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Research Article

Advancements in Population Mean Estimation for Circular Systematic Sampling

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Abstract: Conventional measures of auxiliary variable(s) often yield unreliable results in the presence of outliers or extreme values. This article proposes a generalized estimator for the population mean by incorporating non-conventional measures under Circular Systematic Sampling (CSS). Expressions for the bias and Mean Squared Error (MSE) of the proposed estimators are derived, and the conditions under which they achieve minimum MSE are identified. Theoretical findings demonstrate the superiority of the proposed estimators over traditional unbiased, ratio, product, and regression estimators. Both theoretical and empirical studies indicate that the newly proposed estimators are more efficient than competing ones. The findings of this research will help researchers obtain more precise estimates of the population mean in the presence of outliers under CSS.

Keywords: auxiliary variable, bias, Circular Systematic Sampling (CSS), non-conventional measures, mean squared error

MSC: 62D05, 62F10, 62J05

1. Introduction

It is an indisputable fact that the use of auxiliary information during the estimation phase enhances the efficiency of estimating the population mean. In this context, the ratio estimator suggested by Cochran [1], the product estimator proposed by Murthy [2], and the regression estimator introduced by Watson [3] are widely used. A considerable amount of efficient work has already been done on estimating the finite population mean/total, median, variance, and proportion based on known conventional measures of the auxiliary variable in sample surveys ([4–13] and the references cited therein). One noticeable shortcoming of the existing estimators or classes of estimators under different sampling schemes is their poor performance in the presence of outliers or extreme observations. To address this issue, many researchers have employed various approaches to estimate unknown population parameters. The use of non-conventional measures is a common strategy for handling outliers in survey sampling [14–16]. Examples of such non-conventional measures include the quartile deviation, mid-range, tri-mean, quartile average, and the Hodges-Lehmann estimator.

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Non-conventional measures, Poisson regression, quantile regression, and robust regression methods including Least Absolute Deviation (LAD), Huber-M, Huber-MM, Tukey-M, Least Trimmed Squares (LTS), and Least Median of Squares (LMS) are commonly used when data are contaminated by outliers. To address this issue, several authors [11, 15–31] have developed efficient estimators for unknown population parameters in the presence of outliers under Simple Random Sampling (SRS). However, no research has been found on estimating the population mean under a Circular Sampling Scheme (CSS) when outliers are present. In this article, we propose a generalized estimator for the unknown population mean using non-conventional measures under CSS. The proposed estimator generates a series of estimators, including ratio-type estimators, product-type estimators, ratio-ratio-type exponential estimators, and ratio-product-type exponential estimators.

In Simple Random Sampling (SRS), every unit has an equal probability of being selected, whereas in Systematic Sampling (SS), units are selected based on a predetermined interval. SS offers operational convenience over SRS due to its simplicity and practical accessibility. To illustrate the concept of systematic sampling, consider the following example: Suppose the sampling frame consists of a list of 5,000 voters (the target population), and you want to survey 500 voters about their preference for party "A". If the predetermined interval is 10, then every 10th voter would be included in the sample. Systematic sampling begins with a random start between 1 and k, where the initial unit is selected randomly from a set of N units in the finite population. Subsequently, every k^{th} unit is selected, provided the sample size n satisfies the condition $N = nk$, where k is an integer. This technique is referred to as Linear Systematic Sampling (LSS). Many authors have estimated the population mean using auxiliary variable information under LSS, e.g., [32–37].

When N is not a multiple of n i.e., $N \neq nk$, then LSS can not be used to estimate the unknown population mean. To overcome this limitation, another sampling scheme called Circular Systematic Sampling (CSS), proposed by [38], is used. CSS is applicable in both cases, when $N = nk$ and $N \neq nk$. A comparison between LSS and CSS is presented in Table 1.

Table 1. Comparison between LSS and CSS

Linear Systematic Sampling (LSS)	Circular Systematic Sampling (CSS)
Useful when $N = nk$	Useful for both $N = nk$ and $N \neq nk$
Variable sample size	Constant sample size
Sample mean is biased estimator of population mean	Sample mean is unbiased estimator of population mean

In CSS, all population units N are arranged around a circle, and the random start is chosen from 1 to N, unlike in LSS, where it is chosen from 1 to k. The value of k in CSS is taken as the nearest to $\frac{N}{n}$. The random start unit r is selected first, and then every k^{th} unit around the circle is chosen until the desired sample size is reached. Mathematically, the random start satisfies $1 \leq r \leq N$ where "r" is the random start, followed by the units corresponding to their serial numbers on the circle.

$$\begin{cases} r + jk, \text{ if } r + jk \leq N \\ r + jk - N, \text{ if } r + jk > N \end{cases}$$

where $j = 0,\ 1,\ 2,\ 3,\ \ldots,\ (n-1)$ will be selected in the sample. For example, $N = 14,\ n = 5$ then k will be 3 (nearest to N/n). Suppose 7 is the random start taken from $1 \leq r \leq 14$. (7, 10, 13, 2 and 5) units are selected as samples. This selection procedure can be depicted through the following Figure 1.

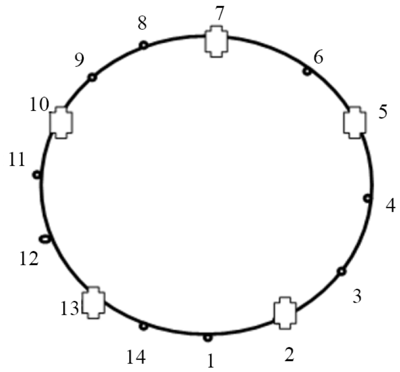

Figure 1. Representation of CSS scheme

The remaining sections are organized as follows: Section 2 presents the important notations. A review of the estimators, along with their expressions for bias and MSE, is provided in section 3. In section 4, we propose a generalized estimator for the estimation of the population mean under CSS, based on non-conventional measures. Mathematical expressions for the bias, MSE, and minimum MSE of the generalized estimator are derived. This generalized estimator can generate a series of estimators. In section 5, theoretical conditions are established under which the generalized estimator is more efficient than the competing estimators. A numerical illustration is presented in section 6, based on a real-life application. The numerical findings confirm the practical applicability of the generalized estimator in the presence of outliers. Finally, concluding remarks are provided in the last section.

2. Important notations

Let's consider a finite population $G = (g_1, g_2, \ldots, g_N)$ is consisting of N different units ranging from 1 to N in any sequence and N be a population size. Suppose (y, x) represent the study and auxiliary variables, respectively. Let, y_{ij} and x_{ij} where $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n$ be the values for j^{th} unit and the i^{th} sample for (y, x), respectively. k samples each of size n will be the total samples, and then select N circular systematic samples, each of size n from $1 \leq r \leq N$. To explain the procedure, Table 2 presents N circular systematic samples.

Table 2. All possible samples under CSS

Sample numbers	1	2	3	r	N
	g_1	g_2	g_3	g_r	g_N
	g_{k+1}	g_{k+2}	g_{k+3}	g_{k+r}	g_{2N}
		.			
		.			
		.			
	$g_{(n-1)k+1}$	$g_{(n-1)k+2}$	$g_{(n-1)k+3}$	$g_{(n-1)k+r}$	g_{nN}

Important measures related to study variable y and auxiliary variable x are described in Table 3.

Remark 1 ρ_y and ρ_x are the intraclass correlation coefficient between pairs of units within the CSS for study and auxiliary variables, respectively. Interaclass correlation coefficient is generally denoted by ρ_c. It lies between $-\dfrac{1}{(n-1)} \leq \rho_c \leq 1$.

Table 3. Measures related to study variable and auxiliary variable

Measures	Study variable	Auxiliary variable
Sample mean	$\bar{y}_{css} = n^{-1}\sum_{j=0}^{n-1} y_{r+jk}$	$\bar{x}_{css} = n^{-1}\sum_{j=0}^{n-1} x_{r+jk}$
Sample variance	$Var(\bar{y}_{css}) = \left(\frac{N-1}{N}\right)[1+(n-1)\rho_y]\frac{S_y^2}{n}$ $= \widetilde{S}_y^2 = \overline{Y}^2\widetilde{C}_y^2$ where $\overline{Y} = N^{-1}\sum_{i=1}^{N} Y_i$ $S_y^2 = \frac{1}{n(N-1)}\sum_{i=1}^{N}\sum_{j=1}^{n}\left(y_{ij}-\overline{Y}\right)^2$	$Var(\bar{x}_{css}) = \left(\frac{N-1}{N}\right)[1+(n-1)\rho_x]\frac{S_x^2}{n}$ $= \widetilde{S}_x^2 = \overline{X}^2\widetilde{C}_x^2$ where $\overline{X} = N^{-1}\sum_{i=1}^{N} X_i$ $S_x^2 = \frac{1}{n(N-1)}\sum_{i=1}^{N}\sum_{j=1}^{n}\left(x_{ij}-\overline{X}\right)^2$
Covariance	$Cov(\bar{y}_{css}, \bar{x}_{css}) = \left(\frac{N-1}{N}\right)[1+(n-1)\rho_y]^{1/2}[1+(n-1)\rho_x]^{1/2}\frac{S_{yx}}{n}$ $= \widetilde{S}_{yx} = \overline{YX}\widetilde{C}_{yx}$ where $S_{yx} = \frac{1}{n(N-1)}\sum_{i=1}^{N}\sum_{j=1}^{n}\left(y_{ij}-\overline{Y}\right)\left(x_{ij}-\overline{X}\right)$	
Intraclass correlation coefficient	$\rho_y = \frac{\sum_{i=1}^{N}\sum_{j\neq \acute{j}=1}^{n}\left(y_{ij}-\overline{Y}\right)\left(y_{i\acute{j}}-\overline{Y}\right)}{kn(n-1)S_y^2}$	$\rho_x = \frac{\sum_{i=1}^{N}\sum_{j\neq \acute{j}=1}^{n}\left(x_{ij}-\overline{X}\right)\left(x_{i\acute{j}}-\overline{X}\right)}{kn(n-1)S_x^2}$

Remark 2 To drive the expressions for the bias, MSE and minimum MSE of the existing and proposed estimators, we consider the following relative error terms along with their expectations as:

$$\zeta_0 = \frac{\bar{y}_{css}-\overline{Y}}{\overline{Y}} \text{ and } \zeta_1 = \frac{\bar{x}_{css}-\overline{X}}{\overline{X}}$$

Such that $E(\zeta_0) = E(\zeta_1) = 0$

$$E\left(\zeta_0^2\right) = \widetilde{C}_y^2,\ E\left(\zeta_1^2\right) = \widetilde{C}_x^2 \text{ and } E(\zeta_0\zeta_1) = \widetilde{C}_{yx}$$

Remark 3 Non-conventional measures related with auxiliary variable are:
Quartile deviation: $Q_D = \frac{Q_3-Q_1}{2}$
Mid-range: $M_R = \frac{x_{(1)}+x_{(N)}}{2}$
Tri-mean: $T_M = \frac{Q_1+2Q_2+Q_3}{4}$

Quartile average: $Q_A = \dfrac{Q_3 + Q_1}{2}$

Hodge-Lehmann estimator: $H_L = Median\left(\dfrac{x_j + x_k}{2}\right),\ 1 \le j \le k \le N.$

3. Reviewing estimators

The traditional ratio, product, and regression estimators to estimate $\overline{Y}$ under SRS scheme are:

$$\overline{y}_r = \overline{y}\left(\frac{\overline{X}}{\overline{x}}\right),\ \overline{x} \neq 0 \tag{1}$$

$$\overline{y}_p = \overline{y}\left(\frac{\overline{x}}{\overline{X}}\right) \tag{2}$$

$$\widehat{\overline{Y}}_{lr} = \overline{y} + b\left(\overline{X} - \overline{x}\right) \tag{3}$$

Expressions for MSE of the $\overline{y}$, $\overline{y}_r$, $\overline{y}_p$ and $\overline{y}_{lr}$ are respectively given by

$$MSE\left(\overline{y}\right) = V\left(\overline{y}\right) = \ \phi\overline{Y}^2 C_y^2 \tag{4}$$

$$MSE\left(\overline{y}_r\right) \cong \ \phi\overline{Y}^2\left[C_y^2 + C_x^2 - 2\rho_{yx}C_yC_x\right] \tag{5}$$

$$MSE\left(\overline{y}_p\right) \cong \ \phi\overline{Y}^2\left[C_y^2 + C_x^2 + 2\rho_{yx}C_yC_x\right] \tag{6}$$

$$MSE\left(\overline{y}_{lr}\right) \cong \ \phi\overline{Y}^2 C_y^2\left[1 - \rho_{yx}^2\right] \tag{7}$$

where $\phi = \left(\dfrac{1}{n} - \dfrac{1}{N}\right)$.

The classical ratio, product, and regression estimators in order to estimate unknown finite population mean $\overline{Y}$ under CSS scheme are as follows:

$$\overline{y}_{rc} = \overline{y}_{css}\left(\frac{\overline{X}}{\overline{x}_{css}}\right) \tag{8}$$

$$\overline{y}_{pc} = \overline{y}_{css}\left(\frac{\overline{x}_{css}}{\overline{X}}\right) \tag{9}$$

$$\overline{y}_{lrc} = \overline{y}_{css} + \widehat{\beta}_{yx}\left(\overline{X} - \overline{x}_{css}\right) \tag{10}$$

where $\widehat{\beta}_{yx} = \dfrac{s_{yx}}{s_x^2}$ represents an estimate of the population's regression coefficient β_{yx} with $s_{yx} = \dfrac{1}{n-1}\sum_{i=1}^{N}\sum_{j=1}^{n}(y_{ij} - \overline{y}_{css})$ $(x_{ij} - \overline{x}_{css})$ and $s_x^2 = \dfrac{1}{n-1}\sum_{i=1}^{N}\sum_{j=1}^{n}(x_{ij} - \overline{x}_{css})^2$.

To the first degree of approximation, the Mean Squared Error (MSE) of $\overline{y}_{rc}$, $\overline{y}_{pc}$ and $\overline{y}_{lrc}$ are given below:

$$MSE\left(\overline{y}_{rc}\right) = \overline{Y}^2\left[\widetilde{C}_y^2 + \widetilde{C}_x^2 - 2\widetilde{C}_{yx}\right] \tag{11}$$

$$MSE\left(\overline{y}_{pc}\right) = \overline{Y}^2\left[\widetilde{C}_y^2 + \widetilde{C}_x^2 + 2\widetilde{C}_{yx}\right] \tag{12}$$

$$MSE\left(\overline{y}_{lrc}\right) = \widetilde{S}_y^2\left(1 - \widetilde{\rho}_{yx}^2\right) \tag{13}$$

where $\widetilde{\rho}_{yx}^2 = \dfrac{\widetilde{S}_{yx}^2}{\widetilde{S}_y^2\widetilde{S}_x^2}$.

Remark 4 $\overline{y}_{lrc}$ is always better than $\overline{y}_{rc}$ and $\overline{y}_{pc}$, therefore we considered $\overline{y}_{lrc}$ as a standard for comparison against newly proposed estimators.

4. Suggested generalized estimator

One noticeable shortcoming of existing estimators is their poor performance in the presence of extreme observations. Singh and Solanki [39] proposed a class of estimators for estimating the unknown population mean under simple random sampling without replacement. Inspired by their work, we propose a generalized estimator for estimating the population mean under the CSS scheme, using non-conventional measures related to the auxiliary variable. The proposed generalized estimator is defined as:

$$L = t_1\overline{y}_{css}\left[\frac{\overline{X}^*}{\overline{x}_{css}^*}\right]^g + t_2\overline{y}_{css}\exp\left[\frac{\delta(\overline{X}^* - \overline{x}_{css}^*)}{(\overline{X}^* + \overline{x}_{css}^*)}\right] \tag{14}$$

where $\overline{X}^* = r\overline{X} + s$ and $\overline{x}_{css}^* = r\overline{x}_{css} + s$ and $r \neq 0$ & s both are either real numbers or functions of the known parameters of the auxiliary variable based on non-conventional measures such as quartile deviation, mid-range, tri-mean, quartile average, and Hodge-Lehmann estimator. $(\delta,\ g)$ are being constant whose values are $(0,\ 1,\ -1)$ to search optimal estimators and $(t_1,\ t_2)$ are the suitable weights.

Expressing generalized estimator "L" in terms of relative error terms, we have

$$L = t_1\overline{Y}(1+\zeta_0)\left[\frac{r\overline{X} + s}{r\overline{X}(1+\zeta_1) + s}\right]^g + t_2\overline{Y}(1+\zeta_0)\exp\left[\frac{-\delta r\overline{X}\zeta_1}{2r\overline{X} + r\overline{X}\zeta_1 + 2s}\right]$$

$$L = t_1\overline{Y}(1+\zeta_0)\left[\frac{(r\overline{X}+s)}{(r\overline{X}+s)\left(1 + \dfrac{r\overline{X}\zeta_1}{r\overline{X}+s}\right)}\right]^g + t_2\overline{Y}(1+\zeta_0)\exp\left[\frac{-\delta r\overline{X}\zeta_1}{(2r\overline{X}+2s)\left(1 + \dfrac{r\overline{X}\zeta_1}{2r\overline{X}+2s}\right)}\right]$$

$$L = t_1\overline{Y}(1+\zeta_0)\left[\frac{1}{1+v\zeta_1}\right]^g + t_2\overline{Y}(1+\zeta_0)\exp\left[\frac{-\delta v\zeta_1}{2\left(1+\frac{v\zeta_1}{2}\right)}\right]$$

where $v = \dfrac{r\overline{X}}{r\overline{X}+s}$

$$L = t_1\overline{Y}(1+\zeta_0)(1+v\zeta_1)^{-g} + t_2\overline{Y}(1+\zeta_0)\exp\left\{-\frac{\delta v\zeta_1}{2}(1+\frac{v\zeta_1}{2})^{-1}\right\}$$

$$L = t_1\overline{Y}(1+\zeta_0)\left(1-gv\zeta_1+\frac{g(g+1)}{2!}v^2\zeta_1^2\right) + t_2\overline{Y}(1+\zeta_0)\exp\left\{-\frac{\delta v\zeta_1}{2}\left(1-\frac{v\zeta_1}{2}+\frac{v^2\zeta_1^2}{4}\right)\right\}$$

After some simplify up to first order of approximation, we obtain

$$L = t_1\overline{Y}\left(1+\zeta_0-gv\zeta_1-gv\zeta_0\zeta_1+\frac{g(g+1)}{2}v^2\zeta_1^2\right) + t_2\overline{Y}(1+\zeta_0)\exp\left\{-\frac{\delta v\zeta_1}{2}+\frac{\delta v^2\zeta_1^2}{4}\right\}$$

Expanding $\exp\left\{-\frac{\delta v\zeta_1}{2}+\frac{\delta v^2\zeta_1^2}{4}\right\}$ up to first order of approximation, we get

$$L = t_1\overline{Y}\left(1+\zeta_0-gv\zeta_1-gv\zeta_0\zeta_1+\frac{g(g+1)}{2}v^2\zeta_1^2\right) + t_2\overline{Y}(1+\zeta_0)\left(1-\frac{\delta v\zeta_1}{2}+\frac{\delta v^2\zeta_1^2}{4}+\frac{\delta^2 v^2\zeta_1^2}{4}\right) \quad (15)$$

Subtract $\overline{Y}$ on both side of Equation (15), we get

$$L-\overline{Y} = \overline{Y}\left[t_1\left\{1+\zeta_0-vg(\zeta_1+\zeta_0\zeta_1)+\frac{g(g+1)}{2}v^2\zeta_1^2\right\} + t_2\left\{1+\zeta_0-\frac{\delta v}{2}(\zeta_1+\zeta_0\zeta_1)+\frac{\delta(\delta+2)}{8}v^2\zeta_1^2\right\}-1\right] \quad (16)$$

Bias of generalized estimator "L" up to the first order of approximation is defined as:

$$Bias(L) \cong E\left(L-\overline{Y}\right)$$

Taking expectation on both side of the Equation (16),

$$\mathrm{E}(L-\bar{Y}) = \bar{Y}\left[t_1\left\{1+\mathrm{E}(\zeta_0)-vg(E(\zeta_1)+\mathrm{E}(\zeta_0\zeta_1))+\frac{g(g+1)}{2}v^2E\left(\zeta_1^2\right)\right\}\right.$$

$$\left.+t_2\left\{1+\mathrm{E}(\zeta_0)-\frac{\delta v}{2}(E(\zeta_1)+\mathrm{E}(\zeta_0\zeta_1))+\frac{\delta(\delta+2)}{8}v^2E\left(\zeta_1^2\right)\right\}-1\right]$$

Bias of generalized estimator "L" can be seen in Equation (17)

$$Bias(L) \cong \overline{Y}\left[t_1\left\{1-vg\widetilde{C}_{yx}+\frac{g(g+1)}{2}v^2\widetilde{C}_x^2\right\}+t_2\left\{1-\frac{\delta v}{2}\widetilde{C}_{yx}+\frac{\delta(\delta+2)}{8}v^2\widetilde{C}_x^2\right\}-1\right] \tag{17}$$

MSE of generalized estimator "L" up to the first order of approximation is defined as:

$$MSE(L) \cong E(L-\overline{Y})^2$$

Taking square on both sides of Equation (16), we get

$$\begin{aligned}(L-\overline{Y})^2 \cong \overline{Y}^2 \Big[1 &+ t_1^2\left\{1+2\zeta_0-2vg\zeta_1+\zeta_0^2-4vg\zeta_0\zeta_1+v^2g(2g+1)\zeta_1^2\right\}\\ &+ t_2^2\left\{1+2\zeta_0-v\delta\zeta_1+\zeta_0^2-2v\delta\zeta_0\zeta_1+\frac{\delta(\delta+2)}{4}v^2\zeta_1^2+\frac{\delta^2v^2\zeta_1^2}{4}\right\}\\ &+2t_1t_2\left\{1+2\zeta_0-\frac{v(2g+\delta)}{2}\zeta_1-v(2g+\delta)\zeta_0\zeta_1+\zeta_0^2+\frac{D^*v^2}{8}\zeta_1^2\right\}\\ &-2t_1\left\{1+\zeta_0-vg\left(\zeta_1+\zeta_0\zeta_1\right)+\frac{g(g+1)}{2}v^2\zeta_1^2\right\}\\ &-2t_2\left\{1+\zeta_0-\frac{\delta v}{2}\left(\zeta_1+\zeta_0\zeta_1\right)+\frac{\delta(\delta+2)}{8}v^2\zeta_1^2\right\}\Big]\end{aligned} \tag{18}$$

where $D^* = \left[(2g+\delta)^2+2(2g+\delta)\right]$.

Taking expectation on both sides of Equation (18),

$$\begin{aligned}E(L-\overline{Y})^2 \cong \overline{Y}^2 \Big[1 &+ t_1^2\left\{1+2E(\zeta_0)-2vgE(\zeta_1)+E(\zeta_0^2)-4vgE(\zeta_0\zeta_1)+v^2g(2g+1)E(\zeta_1^2)\right\}\\ &+t_2^2\left\{1+2E(\zeta_0)-v\delta E(\zeta_1)+E(\zeta_0^2)-2v\delta E(\zeta_0\zeta_1)+\frac{\delta(\delta+2)}{4}v^2E(\zeta_1^2)+\frac{\delta^2v^2}{4}E(\zeta_1^2)\right\}\\ &+2t_1t_2\left\{1+2E(\zeta_0)-\frac{v(2g+\delta)}{2}E(\zeta_1)-v(2g+\delta)E(\zeta_0\zeta_1)+E(\zeta_0^2)+\frac{D^*v^2}{8}E(\zeta_1^2)\right\}\\ &-2t_1\left\{1+E(\zeta_0)-vg\left(E\left(\zeta_1\right)+E\left(\zeta_0\zeta_1\right)\right)+\frac{g(g+1)}{2}v^2E(\zeta_1^2)\right\}\end{aligned}$$

$$-2t_2\left\{1+E(\zeta_0)-\frac{\delta v}{2}\left(E\left(\zeta_1\right)+E\left(\zeta_0\zeta_1\right)\right)+\frac{\delta(\delta+2)}{8}v^2E(\zeta_1^2)\right\}\Bigg]$$

MSE of the generalized estimators "L" can be seen in Equation (19):

$$MSE(L)\cong\overline{Y}^2\left[1+t_1^2\partial_A+t_2^2\partial_C+2t_1t_2\partial_D-2t_1\partial_B-2t_2\partial_E\right] \tag{19}$$

where

$$\partial_A=\left[1+\widetilde{C}_y^2-4vg\widetilde{C}_{yx}+v^2g(2g+1)\widetilde{C}_x^2\right]$$

$$\partial_B=\left[1-vg\widetilde{C}_{yx}+\frac{v^2g(g+1)}{2}\widetilde{C}_x^2\right]$$

$$\partial_C=\left[1+\widetilde{C}_y^2-2v\delta\widetilde{C}_{yx}+\frac{v^2\delta(\delta+1)}{2}\widetilde{C}_x^2\right]$$

$$\partial_D=\left[1+\widetilde{C}_y^2-v(2g+\delta)\widetilde{C}_{yx}+\frac{D^*v^2}{8}\widetilde{C}_x^2\right]$$

$$\partial_E=\left[1-\frac{v\delta}{2}\widetilde{C}_{yx}+\frac{\delta(\delta+2)v^2}{8}\widetilde{C}_x^2\right]$$

To get the optimal values of t_1 and t_2, differentiating Equation (19) w.r.t. t_1 and t_2 and equate to zero, we have

$$t_1=\frac{\partial_B\partial_C-\partial_D\partial_E}{\partial_A\partial_C-\partial_D^2} \tag{20}$$

$$t_2=\frac{\partial_A\partial_E-\partial_B\partial_D}{\partial_A\partial_C-\partial_D^2} \tag{21}$$

After putting optimal values of t_1 and t_2 in Equation (19), we get minimum MSE of the generalized estimators "L" in Equation (22).

$$MSE_{\min}(L)\cong\overline{Y}^2\left[1-\frac{\partial_B^2\partial_C+\partial_A\partial_E^2-2\partial_B\partial_D\partial_E}{\partial_A\partial_C-\partial_D^2}\right] \tag{22}$$

Remark 5 Specifying known population parameters of the auxiliary variable based on non-conventional measures in place of r and s in Equation (14), we can get many optimal estimators. Some of them used in this study are presented in Table 4.

Table 4. Different choices of constants

r	H_L	H_L	H_L	T_M	T_M	Q_D	Q_A	M_R
s	T_M	Q_D	M_R	Q_D	M_R	M_R	T_M	Q_A

Table 5. Some combinations of proposed class of estimators "L"

Ratio-type estimator (g, δ) = (1, 0)	Product-type estimator (g, δ) = (-1, 0)
$L_{r1} = t_1\bar{y}_{css}\left(\frac{H_L\bar{X}+T_M}{H_L\bar{x}_{css}+T_M}\right)+t_2\bar{y}_{css}$	$L_{p1} = t_1\bar{y}_{css}\left(\frac{H_L\bar{x}_{css}+T_M}{H_L\bar{X}+T_M}\right)+t_2\bar{y}_{css}$
$L_{r2} = t_1\bar{y}_{css}\left(\frac{H_L\bar{X}+Q_D}{\bar{x}_{css}+Q_D}\right)+t_2\bar{y}_{css}$	$L_{p2} = t_1\bar{y}_{css}\left(\frac{H_L\bar{x}_{css}+Q_D}{H_L\bar{X}+Q_D}\right)+t_2\bar{y}_{css}$
$L_{r3} = t_1\bar{y}_{css}\left(\frac{H_L\bar{X}+M_R}{H_L\bar{x}_{css}+M_R}\right)+t_2\bar{y}_{css}$	$L_{p3} = t_1\bar{y}_{css}\left(\frac{H_L\bar{x}_{css}+M_R}{H_L\bar{X}+M_R}\right)+t_2\bar{y}_{css}$
$L_{r4} = t_1\bar{y}_{css}\left(\frac{T_M\bar{X}+Q_D}{T_M\bar{x}_{css}+Q_D}\right)+t_2\bar{y}_{css}$	$L_{p4} = t_1\bar{y}_{css}\left(\frac{T_M\bar{x}_{css}+Q_D}{T_M\bar{X}+Q_D}\right)+t_2\bar{y}_{css}$
$L_{r5} = t_1\bar{y}_{css}\left(\frac{T_M\bar{X}+M_R}{T_M\bar{x}_{css}+M_R}\right)+t_2\bar{y}_{css}$	$L_{p5} = t_1\bar{y}_{css}\left(\frac{T_M\bar{x}_{css}+M_R}{T_M\bar{X}+M_R}\right)+t_2\bar{y}_{css}$
$L_{r6} = t_1\bar{y}_{css}\left(\frac{Q_D\bar{X}+M_R}{Q_D\bar{x}_{css}+M_R}\right)+t_2\bar{y}_{css}$	$L_{p6} = t_1\bar{y}_{css}\left(\frac{Q_D\bar{x}_{css}+M_R}{Q_D\bar{X}+M_R}\right)+t_2\bar{y}_{css}$
$L_{r7} = t_1\bar{y}_{css}\left(\frac{Q_A\bar{X}+T_M}{Q_A\bar{x}_{css}+T_M}\right)+t_2\bar{y}_{css}$	$L_{p7} = t_1\bar{y}_{css}\left(\frac{Q_A\bar{x}_{css}+T_M}{Q_A\bar{X}+T_M}\right)+t_2\bar{y}_{css}$
$L_{r8} = t_1\bar{y}_{css}\left(\frac{M_R\bar{X}+Q_A}{M_R\bar{x}_{css}+Q_A}\right)+t_2\bar{y}_{css}$	$L_{p8} = t_1\bar{y}_{css}\left(\frac{M_R\bar{x}_{css}+Q_A}{M_R\bar{X}+Q_A}\right)+t_2\bar{y}_{css}$

Table 6. Some combinations of proposed class of estimators "L"

Ratio-ratio-type exponential estimator (g, δ) = (1, 1)	Ratio-product-type exponential estimator (g, δ) = (1, -1)
$L_{rre1} = t_1\bar{y}_{css}\left(\frac{H_L\bar{X}+T_M}{H_L\bar{x}_{css}+T_M}\right)+t_2\bar{y}_{css}\exp\left[\frac{H_L\left(\bar{X}-\bar{x}_{css}\right)}{H_L\left(\bar{X}+\bar{x}_{css}\right)}\right]$	$L_{rpe1} = t_1\bar{y}_{css}\left(\frac{H_L\bar{X}}{H_L\bar{x}_{css}}\right)+t_2\bar{y}_{css}\exp\left[\frac{H_L\left(\bar{x}_{css}-\bar{X}\right)}{H_L\left(\bar{X}+\bar{x}_{css}\right)}\right]$
$L_{rre2} = t_1\bar{y}_{css}\left(\frac{H_L\bar{X}+Q_D}{H_L\bar{x}_{css}+Q_D}\right)+t_2\bar{y}_{css}\exp\left[\frac{H_L\left(\bar{X}-\bar{x}_{css}\right)}{H_L\left(\bar{X}+\bar{x}_{css}\right)+2Q_D}\right]$	$L_{rpe2} = t_1\bar{y}_{css}\left(\frac{H_L\bar{X}+C_x}{\bar{x}_{css}+C_x}\right)+t_2\bar{y}_{css}\exp\left[\frac{H_L\left(\bar{x}_{css}-\bar{X}\right)}{H_L\left(\bar{X}+\bar{x}_{css}\right)+2C_x}\right]$
$L_{rre3} = t_1\bar{y}_{css}\left(\frac{H_L\bar{X}+M_R}{H_L\bar{x}_{css}+M_R}\right)+t_2\bar{y}_{css}\exp\left[\frac{H_L\left(\bar{X}-\bar{x}_{css}\right)}{H_L\left(\bar{X}+\bar{x}_{css}\right)+2M_R}\right]$	$L_{rpe3} = t_1\bar{y}_{css}\left(\frac{H_L\bar{X}+M_R}{H_L\bar{x}_{css}+M_R}\right)+t_2\bar{y}_{css}\exp\left[\frac{H_L\left(\bar{x}_{css}-\bar{X}\right)}{H_L\left(\bar{X}+\bar{x}_{css}\right)+2M_R}\right]$
$L_{rre4} = t_1\bar{y}_{css}\left(\frac{T_M\bar{X}+Q_D}{T_M\bar{x}_{css}+Q_D}\right)+t_2\bar{y}_{css}\exp\left[\frac{T_M\left(\bar{X}-\bar{x}_{css}\right)}{T_M\left(\bar{X}+\bar{x}_{css}\right)+2Q_D}\right]$	$L_{rpe4} = t_1\bar{y}_{css}\left(\frac{\beta_1(x)\bar{X}+S_x}{\beta_1(x)\bar{x}_{css}+S_x}\right)+t_2\bar{y}_{css}\exp\left[\frac{\beta_1(x)\left(\bar{x}_{css}-\bar{X}\right)}{\beta_1(x)\left(\bar{X}+\bar{x}_{css}\right)+2S_x}\right]$
$L_{rre5} = t_1\bar{y}_{css}\left(\frac{T_M\bar{X}+M_R}{T_M\bar{x}_{css}+M_R}\right)+t_2\bar{y}_{css}\exp\left[\frac{T_M\left(\bar{X}-\bar{x}_{css}\right)}{T_M\left(\bar{X}+\bar{x}_{css}\right)+2M_R}\right]$	$L_{rpe5} = t_1\bar{y}_{css}\left(\frac{\beta_2(x)\bar{X}+M_R}{\beta_2(x)\bar{x}_{css}+M_R}\right)+t_2\bar{y}_{css}\exp\left[\frac{\beta_2(x)\left(\bar{x}_{css}-\bar{X}\right)}{\beta_2(x)\left(\bar{X}+\bar{x}_{css}\right)+2M_R}\right]$
$L_{rre6} = t_1\bar{y}_{css}\left(\frac{Q_D\bar{X}+M_R}{Q_D\bar{x}_{css}+M_R}\right)+t_2\bar{y}_{css}\exp\left[\frac{Q_D\left(\bar{X}-\bar{x}_{css}\right)}{Q_D\left(\bar{X}+\bar{x}_{css}\right)+2M_R}\right]$	$L_{rpe6} = t_1\bar{y}_{css}\left(\frac{\bar{X}+M_R}{\bar{x}_{css}+M_R}\right)+t_2\bar{y}_{css}\exp\left[\frac{\left(\bar{x}_{css}-\bar{X}\right)}{\left(\bar{X}+\bar{x}_{css}\right)+2M_R}\right]$
$L_{rre7} = t_1\bar{y}_{css}\left(\frac{Q_A\bar{X}+T_M}{Q_A\bar{x}_{css}+T_M}\right)+t_2\bar{y}_{css}\exp\left[\frac{Q_A\left(\bar{X}-\bar{x}_{css}\right)}{Q_A\left(\bar{X}+\bar{x}_{css}\right)+2T_M}\right]$	$L_{rpe7} = t_1\bar{y}_{css}\left(\frac{Q_A\bar{X}+T_M}{Q_A\bar{x}_{css}+T_M}\right)+t_2\bar{y}_{css}\exp\left[\frac{Q_A\left(\bar{x}_{css}-\bar{X}\right)}{Q_A\left(\bar{X}+\bar{x}_{css}\right)+2T_M}\right]$
$L_{rre8} = t_1\bar{y}_{css}\left(\frac{M_R\bar{X}+Q_A}{M_R\bar{x}_{css}+Q_A}\right)+t_2\bar{y}_{css}\exp\left[\frac{M_R\left(\bar{X}-\bar{x}_{css}\right)}{M_R\left(\bar{X}+\bar{x}_{css}\right)+2Q_A}\right]$	$L_{rpe8} = t_1\bar{y}_{css}\left(\frac{M_R\bar{X}+Q_A}{M_R\bar{x}_{css}+Q_A}\right)+t_2\bar{y}_{css}\exp\left[\frac{M_R\left(\bar{x}_{css}-\bar{X}\right)}{M_R\left(\bar{X}+\bar{x}_{css}\right)+2Q_A}\right]$

Remark 6 It is worth mentioning that the population parameters appearing in Equation (14) are mostly unknown. However, unknown population parameters can be estimated through the use of preliminary data sets or repeated surveys. Several researchers have discussed the use of pilot/prior information on parameters at the estimation stage.

Remark 7 Generalized estimator "L" can generate series of estimators such as ratio-type, product-type, ratio-ratio-type exponential, and ratio-product-type exponential. All of them are presented in Table 5 and Table 6.

Remark 8 The proposed generalized estimator "L" defined in Equation (14) can be treated as a special case when $t_1 + t_2 = 1$ up to terms of the order n^{-1}:

$$MSE_{\min}(L^*) = \overline{Y}^2 \left[1 - \frac{\partial_B^2 \partial_C - 2\partial_B \partial_D \partial_E + \partial_A \partial_E^2}{\partial_A \partial_C - \partial_D^2}\right]$$

with equality holding if $t_1 = t^*_{1\,(opt)}$.

Hence, the proposed generalized estimator "L" defined in Equation (14) can be written as:

$$L^* = t_1 \overline{y}_{css} \left[\frac{\overline{X}^*}{x^*_{css}}\right]^g + (1 - t_1)\overline{y}_{css} \exp\left[\frac{\delta(\overline{X}^* - x^*_{css})}{(\overline{X}^* + x^*_{css})}\right] \tag{23}$$

where $\overline{X}^* = r\overline{X} + s$ and $\overline{x}^*_{css} = r\overline{x}_{css} + s$.

Bias and MSE of the generalized estimator L^* are as follows:

$$Bias\ (L^*) = \overline{Y}\left(\frac{v}{8}\right)[\delta\{v(\delta+2) - 4k\} + t_1\{4g(v(g+1) - 2k) - \delta(v(\delta+2) - 4k)\}]\widetilde{C}_x^2 \tag{24}$$

where $k = \widetilde{\rho}_{yx}\left(\widetilde{C}_y / \widetilde{C}_x\right)$.

$$MSE\,(L^*) = \overline{Y}^2\left[1 + \partial_C - 2\partial_E + t_1^2(\partial_A + \partial_C - 2\partial_D) - 2t_1(\partial_C - \partial_D + \partial_B - \partial_E)\right] \tag{25}$$

The optimal value of t_1 is obtained as:

$$t_1 = \frac{\partial_C - \partial_D + \partial_B - \partial_E}{\partial_A + \partial_C - 2\partial_D} = t^*_{1\,(opt)} \tag{26}$$

We get minimum MSE of L^* after substituting the optimal value of $t^*_{1\,(opt)}$ in Equation (25)

$$MSE_{\min}(L^*) = \overline{Y}^2\left[1 + \partial_C - 2\partial_E - \frac{(\partial_C - \partial_D + \partial_B - \partial_E)^2}{\partial_A + \partial_C - 2\partial_D}\right] = \widetilde{S}_y^2\left(1 - \widetilde{\rho}_{yx}^2\right) = MSE\,(\overline{y}_{lrc}) \tag{27}$$

5. Efficiency comparison

In this section, theoretical conditions have been proposed that indicate the supremacy of the proposed generalized estimator over the ratio, product, and regression estimators under CSS.

1. By comparing Equation (8) and Equation (22).

$$MSE\left(\bar{y}_{rc}\right) - MSE_{\min}(L) \geq 0$$

$$\bar{Y}^2\left[\widetilde{C}_y^2 + \widetilde{C}_x^2 - 2\widetilde{C}_{yx}^2\right] - \bar{Y}^2\left[1 - \frac{\partial_B^2\partial_C - 2\partial_B\partial_D\partial_E + \partial_A\partial_E^2}{\partial_A\partial_C - \partial_D^2}\right] \geq 0$$

$$\left[1 - \frac{\partial_B^2\partial_C - 2\partial_B\partial_D\partial_E + \partial_A\partial_E^2}{\partial_A\partial_C - \partial_D^2}\right] \leq \left[\widetilde{C}_y^2 + \widetilde{C}_x^2 - 2\widetilde{C}_{yx}^2\right] \tag{28}$$

2. By comparing Equation (9) and Equation (22).

$$MSE\left(\bar{y}_{pc}\right) - MSE_{\min}(L) \geq 0$$

$$\bar{Y}^2\left[\widetilde{C}_y^2 + \widetilde{C}_x^2 - 2\widetilde{C}_{yx}^2\right] - \bar{Y}^2\left[1 - \frac{\partial_B^2\partial_C - 2\partial_B\partial_D\partial_E + \partial_A\partial_E^2}{\partial_A\partial_C - \partial_D^2}\right] \geq 0$$

$$\left[1 - \frac{\partial_B^2\partial_C - 2\partial_B\partial_D\partial_E + \partial_A\partial_E^2}{\partial_A\partial_C - \partial_D^2}\right] \leq \left[\widetilde{C}_y^2 + \widetilde{C}_x^2 + 2\widetilde{C}_{yx}^2\right] \tag{29}$$

3. By comparing Equation (10) and Equation (22).

$$MSE\left(\bar{y}_{lrc}\right) - MSE_{\min}(L) \geq 0$$

$$\widetilde{S}_y^2\left(1 - \widetilde{\rho}_{yx}^2\right) - \bar{Y}^2\left[1 - \frac{\partial_B^2\partial_C - 2\partial_B\partial_D\partial_E + \partial_A\partial_E^2}{\partial_A\partial_C - \partial_D^2}\right] \geq 0$$

$$\bar{Y}^2\left[1 - \frac{\partial_B^2\partial_C - 2\partial_B\partial_D\partial_E + \partial_A\partial_E^2}{\partial_A\partial_C - \partial_D^2}\right] \leq \widetilde{S}_y^2\left(1 - \widetilde{\rho}_{yx}^2\right) \tag{30}$$

When condition Equation (28) or Equation (29) or Equation (30) is satisfied, then the proposed estimators i.e., L_r, L_p, L_{rre}, L_{rpe} are more efficient over the traditional ratio estimator i.e., $\bar{y}_{rc}$, product estimator i.e., $\bar{y}_{pc}$ and regression estimator i.e., $\bar{y}_{lrc}$.

6. Empirical study

In this section, we evaluated the proficiency of the proposed generalized estimator "L" over the traditional estimators under SRS without replacement and CSS. We used a real-life data set for 923 districts of Turkey (Source: Ministry of Education, Republic of Turkey). We considered the number of teachers in both primary and secondary schools as a study variable and the number of students in both primary and secondary schools as an auxiliary variable. Figure 2, illustrates the scatter plot of the data set to check the outliers in the real-life data set. It can be seen clearly from Figure 2. Some potential outliers exist in the real-life data. Histogram is also created to assess the normality of the data, which clearly indicate that dataset is non-normal i.e. right skewed. Due to space constraints, sketch has not been provided here. The efficiency of the estimators is uncertain if we use conventional measures of the auxiliary variable. Therefore, we used non-conventional measures related to the auxiliary variable.

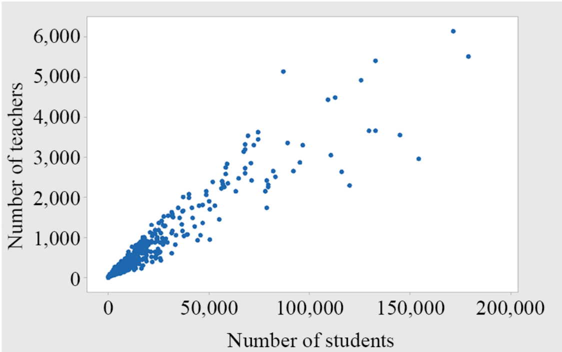

Figure 2. Scatter plot

Important parameters related to the study and auxiliary variables are as follows:

$$N = 923,\ \overline{X} = 11,440.5,\ \overline{Y} = 436.4,\ S_y = 749.94,\ S_x = 21,331.13,$$

$$\rho_y = -0.00255,\ \rho_x = -0.00316,\ C_x = 1.864,\ C_y = 1.718,\ \rho_{xy} = 0.954,$$

$$H_L = 5,407.5,\ T_M = 4,988.625,\ Q_D = 4,120.75,\ Q_A = 5,854.25,\ M_R = 89,530.5.$$

We used different sample sizes i.e., n = 180, 190, and 200 for the data set under study. The Mean Squared Error (MSE) values for the proposed and existing estimators were computed using R software. MSE of traditional estimators under SRS without replacement i.e., $\overline{y}$, $\overline{y}_r$, $\overline{y}_p$, $\overline{y}_{lr}$ and CSS i.e., $\overline{y}_{rc}$, $\overline{y}_{pc}$ and $\overline{y}_{lrc}$ can be seen in Table 7. Procedure to compute the minimum MSE of all the estimators:

1. Select N circular systematic samples, each of size n, for values of r such that $1 \leq r \leq N$.
2. Compute the minimum MSE for each estimator using the samples from Step 1.
3. Repeat Steps 1 and 2 a total of 10,000 times.
4. Obtain 10,000 minimum MSE values for each estimator.
5. Calculate the average of these 10,000 MSE values to determine the overall minimum MSE for each estimator using the following equation:

$$MSE(t) = \frac{\sum_{i=1}^{10,000}\left(t - \overline{Y}\right)^2}{10,000}.\ where\ t = \overline{y},\ \overline{y}_r,\ \overline{y}_p,\ \overline{y}_{lr},\ \overline{y}_{rc},\ \overline{y}_{pc},\ \overline{y}_{lrc},\ L_r,\ L_p,\ L_{rre},\ L_{rpe}. \tag{31}$$

In addition, Percent Relative Efficiency (PRE) is calculated to check the proficiency of our proposed estimators over the traditional estimators under SRS without replacement and CSS. PRE can be calculated through Equation (32) as:

$$PRE = \frac{MSE(i)}{MSE(j)} \times 100 \tag{32}$$

where $i = \bar{y}, \bar{y}_r, \bar{y}_p, \bar{y}_{lr}, \bar{y}_{rc}, \bar{y}_{pc}, \bar{y}_{lrc}$ and $j = L_r, L_p, L_{rre}, L_{rpe}$

Table 7. MSE of traditional estimators

Estimators under SRS	Sample size	$\bar{y}$	$\bar{y}_r$	$\bar{y}_p$	$\bar{y}_{lr}$
MSE	$n = 180$	2,515.171	267.637	10,685.190	224.634
	$n = 190$	2,350.724	250.139	9,986.566	209.946
	$n = 200$	2,202.721	234.389	9,357.807	196.728
Estimators under CSS	**Sample size**	$\bar{y}_c$	$\bar{y}_{rc}$	$\bar{y}_{pc}$	$\bar{y}_{lrc}$
MSE	$n = 180$	1,696.482	151.932	6,433.330	151.516
	$n = 190$	1,531.794	136.816	5,731.036	136.807
	$n = 200$	1,383.575	123.781	5,048.401	123.569

Table 8. PRE's of proposed estimators with respect to competing estimators

Proposed estimators	Sample size	Existing estimators							
		$\bar{y}$	$\bar{y}_r$	$\bar{y}_p$	$\bar{y}_{lr}$	$\bar{y}_c$	$\bar{y}_{rc}$	$\bar{y}_{pc}$	$\bar{y}_{lrc}$
L_{r1}	$n = 180$	1,661.860	176.837	7,060.060	148.423	1,120.920	100.387	4,250.720	100.112
	$n = 190$	1,719.685	182.991	7,305.729	153.587	1,120.593	100.089	4,192.572	100.082
	$n = 200$	1,783.565	189.787	7,577.110	159.293	1,120.295	100.227	4,087.741	100.055
L_{r2}	$n = 180$	1,661.923	176.844	7,060.341	148.429	1,120.967	100.391	4,250.884	100.116
	$n = 190$	1,719.685	182.991	7,305.729	153.587	1,120.593	100.089	4,192.572	100.082
	$n = 200$	1,783.565	189.787	7,577.110	159.293	1,120.295	100.227	4,087.741	100.055
L_{r3}	$n = 180$	1,661.967	176.848	7,060.527	148.433	1,120.996	100.393	4,250.996	100.118
	$n = 190$	1,719.723	182.995	7,305.889	153.591	1,120.617	100.091	4,192.664	100.084
	$n = 200$	1,783.594	189.790	7,577.233	159.295	1,120.313	100.228	4,087.807	100.057
L_{r4}	$n = 180$	1,661.923	176.844	7,060.341	148.429	1,120.967	100.391	4,250.884	100.116
	$n = 190$	1,719.723	182.995	7,305.889	153.591	1,120.617	100.091	4,192.664	100.084
	$n = 200$	1,783.565	189.787	7,577.110	159.293	1,120.295	100.227	4,087.741	100.055
L_{r5}	$n = 180$	1,661.967	176.848	7,060.527	148.433	1,120.996	100.393	4,250.996	100.118
	$n = 190$	1,719.736	182.996	7,305.943	153.592	1,120.625	100.091	4,192.694	100.085
	$n = 200$	1,783.594	189.790	7,577.233	159.295	1,120.313	100.228	4,087.807	100.057
L_{r6}	$n = 180$	1,661.978	176.850	7,060.574	148.434	1,121.004	100.394	4,251.024	100.119
	$n = 190$	1,719.736	182.996	7,305.943	153.592	1,120.625	100.091	4,192.694	100.085
	$n = 200$	1,783.609	189.792	7,577.294	159.297	1,120.322	100.229	4,087.840	100.057
L_{r7}	$n = 180$	1,661.923	176.844	7,060.341	148.429	1,120.967	100.391	4,250.884	100.116
	$n = 190$	1,719.685	182.991	7,305.729	153.587	1,120.593	100.089	4,192.572	100.082
	$n = 200$	1,783.565	189.787	7,577.110	159.293	1,120.295	100.227	4,087.741	100.055
L_{r8}	$n = 180$	1,661.912	176.843	7,060.294	148.428	1,120.959	100.390	4,250.856	100.115
	$n = 190$	1,719.685	182.991	7,305.729	153.587	1,120.593	100.089	4,192.572	100.082
	$n = 200$	1,783.565	189.787	7,577.110	159.293	1,120.295	100.227	4,087.741	100.055

Table 9. PRE's of proposed estimators with respect to competing estimators

Proposed estimators	Sample size	Existing estimators							
		$\bar{y}$	$\bar{y}_r$	$\bar{y}_p$	$\bar{y}_{lr}$	$\bar{y}_c$	$\bar{y}_{rc}$	$\bar{y}_{pc}$	$\bar{y}_{lrc}$
L_{p1}	$n = 180$	1,782.962	189.723	7,574.550	159.239	1,202.607	107.702	4,560.478	107.407
	$n = 190$	1,832.109	194.954	7,783.337	163.628	1,193.851	106.632	4,466.659	106.625
	$n = 200$	1,888.349	200.937	8,022.261	168.651	1,186.111	106.115	4,327.893	105.933
L_{p2}	$n = 180$	1,782.962	189.723	7,574.550	159.239	1,202.607	107.702	4,560.478	107.407
	$n = 190$	1,832.109	194.954	7,783.337	163.628	1,193.851	106.632	4,466.659	106.625
	$n = 200$	1,888.349	200.937	8,022.261	168.651	1,186.111	106.115	4,327.893	105.933
L_{p3}	$n = 180$	1,782.962	189.723	7,574.550	159.239	1,202.607	107.702	4,560.478	107.407
	$n = 190$	1,832.109	194.954	7,783.337	163.628	1,193.851	106.632	4,466.659	106.625
	$n = 200$	1,888.349	200.937	8,022.261	168.651	1,186.111	106.115	4,327.893	105.933
L_{p4}	$n = 180$	1,782.962	189.723	7,574.550	159.239	1,202.607	107.702	4,560.478	107.407
	$n = 190$	1,832.109	194.954	7,783.337	163.628	1,193.851	106.632	4,466.659	106.625
	$n = 200$	1,888.349	200.937	8,022.261	168.651	1,186.111	106.115	4,327.893	105.933
L_{p5}	$n = 180$	1,782.962	189.723	7,574.550	159.239	1,202.607	107.702	4,560.478	107.407
	$n = 190$	1,832.109	194.954	7,783.337	163.628	1,193.851	106.632	4,466.659	106.625
	$n = 200$	1,888.349	200.937	8,022.261	168.651	1,186.111	106.115	4,327.893	105.933
L_{p6}	$n = 180$	1,782.962	189.723	7,574.550	159.239	1,202.607	107.702	4,560.478	107.407
	$n = 190$	1,832.109	194.954	7,783.337	163.628	1,193.851	106.632	4,466.659	106.625
	$n = 200$	1,888.349	200.937	8,022.261	168.651	1,186.111	106.115	4,327.893	105.933
L_{p7}	$n = 180$	1,782.962	189.723	7,574.550	159.239	1,202.607	107.702	4,560.478	107.407
	$n = 190$	1,832.109	194.954	7,783.337	163.628	1,193.851	106.632	4,466.659	106.625
	$n = 200$	1,888.349	200.937	8,022.261	168.651	1,186.111	106.115	4,327.893	105.933
L_{p8}	$n = 180$	1,782.962	189.723	7,574.550	159.239	1,202.607	107.702	4,560.478	107.407
	$n = 190$	1,832.109	194.954	7,783.337	163.628	1,193.851	106.632	4,466.659	106.625
	$n = 200$	1,888.349	200.937	8,022.261	168.651	1,186.111	106.115	4,327.893	105.933

Table 10. PRE's of proposed estimators with respect to competing estimators.

Proposed estimators	Sample size	Existing estimators							
		$\bar{y}$	$\bar{y}_r$	$\bar{y}_p$	$\bar{y}_{lr}$	$\bar{y}_c$	$\bar{y}_{rc}$	$\bar{y}_{pc}$	$\bar{y}_{lrc}$
L_{rre1}	$n = 180$	1,659.554	176.592	7,050.278	148.218	1,119.369	100.247	4,244.825	100.973
	$n = 190$	1,719.623	182.984	7,305.462	153.582	1,120.552	100.085	4,192.418	100.078
	$n = 200$	1,782.093	189.631	7,570.858	159.161	1,119.370	100.144	4,084.368	100.972
L_{rre2}	$n = 180$	1,659.554	176.592	7,050.278	148.218	1,119.369	100.247	4,244.825	100.973
	$n = 190$	1,719.623	182.984	7,305.462	153.582	1,120.552	100.085	4,192.418	100.078
	$n = 200$	1,782.093	189.631	7,570.858	159.161	1,119.370	100.144	4,084.368	100.972
L_{rre3}	$n = 180$	1,659.193	176.553	7,048.743	148.185	1,119.125	100.226	4,243.901	100.951
	$n = 190$	1,719.585	182.980	7,305.301	153.578	1,120.527	100.083	4,192.326	100.076
	$n = 200$	1,782.411	189.664	7,572.205	159.190	1,119.569	100.162	4,085.095	100.990
L_{rre4}	$n = 180$	1,659.554	176.592	7,050.278	148.218	1,119.369	100.247	4,244.825	100.973
	$n = 190$	1,719.623	182.984	7,305.462	153.582	1,120.552	100.085	4,192.418	100.078
	$n = 200$	1,782.093	189.631	7,570.858	159.161	1,119.370	100.144	4,084.368	100.972
L_{rre5}	$n = 180$	1,659.149	176.549	7,048.557	148.181	1,119.096	100.223	4,243.789	100.949
	$n = 190$	1,719.572	182.979	7,305.248	153.577	1,120.519	100.082	4,192.296	100.075
	$n = 200$	1,782.454	189.669	7,572.389	159.193	1,119.597	100.164	4,085.194	100.993
L_{rre6}	$n = 180$	1,659.062	176.539	7,048.185	148.174	1,119.037	100.218	4,243.565	100.943
	$n = 190$	1,719.547	182.976	7,305.141	153.575	1,120.502	100.080	4,192.234	100.074
	$n = 200$	1,782.526	189.677	7,572.695	159.200	1,119.642	100.168	4,085.359	100.997
L_{rre7}	$n = 180$	1,659.554	176.592	7,050.278	148.218	1,119.369	100.247	4,244.825	100.973
	$n = 190$	1,719.623	182.984	7,305.462	153.582	1,120.552	100.085	4,192.418	100.078
	$n = 200$	1,782.093	189.631	7,570.858	159.161	1,119.370	100.144	4,084.368	100.972
L_{rre8}	$n = 180$	1,659.565	176.593	7,050.325	148.218	1,119.376	100.248	4,244.853	100.974
	$n = 190$	1,719.623	182.984	7,305.462	153.582	1,120.552	100.085	4,192.418	100.078
	$n = 200$	1,782.079	189.629	7,570.796	159.160	1,119.361	100.143	4,084.335	100.972

Table 11. PRE's of proposed estimators with respect to competing estimators

Proposed estimators	Sample size	Existing estimators							
		$\bar{y}$	$\bar{y}_r$	$\bar{y}_p$	$\bar{y}_{lr}$	$\bar{y}_c$	$\bar{y}_{rc}$	$\bar{y}_{pc}$	$\bar{y}_{lrc}$
L_{rpe1}	$n = 180$	1,661.583	176.807	7,058.895	148.399	1,120.737	100.370	4,250.013	100.095
	$n = 190$	1,719.698	182.992	7,305.782	153.588	1,120.601	100.089	4,192.602	100.083
	$n = 200$	1,783.204	189.749	7,575.577	159.260	1,120.068	100.206	4,086.914	100.035
L_{rpe2}	$n = 180$	1,661.583	176.807	7,058.895	148.399	1,120.737	100.370	4,250.013	100.095
	$n = 190$	1,719.698	182.992	7,305.782	153.588	1,120.601	100.089	4,192.602	100.083
	$n = 200$	1,783.204	189.749	7,575.577	159.260	1,120.068	100.206	4,086.914	100.035
L_{rpe3}	$n = 180$	1,661.561	176.805	7,058.801	148.397	1,120.722	100.369	4,249.957	100.094
	$n = 190$	1,719.723	182.995	7,305.889	153.591	1,120.617	100.091	4,192.664	100.084
	$n = 200$	1,783.305	189.759	7,576.006	159.269	1,120.131	100.212	4,087.145	100.040
L_{rpe4}	$n = 180$	1,661.583	176.807	7,058.895	148.399	1,120.737	100.370	4,250.013	100.095
	$n = 190$	1,719.698	182.992	7,305.782	153.588	1,120.601	100.089	4,192.602	100.083
	$n = 200$	1,783.204	189.749	7,575.577	159.260	1,120.068	100.206	4,086.914	100.035
L_{rpe5}	$n = 180$	1,661.561	176.805	7,058.801	148.397	1,120.722	100.369	4,249.957	100.094
	$n = 190$	1,719.723	182.995	7,305.889	153.591	1,120.617	100.091	4,192.664	100.084
	$n = 200$	1,783.305	189.759	7,576.006	159.269	1,120.131	100.212	4,087.145	100.040
L_{rpe6}	$n = 180$	1,661.550	176.804	7,058.755	148.396	1,120.715	100.368	4,249.929	100.093
	$n = 190$	1,719.736	182.996	7,305.943	153.592	1,120.625	100.091	4,192.694	100.085
	$n = 200$	1,783.334	189.763	7,576.129	159.272	1,120.149	100.214	4,087.211	100.042
L_{rpe7}	$n = 180$	1,661.583	176.807	7,058.895	148.399	1,120.737	100.370	4,250.013	100.095
	$n = 190$	1,719.698	182.992	7,305.782	153.588	1,120.601	100.089	4,192.602	100.083
	$n = 200$	1,783.204	189.749	7,575.577	159.260	1,120.068	100.206	4,086.914	100.035
L_{rpe8}	$n = 180$	1,661.583	176.807	7,058.895	148.399	1,120.737	100.370	4,250.013	100.095
	$n = 190$	1,719.698	182.992	7,305.782	153.588	1,120.601	100.089	4,192.602	100.083
	$n = 200$	1,783.204	189.749	7,575.577	159.260	1,120.068	100.206	4,086.914	100.035

Some notable findings from Tables 7-11 are:

1. From Table 7, it is evident that the Mean Squared Error (MSE) of the estimator $\bar{y}_{lr}$ is the lowest when compared to the traditional sample mean $\bar{y}$, the product estimator $\bar{y}_p$, and the ratio estimator $\bar{y}_r$ under Simple Random Sampling (SRS) without replacement. This indicates that $\bar{y}_{lr}$ provides a more efficient estimate of the population mean under SRS.

2. Similarly, under Circular Systematic Sampling (CSS), Table 7 shows that the MSE of the proposed estimator $\bar{y}_{lrc}$ is the smallest in comparison to the traditional CSS-based sample mean $\bar{y}_c$, the CSS-based ratio estimator $\bar{y}_{rc}$, and the CSS-based product estimator $\bar{y}_{pc}$. This suggests that $\bar{y}_{lrc}$ is the most efficient among the CSS estimators considered.

3. Furthermore, Table 7 highlights that the MSE values for all CSS-based estimators namely $\bar{y}_c$, $\bar{y}_{rc}$, $\bar{y}_{pc}$ *and* $\bar{y}_{lrc}$ are consistently lower than those of their SRS without replacement counterparts $\bar{y}$, $\bar{y}_p$, $\bar{y}_r$ *and* $\bar{y}_{lr}$. This clearly demonstrates the superiority of the CSS framework in terms of estimation efficiency, particularly in the presence of outliers or extreme values. Analyzing Tables 7 through 11 further supports the observation that both existing and newly proposed estimators under CSS outperform their equivalents under SRS without replacement. This performance gain is attributed to the structure and flexibility of CSS, which allows better handling of population variability.

4. Additionally, as shown in Tables 8 to 11, the Percent Relative Efficiency (PRE) of all estimators both existing and proposed varies depending on the specific choices of the parameters "*r* and *s*", which are based on non-conventional measures. The different configurations of these parameters and their impact on estimator performance are detailed in Table 4.

5. From Table 8, it is observed through numerical results that suggested ratio-type class of estimators i.e., L_r are improved over the competing estimators under both SRS without replacement and CSS schemes. The most prominent estimator is L_{r6} as compared to all proposed estimators i.e.

$$L_{r6} = t_1 \bar{y}_{css} \left(\frac{Q_D \overline{X} + M_R}{Q_D \bar{x}_{css} + M_R} \right) + t_2 \bar{y}_{css}$$

6. From Table 9, it is observed that the proposed product-type estimators, denoted by L_p, demonstrate superior performance compared to the competing estimators considered in the study, under both SRS without replacement and CSS. The MSE values for L_p are consistently lower, indicating higher efficiency.

7. Tables 10 and 11 further highlight the effectiveness of the proposed exponential-type estimators, specifically the ratio-ratio exponential estimator Lrre and the ratio-product exponential estimator Lrpe. Both estimators outperform the conventional alternatives under consideration, underlining their robustness and enhanced accuracy.

8. Moreover, a cross-comparison of Tables 8 through 11 reveals a consistent trend: as the sample size increases, the PRE of all proposed estimators also increases. This trend confirms the scalability and stability of the proposed estimators.

In addition, we determine the confidence interval (lower limit and upper limit) for the population mean through all the existing and proposed estimators under SRS and CSS at a 95% confidence level, using the expression given in Equation (33):

$$T \pm Z_{\alpha/2} \sqrt{MSE(T)} \tag{33}$$

where $T = \bar{y}, \bar{y}_r, \bar{y}_p, \bar{y}_{lr}, \bar{y}_c, \bar{y}_{rc}, \bar{y}_{pc}, \bar{y}_{lrc}, L_r, L_p, L_{rre}$ and L_{rpe}.

Table 12. Confidence interval results for all estimators

$n = 180$									
Existing estimators	CI	Proposed estimators	CI	Proposed estimators	CI	Proposed estimators	CI	Proposed estimators	CI
$\bar{y}$	[330.49, 505.02]	L_{r1}	[418.78, 456.21]	L_{p1}	[391.48, 427.82]	L_{rre1}	[418.36, 455.80]	L_{rep1}	[419.05, 456.47]
$\bar{y}_r$	[269.63, 329.14]	L_{r2}	[418.78, 456.21]	L_{p2}	[391.46, 427.82]	L_{rre2}	[418.37, 455.81]	L_{rep2}	[419.07, 456.41]
$\bar{y}_p$	[401.27, 764.58]	L_{r3}	[418.73, 456.16]	L_{p3}	[391.42, 427.88]	L_{rre3}	[418.39, 455.80]	L_{rep3}	[419.05, 456.48]
$\bar{y}_{lr}$	[289.33, 342.41]	L_{r4}	[418.78, 456.21]	L_{p4}	[391.48, 427.82]	L_{rre4}	[418.31, 455.84]	L_{rep4}	[419.04, 456.49]
$\bar{y}_c$	[262.35, 381.22]	L_{r5}	[418.72, 456.15]	L_{p5}	[391.41, 427.84]	L_{rre5}	[418.34, 455.81]	L_{rep5}	[419.05, 456.47]
$\bar{y}_{rc}$	[424.88, 462.76]	L_{r6}	[418.71, 456.13]	L_{p6}	[391.45, 427.87]	L_{rre6}	[418.32, 455.82]	L_{rep6}	[419.04, 456.46]
$\bar{y}_{pc}$	[116.19, 350.42]	L_{r7}	[418.78, 456.21]	L_{p7}	[391.44, 427.83]	L_{rre7}	[418.36, 455.80]	L_{rep7}	[419.06, 456.47]
$\bar{y}_{lrc}$	[406.98, 444.45]	L_{r8}	[418.78, 456.21]	L_{p8}	[391.48, 427.82]	L_{rre8}	[418.34, 455.83]	L_{rep8}	[419.08, 456.44]
$n = 200$									
Existing estimators	CI	Proposed estimators	CI	Proposed estimators	CI	Proposed estimators	CI	Proposed estimators	CI
$\bar{y}$	[387.39, 564.46]	L_{r1}	[415.61, 464.65]	L_{p1}	[418.81, 467.07]	L_{rre1}	[415.65, 464.69]	L_{rep1}	[415.57, 464.62]
$\bar{y}_r$	[281.22, 334.93]	L_{r2}	[415.63, 464.61]	L_{p2}	[418.82, 467.05]	L_{rre2}	[415.62, 464.63]	L_{rep2}	[415.52, 464.60]
$\bar{y}_p$	[547.25, 923.21]	L_{r3}	[415.68, 464.69]	L_{p3}	[418.81, 467.09]	L_{rre3}	[415.64, 464.63]	L_{rep3}	[415.58, 464.69]
$\bar{y}_{lr}$	[311.25, 355.51]	L_{r4}	[415.61, 464.68]	L_{p4}	[418.87, 467.02]	L_{rre4}	[415.63, 464.69]	L_{rep4}	[415.56, 464.61]
$\bar{y}_c$	[371.58, 512.29]	L_{r5}	[415.60, 464.65]	L_{p5}	[418.89, 467.05]	L_{rre5}	[415.69, 464.64]	L_{rep5}	[415.59, 464.66]
$\bar{y}_{rc}$	[415.61, 465.37]	L_{r6}	[415.69, 464.65]	L_{p6}	[418.81, 467.06]	L_{rre6}	[415.61, 464.69]	L_{rep6}	[415.53, 464.60]
$\bar{y}_{pc}$	[305.12, 586.02]	L_{r7}	[415.68, 464.64]	L_{p7}	[418.83, 467.06]	L_{rre7}	[415.69, 464.67]	L_{rep7}	[415.57, 464.64]
$\bar{y}_{lrc}$	[416.41, 465.49]	L_{r8}	[415.64, 464.61]	L_{p8}	[418.81, 467.04]	L_{rre8}	[415.65, 464.63]	L_{rep8}	[415.56, 464.63]

Using *R* software, we generated simple random samples of sizes 180 and 200. Subsequently, the necessary calculations were performed to compute the confidence intervals using Equation (33). Table 12 confirms that the confidence intervals for the linear regression estimator are narrower compared to those of other traditional estimators.

Afterwards, we generated samples of sizes 180 and 200 using CSS and calculated the confidence intervals for both traditional and proposed estimators. The results indicate that the linear regression estimator under CSS yields better performance.

It is worth mentioning that the confidence intervals for the proposed estimators confirm their superiority, as evidenced by their narrower confidence intervals compared to the existing estimators under CSS.

7. Conclusion

In this study, an improved generalized estimator has been proposed for estimating the unknown population mean under Circular Systematic Sampling (CSS). The generalized estimator can generate a series of estimators such as ratio-type estimators, product-type estimators, ratio-ratio-type exponential estimators, and ratio-product-type exponential. Proposed estimators are based on non-conventional measures of the auxiliary variable such as quartile deviation, mid-range, tri-mean, quartile average, and Hodge-Lehmann estimator. Expressions for the bias and MSE of the proposed estimators have been proposed in section 4. Proposed estimators are found to be more proficient over the competing estimators under both SRS without replacement and CSS. Furthermore, an empirical study is performed through a real population to assess the supremacy of the proposed estimators over the competing estimators in terms of MSE and PRE. Newly proposed estimators are strongly recommended for future applications if data sets contain potential outlier(s). The feasible additions of the current study are to estimate:

- Population means using partial auxiliary information under CSS.
- Population means utilizing complete and partial auxiliary information in the presence of nonresponse under CSS.
- Population means under neutrosophic approach.

Conflicts of interest

The authors declare no conflict of interest.

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