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Review

The Scientific Achievements of Professor Boling Guo

Yufeng Lu[1], Shaobin Tan[2], Baoxiang Wang[1*], Youde Wang[3,4]

[1] School of Sciences, Jimei University, Xiamen, 361021, China
[2] School of Mathematical Sciences, Xiamen University, Xiamen, 361005, China
[3] School of Mathematical Sciences, Guangzhou University, Guangzhou, 510180, China
[4] Institute of Mathematics, Chinese Academy of Mathematics and System Sciences, Beijing, 100190, China
E-mail: wbx@jmu.edu.cn

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Abstract: This paper is dedicated to Professor Boling Guo's 90th birthday by surveying his main contributions to nonlinear evolution equations, infinitely dimensional dynamical systems, soliton theory, geometric Partial Differential Equation (PDE), and related fields. We also point out some open problems related to Professor Guo's work.

***Keywords*:** nonlinear Partial Differential Equation (PDE), infinite dimensional system, soliton, attractor, global solution, blowup

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1. Introduction

Professor Boling Guo is a well-known mathematician. To date, he has published more than 800 (joint) papers and 16 (joint) books [1–15]. His research is involved in the well posedness and blowup of solutions of nonlinear Partial Differential Equation (PDE), infinitely dimensional dynamical systems, numerical analysis, soliton theory, harmonic analysis methods of PDE, geometric PDE, numerical analysis of Nonlinear Equation (NLE), and stochastic PDE, etc.

Professor Boling Guo has studied many important nonlinear evolution equations such as Landau-Lifshitz equations, nonlinear dispersive wave equations including Korteweg-de Vries equation (KdV), Benjemin-Ono, Nonlinear Schrödinger equation (NLS), Zakharov equation, Navier-Stokes equation, nonlinear parabolic equations, and their coupled equations, etc. In 1984–1986, he jointly obtained the first result for the global existence of the generalized solutions of the Landau-Lifshitz equation in higher spatial dimensions. In 1991, he jointly solved the long standing open problem for the global existence for the smooth solution of the Landau-Lifshitz equation in one spatial dimension. In 1995, he study the compactness of attractors for the Benjamin-Ono equation and discovered a new method to show that a weakly compact attractor is strongly compact. In 2000, he pioneered the study of the approximatively integrable infinite-dimensional dynamical system.

In this paper, we survey Professor Guo's main results and also state some of his collaborative works in the above subjects.

The papar is organized as follows. In Section 2, we will introduce Professor Guo's results on local and global wellposedness of some equations including Zakharov equations, Landau-Lifshitz equations, and Benjamin-One equations.

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In Section 3, we will summarize his studies on geometric flows and harmonic maps. In Section 4, we will collect some of his results on (random) attractors. His studies on harmonic analysis and PDE will be surveyed in Section 5. In Section 6, we will state a few of his results on variation methods and blowup solutions. In Section 7, we will briefly introduce his works on rogue waves and solitons. In Section 8, we point out some recent progress and open problems related to Professor Guo's work.

2. Local and global well-posedness results

2.1 *Zakharov equations*

In 1982, Guo and Shen [16] established the existence and uniqueness of global classical solutions to the periodic initial value problem associated with the Zakharov equations as follows:

$$\begin{cases} \partial_t^2 n - \partial_x^2 n = \partial_x^2 |\varepsilon|^2 \\ \mathrm{i}\partial_t \varepsilon + \partial_x^2 \varepsilon = n\varepsilon \\ n(x,\,0) = n_0(x),\, n_t(x,\,0) = n_1(x),\, \varepsilon(x,\,0) = \varepsilon_0(x) \\ n(x+2\pi,\,t) = n(x,\,t),\, \varepsilon(x+2\pi,\,t) = \varepsilon(x,\,t) \end{cases} \tag{1}$$

where the initial value $n_0(x), n_1(x), \varepsilon_0(x)$ are periodic with a period of 2π. Guo and Shen [16] reformulated the problem by introducing a velocity potential φ leading to an equivalent first-order system, which is more amenable to analysis. They used the Galerkin approximation method to obtain the local weak solution in Sobolev space H^s. Combining a priori mass and energy estimates, they obtained the global solution. According to the Sobolev embedding $H^3 \hookrightarrow C^2$, if the initial data are sufficiently smooth ($\varepsilon_0 \in H^6$, $n_0 \in H^4$, $\varphi_0 \in H^4$), then the solution in the Sobolev space is classical solution. Moreover, they proved the uniqueness of the global classical solution, which seems the first result for the global well-posedness for the smooth solutions of the Zakharov equation in 1D:

Theorem 1 ([16], Theorem 5.1) If initial data are periodic with a period of 2π and satisfy

$$(n(x,\,0),\, n_t(x,\,0),\, \varepsilon(x,\,0)) \in H^6(\mathbb{R}) \times H^4(\mathbb{R}) \times H^8(\mathbb{R}),$$

then Problem (1) has a unique global classical solution.

Zakharov Equation is an important and difficult model in PDE, which has been extensively studied in recent years, see Section 8 for its further progress.

2.2 *On Landau-Lifshitz equations*

The Landau-Lifshitz equation is the most important equation in describing the evolution of the microscopic magnetization field in ferromagnetic spin chains, which plays the same role as the Navier-Stokes equation in fluid mechanics:

$$v_t = -\alpha_1 v \times (v \times \Delta v) + \alpha_2 v \times \Delta v, \tag{2}$$

where v is the magnetization field, α_1, α_2 are damping constants, and "$\times$" denotes the cross-product of two 3-dimensional vectors. Here

$$v(x,\, t) = (v^1(x,\, t),\, v^2(x,\, t),\, v^3(x,\, t))\colon \Omega \times [0,\, \infty) \to \mathbb{S}^2,$$

is the unknown microscopic magnetization field on a bounded domain Ω in $\mathbb{R}^m$. This system is also named as the ferromagnetic chain equation. This system is derived from the conservation of energy and the magnitude of v, and it represents a formulation that leads to a continuum spin wave theory. When $\alpha_1 = 0$, $m = 1$, this system is an integrable system, and has N soliton solutions. When $\alpha_1 > 0$, it becomes a strongly coupled degenerate quasilinear parabolic system.

In 1984, Zhou and Guo [17] established the existence of weak solutions for boundary value problems associated with the ferromagnetic chain system, a degenerate parabolic system modeling spin dynamics in condensed matter physics. A generalized version of the ferromagnetic chain system is

$$Z_t = Z \times Z_{xx} + f(x,\, t,\, Z), \tag{3}$$

where $Z = (u,\, v,\, w)$ is an unknown 3-dimensional vector-valued function, $f(x,\, t,\, Z)$ is a given 3-dimensional vector-valued function of x, t and Z, and "$\times$" denotes the cross-product of two 3-dimensional vectors. The equation generalizes the Landau-Lifshitz equation for isotropic Heisenberg chains.

Zhou and Guo considered the boundary problem for Equation (3) in a rectangular domain $Q_T = \{0 \leqslant x \leqslant l,\, 0 \leqslant t \leqslant T\}$. They studied three types of boundary conditions:

1. Dirichlet: $Z(0,\, t) = Z(l,\, t) = 0$,
2. Neumann: $Z_x(0,\, t) = Z_x(l,\, t) = 0$,
3. Mixed: $Z_{0,\, t} = Z_x(l,\, t) = 0$ or $Z_x(0,\, t) = Z(l,\, t) = 0$,

and the initial condition $Z(x,\, 0) = Z_0(x)$.

Equation (3) can be regarded as the strongly degenerate parabolic system, since the coefficient matrix of second-order derivatives is singular. Therefore, Zhou and Guo introduced a system with a small diffusion term, named spin system.

$$U_t = \varepsilon U_{xx} + U \times U_{xx} + f(x,\, t,\, U). \tag{4}$$

They used Leray-Schauder fixed point theorem to obtain the unique weakly global solution of Equation (4). The viscosity term εU_{xx} with $\varepsilon > 0$ can guarantee the solutions of Equation (4) are sufficiently smooth at any $t > 0$, so that one can directly use the equation making the "a priori" estimates and obtaining the uniform bounds which are independent of $\varepsilon > 0$. Finally, taking $\varepsilon \to 0$, the existence of the weak solutions of Equation (3) can be obtained under the following assumptions:

1. $f(x,\, t,\, U)$ is continuously differentiable with respect to x and U, with

$$\xi \cdot f_U(x,\, t,\, U)\xi \leqslant b|\xi|^2,\, \forall \xi \in \mathbb{R}^3,$$

holds for any $(x,\, t,\, U) \in Q_T \times \mathbb{R}^3$.

2. For any $(x,\, t,\, U) \in Q_T \times \mathbb{R}^3$, there is the inequality

$$|f_x(x,\, t,\, U)| \leqslant c(x,\, t)|U|^3 + d(x,\, t),$$

where $c(x, t) \in L_\infty(Q_T)$, $d(x, t) \in L_2(Q_T)$.

3. $U_0(x) \in w_2^1(0, l)$.

Definition 1 ([17], Definition 1) The 3-dimensional vector-valued function $U(x, t) \in L_2((0, T); w_2^{(1)}(0, l)) \cap C(Q_T)$ is called the weak solution of the boundary problem for Equation (3), if for any test function $\varphi(x, t) \in \Phi$, the integral relation

$$\iint_{Q_T} [\varphi_t U - \varphi_x (U \times U_x) + \varphi f(x, t, U)] dxdt + \int_0^l \varphi(x, 0) U_0(x) = 0,$$

holds, where $\varphi \in C^{(1)}(Q_T)$, $\varphi(x, T) = 0$, and $\varphi(0, t) = 0$ (or $\varphi(l, t) = 0$) when the boundary condition at $x = 0$ (or $x = l$) is $U(0, t) = 0$ (or $U(l, t) = 0$).

Theorem 2 ([17], Theorem 5) Suppose that Equation (3) and the initial vector-valued data U_0 satisfy the assumptions above. Then the Neumann boundary problem of system (3) has at least one global weak solution:

$$U(x, t) \in L_\infty((0, T); W_2^{(1)}(0, l)) \cap C^{(1/2, 1/4)}(Q_T), \tag{5}$$

where $U(x, t) \in C^{(\alpha, \beta)}(Q_T)$ means that $U(x, t)$ is α-, β-order Hölder continuous on x, t respectively.

Theorem 3 ([17], Theorem 6) Suppose that the homogeneous Equation (3) (i.e. $f(x, t, 0) \equiv 0$) and the initial vector-valued data U_0 satisfy the assumptions above. Then the Dicichlet and mixed problem of Equation (3) has at least one global weak solution:

$$U(x, t) \in L_\infty((0, T); W_2^{(1)}(0, l)) \cap C^{(1/2, 1/4)}(Q_T).$$

Remark 1 Zhou and Guo [17] also established the existence of weak solutions for boundary value problems associated with the ferromagnetic chain system in a semi-infinite domain $Q_T^* = \{x \in \mathbb{R}^+, 0 \leqslant t \leqslant T\}$. Then they generalized the existence of weak solutions into higher spatial dimensions and obtain similar results to 1D cases.

Later in 1991, Zhou, Guo, and Tan [18] investigated the Cauchy problem for a system of ferromagnetic chains incorporating the Gilbert damping term:

$$Z_t = -\varepsilon Z \times (Z \times Z_{xx}) + Z \times Z_{xx}, \tag{6}$$

where $Z = (u, v, w)$ is a unit vector field ($|Z| = 1$), and $\varepsilon \geqslant 0$ is the damping coefficient. This system combines nonlinear dispersion ($Z \times Z_{xx}$) and degenerate dissipation ($-\varepsilon Z \times (Z \times Z_{xx})$). Zhou, Guo, and Tan estibalished the existence and uniqueness of smooth solution for Cauchy problems of Equation (6) by employing the technique of spatial differences and essential a priori estimates of higher-order derivatives in Sobolev spaces, which is the first result for the global smooth solution with large data for the Landau-Lifshitz equations:

Theorem 4 ([18], Theorem 5) Suppose that $Z_0(x) \in H^a(\mathbb{R})$ for some $a \geqslant 4$. Then the Cauchy problem for the system of ferromagnetic chains described in equation (6) possesses a unique global smooth solution such that

$$Z(x, t) \in \bigcap_{s=0}^{[a/2]} W_\infty^s(0, T; H^{a-2s}(\mathbb{R})),$$

with $a-2s \geqslant 0$.

In 2011, Pu and Guo [19] considered the following generalized periodic fractional Landau-Lifshitz-Gilbert equation.

$$\begin{cases} v_t = av \times \Lambda^{2\alpha} v + bv(v \times \Lambda^{2\alpha} v), \\ v(0) = v_0 \in H^\alpha, \end{cases} \tag{7}$$

where $v(x, t)$ is a three-dimensional vector that represents the magnetization, and $a, b \geqslant 0$ are real numbers. The square root of the Laplacian, $\Lambda = (\Delta)^{1/2}$, is the Zygmund operator, and $\times$ denotes the cross product for $\mathbb{R}^3$-valued vectors. The last term $v \times (v \times \Lambda^{2\alpha} v)$ is usually refered to as the Gilbert damping term and hence $b > 0$ is called the Gilbert damping parameter. Specially, when $b = 1$, Equation (7) becomes the standard Landau-Lifshitz equation. When $b = 0$, Equation (7) corresponds to the fractional Heisenberg equation. When $a = 1$, Equation (7) can be transformed into the harmonic map heat flow on the unit sphere.

Pu and Guo [19] demonstrated the existence of a global weak solution to the Equation (7) using the Ginzburg-Landau and the Galerkin approximation. Given that the nonlinear term is nonlocal and of full order, specific structural properties of the equation, commutator estimates, and various fractional calculus inequalities were utilized to establish the convergence of the approximating solutions. Furthermore, this equation can be viewed as a generalization of the heat flow of harmonic maps to fractional order.

Definition 2 ([19], Definition 1) Denote $\Omega = [0, 2\pi]^d$. Let $\alpha \in (0, 1)$, $v_0 \in H^\alpha$, $|u_0| = 1$ a.e. We say that v is a weak solution of Equation (7) if

- for all $T > 0$, $u \in L^\infty(0, T; H^\alpha(\Omega))$;
- for all $\varphi \in C^\infty(Q_T)$, there holds when $b = 0$,

$$\int_{Q_T} v_t \varphi = -a \int_{Q_T} \Lambda^\alpha v \cdot \Lambda^\alpha (v \times \varphi) dxdt,$$

where $Q_T = (0, T) \times \Omega$; or when $b > 0$,

$$\int_{Q_T} v_t \varphi = a \int_{Q_T} \left(v \times \Lambda^{2\alpha} v\right) \cdot \varphi dxdt - \int_{Q_T} b \left(v \times \Lambda^{2\alpha} v\right) \cdot (v \times \varphi) dxdt.$$

Theorem 5 ([19], Theorem 1) Let $\alpha \in (0, 1)$. Then for all $v_0 \in H^\alpha(\Omega)$, $|v_0| = 1$ a.e., there exists at least a weak solution for the fractional Landau-Lifshitz-Gilbert equation (7) such that

1. when $b = 0$,

$$v(x, t) \in L^\infty(0, T; H^\alpha(\Omega)) \bigcap C^{0, \frac{\alpha}{\alpha+m}}(0, T; L^2(\Omega)),$$

where $m > \alpha + d/2$;

2. when $b > 0$,

$$v(x, t) \in L^\infty(0, T; H^\alpha(\Omega)) \bigcap C^{0, \frac{r-1}{r}}(0, T; L^r(\Omega)),$$

where $r < 2$, $1 \leqslant r \leqslant \frac{d}{d-\alpha}$ or $r = 2$, $d = 1$, $\alpha > 1/2$.

Later in 2013, Pu and Guo [20] considered the following fractional Landau-Lifshitz equation without Gilbert damping.

$$\begin{cases} U_t = U \times \Lambda^{2\alpha} U, \\ U(x, 0) = U_0(x). \end{cases} \tag{8}$$

Here $U(x, t)$: $\mathscr{T}^d \times \mathbb{R}^+ \to S^2$ is a three-dimentional vector representing the magnectization, where $\mathscr{T}^d$ is the d-dimensional torus with $d \leqslant 3$.

In [20], Pu and Guo demonstrated the local existence of classical solutions by integrating Kato's method with the vanishing viscosity approach and by meticulously selecting the appropriate function space. Considering that Equation (8) is both strongly degenerate and nonlocal, and that no regularizing effect is present, extending this smooth solution to a global one presents a significant challenge. In this paper, they established several regularity criteria demonstrating that the solution is global under the assumption of additional regularity, which appears minimal from the perspective of dimensional analysis. Due to the unique structure of the equation, the conventional integer-order Sobolev space H^m is inadequate, as it lacks a divergence-free condition, which is typically present in equations from incompressible fluid mechanics. Instead, the authors employed the fractional space $H^{m\alpha}$ as the working space to establish the local existence of a solution. However, extending the local classical solution to a global one is challenging due to its nonlocal nature, strong degeneracy, and the absence of a regularizing effect. Unlike standard equations, to ensure the convergence of approximate solutions, the authors introduced a commutator structure to address the nonlocal properties of the fractional Laplacian, thereby facilitating the attainment of a global solution.

Theorem 6 ([20], Theorem 3) Let $d \leqslant 3$, $\alpha \in (0, 1/2]$, and $U_0 \in H^{s+\alpha}$ with $s \geqslant 4$. Then there exists a $T^* > 0$ depending only on U_0 such that Equation (8) possesses a unique classical solution $U \in C([0, T^*]; H^{s+\alpha} \cap C^2)$.

Theorem 7 ([20], Theorem 4) Let $d \leqslant 3$, $\alpha \in (0, 1/2]$ and $U_0 \in H^{s+\alpha}$ with $s \geqslant 4$, such that there exists a classical solution $U \in C([0, T^*]; H^{s+\alpha} \cap C^2)$ to Equation (8). Then for any $0 < T < \infty$, if when $\alpha = 1/2$ that

$$\int_0^T \|\nabla U\|_{L^\infty} dt < \infty, \ \int_0^T \|\Lambda U\|_{L^\infty} dt < \infty,$$

or when $0 < \alpha < 1/2$ that

$$\int_0^T \|\nabla U\|_{L^p} dt < \infty, \ \int_0^T \left\|\Lambda^{2\alpha} U\right\|_{L^\infty} dt < \infty,$$

for some $p > 1$ satisfying $2\alpha + d/p \leqslant 1$, then the solution U exists globally in time, i.e. $U \in C([0, \infty); H^{1+\alpha})$.

Let $U(x, t)$ be a solution of Equation (8), then the scaling $U_\lambda(x, t) = U(\lambda x, \lambda^{2\alpha} t)$ is also a solution. Notice that

$$\left\|\Lambda^\beta U_\lambda\right\|_{L^r(0, T; L^s)} = \left\|\Lambda^\beta U\right\|_{L^r(0, T; L^s)},$$

holds if and only if

$$\beta = \frac{2\alpha}{r} + \frac{d}{s}. \tag{9}$$

Therefore, the regularity criteria may involve the finiteness of $\left\|\Lambda^\beta U\right\|_{L^{r,s}}$ for β, r, s satisfying (9). Therefore, Pu and Guo also obtained a regularity criteria concerned with $\left\|\Lambda^\beta U\right\|_{L^{r,s}}$:

Theorem 8 Let $d \leqslant 3$, $\alpha \in (0, 1/2]$ and $U_0 \in H^{s+\alpha}$ with $s \geqslant 4$, so that there exists a classical solution $U \in C([0, T^*]; H^{s+\alpha} \cap C^2)$ to Equation (8). Then for any $0 < T < \infty$, if

$$\int_0^T \left\|\Lambda^{2\alpha} U(t)\right\|_{L^\infty} < \infty, \tag{10}$$

$$\int_0^T \left\|\Lambda^\beta U(\cdot, t)\right\|_{L^s} dt < \infty, \tag{11}$$

for some $\beta \geqslant 2\alpha + d/s$ and $1 < s < \infty$, then the local solution can be extended into a global classical solution and remains in $L^\infty(0, T; H^{m+\alpha})$.

Definition 3 ([20], Definition 1) Let $d \leqslant 3$, $\alpha \in (0, 1/2]$ and $U_0 \in H^\alpha$. We say that U is a weak solution of Equation (8) if

1. for all $T > 0$, $U \in L^\infty(0, T; H^\alpha(\mathscr{T}^d))$;
2. for all three-dimensional vectors $\phi \in C^\infty(\mathscr{T}^d_T)$, there holds

$$\int_{\mathscr{T}^d} U \cdot \phi_t - \int_{\mathscr{T}^d_T} \Lambda^\alpha U \cdot \Lambda^\alpha (U \times \varphi) - \int_{\mathscr{T}^d} \phi(x, 0) \cdot U_0 = 0,$$

where $\mathscr{T}^d_T = (0, T) \times \mathscr{T}^d$.

Theorem 9 ([20], Theorem 6) Let $d \leqslant 3$, $\alpha \in (0, 1/2]$ and $U_0 \in H^\alpha(\mathscr{T}^d)$. Then there exists at least one global weak solution for Equation (8) in the sense of Definition 3, such that for any $T > 0$,

$$U \in L^\infty\left(0, T; H^\alpha\left(\mathscr{T}^d\right)\right) \cap C^{0, \frac{\alpha}{\alpha+s}}\left(0, T; L^2\left(\mathscr{T}^d\right)\right), \forall s > \max\{2, \alpha + d/2\},$$

and satisfy

$$\sup_{0 \leqslant t \leqslant T} \|U\|_{H^\alpha} \leqslant \|U_0\|_{H^\alpha}.$$

In 2004, Ding and Guo [21] rewrote Equation (2) as

$$\frac{1}{2} v_t - \frac{1}{2}(v \times v_t) = \Delta v + v|\nabla v|^2 + H(v) - H(v)v, \ \text{ in } B^3 \times \mathbb{R}^+, \tag{12}$$

where B^3 is the unit ball in $\mathbb{R}^3$, centered at 0; $H(v)$ is the nonlocal term satisfying the quasi-steady state Maxwell equations as follows:

$$\textbf{curl}\, H(v) = 0, \text{ in } \mathscr{D}'(\mathbb{R}^3), \tag{13}$$

$$\operatorname{div}(H(v) + \bar{v}) = 0, \text{ in } \mathscr{D}'(\mathbb{R}^3). \tag{14}$$

They improsed on Eqs. (12)–(14) the initial coondition

$$v(x, 0) = v_0(x), \tag{15}$$

and the boundary condition

$$\frac{\partial v}{\partial n}|_{\partial B^3} = 0. \tag{16}$$

in Equation (15), $|v_0(x)| \equiv 1$; in Equation (16), n is the unit outer normal to the boundary of B^3, $\bar{v}$ is the zero extension of v from B^3 to $\mathbb{R}^3$.

Ding and Guo studied the partial regularity property of stationary solutions to this Landau-Lifshitz equations, incorporating a nonlocal term in three dimensions (Equation (12)), and analyzed the Hausdorff measure of their singular sets. The Landau-Lifshitz equations describe the dynamics of ferromagnetic spin chains, making their regularity a crucial topic in mathematical physics. By introducing the concept of "stationary solutions" and utilizing energy estimates and monotonicity inequalities, they demonstrated that the singular set possesses zero Hausdorff measure under specific conditions. In contrast to the conventional approach to harmonic map heat flow, this method does not require the theories of Hodge decompositions, Hardy space, BMO space and Hardy maximal functions. Only the Hélein technique is used.

Definition 4 ([21], Definition 3.1) A weak solution v of Equation (12) is called a stationary solution if for any $\eta(x, t) \in C_0^1(B^3 \times R^+, \mathbb{R}^3)$, $\gamma(x, t) \in C_0^1(B^3 \times R^+, \mathbb{R}^3)$ with $\eta(x, t)$, $\gamma(x, t)$, $\nabla_{(x, t)}\eta$, $\nabla_{(x, t)}\gamma$ bounded on $B^3 \times \mathbb{R}^+$ and $\eta, \gamma \equiv 0$ for $t = 0$ and $t \geqslant t^* > 0$ such that $x + \tau\eta|_{\partial B^3} = Id$, $t + \tau\gamma|_{\partial B^3} = Id$, there holds

$$\int_0^\infty \int_{B^3} (\frac{1}{2}v_t - \frac{1}{2}v \times v_t)(\frac{\partial v^\tau}{\partial \tau})_{\tau=0} + \partial_\tau^+ \int_0^\infty \int_{B^3} e(v^\tau) + |H(v^\tau)|^2 dxdt \leqslant 0,$$

where $v^\tau(x, t) = v(x + \tau\eta(x, t), t + \tau\tau(x, t))$, $e(v) = \frac{1}{2}|\nabla v(x, t)|^2$.

Denote $B_\rho = B_\rho(0)$, $Q_\rho(z) = B_\rho(x) \times (t - \rho^2, t + \rho^2)$ for $z = (x, t)$.

Theorem 10 ([21], Theorem 5.1) There exist constants $c_0 > 0$ and $C_{k, l} > 0$ such that any stationary solution $v \in H^1(Q_r(z_0))$ of Eqs. (12)–(15) satisfying the small energy condition:

$$r^{-3} \int_{Q_r(z_0)} |\nabla v|^2 dz \leqslant c^2 \leqslant c_0^2,$$

is smooth in $Q_{r/2}(z_0)$ and

$$\left\| \partial_t^l \nabla^k v \right\|_{L^\infty(Q_{r/2}(z_0))} \leqslant C_{k, l} r^{-k-2l} c, \ k, l = 0, 1, 2 \cdots.$$

Theorem 11 ([21], Theorem 5.2) Let $v \in H^1(B^3 \times (0, T); S^2)$ be a stationary solution of Eqs. (12)–(15). There is an open set $\Omega \subseteq B^3 \times (0, T)$ such that v is smooth in Ω and

$$\mathscr{H}^3(B^3 \times (0, T) \setminus \Omega) = 0,$$

where

$$B^3 \times (0, T)\backslash\Omega = \left\{ z = (x, t) \left| \liminf_{r \to 0} r^{-3} \int_{P_r(z)} \left| \nabla v \right|^2 dz \geqslant c_0 \right. \right\}.$$

2.3 *On (generalized) Benjamin-Ono equations*

In 1988, Zhou and Guo [22] studied the initial value problem for a nonlinear singular integral-differential equation of deep water:

$$v_t + 2vv_x + Hv_{xx} + b(x, t)v_x + c(x, t)v = f(x, t), \qquad v(0, x) = \phi(x), \tag{17}$$

For $b = c = 0$, Equation (17) reduces to the Benjamin-Ono equation:

$$v_t + 2vv_x + Hv_{xx} = 0, \qquad v(0, x) = \phi(x). \tag{18}$$

The study of these equations is of significant interest from both physical and mathematical perspectives. Zhou and Guo [22] introduced a diffusion term εv_{xx} to convert the Equation (17) into a parabolic form. They combined the energy methods, fixed-point theory, and compactness to handle singular integrals and nonlinearity. The unique solution of Equation (17) was built up by the limiting process of the vanishing of the diffusion coefficient $\varepsilon \to 0$. Moreover, they obtained the estimates of the convergence speed in relation to the diffusion coefficient ε.

Theorem 12 ([22], Theorem 8) Denote $Q_T = \{x \in \mathbb{R},\ 0 \leqslant t \leqslant T\}$. Suppose that $b(t, x) \in W_\infty^{2,1}(Q_T)$, $c(x, t) \in W_\infty^{2,0}(Q_T)$ and $f(x, t) \in W_2^{2,0}(Q_T)$ hold for any $T > 0$, and suppose also that $\phi \in H^2(\mathbb{R})$. Then the initial value problem for Equation (17) has a unique generalized global solution

$$v \in L^\infty_{\text{loc}}(\mathbb{R}^+; H^2(\mathbb{R})) \bigcap W^{1,\infty}_{\text{loc}}(\mathbb{R}^+; L^2(\mathbb{R})).$$

Theorem 13 ([22], Theorem 9) Suppose that $\phi \in H^K(\mathbb{R})$ for $K \geqslant 2$. The initial value problem for Equation (18) has a unique global solution

$$v \in \bigcap_{s=0}^{[K/2]} W^{k,\infty}_{\text{loc}}(\mathbb{R}^+; H^{K-2s}(\mathbb{R})).$$

In 1996, Zhou and Guo [23] established the global well-posedness and considered the large time behavior of the global solution for the generalized Benjamin-Ono equations (deep water-type equations), including the existence of attractors and their dimension estimates. These equations characterize the propagation of internal waves in deep stratified fluids and incorporate nonlinearity alongside singular integral operators. The general form is:

$$v_t + 2vv_x + \alpha H v_{xx} - \beta H v_x + \gamma(x, t) H v + b(x, t) v_x + c(x, t) v = f(x, t), \tag{19}$$

where H is the Hilbert transform, $\alpha > 0$, $\beta \geqslant 0$ are constants, and

$$(x, t) \in Q_T = \{x \in \mathbb{R}, 0 \leqslant t \leqslant T\}.$$

Zhou and Guo [23] introduced a diffusion term εv_{xx} to convert the equation into a parabolic form. The existence of these nonlinear parabolic equations were proved using energy estimates and fixed-point theorems. Then a priori estimates independent of ε ensure the feasibility of the limit $\varepsilon \to 0$. Weak convergence and compactness lemmas were employed to pass from nonlinear parabolic solutions to the original equation (19). Moreover, they introduced the weighted Sobolev spaces. Combining with energy estimates and compactness argument, they obtained that with appropriate damping and nonlinearity conditions, the Cauchy problem possesses a compact global attractor and the Hausdorff and fractal dimensions of the attractor are finite.

Theorem 14 ([23], Theorem 4) Suppose that $\alpha > 0$, $\beta \geqslant 0$, $b(t, x) \in W_\infty^{(2,1)}(Q_T)$, $c(x, t)$, $\gamma(x, t) \in W_\infty^{(2,0)}(Q_T)$, $f(t, x) \in W_2^{(2,0)}(Q_T)$, and assume that the initial data $v(x, 0) \in H^2(\mathbb{R})$. Then Equation (19) has at least one generalized global solution

$$v(x, t) \in Z := L_\infty(0, T; H^2(\mathbb{R})) \cap W_\infty^{(1)}(0, T; L_2(\mathbb{R})),$$

which satisfies the Equation (19) in generalized sense and satisfies the initial condition in classical sense.

Theorem 15 ([23], Theorem 5) Suppose that $b(x, t) \in W_\infty^{(1,0)}(Q_T)$ and $c(x, t)$, $\gamma(x, t) \in L_\infty(Q_T)$. The generalized global solution $v(x, t) \in Z$ for the Cauchy problem of the nonlinear singular integral-differential equation (19) is unique.

3. Geometric flows and harmonic maps

Generally, the Landau-Lifshitz equation (2) can be generalized to a Riemannian manifold. In this Section we use the following notations. Let (M, g) be an m-dimensional Riemannian manifold with metric g, "$\wedge$" denotes the exterior operator in $\mathbb{R}^n$, "$*$" denotes the Hodge star operator of $\mathbb{R}^n$. Δ_M denotes the Laplace-Beltrami operator with respect to the metric g of M. The Laplace-Beltrami operator and the norm $\|\nabla u\|$ are expressed by

$$\Delta_M u = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{g} \frac{\partial u}{\partial x^j} \right) = g^{ij} \left(\frac{\partial^2 u}{\partial x^i \partial x^j} - \Gamma|_{ij}^k \frac{\partial u}{\partial x^k} \right)$$

$$\|\nabla u(x)\|^2 = \sum_{i,j} \sum_k g^{ij} \frac{\partial u^k}{\partial x^i} \frac{\partial u^k}{\partial x^j}.$$

In 1993, Guo and Hong [24] studied the following Landau-Lifshitz type equation from Riemannian manifolds M into the unit sphere S^2.

$$v_t = -\alpha_1 v \times (v \times \Delta_M v) + \alpha_2 v \times \Delta_M v, \tag{20}$$

where $\alpha_1 > 0$ and α_2 are constants. The Equation (20) plays a fundamental role in understanding nonequilibrium magnetism. Much research has contributed to the study of solitons in the Landau-Lifshitz equation for one-dimensional spin chains. However, little is known about higher-dimensional motion in the context of the Heisenberg spin chain in physics.

Guo and Hong [24] proved the global existence and regularity of Equation (20) and established some profound connections between this equation and harmonic map theory by the methods of extended Sobolev inequalities, local energy monotonicity and Galerkin approximation method.

Theorem 16 ([24], Theorem 2.6) Let Ω be the flat torus $\mathscr{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Suppose that $\nabla v_0(x)$ is a given initial value in $H^s(\Omega, S^2)$ satisfting Equation (20) where s is large enough. Then there exists a constant $C > 0$ such that the periodic value problem

$$v|_{t=0} = v(x, 0) = v_0(x),$$

with the initial value v_0, has a smooth global solution $v(x, t)$ provided $\|\nabla v_0\| \leqslant C$.

Remark 2 Theorem 16 is also true for the Cauchy problem on $\mathbb{R}^2$.

Theorem 17 ([24], Theorem 3.13) Let M be a closed Riemannian surface. For any initial value $v_0 \in H^{1,2}(M; S^2)$ there exists a unique solution v of Equation (20) on $M \times [0, \infty)$ which is regular on $M \times (0, \infty)$ with exception of at most finitely many points (x^m, T^m), $1 \leqslant m \leqslant L$, characterized by the condition that

$$\limsup_{T \to T^{m-}} E_R(v(\cdot, T), x^m) > \varepsilon_1, \ \forall R \in (0, R_0].$$

In higher dimensions, suppose that M be a compact d-dimensional Riemannian manifold without boundry, and $d \geqslant 3$. Guo and Hong proved that Equation (20) is equivalent to the following equation

$$\frac{\alpha_1}{\alpha_1^2 + \alpha_2^2} v_t - \frac{\alpha_2}{\alpha_1^2 + \alpha_2^2} v \times v_t = \Delta v + |\nabla v|^2 v. \tag{21}$$

Definition 5 ([24], Weak solution) A vector function $v(x, t)$ is said to be a global weak solution of Equation (21), if v is defined a.e. $M \times R^+$ such that

1. $v \in L^\infty(0, \infty; H^{1,2}(M))$ and $v_t \in L^2((0, \infty); L^2(M))$;
2. $|v(x, t)|^2 = 1$ a.e. on $M \times \mathbb{R}^+$;
3. Equation (21) holds in the sense of distribution;
4. $v(x, 0) = v_0(x)$ in the trace sense.

Theorem 18 ([24], Theorem 4.2) Let $\alpha_1 > 0$ in (20). For any v_0 in $H^{1,2}(M, S^2)$, there exists a global weak solution of Equation (21).

Later in 1996, Guo and Wang [25] considered the generalized Landau-Lifshitz:

$$v_t = \alpha_1(\Delta_M v + |\nabla v|^2 v) + \alpha_2 * [v \wedge a_2(v) \wedge \cdots \wedge a_{n-2}(v) \wedge \Delta_M v], \tag{22}$$

where $a_i(v)$, $(i=2, \cdots, n-2)$ are smooth vector function from S^{n-1} to $\mathbb{R}^n$. They constructed local solutions of Cauchy problem for Equation (22) via local energy control and compactness methods. They established connections between harmonic maps and solutions of the generalized Landau-Lifshitz equation by employing methods analogous to those used to demonstrate the existence of heat flow for harmonic mappings.

Theorem 19 ([25], Theorem 1.2) Assume that M is a smooth closed Riemannian manifold, initial data $v_0 = v(x, 0) \in C^2(M, S^{n-1})$, and that $a_i(v)(i=2, \cdots, n-2)$ are smooth vector functions. Then the Cauchy problem for Equation (22) admits a unique smooth maximal solution defined on the subinterval $[0, \omega] \subseteq [0, T]$.

Combining the energy inequality and higher-order regularity estimates, they extended local solutions globally if the initial energy is small enough on $\mathscr{T}^2$.

Theorem 20 ([25], Theorem 2.1) Suppose $M = \mathscr{T}^2$, $v_0 \in H^k(\mathscr{T}^2, S^{n-1})(k \geqslant 4)$ and $E(v_0) := \frac{1}{2}|\nabla v|^2$ is small enough. Then Equation (22) admits a global classical solution.

Moreover, Guo and Wang analyzed singularities in harmonic map heat flow using energy concentration and Struwe's methods, demonstrating that the solution is regular except at a finite number of points. Assuming that the initial energy is small, the solution remains globally regular.

Theorem 21 ([25], Theorem 3.1) Let $v_0 \in H^1(M, S^{n-1})$, there exists a unique solution to Equation (22) on $M \times [0, \infty)$, which is regular on $M \times [0, \infty)$ with exception of at most finitely points (x^m, T^m), $1 \leqslant m \leqslant L$, characterized by the condition that

$$\limsup_{T \to T^{m-}} E_a(v(\cdot, T), x^m) > \varepsilon \text{ for all } a \in (0, R_0].$$

At a singularity point $(\bar{x}, \bar{t})$, a smooth harmonic map $\bar{v}$: $S^2 \to S^{n-1}$ separates in the sense that for sequence $x_m \to \bar{x}$, $t_m \nearrow \bar{t}$, $R_m \searrow 0$ as $m \to \infty$, the family

$$v_m(x) \equiv v(\exp(R_m x), t_m) \to \tilde{v} \text{ in } H^2_{loc}(\mathbb{R}^2, S^{n-1}),$$

where $\tilde{v}$ has finite energy and extends to a smooth harmonic map $\bar{v}$: $S^2 \to S^{n-1}$. Finally, for suitable sequence $t_m \to \infty$ the sequence of maps $v(\cdot, t_m)$ converges weakly in $H^1(M, S^{n-1})$ to a smooth v_∞: $M \to S^{n-1}$.

Theorem 22 ([25], Theorem 3.2) Let

$$b = \inf\left\{E(v) \colon v \colon S^2 \to S^{n-1} \text{ be a nonconstant smooth harmonic map}\right\}.$$

If $E(v_0) < b$, then the solution is globally regular on $M \times [0, \infty)$.

4. (Random) attractors

Global or pullback attractors are crucial for comprehending the long-term dynamics of Partial Differential Equations (PDEs). Once the existence of random attractors in random dynamical systems or stochastic PDEs is established, it becomes imperative to analyze their finite fractal or Hausdorff dimension to explore the infinite-dimensional dynamics of stochastic PDEs.

In 2009, Guo and Huang [26] investigated the global well-posedness and the existence of random attractors for the three-dimensional viscous stochastic primitive equations that model large-scale oceanic motion under random forcing.

They examined the initial boundary value problem for these equations as follows. Given the stochastic nature of the nonlinear evolution equations, it is natural to study their random attractors.

$$\begin{cases} \frac{\partial w}{\partial t}+(w\cdot\nabla)w+\Phi(w)\frac{\partial w}{\partial z}+fk\times w+\nabla p_b-\int_{-1}^{z}\nabla T dz'-\Delta w-\frac{\partial^2 w}{\partial z^2}=\Psi, \\ \frac{\partial T}{\partial t}+(w\cdot\nabla)T+\Phi(w)\frac{\partial T}{\partial z}-\Delta T-\frac{\partial^2 T}{\partial z^2}=Q, \\ \int_{-1}^{0}\nabla\cdot w dz=0, \\ \frac{\partial w}{\partial z}=0,\ \frac{\partial T}{\partial z}=-\alpha_u T, \quad \text{on } \Gamma_u, \\ \frac{\partial w}{\partial z}=0,\ \frac{\partial T}{\partial z}=0, \quad \text{on } \Gamma_b, \\ w\cdot\vec{n}=0,\ \frac{\partial w}{\partial \vec{n}}\times\vec{n}=0,\ \frac{\partial T}{\partial \vec{n}}=0,\ \text{on } \Gamma_l, \\ U|_{t=t_0}=\left(w|_{t=t_0},\, T|_{t=t_0}\right)=U_{t_0}=\left(w_{t_0},\, T_{t_0}\right), \end{cases} \tag{23}$$

where the unknown functions are $w,\, T,\, p.$ $w=(w^{(1)},\, w^{(2)})$ is the horizontal velocity, p is the pressure, k is the vertical unit vector T is the temperature, $f=f_0(\beta+y)$ is the Corilis parameter, $Q(x,\, y,\, z)$ is a given heat source, α_u is a positive constant, $\overrightarrow{n}$ is the norm vector of Γ_l, p_b is a certain unknown function at Γ_b, and

$$\Phi(w)(t,\, x,\, y,\, z)=-\int_{-1}^{z}\nabla\cdot w(t,\, x,\, y,\, z')dz'.$$

The domain of (23) is

$$\Omega=\{(x,\, y,\, z)\colon (x,\, y)\in M,\, z\in(-1,\, 0)\},$$

where M is a smooth bounded domain in $\mathbb{R}^2$. $\Gamma_u=M\times\{0\}$, $\Gamma_b=M\times\{-1\}$, $\Gamma_l=\partial M\times[-1,\, 0]$. $\Psi(t,\, x,\, y,\, z)$ is an additive white noise with the form

$$\Psi(t,\, x,\, y,\, z)=G\frac{\partial V}{\partial t},$$

where the derivative is in the It$\hat{o}$ integral sense, the random process V is a two-sided in time cylindrical Wiener process in H_1 with the form

$$V(t)=\sum_{i=1}^{+\infty}v_i(t,\, \omega)e_i,$$

and G is a Hilbert-Schimidt operator from H^1 to $H^{1+2c_0}(\Omega)\times H^{1+2c_0}(\Omega)$ for some $c_0>0$, i.e.

$$\sum_{i=1}^{+\infty} \|Ge_i\|_{1+2c_0}^2 < +\infty.$$

Here $v_1, v_2, \cdots$ is a sequence of independent standard one-dimensional Brownian motions on a complete probability space $(\Omega, \mathscr{F}, P)$ with expectation denoted by E, and $H^{1+2c_0}(\Omega)$ is the usual Sobolev space with non-integer order.

The work spaces used in [26] are as follows.

$$\mathscr{V}_1 = \{w \in C^\infty(\Omega)^2;\ \frac{\partial w}{\partial z}|_{\Gamma_w, \Gamma_b\}=0,\ w\cdot\vec{n}|_{\Gamma_l}=0,\ \frac{\partial w}{\partial \vec{n}}\times\vec{n}|_{\Gamma_l}=0,\ \int_{-1}^{0}\nabla\cdot w dz=0},$$

$$\mathscr{V}_2 = \{T \in C^\infty(\Omega);\ \frac{\partial T}{\partial z}|_{\Gamma_w\}=-\alpha_w T,\ \frac{\partial T}{\partial z}|_{\Gamma_b}=0,\ \frac{\partial T}{\partial \vec{n}}|_{\Gamma_l}=0},$$

$$V_1 = \text{ closure of } \mathscr{V}_1 \text{ with respect to the norm } \|\cdot\|_1,$$

$$V_2 = \text{ closure of } \mathscr{V}_2 \text{ with respect to the norm } \|\cdot\|_1,$$

$$W = V_1 \times V_2.$$

Guo and Huang's results [26] are as follows.

Theorem 23 ([26], Theorem 1.1) If $Q \in H^1(\Omega)$ and $W_{t_0} \in W$, then for any given $T > t_0$, there exists a unique strong solution W of the system (23) on the interval $[t_0, T]$; moreover, the strong solution W is dependent continuously on the initial data.

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space, $\{\theta_t\colon \Omega \to \Omega, t \in \mathbb{R}\}$ a family of measure preserving transformation such that $\theta_0 = \mathrm{id}_\Omega$ and $\theta_{t+s} = \theta_t \circ \theta_s$ for all $t, s \in \mathbb{R}$. $\{\theta_t\}$ is called a metric dynamical system on Ω, which represents the noise driving a random dynamical system. Assume that θ_t is ergodic under P.

Definition 6 ([26], Random dynamical system; Definition 6.1) A measurable map $\varphi\colon \mathbb{R}^+ \times \Omega \times X \to X$, $(t, \omega, U) \to \varphi(t, \omega)U$ is called a random dynamical system if φ satisfies the cocycle property: $\varphi(0, \omega) = \mathrm{id}_X$, $\varphi(t+s, \omega) = \varphi(t, \theta_s\omega)\varphi(s, \omega)$ for all $t, s \in \mathbb{R}^+$ and $P-a.s.$ $\omega \in \Omega$. If $\varphi(t, \omega)\colon X \to X$ is continuous, then φ is called a continuous random dynamical system.

Definition 7 ([26], Random compact set; Definition 6.2) Let $L\colon \Omega \to 2^X$, 2^X be the set of all subsets of X. L is called a random compact set if $L(\omega)$ is compact $P-a.s.$ and the map $\omega \to d(V, L(\omega))$ is measurable for any $U \in X$, where

$$d(V, L(\omega)) = \inf_{V_1 \in L(\omega)} d(V, V_1).$$

Definition 8 Let $A(\omega)$, $B(\omega)$ be two random sets.

1. $A(\omega)$ attracts $B(\omega)$ if $\lim_{t\to+\infty} d(\varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega), A(\omega)) = 0$, $P-a.s.$
2. $A(\omega)$ absorbs $B(\omega)$ if there exists $t_B(\omega)$ such that for all $t \geqslant t_B(\omega)$,

$$\varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega) \subseteq A(\omega),\ \ P-a.s.$$

Definition 9 ([26], Random attractor; Definition 6.4) A random set $\mathscr{A}(\omega)$ is said to be a random attractor for the random dynamical system φ if $P-a.s.$

1. $\mathscr{A}(\omega)$ is a random compact set.
2. $\mathscr{A}(\omega)$ is invariant, that is, $\varphi(t,\, \omega)\mathscr{A}(\omega) = \mathscr{A}(\theta_t\omega),$ for $\forall t \geqslant 0$.
3. $\mathscr{A}(\omega)$ attracts all deterministic bound sets $B \subseteq X$, i.e.

$$\lim_{t\to+\infty} d(\varphi(t,\, \theta_{-t}\omega)B,\, A(\omega)) = 0,\ P-a.s.$$

Remark 3 ([26], Remark 6.5) $\varphi(t,\, \theta_{-t}\omega)U$ can be interpreted as follows. When t is in motion, the trajectory $\varphi(t,\, \theta_{-t}\omega)U$ consistently corresponds to the position at $t = 0$. Consequently, the random attractor is also referred to as the random pull-back attractor.

Theorem 24 ([26], Theorem 1.2) The system (23) has a unique random pull-back attractor $\mathscr{A}(\omega)$ which captures all trajectories initiated at time $-\infty$ and evolved, under the action of the shift θ_t from $t = -\infty$ to $t = 0$. The attractor $\mathscr{A}(\omega)$ enjoys the following:

1. $\mathscr{A}(\omega)$ is bounded and weakly closed in W;
2. $\mathscr{A}(\omega)$ is invariant in the following sense: $\varphi(t,\, \omega)\mathscr{A}(\omega) = \mathscr{A}(\theta_t\omega)$, $\forall t \geqslant 0$;
3. $\mathscr{A}(\omega)$ is attracting which means that, for any deterministic bounded set D in W, the set $\varphi(t,\, \theta_{-t}\omega)D$ converge to $\mathscr{A}(\theta_t\omega)$ with respect to W-weak topology as $t \to +\infty$, i.e.,

$$\lim_{r\to+\infty} d_W^{\omega}(\varphi(t,\, \theta_{-t}\omega)D,\, \mathscr{A}(\omega)) = 0,\ P-a.s.,$$

where the distance d_W^{ω} is induced by the W-weak topology.

Later in 2011, Guo and Huang [27] considered the global well-posedness and the existence of random attractors for the three-dimensional viscous primitive equations of the large-scale moist atmosphere. They studied the initial boundary value problem of 3D viscous stochastic primitive equations

$$\begin{cases} \frac{\partial u}{\partial t} + \nabla_u u + W(u)\frac{\partial u}{\partial \xi} + \frac{f}{R_1} k \times u + \text{grad}\ \phi_s + \int_{\xi}^{1} \frac{bP}{p} \text{gard}\ [(1+aQ)T] d\xi' - \Delta u - \frac{\partial^2 u}{\partial \xi^2} = 0, \\ \frac{\partial T}{\partial t} + \nabla_u T + W(u)\frac{\partial T}{\partial \xi} - \frac{bP}{p}(1+aQ)W(u) - \Delta T - \frac{\partial^2 T}{\partial \xi^2} = Q_1, \\ \frac{\partial Q}{\partial t} + \nabla_u Q + W(u)\frac{\partial Q}{\partial \xi} - \Delta Q - \frac{\partial^2 Q}{\partial \xi^2} = Q_2, \\ \int_0^1 \text{div}\ u\ d\xi = 0, \\ \xi = 1\colon \frac{\partial u}{\partial \xi} = 0,\ \frac{\partial T}{\partial \xi} = -\alpha_s T,\ \frac{\partial Q}{\partial \xi} = -\beta_s Q, \\ \xi = 0\colon \frac{\partial u}{\partial \xi} = 0,\ \frac{\partial T}{\partial \xi} = 0,\ \frac{\partial Q}{\partial \xi} = 0, \\ U|_{t=0} = (u|_{t=0},\, T|_{t=0},\, Q|_{t=0}) = U_0 = (u_0,\, T_0,\, Q_0), \end{cases} \tag{24}$$

where $u = (u_\theta, u_\varphi)$ denotes the horizontal velocity, ω stands for the vertical velocity in p-coordinate system, ϕ is the geopotential, Q is the mixing ratio of water vapor in the air, T is temperature, $f = 2\cos\theta$ is the Coriolis parameter, R_1 is the Rossby number, k is vertical unit vector, P is an approximate value of pressure at the surface of the earth, p_0 is the pressure of the upper atmosphere and $p_0 > 0$, the variable ξ satisfies $p = (P - p_0)\xi + p_0 (0 < p_0 \leqslant p \leqslant P)$, Q_1, Q_2 are known functions on $S^2 \times (0, 1)$, a is a positive constant $a \approx 0.618$, b is a positive constant. The space domain of the system (24) is $\Omega = S^2 \times (0, 1)$, and

$$\omega(t; \theta, \varphi, \xi) = W(u)(t; \theta, \varphi, \xi) = \int_\xi^1 \operatorname{div} u(t; \theta, \varphi, \xi')d\xi',$$

$$\phi(t; \theta, \varphi, \xi) = \phi_s(t; \theta, \varphi) + \int_\xi^1 \frac{bP}{p}(1 + aQ)T d\xi'.$$

The work spaces used in [27] are as follows.

$$\widetilde{\mathscr{V}_1} = \{u\colon u \in C^\infty(T\Omega|TS^2),\ \frac{\partial u}{\partial \xi}|_{\xi=0} = 0,\ \frac{\partial u}{\partial \xi}|_{\xi=1} = 0,\ \int_0^1 \operatorname{div} u\, d\xi = 0\},$$

$$\widetilde{\mathscr{V}_2} = \{T\colon T \in C^\infty(\Omega),\ \frac{\partial T}{\partial \xi}|_{\xi=0} = 0,\ \frac{\partial T}{\partial \xi}|_{\xi=1} = -\alpha_s T\},$$

$$\widetilde{\mathscr{V}_3} = \{Q\colon Q \in C^\infty(\Omega),\ \frac{\partial Q}{\partial \xi}|_{\xi=0} = 0,\ \frac{\partial Q}{\partial \xi}|_{\xi=1} = -\beta_s Q\},$$

$$\widetilde{V_1} = \text{ the closure of } \widetilde{\mathscr{V}_1} \text{ with respect to the norm } \|\cdot\|_1,$$

$$\widetilde{V_2} = \text{ the closure of } \widetilde{\mathscr{V}_2} \text{ with respect to the norm } \|\cdot\|_1,$$

$$\widetilde{V_3} = \text{ the closure of } \widetilde{\mathscr{V}_3} \text{ with respect to the norm } \|\cdot\|_1,$$

$$\widetilde{V_0} = \widetilde{V_1} \times \widetilde{V_2} \times \widetilde{V_3}.$$

Guo and Huang's results in [27] are as follows.

Proposition 1 ([27], Existence; Proposition 3.1) Let Q_1, $Q_2 \in H^1(\Omega)$ and $U_0 = (u_0, T_0, Q_0) \in \widetilde{V_0}$. Then for any $\mathscr{T} > 0$ given, there exists a strong solution u of the system (24) on the interval $[0, \mathscr{T}]$.

Proposition 2 ([27], Uniqueness; Proposition 3.2) Let $Q_1, Q_2 \in H^1(\Omega)$, $U_0 = (u_0, T_0, Q_0) \in \widetilde{V_0}$. Then for any $T > 0$ given, the strong solution u of the system (24) on the interval $[0, T]$ is unique. Moreover, the strong solution u is dependent continuously on the initial data.

Theorem 25 ([27], Theorem 3.4) The system (24) possesses a weak universal attractor $\mathscr{A} = \bigcap_{s\geqslant 0} \overline{\bigcup_{t\geqslant s} S(t)B_\rho}$ that encompasses all trajectories, with closures defined in the context of $\widetilde{V_0}$-weak topology. The $\widetilde{V_0}$-weak universal attractor $\mathscr{A}$ exhibits the following properties:

1. (Weak compact) $\mathscr{A}$ is bounded and weakly closed in $\widetilde{V_0}$;
2. (invariant) $S(t)\mathscr{A} = \mathscr{A}$, $\forall t \geqslant 0$;

3. (attracting) for every bounded set D in $\widetilde{V}_0$, the set $S(t)D$ converge to $\mathscr{A}$ with respect to $\widetilde{V}_0$-weak topology as $t \to +\infty$, i.e.,

$$\lim_{t \to +\infty} d^{\omega}_{\widetilde{V}_0}(S(t)D, \mathscr{A}) = 0,$$

where the distance $d^{\omega}_{\widetilde{V}_0}$ is induced by the $\widetilde{V}_0$-weak topology.

In 2024, Wang et al. [28] presented comprehensive and unified results concerning the existence, regularity, and finite fractal dimension estimates of pullback random attractors for a wide range of non-autonomous stochastic hydrodynamical systems derived from fluid dynamics. They considered the Cauchy problem of the following abstract stochastic hydrodynamical system in H for $t > s$ with $s \in \mathbb{R}$

$$du(t) + Au(t)dt + B(u(t), u(t))dt + R(t, u(t))dt = g(t)dt + hdW(t), \tag{25}$$

$$u(s) = u_s \in H. \tag{26}$$

Here, H is a separable Hlibert space with the inner product and norm $(\cdot, \cdot)$ and $|\cdot|$, respectively. A is an unbounded self-adjoint positive linear operator on H such that $U \subseteq H \subseteq U'$ is a Gelfand triple, where $U = \mathrm{Dom}(A^{1/2})$ with norm $\|\cdot\| = |A^{1/2} \cdot|$. Let $\langle \cdot, \cdot \rangle$ be the duality between U' and U. Assume that $R(s, \cdot)$: $H \to H$ is a bounded linear operator uniformly for $s \in \mathbb{R}$, i.e., there exists a constant $\ell_1 > 0$ independent of s such that

$$|R(s, u)| \leqslant \ell_1 |u|, \ \forall u \in H.$$

Assume that B: $U \times U \to U'$ is a bilinear continuous mapping with

$$\langle B(u, w), v \rangle = -\langle B(u, v), w \rangle, \forall u, v, w \in U$$

$$|\langle B(u, v), w \rangle| \leqslant \ell_2 |u|^{1/2} \|u\|^{1/2} \|v\|^{1/2} |w|^{1/2} \|w\|^{1/2}, \forall u, v, w \in U$$

where $\ell_2 > 0$ is a constant. Assume that $h \in U$, $g \in L^2_{loc}(\mathbb{R}, U')$ and the two-sided real-valued Wiener process T is defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $\{\theta_t\}_{t \in \mathbb{R}}$ be a family of shift operators on Ω defined by $\theta_t \omega(\cdot) = T(\cdot + t) - \omega(t)$ for $(\omega, t) \in \Omega \times \mathbb{R}$. Then $\left(\Omega, \mathscr{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}}\right)$ is a ergodic metric dynamical system.

By introducing the weakly tempered condition to replace traditional tempered conditions, combining with spectral decomposition and Lipschitz projection techniques, Wang et al. [28] obtained an improved upper bound estimate for the fractal dimension of random invariant sets in both Banach space and its subspaces. They also proved asymptotic compactness of solutions via energy equation method and trajectory estimates, leading to the existence of random attractors.

Theorem 26 ([28], Theorem 3.8) Assume that $g \in L^2_{loc}(\mathbb{R}, U')$ satisfies

$$\int_{-\infty}^{s} e^{\frac{1}{16}\sigma_1 r} \|g(r)\|^2_{U'} dr < \infty, \ \forall s \in \mathbb{R}.$$

The continuous cocycle Φ for (25)–(26) admits a unique $\mathfrak{D}$-pullback random attractor $\mathscr{A} = \{\mathscr{A}(s, \omega): s \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$ in H satisfying

$$\mathscr{A}(s, \omega) \subseteq \mathscr{K}(s, \omega), \ \forall (s, \omega) \in \mathbb{R} \times \Omega,$$

$$|\mathscr{A}(s, \omega)|^2 \leqslant |\mathscr{K}(s, \omega)| \leqslant \mathscr{R}(s, \omega), \ \forall (s, \omega) \in \mathbb{R} \times \Omega.$$

Assume that there are constants ℓ_3, ℓ_4 such that the bilinear continuous mapping B satisfies

$$|\langle B(u, v), w\rangle| \leqslant \ell_3 |u|^{1/2} |Au|^{1/2} \|v\| |w|, \quad \forall u \in \mathrm{Dom}(A), v \in U, w \in H; \tag{27}$$

$$|\langle B(u, v), w\rangle| \leqslant \ell_4 |u|^{1/2} \|u\|^{1/2} \|v\|^{1/2} |Av|^{1/2} |w|, \quad \forall u \in U, v \in \mathrm{Dom}(A), w \in H. \tag{28}$$

Wang et al. [28] obtained the finite fractal dimension of the random attractors in U and H.

Theorem 27 ([28], Theorem 5.5) Let $g \in L^\infty(\mathbb{R}, H)$, Equation (27) and Equation (28) hold. For each $s \in \mathbb{R}$ and $\omega \in \Omega$, we have:

1. The fractal dimension of $\mathscr{A}(s, \omega)$ has a finite upper bound in H:

$$\dim_H \mathscr{A}(s, \omega) \leqslant \frac{3m_0}{\ln 2} \ln(8\sqrt{m_0} + 1) < \infty,$$

where $m_0 \in \mathbb{N}$ is independent of s and ω.

2. The fractal dimension of $\mathscr{A}(s, \omega)$ has a finite upper bound in U:

$$\dim_V \mathscr{A}(s, \omega) \leqslant \frac{3m}{\ln 2} \ln\left(8\sqrt{m} + 1\right) < \infty,$$

where $m \in \mathbb{N}$ is independent of s and ω, which is large than m_0.

The upper bound here is smaller than that of Zhou et al. [29, Theorem 2.1], which is

$$\frac{16m}{5\ln 2} \ln\left(\sqrt{m}/\delta + 1\right),$$

where $0 < \delta < 1/16$.

5. Harmonic analysis and PDE

5.1 *Finite depth fluid equation*

In 1994, Guo and Tan [30] studied Cauchy problem for the generalized equation governing finite-depth fluids

$$\partial_t v - G(\partial_x^2 v) - \partial_x \left(\frac{v^p}{p}\right) = 0, \quad v(0, x) = v_0(x), \tag{29}$$

where

$$G(f) = -i\mathscr{F}^{-1}\left(\coth(2\pi\delta\xi) - \frac{1}{2\pi\delta\xi}\right)\widehat{f},$$

is a singular integral, and p is an integer larger than 1. Guo and Tan [30] obtained that the solutions to the nonlinear problem with small initial data for $p > 5/2+\sqrt{21}/2$ decay over time and asymptotically approach the solutions of the linear problem.

Theorem 28 Let $\delta \in (0, \infty)$, $q = 2p$, and $p > 5/2+\sqrt{21}/2$. Assume that the initial data $v_0 \in H^3(\mathbb{R}) \cap W^{2, 2p/(2p-1)}(\mathbb{R})$ is sufficiently small. Then the solution v of nonlinear problem (29) satisfying

$$\|v(t)\|_{W^{2, q}} \leqslant C(1+|t|)^{-(1-2/q)/3}, \tag{30}$$

for all $t > 0$, where the constant C is independent of v and t.

Noticing that Equation (30) has implied that the solution is scattering in $H^2(\mathbb{R})$.

5.2 *Fractional NLS*

In 2011, Guo and Huo [31] consider the Cauchy problem for the fractional NLS

$$\mathrm{i}v_t + (-\Delta)^{\alpha} v + |v|^2 v = 0, \quad v(0, x) = v_0, \tag{31}$$

and they obtain the following

Theorem 29 Let $1/2 < \alpha < 1$, $u_0 \in L^2(\mathbb{R})$. Then (31) is globally well-posed in $L^2(\mathbb{R})$.

5.3 *Frequency uniform decomposition methods*

In [32], Feichtinger introduced the notion of modulation spaces $M^s_{p, q}$ via short time Fourier transform V_g, whose norm is defined by

$$\|f\|_{M^s_{p, q}} = \|\langle\xi\rangle^s V_g f\|_{L^{p, q}(\mathbb{R}^{2d})}, \quad V_g f(x, \xi) := \int_{\mathbb{R}^d} f(t)\overline{g(x-t)}e^{-it\xi}dt$$

for a smooth cut-off function g. Frequency uniform decompositions can be regarded as the frequency-discrete version of V_g which are defined as follows. Let ρ be a smooth cut-off function adapted to the unit cube $[-1/2, 1/2]^d$ and $\rho = 0$ outside the cube $[-3/4, 3/4]^d$.

We writet $\rho_k = \rho(\cdot - k)$ and assume that

$$\sum_{k\in\mathbb{Z}^d}\rho_k(\xi)\equiv 1,\quad \forall\,\xi\in\mathbb{R}^d.$$

The frequency uniform decomposition operators are defined as follows:

$$\Box_k:=\mathscr{F}^{-1}\rho_k\mathscr{F},\quad k\in\mathbb{Z}^d.$$

Let $1\leqslant p,\, q\leqslant\infty,\ s\geqslant 0$, we can define the modulation spaces which have the exponential regularity,

$$\|f\|_{E^s_{p,\,q}}:=\left\|\{2^{s|k|}\Box_k f\}_{k\in\mathbb{Z}^d}\right\|_{\ell^q(L^p)}$$

and $M^s_{p,\,q}$ has an equivalent norm $\|f\|_{M^s_{p,\,q}}:=\left\|\{\langle k\rangle^s\Box_k f\}_{k\in\mathbb{Z}^d}\right\|_{\ell^q(L^p)}$ (cf. [33]).

In 2006, Wang, Zhao and Guo [34] first applied the frequency uniform decomposition techniques to study nonlinear PDE, which have been developed to a systematic method in the past two decades. Roughly speaking, using modulation spaces to study nonlinear evolution equation, an advantage is that the regularity of modualtion spaces are much lower than that of Sobolev spaces, for instance, $H^{d/2+}\subset M^0_{2,\,1}=E^0_{2,\,1}\subset L^\infty\cap L^2$ are optimal embeddings, where $\|f\|_{H^s}=\|\langle\xi\rangle^s\hat{f}\|_{L^2}$. The initial value problem for the Complex Ginzburg-Landau (CGL) equation was studied by using $E^s_{p,\,q}$:

$$v_t-(a+\mathrm{i}\alpha)\Delta v+(b+\mathrm{i}\beta)|v|^{2\kappa}v+\nu v=0,\quad v(0,\,x)=v_0(x),\tag{32}$$

where $v(t,\,x)$ is a complex valued function of $(t,\,x)\in\mathbb{R}^+\times\mathbb{R}^d$, $\mathbb{R}^+=[0,\,\infty)$. $a>0,\ b\geqslant 0,\ \kappa\in\mathbb{N},\ \alpha,\ \beta,\ \nu\in\mathbb{R}$. v_0 is a complex valued function of $x\in\mathbb{R}^d$. Denote

$$G_1(\mathbb{R}^d)=\left\{f\in C^\infty(\mathbb{R}^d)\colon\ \exists\rho,\,M>0\ s.t.\ \|f\|_{\dot{H}^m}\leq M\frac{m!}{\rho^m},\ \forall m\in\mathbb{Z}_+,\right\}$$

and $G_1(\mathbb{R}^d)$ is said to be the Gevrey 1-class.

One can show that $G_1(\mathbb{R}^d)$ is the collection of all $E^\lambda_{2,\,1}$, $\lambda>0$, that is $G_1(\mathbb{R}^d)=\cup_{\lambda>0}E^\lambda_{2,\,1}$. Wang et al. [34] obtained the following result:

Theorem 30 Let $a>0,\ \kappa\in\mathbb{N};\ b,\ \alpha,\ \beta,\ \nu\in\mathbb{R};\ u_0\in E^0_{2,\,1}(\mathbb{R}^d)$, $n\geq 1$. Then there exists $T^*:=T^*(\|u_0\|_{E^0_{2,\,1}})>0$ such that Equation (32) has a unique solution

$$u\in C_{\mathrm{loc}}([0,\,T^*);\,E^0_{2,\,1}(\mathbb{R}^d)).\tag{33}$$

Moreover, this solution has the Gevrey 1-class regularity effect: there exists $t_0>0$ such that

$$\begin{cases} u(t) \in E^{ct}_{2,\,1} \subset G_1(\mathbb{R}^d) \subset C^\infty(\mathbb{R}^d), \quad \forall\, t \in [0,\, t_0], \\ \\ u(t) \in E^{ct_0}_{2,\,1} \subset G_1(\mathbb{R}^d) \subset C^\infty(\mathbb{R}^d), \quad \forall\, t_0 < t < T^*, \end{cases} \tag{34}$$

and for any $T < T^*$,

$$\sup_{0 \le t \le T} \|u(t)\|_{E^{c(t_0 \wedge t)}_{2,\,1}} \le C(T,\, \|u_0\|_{E^0_{2,\,1}}). \tag{35}$$

Further, if $b > 0$ and $|\alpha|(d\kappa - 2)/a < 2\sqrt{d\kappa - 1}$, then the above solution is a global one, i.e. $T^* = \infty$ in (33)–(35).

The proof of the above Theorem is based on the frequency uniform decomposition and contraction mapping, the solutions map $u_0 \to u(t)$ is continuous from $E^0_{2,\,1}$ to $E^{c(t_0 \wedge t)}_{2,\,1}$, $\forall\, 0 < t \leqslant T$, so the regularity of the solution in $E^{c(t_0 \wedge t)}_{2,\,1}$ is preserved if the initial data have a small perturbation $E^0_{2,\,1}$.

5.4 *On Davey-Stewartson system*

In 1999, Guo and Wang [35] considered the Cauchy problem for the generalized Davey-Stewartson system, and studied the Cauchy problem of the following generalized Davey–Stewartson systems:

$$\begin{cases} \mathrm{i}u_t + \triangle u = a|u|^\alpha u + b_1 u v_{x_1}, \\ \\ \triangle v = b_2(|u|^2)_{x_1}, \\ \\ u(0,\, x) = u_0(x), \end{cases} \tag{36}$$

where $u(t,\, x)$ and $v(t,\, x)$ $(x = (x_1,\, \ldots,\, x_d))$ are complex valued functions of $(t,\, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $\triangle$ is the Laplace operator on $\mathbb{R}^n$, $a,\, b_1,\, b_2 \in \mathbb{R}$. Guo and Wang [35] obtained the following global well-posedness results

Theorem 31 Let $d = 2$. $1 \leqslant s \leqslant 2$ and $u_0 \in H^s$. Suppose that one of the following conditions holds:

(i) $a > 0,\, 2 < \alpha < \infty$;

(ii) $\alpha = 2,\ a \geqslant \max(0,\, b_1 b_2)$;

(iii) $\alpha = 2,\ b_1 b_1 \geqslant 0$ and $(b_1 b_2 - a)\|u_0\|^2_{L^2} < 4$;

(iv) $\alpha = 2,\ b_1 b_1 < 0$ and $-a\|u_0\|^2_{L^2} < 4$;

(v) $1 \leqslant \alpha < 2,\, b_1 b_2 \|u_0\|^2_{L^2} < 4$.

Then, Equation (36) has a unique solution $u \in C_{\mathrm{loc}}(0,\, \infty;\, H^s) \cap L^{\gamma(r)}_{\mathrm{loc}}(0,\, \infty;\, H^{s,\, r}) \cap C(0,\, \infty;\, H^1)$ for any $r \in [2,\, \infty)$ and $2/\gamma(r) = d(1/2 - 1/r)$.

Theorem 32 Let $d = 3$. $1 \leqslant s \leqslant 2$ and $u_0 \in H^s$. Suppose that one of the following conditions holds:

(i) $a > 0,\, 2 < \alpha < 4$;

(ii) $\alpha = 2,\ a > 0$ and $a \leqslant b_1 b_2$.

Then, (1.4) has a unique solution $u \in C_{\mathrm{loc}}(0,\, \infty;\, H^s) \cap L^{\gamma(r)}_{\mathrm{loc}}(0,\, \infty;\, H^{s,\, r}) \cap C(0,\, \infty;\, H^1)$ for any $r \in [2,\, 6)$ and $2/\gamma(r) = d(1/2 - 1/r)$.

6. Variation methods and blowup solutions

In 2013, Gan et al. [36] constructed a kind of blow-up solutions of the generalized Zakharov system with magnetic fields

$$\begin{cases} \mathrm{i}\partial_t \mathbf{E} + \Delta \mathbf{E} - n\mathbf{E} + \mathrm{i}(\mathbf{E}\wedge\mathbf{B}) = 0, \\ n_{tt} - c_0^2\Delta n = c_0^2\Delta|\mathbf{E}|^2, \\ \Delta\mathbf{B} - i\eta\nabla\times(\nabla\times(\mathbf{E}\wedge\overline{\mathbf{E}})) + \beta\mathbf{B} = 0, \\ \mathbf{E}(0,x) = E_0(x),\, n(0,x) = n_0(x),\, n_t(0,x) = n_1(x), \end{cases} \tag{37}$$

where $\eta > 0$, $\beta \geqslant 0$, $\mathbf{E}$: $\mathbb{R}_+\times\mathbb{R}^2 \to \mathbb{C}^3$, n: $\mathbb{R}_+\times\mathbb{R}^2\to\mathbb{R}$, B: $\mathbb{R}_+\times\mathbb{R}^2\to\mathbb{R}^3$. Take $\mathbf{E} = (E_1, E_2, 0)$ and $\mathbf{B} = -i\eta\mathscr{F}^{-1}\frac{|\xi|^2}{|\xi|^2-\beta}\mathscr{F}(\mathbf{E}\wedge\overline{\mathbf{E}})$. For $n_1\in H^{-1}$, one can find $w_0\in L^2$ and $\mathbf{v}_0\in L^2$ such that $n_1 = -\operatorname{div}\mathbf{v}_0 + w_0$. Then the system (37) can be re-written as

$$\begin{cases} \mathrm{i}\partial_t \mathbf{E} + \Delta \mathbf{E} - n\mathbf{E} + \mathrm{i}(\mathbf{E}\wedge\mathbf{B}(\mathbf{E})) = 0 \\ n_t = -\operatorname{div}\mathrm{v} + w_0 \\ \mathbf{v}_t = -c_0^2\nabla\left(n+|\mathbf{E}|^2\right) \\ \mathbf{E}(0,x) = E_0(x),\, n(0,x) = n_0(x),\, \mathbf{v}(0,x) = \mathbf{v}_0(x), \end{cases} \tag{38}$$

The above generalized Zakharov System describes the spontaneous generation of a magnetic field in a cold plasma by investigating two time-scales which refer to the fast electron motions on a time-scale corresponding to the plasma frequency cope and to the ion motion, respectively, cf. [37].

Let $\lambda = 1/\omega c_0$, $\omega > 0$. We consider the solution of the following equations

$$\begin{cases} \Delta P - P + \frac{\eta P}{1+\eta}\mathscr{F}^{-1}\frac{|\xi|^2}{|\xi|^2-\beta c_0^2(T-t)^2\lambda^2}\mathscr{F}P^2 = \frac{PN}{1+\eta}, \\ \lambda^2(r^2N_{rr} + 6rN_r + 6N_r) - \Delta N = \Delta|P|^2, \end{cases} \tag{39}$$

By resorting to the solutions (P, N) of the system (39), Gan et al. [36] constructed a kind of blow-up solutions of the system (37) and they obtained the following:

Theorem 33 Let $0 < T < \infty$. There exists $\lambda_T > 0$ such that for any $\lambda\in(0,\lambda_T)$, the system (39) has a solution $(P_{\lambda,T-t}, N_{\lambda,T-t})$ which leds to a blowup solution $(\mathbf{E}, n, \mathbf{B})$ of the system (37) satisfying

$$\mathbf{E} = (E_1, -iE_1, 0), \quad n(t,x) = \frac{\omega^2}{(T-t)^2}\tilde{N}\left(\frac{x\omega}{T-t}\right),$$

with

$$E_1 = \frac{1}{\sqrt{2}} \frac{\omega}{T-t} e^{i\left(\theta - \frac{|x|^2}{4(T-t)^2} + \frac{\omega^2}{T-t}\right)} \tilde{P}\left(\frac{x\omega}{T-t}\right),$$

$$\mathbf{B} = \left(0,\, 0,\, \frac{\omega^2}{(T-t)^2} \tilde{B}\left(\frac{x\omega}{T-t}\right)\right),$$

where $(\tilde{P}, \tilde{N}) = (P_{\lambda,\, T-t}/(1+\eta)^{1/2},\, N_{\lambda,\, T-t}/(1+\eta))$. Moreover, we have

$$\lim_{t \to T} \|\mathbf{E}(t)\|_{H^1} + \|n(t)\|_{L^2} + \|n_t(t)\|_{\hat{H}^{-1}} = \infty, \tag{40}$$

where

$$\hat{H}^{-1} = \{u\colon \exists w \in L^2 \text{ such that } u = -\nabla \cdot w,\ \ \|u\|_{\hat{H}^{-1}} = \|w\|_{L^2}\}.$$

Equation (33) indicates that the solutions of the system (37) blows up at finite time in energy spaces, which is the self-similar blowup solutions of type-I.

Using a sophisticated variational argument, Gan et al. [38] investigated the Klein-Gordon-Zakharov system with nonlinearities of varying degrees in two and three spatial dimensions, proving the existence of standing waves with ground states. Subsequently, by introducing an auxiliary functional and an equivalent minimization problem, they identified two invariant manifolds associated with the solution flow generated by the Cauchy problem for the Klein-Gordon-Zakharov system. Furthermore, by constructing a constrained variational problem and utilizing the two invariant manifolds, along with applying the potential well argument and the concavity method, they established a sharp threshold for global existence and blowup; see [38] for details.

7. Rogue waves and solitons

Ling et al. [39] investigated the generation mechanism of fundamental rogue wave structures in N-component coupled systems by utilizing analytical solutions of the nonlinear Schrödinger equation and conducting a modulational instability analysis. Their findings indicate that the pattern of a fundamental rogue wave is determined by the evolution energy and the growth rate of the resonant perturbation responsible for its formation. This finding enables the prediction of rogue wave patterns without the necessity of solving the N-component coupled nonlinear Schrödinger equation. Furthermore, they demonstrated that N-component coupled nonlinear Schrödinger systems may possess at most N different fundamental rogue wave patterns.

Ling et al. [40] developed a uniform Darboux transformation for multi-component coupled Nonlinear Schrödinger (NLS) equations, which encompasses all previously presented Darboux transformations. Utilizing this uniform Darboux transformation, they derived solutions for single dark solitons and multi-dark solitons in both the defocusing case and the mixed focusing-defocusing case. Additionally, they illustrated several exact single and two-dark solitons of the three-component NLS equation through graphical representations.

8. Recent progress and open questions

As the end of this paper, we point out some recent progress and present some open questions related to the topics of this paper. Roughly speaking, if the energy and the "a priori" estimates for the equations cannot provide the upper bounds of local solutions in H^s, so far we have no systematic method to get the global well-posedness for the equations.

8.1 *On (generalized) Zakharov system*

The Zakharov system

$$\begin{cases} \partial_t^2 v - \Delta v = \Delta\left(|u|^2\right), \\ \mathrm{i}\partial_t u + \Delta u = uv, \end{cases} \tag{41}$$

in higher spatial dimensions $d \geqslant 4$ seems to be very difficult problem, up to now the global well-posedness for large data seems to be open. The Zakharov system is Lagrangian, and formally the L^2-norm of u and the energy

$$E(u, v, v_t) = \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla u(t)|^2 + \frac{1}{4}||\nabla|^{-1} v_t(t)|^2 + \frac{1}{4}|v(t)|^2 + \frac{1}{2}v(t)|u(t)|^2\right) dx$$

are constant in time, where the existence of local solutions needs more regularity than that of the energy provided in higher dimensions $d \geqslant 4$ [41].

For the Generalized Zakharov System (ZSM) considered in Section 6, in 2024, Gan et al. obtained an optimal lower bound for the blowup rate [42].

8.2 *Landau-Lifshitz equation and Schrödinger map*

Considering a difficult case of Landau-Lifshitz equation by taking $\alpha_1 = 0,\ \alpha_2 = 1$ in Equation (2), then the Landau-Lifshitz equation reduces to

$$Z_t = Z \times \Delta Z, \tag{42}$$

which has the energy $E(Z) = \int_{\mathbb{R}^d} |\nabla Z(t, x)| dx$, however, in higher dimensional case $d \geqslant 3$, the energy cannot control the nonlinear interactions in $Z \times \Delta Z$. On the other hand, using the stereographic projection, we see that Equation (42) is equivalent to the following derivative NLS

$$i\partial_t u + \Delta u = \frac{2\bar{u}(\nabla u)^2}{1 + |u|^2}. \tag{43}$$

The above derivative NLS has a scaling-critical space $\dot{H}^{d/2}$. The global well-posedness of Equation (43) with large data is open in 3D and higher, and the "a priori" upper bounds in $H^{d/2}$ for the smooth solutions of Equation (43) seem to be a challenge work.

The Schrödinger map which is closely related to Equation (42) and Equation (43) has the same questions and we do not know if the Schrödinger map is globally well-posed for the smooth large data in higher dimensional cases [43].

8.3 *On BO equation*

For the BO equation $u_t - Hu_{xx} + 2uu_x = 0$, Killip, Laurens and Visan recently obtained the global wellposedness of BO equation in the Sobolev spaces $H^s(\mathbb{R})$ for $s > -1/2$, both on the line and on the circle. Noticing that $H^{-1/2}$ is the scaling-critical space of BO equation, so their result is optimal [44].

8.4 *On harmonic analysis and PDE*

Using harmonic analysis method to study wave equations, it goes back to the work of R. Strichartz in 1976 and has been a very powerful tool in the study of nonlinear evolution equations. Using modulation spaces to consider nonlinear evolution equations, one can refer to Wang et al. [33], and Bényi and Okoudjou [45].

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Conflict of interest

The authors declare no competing financial interest.

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