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Research Article

On Non-Semisimple Non-Pointed Hopf Algebras

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Abstract: In this paper, a family of Hopf algebra $\mathscr{H}_{n,\,d}$ which is neither pointed nor semisimple is introduced. It is shown that $\mathscr{H}_{n,\,d}$ is quasi-triangular if and only if n is even and $d = 2$. The indecomposable modules of $\mathscr{H}_{n,\,d}$ are completely classified. Moreover, the decomposition formulas of tensor product of two $\mathscr{H}_{n,\,d}$-modules are established.

***Keywords*:** indecomposable module, non-pointed Hopf algebra, tensor product

MSC: 16D70, 16T10

1. Introduction

Suppose that $\Bbbk$ is always an algebraically closed field whose characteristic is zero, and our work is to discuss the modules of a nd-dimensional Hopf algebra $\mathscr{H}_{n,\,d}$ with Chevalley property (see Definition 1, also can see [1]), which is neither pointed nor semi-simple. And $\mathscr{H}_{n,\,d}$ is just the dual Hopf algebra of Radford algebra up to isomorphim [2, 3].

The Hopf algebra $\mathscr{H}_{n,\,d}$ plays a significant role in constructing and classifying Hopf algebras and Nichols algebras. Note that $\mathscr{H}_{4,\,2}$ and $\mathscr{H}_{6,\,2}$ are the unique neither pointed nor semi-simple 8-dimension and 12-dimension Hopf algebras (see [4, 5]). In [6], García and Giraldi provided all finite dimensional Hopf algebras over the field $\Bbbk$, of which the coradicals generate $\mathscr{H}_{4,\,2}$ up to isomorphism. Basing on this, the author [7] determined all finite dimensional Nichols algebras in the category of Yetter-Drinfied module ${}^{\mathscr{H}_{4,\,2}}_{\mathscr{H}_{4,\,2}}\mathscr{YD}$. Furthermore, some new Hopf algebras without the dual Chevalley property were obtained. In addition, some new Nichols algebras in ${}^{\mathscr{H}_{6,\,2}}_{\mathscr{H}_{6,\,2}}\mathscr{YD}$ which are not of diagonal type were given by the authors in [8, 9]. Several families of new Hopf algebras were provided, they are dimension 216. It is noticed that the classification of finite dimensional Nichols algebras over $\mathscr{H}_{2n,\,2}$ has been established in [10], and finite dimensional Hopf algebras over $\mathscr{H}_{2n,\,2}$ in general case has been given [11].

To describe the decomposition of the tensor product of finite dimensional modules is a classic problem in the representation theory of Hopf algebras. For example, the author calculated the decomposition formulas of tensor product of two indecomposable modules of $k\mathbb{Z}_n(q)/I_d$ in [12]. Gunnlaugsdóttir [13] considered the representation theory of the half part u_q^+ of a class of quantum group u_q, the decomposition formulas of the tensor product of two indecomposable u_q^+-modules have also been constructed. More research of Hopf algebra can see [14]. It is remarked that the representations of Radford algebra $\mathscr{R}_{n,\,d}$ have been researched [15, 16], the authors not only provided the finite dimensional representations

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of $\mathscr{R}_{n,\ d}$, but also described its representation ring, which is characterized by generators with relations. In [4], the representations and representation ring of $\mathscr{H}_{4,\ 2}$ have been given. Chen et al. [17] researched the representations and representation rings of $\mathscr{H}_{2n,\ 2}$. However, the representations of $\mathscr{H}_{n,\ d}$ have not been studied yet for more general situations. By this motivation, the aim of this paper is to classify all indecomposable modules over $\mathscr{H}_{n,\ d}$, and provide the decomposition formulas of the tensor product of two indecomposable modules of $\mathscr{H}_{n,\ d}$. Moreover, the representation ring of $\mathscr{H}_{n,\ d}$ is described by two generators with two generating relations.

In this paper, we focus on determining the decomposition formulas of the tensor product of finite dimensional $\mathscr{H}_{n,\ d}$-modules. In Section 2, some necessary symbols and concepts are introduced. The definition of the Hopf algebras $\mathscr{H}_{n,\ d}$ is given, the readers can see [1, 3] for more details. It is shown that $\mathscr{H}_{n,\ d}$ is quasi-triangular if and only if n is even and $d = 2$. In Section 3, all indecomposable $\mathscr{H}_{n,\ d}$-modules are listed. It is shown that there are nd finite dimensional indecomposable $\mathscr{H}_{n,\ d}$-modules up to isomorphism. The decomposition formulas of the tensor product of $\mathscr{H}_{n,\ d}$-modules are established. Finally, the representation ring of $\mathscr{H}_{n,\ d}$ is characterized. The representation ring is an invariant of a monoidal category, it plays a significant role in the classification of monoidal categories. Our conclusion adds an example to research in this field, and it also provides some insights for representation theory of the Hopf algebras and Nichols algebras over $\mathscr{H}_{n,\ d}$.

2. The Hopf algebra $\mathscr{H}_{n,\ d}$

Throughout, we work on an algebraically closed field $\Bbbk$ whose characteristic is 0. Unless otherwise stated. Every algebra, Hopf algebra and module are defined on $\Bbbk$; every module is left module and finite dimensional; every map is $\Bbbk$-linear; $\dim_{\Bbbk}$, $\otimes_{\Bbbk}$ and $\hom_{\Bbbk}$ are abbreviated as dim, $\otimes$ and hom. Some elementary knowledge and theory about Hopf algebras and quantum groups, we refer to [18, 19].

Let n and d be two integers, such that $n,\ d \geq 2$ and $d \mid n$. We set $m = \dfrac{n}{d}$ and q be a parameter,

$$\begin{aligned} &(n)_q = 1 + q + \cdots + q^{n-1}, \\ &(n)_q! = (n)_q (n-1)_q \cdots (1)_q, \\ &\binom{n}{i}_q = \frac{(n)_q!}{(i)_q!(n-i)_q!}. \end{aligned} \tag{1}$$

Suppose that $\omega \in \Bbbk$ is a primitive n-th root of unity. First of all, we recall the concept of the Hopf algebra $\mathscr{H}_{n,\ d}$.

Definition 1 The Hopf algebra $\mathscr{H}_{n,\ d}$ is generated by two elements g and x subject to the relations

$$g^n = 1, \quad x^d = 0, \quad gx = \omega xg, \tag{2}$$

as an algebra, whose coalgebra structure and an antipode are as follows:

$$\Delta(g) = g \otimes g + \sum_{k=1}^{d-1} \frac{\omega^d - 1}{(d-k)!_{\omega^m} (k)!_{\omega^m}} x^{d-k} g^{km+1} \otimes x^k g, \quad \Delta(x) = x \otimes 1 + g^m \otimes x, \tag{3}$$

$$\varepsilon(g) = 1, \quad \varepsilon(x) = 0, \tag{4}$$

$$S(g) = g^{-1}, \quad S(x) = -g^{-m}x. \tag{5}$$

Note that if $m = 1$, then $\mathscr{H}_{n,\ d}$ is just n^2-dimensional Taft algebra $H_n(q)$ [20], which is a special case of Hopf algebra $\mathscr{H}_{n,\ d}$. It is easy to see that the set

$$\left\{x^i g^j \mid 0 \le i \le d-1,\ 0 \le j \le n-1\right\} \tag{6}$$

is a basis of $\mathscr{H}_{n,\ d}$.

Recall that the Radford algebra $\mathscr{R}_{n,\ d}$ [2], as an algebra, is generated by a and b subjecting to the relations

$$a^n = 1, \quad b^d = a^d - 1, \quad ab = \omega^m ba. \tag{7}$$

In fact, $\mathscr{R}_{n,\ d}$ is a Hopf algebra whose coalgebra structure and antipode are as follows:

$$\Delta(a) = a \otimes a, \quad \Delta(b) = b \otimes 1 + a \otimes b, \tag{8}$$

$$\varepsilon(a) = 1, \quad \varepsilon(b) = 0, \quad S(a) = a^{-1}, \quad S(b) = -ba^{-1}. \tag{9}$$

It is well-known that $\mathscr{R}_{n,\ d}$ has the basis

$$\{b^s a^t | 0 \le s \le d-1,\ 0 \le t \le n-1\}. \tag{10}$$

Consider the linear forms h and f on $\mathscr{R}_{n,\ d}$ defined on the basis $\{b^s a^t | 0 \le s \le d-1,\ 0 \le t \le n-1\}$ by

$$\left\langle h,\ b^s a^t \right\rangle = \delta_{s,\ 0}\omega^t \ \text{ and } \ \left\langle f,\ b^s a^t \right\rangle = \delta_{s,\ 1}, \tag{11}$$

for $h,\ f \in \mathscr{R}^*_{n,\ d}$, it follows that $\mathscr{H}_{n,\ d}$ is just the dual Hopf algebra $\mathscr{R}^*_{n,\ d}$ of Radford algebra up to isomorphism.

Lemma 2 For $1 \le i \le m-1$ and $0 \le j,\ k \le d-1$, the following equations hold:

$$\Delta(g^i) = g^i \otimes g^i + \sum_{k=1}^{d-1} \frac{\omega^{id} - 1}{(d-k)!_{\omega^m}(k)!_{\omega^m}} x^{d-k} g^{km+i} \otimes x^k g^i; \tag{12}$$

$$\Delta(x^j) = \sum_{k=0}^{j} \binom{j}{k}_{\omega^m} x^{j-k} g^{km} \otimes x^k; \tag{13}$$

$$\Delta(x^j g^{km+i}) = \sum_{l=0}^{j} \binom{j}{l}_{\omega^m} x^{j-l} g^{(l+k)m+i} \otimes x^l g^{km+i} \tag{14}$$

$$+ \sum_{l=0}^{j} \sum_{t=1}^{d-1} \binom{j}{l}_{\omega^m} \frac{\omega^{(km+i)d} - 1}{(d-t)!_{\omega^m} (t)!_{\omega^m}} \omega^{-ltm} x^{d+j-l-t} g^{(t+l+k)m+i} \otimes x^{l+t} g^{km+i}.$$

Proof. We prove the equation (12) by induction on i. Since $x^d = 0$, one can get

$$\Delta(g)\Delta(g^i) = \left(g \otimes g + \sum_{k=1}^{d-1} \frac{\omega^d - 1}{(d-k)!_{\omega^m} (k)!_{\omega^m}} x^{d-k} g^{km} \otimes x^k g \right) \left(g^i \otimes g^i + \sum_{k=1}^{d-1} \frac{\omega^{id} - 1}{(d-k)!_{\omega^m} (k)!_{\omega^m}} x^{d-k} g^{km+i} \otimes x^k g^i \right)$$

$$\begin{aligned} &= g^{i+1} \otimes g^{i+1} + \sum_{k=1}^{d-1} \frac{\omega^{id} - 1}{(d-k)!_{\omega^m} (k)!_{\omega^m}} g x^{d-k} g^{km+i} \otimes g x^k g^i \\ &+ \sum_{k=1}^{d-1} \frac{\omega^d - 1}{(d-k)!_{\omega^m} (k)!_{\omega^m}} x^{d-k} g^{km+1+i} \otimes x^k g^{i+1} \\ &= g^{i+1} \otimes g^{i+1} + \sum_{k=1}^{d-1} \frac{\omega^{(i+1)d} - 1}{(d-k)!_{\omega^m} (k)!_{\omega^m}} x^{d-k} g^{km+i+1} \otimes x^k g^{i+1}. \end{aligned} \tag{15}$$

Therefore, the equation (12) holds. In a similar way, we can show that the equation (13) holds by induction on j. According to the equations (12), (13) and the relations of $\mathscr{H}_{n,\,d}$, one can get the equation (14). □

Let $B_i (1 \le i \le m-1)$ (resp. T_d) be the space spanned by the basis

$$\left\{ x^j g^{km+i} \mid 0 \le k,\ j \le d-1 \right\} \left(resp. \left\{ g^{ms'} x^{t'} \mid 0 \le s',\ t' \le d-1 \right\} \right). \tag{16}$$

By Lemma 2, we have the following Lemma.

Lemma 3 As coalgebras

$$\mathscr{H}_{n,\,d} = \bigoplus_{i=1}^{m-1} B_i \oplus T_d. \tag{17}$$

T_d is isomorphic to d^2-dimensional Taft algebra $H_d(q)$ as Hopf algebras, and B_i is a simple subcoalgebra of $\mathscr{H}_{n,\,d}$.

Proof. In T_d, we have

$$(g^m)^d = 1, \quad g^m x = \omega^m x g^m, \quad x^d = 0. \tag{18}$$

$$\Delta(g^m) = g^m \otimes g^m, \quad \Delta(x) = x \otimes 1 + g^m \otimes x, \tag{19}$$

$$\varepsilon(g^m) = 1, \quad \varepsilon(x) = 0, \quad S(g^m) = (g^m)^{-1}, \quad S(x) = -g^{-m}x. \tag{20}$$

Therefore, T_d is isomorphic to d^2-dimensional Taft algebra as Hopf algebras.

In $B_i (1 \leq i \leq m-1)$, one can get that $\Delta(x^j g^{km+i}) \in B_i \otimes B_i$ by equation (14). Hence, B_i is a subcoalgebra of $\mathscr{H}_{n,\,d}$. Now let $0 \neq a = \sum_{j,\,k=0}^{d-1} b_{j,\,k} x^j g^{km+i} \in B_i$, there are $0 \neq b_{j,\,k} \in \Bbbk$. Let

$$j_0 = \min\{j | b_{j,\,k} \neq 0\}, \quad k_0 = \min\{k | b_{j_0,\,k} \neq 0\}. \tag{21}$$

We have $b_{j_0,\,k_0} \neq 0$ and yield that $g^{k_0 m+i} \in B_i$ by equation (14). And we conclude that $x^l g^{k_0 m+i} \in B_i$ for all $0 \leq l \leq d-1$ by equation (12). Repeating the application of equation (14) and (12), one can get that $g^{sm+i} \in B_i$ for all $0 \leq s \leq d-1$. and all $x^j g^{sm+i} \in B_i$ for all $0 \leq j \leq d-1$ and $0 \leq s \leq d-1$, it implies that B_i has no subcoalgebra. Hence, B_i is a simple subcoalgebra of $\mathscr{H}_{n,\,d}$.

The proof is complete. □

On the basis of Lemma 3, we can induce the coradical and Jacobson radical of $\mathscr{H}_{n,\,d}$. Let C_0 and $\mathrm{rad}(\mathscr{H}_{n,\,d})$ denote the coradical and Jacobson radical of $\mathscr{H}_{n,\,d}$, respectively. we have

$$C_0 = \bigoplus_{i=1}^{m-1} B_i \oplus \bigoplus_{k=0}^{d-1} \Bbbk g^{km}$$

and $\mathrm{rad}(\mathscr{H}_{n,\,d})$ is the space spanned by the set $\left\{g^i x^j \mid 0 \leq i \leq n-1,\ 1 \leq j \leq d-1\right\}$. Moreover $g^{km} (0 \leq k \leq d-1)$ are all group-likes of $\mathscr{H}_{n,\,d}$. This implies that if $m \neq 1$, $\mathscr{H}_{n,\,d}$ is a neither pointed nor semisimple Hopf algebra.

Proposition 4 The Hopf algebra $\mathscr{H}_{n,\,d}$ is quasi-triangular if and only if n is even and $d = 2$. In other cases the Hopf algebras are not almost cocommutative.

Proof. Let

$$R = \sum_{\substack{0 \leq i,\ j \leq n-1 \\ 0 \leq u,\ v \leq d-1}} d_{i,\,j}^{u,\,v} g^i x^u \otimes g^j x^v \in \mathscr{H}_{n,\,d} \otimes \mathscr{H}_{n,\,d} \tag{22}$$

be an invertible element such that $\Delta' R = R\Delta$, where Δ' stands for the opposite comultiplication and $d_{i,\,j}^{u,\,v} \in \Bbbk$. Since $\Delta'(g^m)R = R\Delta(g^m)$, we have

$$\sum d_{i,\,j}^{u,\,v} g^{m+i} x^u \otimes g^{m+j} x^v = \sum d_{i,\,j}^{u,\,v} \omega^{-m(u+v)} g^{m+i} x^u \otimes g^{m+j} x^v. \tag{23}$$

It means that $d_{i,\,j}^{u,\,v} = 0$ if $d \nmid u+v$. It is noted that $\Delta(x) = x \otimes 1 + g^m \otimes x$, we have following

$$\sum \omega^{-j} d_{i,\,j}^{u,\,v} g^i x^u \otimes g^j x^{v+1} + \sum \omega^{-i} d_{i,\,j}^{u,\,v} g^i x^{u+1} \otimes g^{m+j} x^v$$
$$= \sum d_{i,\,j}^{u,\,v} g^i x^{u+1} \otimes g^j x^v + \sum \omega^{-mu} d_{i,\,j}^{u,\,v} g^{m+i} x^u \otimes g^j x^{v+1}, \tag{24}$$

if $u = v = 0$, then

$$\omega^{-i} d_{i,\,j}^{0,\,0} = d_{i,\,j+m}^{0,\,0}, \quad \omega^{-j} d_{i+m,\,j}^{0,\,0} = d_{i,\,j}^{0,\,0}, \tag{25}$$

and if $u \neq 0,\ v \neq 0,\ u+v = d$, then

$$\omega^{-j} d_{i,\,j}^{u,\,v-1} + \omega^{-i} d_{i,\,j-m}^{u-1,\,v} = d_{i,\,j}^{u-1,\,v} + \omega^{-mu} d_{i-m,\,j}^{u,\,v-1}. \tag{26}$$

In particular, we have $d_{0,\,0}^{0,\,0} = d_{0,\,m}^{0,\,0}$ and $d_{0,\,0}^{0,\,0} = d_{m,\,0}^{0,\,0}$. But $\omega^{-m} d_{m,\,m}^{0,\,0} = d_{0,\,m}^{0,\,0}$ and $d_{m,\,m}^{0,\,0} = \omega^{-m} d_{m,\,0}^{0,\,0}$, so either $(\omega^m)^2 = 1$ or $d_{i,\,j}^{0,\,0} = 0$ for all $(i,\ j)$. If $d_{i,\,j}^{0,\,0} = 0$ for all $(i,\ j)$, then

$$R = \sum_{\substack{0 \leq i,\ j \leq n-1 \\ 1 \leq u,\ v \leq d-1}} d_{i,\,j}^{u,\,v} g^i x^u \otimes g^j x^v, \tag{27}$$

according to $x^d = 0$, it follows that R is a nilpotent element, and it is not invertible. Hence, at least one $d_{i,\,j}^{0,\,0}$ is non-zero, it follows that $\omega^{2m} = 1$ and $n \mid 2m$ by ω is a primitive n-th root of unity. Therefore, $d = 2$ and $n = 2m$. On the other hand, all universal R-matrices of $\mathscr{H}_{2m,\,2}$ are given in [17, Theorem 1.6]. □

3. Representations of $\mathscr{H}_{n,\,d}$

In this section, we mainly focus on classifying all representations of $\mathscr{H}_{n,\,d}$.

Since $\mathscr{H}_{n,\,d}$ is a Nakayama algebra, $\mathscr{H}_{n,\,d}$ is of finite representation type. Suppose that $\mathscr{M}(l,\ i)(0 \leq l \leq d-1,\ i \in \mathbb{Z}_n)$ is a vector space, and $\{x_0^{(i)},\ \cdots,\ x_l^{(i)}\}$ is a basis of $\mathscr{M}(l,\ i)$, then $\mathscr{M}(l,\ i)$ is a $\mathscr{H}_{n,\,d}$-module as follows:

$$g \cdot x_j^{(i)} = \omega^{j-i} x_j^{(i)}, \quad 0 \leq j \leq l.$$
$$x \cdot x_j^{(i)} = \begin{cases} x_{j+1}^{(i)}, & 0 \leq j \leq l-1, \\ 0, & j = l, \end{cases} \tag{28}$$

and $\mathscr{M}(l,\ i)$ is an indecomposable uniserial $\mathscr{H}_{n,\,d}$-module.

Proposition 5 Up to isomorphism, the following set

$$\{\mathscr{M}(l,\ i) \mid 0 \le l \le d-1,\ i \in \mathbb{Z}_n\} \tag{29}$$

consists of all indecomposable $\mathscr{H}_{n,\ d}$-modules.

Let $\mathscr{M}$ and $\mathscr{N}$ be two $\mathscr{H}$-modules, where $\mathscr{H}$ is a Hopf algebra, then $\mathscr{M} \otimes \mathscr{N}$ is also an $\mathscr{H}$-module, the action of $\mathscr{H}$ on $\mathscr{M} \otimes \mathscr{N}$ as follows:

$$h \cdot (u \otimes v) = \sum_{(h)} h_{(1)} \cdot u \otimes h_{(2)} \cdot v \tag{30}$$

for all $h \in \mathscr{H}$ and $u \in \mathscr{M},\ v \in \mathscr{N}$.

Let $\mathscr{I}$ and $\mathscr{J}_d$ be two sets, such that $\mathscr{I} = \{0\}$ or $\mathscr{I} = \mathbb{Z}_n$, and if $d \in \mathbb{N}$, then $\mathscr{J}_d = \{0,\ \cdots,\ d-1\}$; if $d = \infty$, then $\mathscr{J}_d = \mathbb{N}$. Next we consider a ring, which is a free commutative group, and generated by the elements $[\theta,\ \sigma]$, where $(\theta,\ \sigma) \in \mathscr{I} \times \mathscr{J}_d$, it has a $\mathbb{Z}$-basis

$$\{[\theta,\ \sigma] | (\theta,\ \sigma) \in \mathscr{I} \times \mathscr{J}_d\}. \tag{31}$$

The addition law is denoted by $\oplus$. One can endow with a multiplication structure to this group, and the multiplication law is denoted by $\otimes$. We stipulate that if $\sigma < 0$, then $[\theta,\ \sigma] = 0$.

We need assume that the following equations hold true:

$$\begin{aligned} &[\theta,\ 0] \otimes [\delta,\ 0] = [\theta + \delta,\ 0], \\ &[0,\ 1] \otimes [\delta,\ \rho] = [\delta,\ \rho + 1] \oplus [\delta - 1,\ \rho - 1] \ \text{ for } \ 0 \le \rho \le d-2, \end{aligned} \tag{32}$$

and

$$[0,\ 1] \otimes [\delta,\ d-1] = [\delta,\ d-1] \oplus [\delta - 1,\ d-1] \tag{33}$$

where $\theta,\ \delta \in \mathscr{I}$ and $\rho \in \mathscr{J}_d$, and these equations are symmetrical about multiplication law.

Lemma 6 For $\theta,\ \delta \in \mathscr{I}$ and $\sigma,\ \rho \in \mathscr{J}_d$, the equations are correct:

1. $[\theta,\ \sigma] \otimes [\delta,\ \rho] = \bigoplus_{l=0}^{min(\sigma,\ \rho)} [\theta + \delta - l,\ \sigma + \rho - 2l]$ for $u + v \le d-1$.
2. $[\theta,\ \sigma] \otimes [\delta,\ \rho] = \bigoplus_{l=0}^{e} [\theta + \delta - l,\ d-1] \oplus \bigoplus_{l=e+1}^{min(\sigma,\ \rho)} [\theta + \delta - l,\ \sigma + \rho - 2l]$ for $\sigma + \rho \ge d-1$ where $e = \sigma + \rho - (d-1)$.

Proof. In [13, Proposition 3.1], $i+1$ is replaced by $\theta - 1$; $i + j + 1$ is replaced by $\theta + \delta - 1$; m is replaced by d; $i + j + l$ is replaced by $\theta + \delta - l$; $j + 1$ is replaced by $\delta - 1$, and $i + j + l + 1$ is replaced by $\theta + \delta - l - 1$. Then the proof is the same as the proof of [13, Proposition 3.1]. □

Now we return to the decomposition of the tensor product of $\mathscr{H}_{n,\ d}$-modules, we have following several lemmas.

Lemma 7 If $0 \le r \le d-1$ and $\tau,\ i,\ \tau + i \in \mathbb{Z}_n$, then we have

1. $\mathscr{M}(0,\ i) \otimes \mathscr{M}(r,\ \tau) \cong \mathscr{M}(r,\ \tau) \otimes \mathscr{M}(0,\ i) \cong \mathscr{M}(r,\ \tau + i)$;
2. $\mathscr{M}(r,\ \tau) \cong \mathscr{M}(0,\ \tau) \otimes \mathscr{M}(r,\ 0) \cong \mathscr{M}(r,\ 0) \otimes \mathscr{M}(0,\ \tau)$.

Proof. In order to computer the decomposition formulas of $\mathscr{M}(0,\ i)\otimes\mathscr{M}(r,\ \tau)$ and $\mathscr{M}(r,\ \tau)\otimes\mathscr{M}(0,\ i)$, where $0\le r\le d-1$ and $i,\ \tau\in\mathbb{Z}_n$, we have

$$g\cdot(v_0^{(i)}\otimes u_j^{(\tau)})=\omega^{j-\tau-i}v_0^{(i)}\otimes u_j^{(\tau)},\quad 0\le j\le r.$$

$$x\cdot(v_0^{(i)}\otimes u_j^{(\tau)})=\begin{cases}\omega^{-mi}v_0^{(i)}\otimes u_{j+1}^{(\tau)}, & 0\le j\le r-1,\\ 0, & j=r.\end{cases}\tag{34}$$

we set

$$a_j=(\omega^{-mi})^{(j-1)}v_0^{(i)}\otimes u_j^{(\tau)}.\tag{35}$$

Then $\mathscr{H}_{n,\ d}$-module $M(0,\ i)\otimes M(r,\ s)$ has a new basis $\{a_j\mid 0\le j\le r\}$ as a vector space, and

$$g\cdot a_j=\omega^{j-\tau-i}a_j,\quad 0\le j\le r.$$

$$x\cdot a_j=\begin{cases}a_{j+1}, & 0\le j\le r-1,\\ 0, & j=r.\end{cases}\tag{36}$$

Moreover,

$$g\cdot(u_j^{(\tau)}\otimes v_0^{i})=\omega^{j-\tau-i}u_j^{(\tau)}\otimes v_0^{(i)},\quad 0\le j\le r.$$

$$x\cdot(u_j^{(\tau)}\otimes v_0^{(i)})=\begin{cases}u_{j+1}^{(\tau)}\otimes v_0^{(i)}, & 0\le j\le r-1,\\ 0, & j=r.\end{cases}\tag{37}$$

Therefore, the first conclusion holds. In a similar way, one can prove the statement (2). □

Lemma 8 Suppose that $1\le l\le d-2$ and $d\ge 2$, then we have

$$\mathscr{M}(1,\ 0)\otimes\mathscr{M}(l,\ 0)\cong\mathscr{M}(l,\ 0)\otimes\mathscr{M}(1,\ 0)\cong\mathscr{M}(l+1,\ 0)\oplus\mathscr{M}(l-1,\ -1).\tag{38}$$

Proof. In order to consider the decomposition formulas of $\mathscr{M}(1,\ 0)\otimes\mathscr{M}(l,\ 0)$ and $\mathscr{M}(l,\ 0)\otimes\mathscr{M}(1,\ 0)$, where $1\le l\le d-2$ and $d\ge 2$, we have

$$g\cdot(v_0^{(0)}\otimes u_j^{(0)})=\omega^j v_0^{(0)}\otimes u_j^{(0)},\quad 0\le j\le l.$$

$$x\cdot(v_0^{(0)}\otimes u_j^{(0)})=\begin{cases} v_1^{(0)}\otimes u_j^{(0)}+v_0^{(0)}\otimes u_{j+1}^{(0)}, & 0\le j\le l-1,\\ v_1^{(0)}\otimes u_l^{(0)}, & j=l.\end{cases}$$

$$g\cdot(v_1^{(0)}\otimes u_j^{(0)})=\omega^{j+1} v_1^{(0)}\otimes u_j^{(0)},\quad 0\le j\le l. \tag{39}$$

$$x\cdot(v_1^{(0)}\otimes u_j^{(0)})=\begin{cases} \omega^m v_1^{(0)}\otimes u_{j+1}^{(0)}, & 0\le j\le l-1,\\ 0, & j=l.\end{cases}$$

Set

$$\beta_j=\frac{1-\omega^{mj}}{1-\omega^m}v_1^{(0)}\otimes u_{j-1}^{(0)}+v_0^{(0)}\otimes u_j^{(0)},\ \ 0\le j\le l,$$

$$\beta_{l+1}=\frac{1-\omega^{m(l+1)}}{1-\omega^m}v_1^{(0)}\otimes u_l^{(0)}, \tag{40}$$

$$\gamma_t=-\omega^m v_0^{(0)}\otimes u_{t+1}^{(0)}+\frac{1+\omega^m+\cdots+\omega^{(l-1-t)m}}{\omega^{(l-1-t)m}}v_1^{(0)}\otimes u_t^{(0)},\ \ 0\le t\le l-1.$$

Obviously, as a vector space, $\{\beta_j,\ \gamma_t\mid 0\le j\le l+1,\ 0\le t\le l-1\}$ is a new basis of $\mathscr{H}_{n,\ d}$-module $\mathscr{M}(1,\ 0)\otimes\mathscr{M}(l,\ 0)$, and

$$g\cdot\beta_j=\omega^j\beta_j,\quad 0\le j\le l+1.$$

$$x\cdot\beta_j=\begin{cases}\beta_{j+1}, & 0\le j\le l,\\ 0, & j=l+1.\end{cases}$$

$$g\cdot\gamma_t=\omega^{t+1}\gamma_t,\quad 0\le t\le l-1. \tag{41}$$

$$x\cdot\gamma_t=\begin{cases}\gamma_{t+1}, & 0\le j\le l-2,\\ 0, & j=l-1.\end{cases}$$

It follows that $\mathscr{M}(1,\ 0)\otimes\mathscr{M}(l,\ 0)\cong\mathscr{M}(l+1,\ 0)\oplus\mathscr{M}(l-1,\ -1)$ for all $1\leq l\leq d-2$ and $d\geq 2$. On the other hand,

$$
\begin{aligned}
&g\cdot(u_j^{(0)}\otimes v_0^{(0)})=\omega^j u_j^{(0)}\otimes v_0^{(0)},\quad 0\leq j\leq l.\\
&x\cdot(u_j^{(0)}\otimes v_0^{(0)})=\begin{cases} u_{j+1}^{(0)}\otimes v_0^{(0)}+\omega^{mj}u_j^{(0)}\otimes v_1^{(0)}, & 0\leq j\leq l-1,\\ \omega^{ml}u_l^{(0)}\otimes v_1^{(0)}, & j=l.\end{cases}\\
&g\cdot(u_j^{(0)}\otimes v_1^{(0)})=\omega^{j+1} u_j^{(0)}\otimes v_1^{(0)},\quad 0\leq j\leq l.\\
&x\cdot(u_j^{(0)}\otimes v_1^{(0)})=\begin{cases} u_{j+1}^{(0)}\otimes v_1^{(0)}, & 0\leq j\leq l-1,\\ 0, & j=l.\end{cases}
\end{aligned}
\tag{42}
$$

We set

$$
\begin{aligned}
&\delta_0=u_0^{(0)}\otimes v_0^{(0)},\ \ \delta_j=u_j^{(0)}\otimes v_0^{(0)}+(1+\omega^m+\cdots+\omega^{m(j-1)})u_{j-1}^{(0)}\otimes v_1^{(0)},\ \ 1\leq j\leq l,\\
&\delta_{l+1}=(1+\omega^m+\cdots+\omega^{lm})u_l^{(0)}\otimes v_1^{(0)},\\
&b_t=u_{t+1}^{(0)}\otimes v_0^{(0)}-(\omega^{(t+1)m}+\omega^{(t+2)m}+\cdots+\omega^{lm})u_t^{(0)}\otimes v_1^{(0)},\ \ 0\leq t\leq l-1.
\end{aligned}
\tag{43}
$$

It is easy to check that $\{\delta_j,\ b_t\mid 0\leq j\leq l+1,\ 0\leq t\leq l-1\}$ forms a basis of $M(l,\ 0)\otimes M(1,\ 0)$, and

$$
\begin{aligned}
&g\cdot\delta_j=\omega^j\delta_j,\quad 0\leq j\leq l+1.\\
&x\cdot\delta_j=\begin{cases}\delta_{j+1}, & 0\leq j\leq l,\\ 0, & j=l+1.\end{cases}\\
&g\cdot b_t=\omega^{t+1}b_t,\quad 0\leq t\leq l-1.\\
&x\cdot b_t=\begin{cases} b_{t+1}, & 0\leq t\leq l-2,\\ 0, & t=l-1.\end{cases}
\end{aligned}
\tag{44}
$$

Hence, we have

$$\mathscr{M}(l,\ 0)\otimes\mathscr{M}(1,\ 0)\cong\mathscr{M}(l+1,\ 0)\oplus\mathscr{M}(l-1,\ -1) \tag{45}$$

for all $1\leq l\leq d-2$ and $d\geq 2$. □

Lemma 9 $\mathscr{M}(1,\ 0)\otimes\mathscr{M}(d-1,\ 0)\cong M(d-1,\ 0)\otimes\mathscr{M}(1,\ 0)\cong\mathscr{M}(d-1,\ 0)\oplus\mathscr{M}(d-1,\ -1)$.

Proof. Considering the tensor product $\mathscr{M}(1,\ 0)\otimes\mathscr{M}(d-1,\ 0)$ and $\mathscr{M}(d-1,\ 0)\otimes\mathscr{M}(1,\ 0)$, we have

$$\begin{aligned}
&g\cdot(v_0^{(0)}\otimes u_0^{(0)})=v_0^{(0)}\otimes u_0^{(0)}+\frac{\omega^d-1}{(d-1)!_{\omega^m}}v_1^{(0)}\otimes u_{d-1}^{(0)}\\
&g\cdot(v_0^{(0)}\otimes u_j^{(0)})=\omega^j v_0^{(0)}\otimes u_j^{(0)},\quad 1\leq j\leq d-1.\\
&x\cdot(v_0^{(0)}\otimes u_j^{(0)})=\begin{cases}v_1^{(0)}\otimes u_j^{(0)}+v_0^{(0)}\otimes u_{j+1}^{(0)}, & 0\leq j\leq d-2,\\ v_1^{(0)}\otimes u_{d-1}^{(0)}, & j=d-1.\end{cases}\\
&g\cdot(v_1^{(0)}\otimes u_j^{(0)})=\omega^{j+1}v_1^{(0)}\otimes u_j^{(0)},\quad 0\leq j\leq d-1.\\
&x\cdot(v_1^{(0)}\otimes u_j^{(0)})=\begin{cases}\omega^m v_1^{(0)}\otimes u_{j+1}^{(0)}, & 0\leq j\leq d-2,\\ 0, & j=d-1.\end{cases}
\end{aligned} \tag{46}$$

We set

$$\begin{aligned}
&\beta_0'=v_0^{(0)}\otimes u_0^{(0)}-\frac{1}{(d-1)!_{\omega^m}}v_1^{(0)}\otimes u_{d-1}^{(0)},\\
&\beta_j'=v_0^{(0)}\otimes u_j^{(0)}+(1+\omega^m+\cdots+\omega^{m(j-1)})v_1^{(0)}\otimes u_{j-1}^{(0)},\ \ 1\leq j\leq d-1,\\
&\gamma_t'=\omega^m v_0^{(0)}\otimes u_{t+1}^{(0)}+\frac{1-\omega^m-\cdots-\omega^{(d-1-t)m}}{\omega^{(d-1-t)m}}v_1^{(0)}\otimes u_t^{(0)},\ \ 0\leq t\leq d-2,\\
&\gamma_{d-1}'=v_1^{(0)}\otimes u_{d-1}^{(0)}.
\end{aligned} \tag{47}$$

It is easy to check that $\{\beta_j',\ \gamma_t'\mid 0\leq j\leq d-1,\ 0\leq t\leq d-1\}$ forms a new basis of $\mathscr{H}_{n,\ d}$-module $M(1,\ 0)\otimes M(d-1,\ 0)$, and

$$
\begin{aligned}
&g\cdot\beta'_j=\omega^j\beta'_j,\quad 0\le j\le d-1.\\
&x\cdot\beta'_j=\begin{cases}\beta'_{j+1}, & 0\le j\le d-2,\\ 0, & j=d-1.\end{cases}\\
&g\cdot\gamma'_t=\omega^{t+1}\gamma'_\tau,\quad 0\le t\le d-1.\\
&x\cdot\gamma'_t=\begin{cases}\gamma'_{t+1}, & 0\le t\le d-2,\\ 0, & t=d-1.\end{cases}
\end{aligned}
\tag{48}
$$

It follows that $\mathscr{M}(1,\,0)\otimes\mathscr{M}(d-1,\,0)\cong\mathscr{M}(d-1,\,0)\oplus\mathscr{M}(d-1,\,-1)$.
On the other hand,

$$
\begin{aligned}
&g\cdot(u_0^{(0)}\otimes v_0^{(0)})=u_0^{(0)}\otimes v_0^{(0)}+\frac{\omega^d-1}{(d-1)!_{\omega^m}}u_{d-1}^{(0)}\otimes v_1^{(0)},\\
&g\cdot(u_j^{(0)}\otimes v_0^{(0)})=\omega^j u_j^{(0)}\otimes v_0^{(0)},\\
&x\cdot(u_j^{(0)}\otimes v_0^{(0)})=\begin{cases}u_{j+1}^{(0)}\otimes v_0^{(0)}+\omega^{mj}u_j^{(0)}\otimes v_1^{(0)}, & 0\le j\le d-2,\\ \omega^{m(d-1)}u_{d-1}^{(0)}\otimes v_1^{(0)}, & j=d-1.\end{cases}\\
&g\cdot(u_j^{(0)}\otimes v_1^{(0)})=\omega^{j+1}u_j^{(0)}\otimes v_1^{(0)},\quad 0\le j\le d-1.\\
&x\cdot(u_j^{(0)}\otimes v_1^{(0)})=\begin{cases}u_{j+1}^{(0)}\otimes v_1^{(0)}, & 0\le j\le d-2,\\ 0, & j=d-1.\end{cases}
\end{aligned}
\tag{49}
$$

We set

$$
\delta'_0=u_0^{(0)}\otimes v_0^{(0)}-\frac{1}{(d-1)!_{\omega^m}}u_{d-1}^{(0)}\otimes v_1^{(0)},
$$

$$
\delta'_j=u_j^{(0)}\otimes v_0^{(0)}+(1+\omega^m+\cdots+\omega^{m(j-1)})u_{j-1}^{(0)}\otimes v_1^{(0)},\ \ 1\le j\le d-1,
$$

$$\begin{aligned} b'_t &= u^{(0)}_{\tau+1} \otimes v^{(0)}_0 - (\omega^{(\tau+1)m} + \omega^{(\tau+2)m} + \cdots + \omega^{md}) u^{(0)}_t \otimes v^{(0)}_1, \ \ 0 \le t \le d-2, \\ b'_{d-1} &= -u^{(0)}_{d-1} \otimes v^{(0)}_1. \end{aligned} \tag{50}$$

It is easy to check that $\{\delta'_j,\ b'_\tau \mid 0 \le j \le d-1,\ 0 \le t \le d-1\}$ forms a new basis of $\mathscr{M}(d-1,\ 0) \otimes \mathscr{M}(1,\ 0)$, and

$$\begin{gathered} g \cdot \delta'_j = \omega^j \delta'_j, \quad 0 \le j \le d-1. \\ x \cdot \delta'_j = \begin{cases} \delta'_{j+1}, & 0 \le j \le d-2, \\ 0, & j = d-1. \end{cases} \\ g \cdot b'_t = \omega^{t+1} b'_t, \quad 0 \le t \le d-1. \\ x \cdot b'_t = \begin{cases} b'_{t+1}, & 0 \le t \le d-2, \\ 0, & t = d-1. \end{cases} \end{gathered} \tag{51}$$

It follows that $\mathscr{M}(d-1,\ 0) \otimes \mathscr{M}(1,\ 0) \cong \mathscr{M}(d-1,\ 0) \oplus \mathscr{M}(d-1,\ -1)$. □

Example 10 Suppose that $n = 4$ and $d = 2$, then $\mathscr{H}_{4,\ 2}$ is just $(A''_{C_4})^*$. In [4], Lemma 3.3 has decomposed the tensor product of $(A''_{C_4})^*$-modules. We also have the following decomposition formulas of the tensor product of $(A''_{C_4})^*$-modules by Lemma 7-9:

1. $\mathscr{M}(0,\ i) \otimes \mathscr{M}(0,\ j) \cong \mathscr{M}(0,\ j) \otimes \mathscr{M}(0,\ i) \cong \mathscr{M}(0,\ i+j)$;
2. $\mathscr{M}(0,\ i) \otimes \mathscr{M}(1,\ j) \cong \mathscr{M}(1,\ j) \otimes \mathscr{M}(0,\ i) \cong \mathscr{M}(1,\ i+j)$;
3. $\mathscr{M}(1,\ i) \otimes \mathscr{M}(1,\ j) \cong \mathscr{M}(1,\ j) \otimes \mathscr{M}(1,\ i) \cong \mathscr{M}(1,\ i+j) \oplus \mathscr{M}(1,\ i+j-1)$,

where $i,\ j,\ i+j,\ i+j-1 \in \mathbb{Z}_4$.

Next, let's summarize the main conclusions of this section as follows.

Theorem 11 Suppose that $0 \le \sigma,\ \rho \le d-1$ and $i,\ j \in \mathbb{Z}_n$, then

1. if $\sigma + \rho \le d-1$, then

$$\mathscr{M}(\sigma,\ i) \otimes \mathscr{M}(\rho,\ j) \cong \bigoplus_{k=0}^{\min(\sigma,\ \rho)} \mathscr{M}(\sigma+\rho-2k,\ i+j-k); \tag{52}$$

2. if $\sigma + \rho \ge d-1$, set $t = \sigma + \rho - (d-1)$, then

$$\mathscr{M}(\sigma,\ i) \otimes \mathscr{M}(\rho,\ j) \cong \bigoplus_{k=0}^{t} \mathscr{M}(d-1,\ i+j-k) \oplus \bigoplus_{k=t+1}^{\min(\sigma,\ \rho)} \mathscr{M}(\sigma+\rho-2k,\ i+j-k). \tag{53}$$

Proof. In the category of $\mathscr{H}_{n,\,d}$-modules, by Lemma 6, we have that the decomposition formulas of the tensor product of simple $\mathscr{H}_{n,\,d}$-modules, it implies all simple $\mathscr{H}_{n,\,d}$-modules can be generated by $[\mathscr{M}(0,\,1)]$. By Lemma 7 and 9, one can get that the decomposition formulas of the tensor product of indecomposable $\mathscr{H}_{n,\,d}$-modules, it follows that arbitrary indecomposable $\mathscr{H}_{n,\,d}$-modules is generated by $[\mathscr{M}(0,\,1)]$ and $[\mathscr{M}(1,\,0)]$. And the proof follows from Lemma 6 and Lemma 7-9. □

As everyone knows that if Hopf algebra $\mathscr{H}$ is quasi-triangular, then the representation ring of $\mathscr{H}$ is a commutative ring. In general, $\mathscr{H}_{n,\,d}$ is not quasi-triangular. However, Theorem 11 implies that the multiplication law $\otimes$ is commutative in the category of $\mathscr{H}_{n,\,d}$-modules. Therefore, we have following corollary.

Corollary 12 Let $\mathscr{M}$ and $\mathscr{N}$ be two arbitrary $\mathscr{H}_{n,\,d}$-module, then $\mathscr{M}\otimes\mathscr{N}\cong\mathscr{N}\otimes\mathscr{M}$.

By Theorem 11 we have that the representation ring $r(\mathscr{H}_{n,\,d})$ of $\mathscr{H}_{n,\,d}$ is generated by $[\mathscr{M}(0,\,1)]$ and $[\mathscr{M}(1,\,0)]$ with the relations:

$$[\mathscr{M}(0,\,1)]^n=[\mathscr{M}(0,\,0)], \tag{54}$$

$$([\mathscr{M}(0,\,1)]-[\mathscr{M}(1,\,0)]-[\mathscr{M}(0,\,0)])\sum_{i=0}^{[\frac{d-1}{2}]}(-1)^i\binom{d-1-i}{i}[\mathscr{M}(0,\,1)]^i[\mathscr{M}(1,\,0)]^{d-1-2i}=0, \tag{55}$$

where $[\mathscr{M}(0,\,0)]$ is the unit element of $r(\mathscr{H}_{n,\,d})$.

The analogous statements as in [21], the representation ring $r(\mathscr{H}_{n,\,d})$ of $\mathscr{H}_{n,\,d}$ can be easily characterized as follows.

Remark 13 As a ring, we have $r(\mathscr{H}_{n,\,d})\cong\mathbb{Z}[y,\,z]/\mathscr{I}$, where $\mathscr{I}$ is the ideal generated by

$$y^n-1,\quad (z-y-1)\sum_{i=0}^{[\frac{d-1}{2}]}(-1)^i\binom{d-1-i}{i}y^iz^{d-1-2i}. \tag{56}$$

Based on the above remark, we can give several representation rings of Hopf algebras as follows.

Example 14

If $n=2$ and $d=2$, then $r(\mathscr{H}_{2,\,2})\cong\mathbb{Z}[y,\,z]/\langle y^2-1,\,(z-y-1)z\rangle$ (also can see [22]);

If $n=4$ and $d=2$, then $r(\mathscr{H}_{4,\,2})\cong\mathbb{Z}[y,\,z]/\langle y^4-1,\,(z-y-1)z\rangle$ (also can see [4]);

If $n=6$ and $d=3$, then $r(\mathscr{H}_{6,\,3})\cong\mathbb{Z}[y,\,z]/\langle y^6-1,\,(z-y-1)(z^2-y)\rangle$;

If $n=9$ and $d=3$, then $r(\mathscr{H}_{9,\,3})\cong\mathbb{Z}[y,\,z]/\langle y^9-1,\,(z-y-1)(z^2-y)\rangle$;

If $n=12$ and $d=4$, then $r(\mathscr{H}_{12,\,4})\cong\mathbb{Z}[y,\,z]/\langle y^{12}-1,\,(z-y-1)(z^3-2yz)\rangle$.

Remark 15 If n is a odd or $d\neq 2$, then $\mathscr{H}_{n,\,d}$ is not quasi-triangular, however, the representation ring of $\mathscr{H}_{n,\,d}$ is a commutative ring. It begs a question: if the representation ring of a Hopf algebra H is a commutative ring, then What are the properties of H?

Author contributions

The author wrote and reviewed the manuscript.

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Conflict of interest

The authors declare that they have no competing interests.

References

[1] Mombelli M. Families of finite-dimensional Hopf algebras with the Chevalley property. *Algebra and Representation Theory*. 2013; 16(2): 421-435. Available from: https://doi.org/10.1007/s10468-011-9313-3.

[2] Radford DE. On a coradical of a finite-dimensional Hopf algebra. *Proceedings of the American Mathematical Society*. 1975; 53(1): 9-15. Available from: https://doi.org/10.1090/s0002-9939-1975-0396652-0.

[3] Andruskiewitsch N, Cuadra J, Etingof P. On two finiteness conditions for Hopf algebras with nonzero integral. *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze [Annals of the Scuola Normale Superiore di Pisa, Class of Sciences]*. 2015; 14(2): 401-440. Available from: https://doi.org/10.2422/2036-2145.201206_011.

[4] Wakui M. Various structures associated to the representation categories of eight-dimensional nonsemisimple Hopf algebras. *Algebra and Representation Theory*. 2004; 7(5): 491-515. Available from: https://doi.org/10.1023/B:ALGE.0000048318.64013.db.

[5] Natale S. Hopf algebras of dimension 12. *Algebra and Representation Theory*. 2002; 5(2): 445-455. Available from: https://doi.org/10.1023/A:1020504123567.

[6] García AG, Giraldi JMJ. On Hopf algebras over quantum subgroups. *Journal of Pure and Applied Algebra*. 2019; 223(2): 738-768. Available from: https://doi.org/10.1016/j.jpaa.2018.04.018.

[7] Xiong R. Finite-dimensional Hopf algebras over the smallest non-pointed basic Hopf algebra. *Frontiers of Mathematics*. 2024; 19(5): 865-890. Available from: https://doi.org/10.1007/s11464-022-0272-2.

[8] Hu N, Xiong R. Some Hopf algebras of dimension 72 without the Chevalley property. *arXiv:1612.04987*. 2016. Available from: https://doi.org/10.48550/arXiv.1612.04987.

[9] Xiong R. On Hopf algebras over the unique 12-dimensional Hopf algebra without the dual Chevalley property. *Communications in Algebra*. 2019; 47(4): 1516-1540. Available from: https://doi.org/10.1080/00927872.2018.1508582.

[10] Márquez O, Bagio D, Giraldi JMJ, García GA. Finite-dimensional Nichols algebras over dual Radford algebras. *Journal of Algebra and Its Applications*. 2021; 20(1): 2140001. Available from: https://doi.org/10.1142/S0219498821400016.

[11] Xiong R, Hu N. Classification of finite-dimensional Hopf algebras over dual Radford algebras. *Bulletin of the Belgian Mathematical Society-Simon Stevin*. 2021; 28(5): 633-688. Available from: https://doi.org/10.36045/j.bbms.210612.

[12] Cibils C. A quiver quantum group. *Communications in Mathematical Physics*. 1993; 157(3): 459-477. Available from: https://doi.org/10.1007/BF02096879.

[13] Gunnlaugsdóttir E. Monoidal structure of the category of u_q^+-modules. *Linear Algebra and Its Applications*. 2003; 365: 183-199. Available from: https://doi.org/10.1016/S0024-3795(02)00484-6.

[14] Karasik Y. Kemer's theory for H-module algebras with application to the PI exponent. *Journal of Algebra*. 2016; 457: 194-227. Available from: https://doi.org/10.1016/j.jalgebra.2016.02.021.

[15] Yang S. Representations of simple pointed Hopf algebras. *Journal of Algebra and Its Applications*. 2004; 3(1): 91-104. Available from: https://doi.org/10.1142/S021949880400071X.

[16] Wang Z, Li L, Zhang Y. Green rings of pointed rank one Hopf algebras of non-nilpotent type. *Journal of Algebra*. 2016; 449: 108-137. Available from: https://doi.org/10.1016/j.jalgebra.2015.11.002.

[17] Chen J, Yang S, Wang D, Xu Y. On 4n-dimensional neither pointed nor semisimple Hopf algebras and the associated weak Hopf algebras. *arXiv:1809.00514*. 2018. Available from: https://doi.org/10.48550/arXiv.1809.00514.

[18] Kassel C. *Quantum Groups*. New York: Springer; 1995. Available from: https://doi.org/10.1007/978-1-4612-0783-2.

[19] Sweedler ME. *Hopf Algebras*. New York: Benjamin; 1969.

[20] Taft EJ. The order of the antipode of finite-dimensional Hopf algebra. *Proceedings of the National Academy of Sciences of the United States of America*. 1971; 68(11): 2631-2633. Available from: https://doi.org/10.1073/pnas.68.11.2631.

[21] Li L, Zhang Y. The Green rings of the generalized Taft Hopf algebras. *Contemporary Mathematics*. 2013; 585: 275-288. Available from: https://doi.org/10.1090/conm/585.

[22] Chen H, Van Oystaeyen F, Zhang Y. The Green rings of Taft algebras. *Proceedings of the American Mathematical Society*. 2014; 142(3): 765-775. Available from: https://doi.org/10.1090/S0002-9939-2013-11823-X.