_{L^2(\Omega)} = 0$. Recalling that $u_f(x) = u_h(x) = 0$ on $\partial\Omega$, we get $u_f(x) = u_h(x)$ in Ω. Finally, using the equations for u_f and u_h, we find $f(x) = h(x)$ almost everywhere in Ω. The theorem is proved.

References

[1] Alvino A, Trombetti G, Lions PL. On optimization problems with prescribed rearrangements. *Nonlinear Analysis: Theory, Methods & Applications.* 1989; 13(2): 185-220.

[2] Burton GR. Rearrangements of functions, maximization of convex functionals and vortex rings. *Mathematische Annalen.* 1987; 276: 225-253.

[3] Burton GR. Variational problems on classes of rearrangements and multiple configurations for steady vortices. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire.* 1989; 6(4): 295-319.

[4] Cuccu F, Emamizadeh B, Porru G. Design of a composite membrane with patches. *Applied Mathematics & Optimization.* 2010; 62(2): 169-184

[5] Cuccu F, Porru G, Sakaguchi S. Optimization problems on general classes of rearrangements. *Nonlinear Analysis: Theory, Methods & Applications.* 2011; 74(16): 5554-5565.

[6] Qiu C, Huang YS, Zhou Y. A class of rearrangement optimization problems involving the p-Laplacian. *Nonlinear Analysis: Theory, Methods & Applications.* 2015; 112: 30-42.

[7] Qiu C, Huang YS, Zhou Y. Optimization problems involving the fractional Laplacian. *Electronic Journal of Differential Equations.* 2016; 2016(98): 1-15.

[8] Emamizadeh B, Farjudian A, Liu Y. Optimal harvesting strategy based on rearrangements of functions. *Applied Mathematics and Computation.* 2018; 320: 677-690.

[9] Feyissa K. *Optimization problems in classes of rearrangements*. Ph.D. Thesis. Addis Ababa University; 2019.

[10] Feyissa K. Optimization problems in classes of rearrangements for (p,q)-Laplace equations. *Libertas Mathematica (new series).* 2018; 38: 1-15.

[11] Feyissa K, Tadesse A, Porru G. Placement of a source or a well for optimizing the energy integral. *Electronic Journal of Differential Equations.* 2018; 2018(104): 1-12.

[12] Amiri N, Zivari-Rezapour M. Maximization and minimization problems related to an equation with the p-Laplacian. *Indian Journal of Pure and Applied Mathematics.* 2020; 51: 777-788.

[13] Emamizadeh B, Liu Y. Bang-bang and multiple valued optimal solutions of control problems related to quasi-linear elliptic equations. *SIAM Journal on Control and Optimization.* 2020; 58(2): 1103-1117.

[14] Anedda C, Cuccu F, Frassu S. Steiner symmetry in the minimization of the first eigenvalue of a fractional eigenvalue problem with indefinite weight. *Canadian Journal of Mathematics.* 2020; 1-23. Available from: https://doi.org/10.4153/S0008414X20000267.

[15] Kawohl B. *Rearrangements and convexity of level sets in PDE*. Lectures Notes in Mathematics. Berlin; 1985.

[16] Gilbarg D, Trudinger NS. *Elliptic partial differential equations of second order* [*Theorem 8.19*]. Classics in Mathematics. Berlin: Springer-Verlag; 2001.

[17] Krein M, Milman D. On extreme points of regular convex sets. *Studia Mathematica.* 1940; 9(1): 133-138.