Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

Research Article

Some New Inequalities Involving Generalized Convex Functions in the Katugampola Fractional Setting

Sikander Mehmood[1], Dumitru Baleanu[2], Majeed Ahmad Yousif[3], Pshtiwan Othman Mohammed[4,5*], Abdullah Abbas[1], Nejmeddine Chorfi[6]

[1] Department of Mathematics, Government Graduate College, Sahiwal, 57000, Pakistan
[2] Department of Computer Science and Mathematics, Lebanese American University, Beirut, 11022801, Lebanon
[3] Department of Mathematics, College of Education, University of Zakho, Zakho, 42002, Iraq
[4] Department of Mathematics, College of Education, University of Sulaimani, Sulaymaniyah, 46001, Iraq
[5] Research Center, University of Halabja, Halabja, 46018, Iraq
[6] Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh, 11451, Saudi Arabia
E-mail: pshtiwansangawi@gmail.com

Received: 2 May 2025; **Revised:** 9 June 2025; **Accepted:** 10 June 2025

Abstract: In this study, we explore a new class of convex functions termed *cr*-log-*h*-convex functions within the framework of interval-valued functions and the *cr*-order. We introduce and analyze fundamental properties of these functions and establish several Hermite-Hadamard inequalities by employing Katugampola fractional integrals. To illustrate the theoretical results, we present numerical examples that validate the proposed inequalities. This work extends the understanding of convexity concepts and their applications, offering a broader perspective on inequalities in real analysis and fuzzy systems.

***Keywords*:** integral inequalities, generalized convex functions, Katugampola fractional operators, Hermite-Hadamard inequality, symmetric function

1. Introduction

Convexity is a key concept in many areas of science and has led to innovations in fields like geometry, information theory, control theory, optimization, operations research, functional analysis and game theory. It is also important in economics, finance, engineering and management. Because of its wide relevance, researchers are very interested in studying generalized convexity of functions [1–4]. To solve real-world problems, many new forms of weaker generalized convexities have been developed. The well-known inequality is the Hermite-Hadamard inequality, it connects integral inequalities with convex function theory. Since it is seen as a representation of convex functions, creating a Hermite-Hadamard type inequality requires using generalized convexity principles [5, 6].

Recent studies have focused on advancing numerical modeling techniques for vibrating circular cylinders, with particular emphasis on validation through experimental data. These developments aim to improve the accuracy and reliability of predictions in structural dynamics and acoustic applications. Several works [7–9] have demonstrated effective integration of numerical simulations with experimental results, offering deeper insights into vibration characteristics

DOI: https://doi.org/10.37256/cm.6420257169

and resonance behaviors. Such contributions significantly support the practical implementation of computational methods in engineering design and analysis.

The motivation for this study stems from the growing need to extend classical convexity concepts to more generalized settings, particularly for interval-valued functions and fractional calculus. Classical convexity often fails to capture the nuanced behaviors of such functions, especially in the context of advanced mathematical structures and applications. This has inspired the introduction of a broader class of convex functions, namely *cr*-log-*h*-convex functions, which provide a robust framework for exploring new inequalities and their implications.

The main objective of this work is to establish and analyze various inequalities for Katugampola fractional integrals and their weighted counterparts, specifically for the newly introduced *cr*-log-*h*-convex functions. By investigating the fundamental properties of this class, we aim to generalize existing results and provide a systematic approach to deriving Hermite-Hadamard-type inequalities. Furthermore, numerical examples are included to validate the theoretical results and demonstrate their practical relevance.

The main advantage of the method employed in this study lies in its ability to unify and generalize a wide range of classical inequalities using the framework of *cr*-log-*h*-convex functions combined with the Katugampola fractional integral. Unlike traditional convexity approaches that are limited to real-valued and single-variable functions, our method extends to interval-valued functions under the *cr*-order, which is essential for handling uncertainty in fuzzy systems and real-world applications. Additionally, the Katugampola fractional integral encompasses both Riemann-Liouville and Hadamard integrals as special cases, offering greater flexibility and broader applicability in modeling memory and hereditary properties. This dual generalization-both in the class of convexity and in the fractional operator-provides a more powerful and comprehensive tool for deriving new inequalities and revisiting classical results from a modern perspective.

The novelty of this research lies in the introduction of *cr*-log-*h*-convex functions as a new class of convexity that accommodates interval-valued functions within the *cr*-order framework. The application of Katugampola fractional integrals and their weighted forms to derive inequalities for these functions represents a significant advancement in fractional calculus and convex analysis. These results not only extend classical inequalities but also open new avenues for research in real analysis, fuzzy systems, and related mathematical fields.

Given a convex function $\mathtt{C}$: $X \to \mathbb{R}$ defined on interval X, which is set of real numbers and ϕ_1, ϕ_2 these are both within X with $\phi_1 < \phi_2$, then the Hermite-Hadamard condition for convex functions is stated as:

$$\mathtt{C}\left(\frac{\phi_1+\phi_2}{2}\right) \leq \frac{1}{\phi_2-\phi_1}\int_{\phi_1}^{\phi_2}\mathtt{C}(w)dw \leq \frac{\mathtt{C}(\phi_1)+\mathtt{C}(\phi_2)}{2}. \tag{1}$$

The generalized version of inequality (1), as presented by Fejér [10], is as follows:

$$\mathtt{C}\left(\frac{\phi_1+\phi_2}{2}\right)\int_{\phi_1}^{\phi_2}\tilde{H}(w)dw \leq \int_{\phi_1}^{\phi_2}\tilde{H}(w)\mathtt{C}(w)dw \leq \frac{\mathtt{C}(\phi_1)+\mathtt{C}(\phi_2)}{2}\int_{\phi_1}^{\phi_2}\tilde{H}(w)dw,$$

holds, where $\tilde{H} : [\phi_1, \phi_2] \to \mathbb{R}$ is an integrable, nonnegative function that is symmetric about $w = \dfrac{\phi_1+\phi_2}{2}$.

These inequalities were first introduced independently by Charles Hermite and Jacques Hadamard in the late 19th century [11] and has since found numerous applications.

The Hermite-Hadamard inequality is widely used in engineering, particularly in relation to 3D printing technology, to approximate the fastest and slowest printing speeds. This is due to the inherent challenge of accurately predicting printing speeds, e.g. [12, 13].

Antczak [14], developed the notion of invex sets. Consequently, this concept laid the foundation for the definition of preinvex functions.

Definition 1 Suppose that $\Im$ is a function such that $\Im: X\times X\rightarrow\mathbb{R}$. A set $X\subseteq\mathbb{R}$ is called an invex w.r.t $\Im$ if for all $\phi_1,\ \phi_2\in X$ and $\mu\in[0,\ 1]$, $\phi_2+\mu\Im(\phi_1,\ \phi_2)\in X$.

Preinvex functions belong to a category of generalized convex functions. The theory of preinvex functions was introduced by Weir et al. in [15] and it is defined as:

Definition 2 Suppose $X\subseteq\mathbb{R}$ be an $\Im$-connected set. A function $\complement: X\rightarrow\mathbb{R}$ is called preinvex w.r.t. $\Im$ and if

$$\complement(\phi_2+\mu\Im(\phi_1,\ \phi_2))\leq\mu\complement(\phi_1)+(1-\mu)\complement(\phi_2).$$

Remark 1 The preinvex functions become convex for $\Im(\phi_1,\ \phi_2)=\phi_1-\phi_2,\ \forall\ \phi_1,\ \phi_2\in X$ and $\mu\in[0,\ 1]$.

Definition 3 A function is defined as log-convex if $\log\complement$ is convex or equivalently, $\forall\ \phi_1,\ \phi_2\in X$ and $\mu\in[0,\ 1]$, one has the inequality:

$$\complement(\mu\phi_1+(1-\mu)\phi_2)\leq[\complement(\phi_1)]^{\mu}[\complement(\phi_2)]^{(1-\mu)}.$$

There are numerous enhancements, generalizations and extensions of log-convex functions available (See [16–18]). In [19], Varosanec, gave the concept of h-convexity:

Definition 4 A non-negative function defined as $h: X\rightarrow\mathbb{R}$ is called h-convex, if for non-negative function $\complement: I\rightarrow(0,\ \infty)$ and $\forall\ \phi_1,\ \phi_2\in X$ and $\mu\in[0,\ 1]$, one has the inequality:

$$\complement(\mu\phi_1+(1-\mu)\phi_2)\leq h(\mu)\complement(\phi_1)+h(1-\mu)\complement(\phi_2).$$

In [20], Noor et al. described log-h-convex functions in the following manner:

Definition 5 A non-negative function defined as $h: X\rightarrow\mathbb{R}$ is called log-h-convex, if for non-negative function $\complement: I\rightarrow(0,\ \infty)$ and $\forall\ \phi_1,\ \phi_2\in X$ and $\mu\in[0,\ 1]$, one has the inequality:

$$\complement(\mu\phi_1+(1-\mu)\phi_2)\leq[\complement(\phi_1)]^{h(\mu)}[\complement(\phi_2)]^{h(1-\mu)}. \tag{2}$$

For every $\zeta,\ \sigma\in[0,\ 1]$, the function h is referred to as a super multiplicative function if $h(\zeta)h(\sigma)\leq h(\zeta\sigma)$.

Remark 2 If $h(\mu)=1$ then (2) reduces to:

$$\complement(\mu\phi_1+(1-\mu)\phi_2)\leq[\complement(\phi_1)]\times[\complement(\phi_2)]. \tag{3}$$

Remark 3 If $h(\mu)=\mu$ then (2) reduces to:

$$\complement(\mu\phi_1+(1-\mu)\phi_2)\leq[\complement(\phi_1)]^{\mu}\times[\complement(\phi_2)]^{1-\mu}. \tag{4}$$

Remark 4 If $h(\mu)=\mu^s$ then (2) reduces to:

$$\complement(\mu\phi_1+(1-\mu)\phi_2)\leq[\complement(\phi_1)]^{\mu^s}\times[\complement(\phi_2)]^{(1-\mu)^s}. \tag{5}$$

Researchers are actively investigating Hermite-Hadamard inequalities and fractional calculus because of their widespread applications in a variety of scientific fields. Recent developments in the field show that this study direction has some momentum (see [21–24]).

Definition 6 For $\mathsf{C} \in L_1[\phi_1,\ \phi_2]$, the left- sided Riemann-Liouville integral operator of order $\alpha\ > 0$ is given as:

$$J^{\alpha}_{\phi_1^+}\mathsf{C}(w) = \frac{1}{\Gamma(\alpha)}\int_{\phi_1}^{w}(w-\zeta)^{\alpha-1}\mathsf{C}(\zeta)d\zeta,\ (0 \leq \phi_1 < w \leq \phi_2),$$

The right- sided Riemann-Liouville integral operator of order $\alpha\ > 0$ is given as:

$$J^{\alpha}_{\phi_2^-}\mathsf{C}(w) = \frac{1}{\Gamma(\alpha)}\int_{w}^{\phi_2}(\zeta-w)^{\alpha-1}\mathsf{C}(\zeta)d\zeta,\ (0 \leq \phi_1 < w < \phi_2),$$

where $\Gamma(\alpha)$ is the Gamma function, which is defined as:

$$\Gamma(\alpha) = \int_{0}^{\infty}\zeta^{\alpha-1}e^{-\zeta}d\zeta,\quad Re(\alpha) > 0.$$

Hermite-Hadamard inequalities for fractional integrals developed is given as:.

Theorem 1 Suppose $\mathsf{C} : [\phi_1,\ \phi_2] \subseteq \mathbb{R} \to \mathbb{R}$ where $\mathsf{C} \in L_1[\phi_1,\ \phi_2]$ with $0 \leq \phi_1 \leq \phi_2$ and, if C is convex function and positive on $[\phi_1,\ \phi_2]$, then we have the following inequalities for fractional integrals:

$$\mathsf{C}\left(\frac{\phi_1+\phi_2}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(\phi_2-\phi_1)^{\alpha}}\left[J^{\alpha}_{\phi_1^+}\mathsf{C}(\phi_2)+J^{\alpha}_{\phi_2^-}\mathsf{C}(\phi_1)\right] \leq \frac{\mathsf{C}(\phi_1)+\mathsf{C}(\phi_2)}{2},$$

with $\alpha > 0$. The set of all Lebesgue integrable functions on $[\phi_1,\ \phi_2]$ is denoted by $L_1[\phi_1,\ \phi_2]$.

Definition 7 Suppose $[\phi_1,\ \phi_2] \subset \mathbb{R}$ is a finite interval. Then the left and right-sided Katugampola fractional integrals of order $\alpha(> 0)$ of $\mathsf{C} \in X^{c}_{p}(\phi_1,\ \phi_2)$ are defined by [25]:

$${}^{\xi}K^{\alpha}_{\phi_1^+}\mathsf{C}(w) = \frac{\xi^{1-\alpha}}{\Gamma(\alpha)}\int_{\phi_1}^{w}\frac{\zeta^{\xi-1}}{(w^{\xi}-\zeta^{\xi})^{1-\alpha}}\mathsf{C}(\zeta)d\zeta,\ (0 \leq \phi_1 < w \leq \phi_2),$$

and

$${}^{\xi}K^{\alpha}_{\phi_2^-}\mathsf{C}(w) = \frac{\xi^{1-\alpha}}{\Gamma(\alpha)}\int_{w}^{\phi_2}\frac{\zeta^{\xi-1}}{(\zeta^{\xi}-w^{\xi})^{1-\alpha}}\mathsf{C}(\zeta)d\zeta,\ (0 \leq \phi_1 < w < \phi_2).$$

Let $\mathbb{R}$ represents the collection of all real numbers, where the collection of all positive real numbers are denoted by $\mathbb{R}^+$. The set $\mathbb{R}_I$ is collection of all closed intervals within $\mathbb{R}$. For $[\underline{s},\ \overline{s}] \in \mathbb{R}I$, if $\underline{s} > 0$, then $[\underline{s},\ \overline{s}]$ will be a positive interval. The set $\mathbb{R}_I^+$ is the collection of all positive intervals.

Definition 8 For any $\mu \in \mathbb{R}$, $s = [\underline{s},\ \overline{s}]$, $\sigma = [\underline{\sigma},\ \overline{\sigma}] \in \mathbb{R}_I$, the operations of Minkowski addition, multiplication and scalar multiplication on intervals are defined as follows:

$$s+\sigma=[\underline{s},\ \overline{s}]+[\underline{\sigma},\ \overline{\sigma}]=[\underline{s}+\underline{\sigma},\ \overline{s}+\overline{\sigma}],$$

and

$$[\underline{s},\ \overline{s}]\times[\underline{\sigma},\ \overline{\sigma}]=\begin{bmatrix}\min\{\underline{s}\underline{\sigma},\ \overline{s}\underline{\sigma},\ \underline{s}\overline{\sigma},\ \overline{s}\overline{\sigma}\},\\ \max\{\underline{s}\underline{\sigma},\ \overline{s}\underline{\sigma},\ \underline{s}\overline{\sigma},\ \overline{s}\overline{\sigma}\}\end{bmatrix},$$

and

$$\mu s=\mu[\underline{s},\ \overline{s}]=\begin{cases}[\mu\underline{s},\ \mu\overline{s}] & \mu>0\\ [0,\ 0] & \mu=0\\ [\mu\overline{s},\ \mu\underline{s}] & \mu<0\end{cases}.$$

Let $s=[\underline{s},\ \overline{s}]\in\mathbb{R}_I$, centre of s is defined as $s_c=\dfrac{\underline{s}+\overline{s}}{2}$ while radius of s is given as $s_r=\dfrac{\overline{s}-\underline{s}}{2}$. Then $s=[\underline{s},\ \overline{s}]$ can also be presented in cente-radius form as

$$s=\left\langle\frac{\underline{s}+\overline{s}}{2},\ \frac{\overline{s}-\underline{s}}{2}\right\rangle=\langle s_c,\ s_r\rangle.$$

Definition 9 [26] Let $s=[\underline{s},\ \overline{s}]=\langle s_c,\ s_r\rangle$, $\sigma=[\underline{\sigma},\ \overline{\sigma}]=\langle\sigma_c,\ \sigma_r\rangle\in\mathbb{R}_I$, then the cente-radius order relation is given as

$$s\preceq_{cr}\sigma\Longleftrightarrow\begin{cases}s_c<\sigma_c, & \text{if } s_c\neq\sigma_c,\\ s_r\leq\sigma_r, & \text{if } s_c=\sigma_c.\end{cases}$$

Obviously, for any two intervals $s,\ \sigma\in\mathbb{R}_I$, either $s\leq_{cr}\sigma$ or $\sigma\leq_{cr}s$.

Remark 5 [27] The relation "$\leq$ " defined on $\mathbb{R}_I$ by $[\underline{s},\ \overline{s}]\leq[\underline{\sigma},\ \overline{\sigma}]$ if and only if $\underline{s}\leq\underline{\sigma}$, $\overline{s}\leq\overline{\sigma}\ \forall\ [\underline{s},\ \overline{s}],\ [\underline{\sigma},\ \overline{\sigma}]\in\mathbb{R}_I$. Also $[\underline{s},\ \overline{s}],\ [\underline{\sigma},\ \overline{\sigma}]\in\mathbb{R}_I$, the inclusion "$\subseteq$" is defined by $[\underline{s},\ \overline{s}]\subseteq[\underline{\sigma},\ \overline{\sigma}]$ if and only if $\underline{\sigma}\leq\underline{s}$, $\overline{s}\leq\overline{\sigma}$.

Remark 6 Let $\mathtt{C}(s)$ be the real-valued function , then we have obtained extension $\mathtt{C}([\underline{s},\ \overline{s}])$ of $\mathtt{C}(s)$ by substituting an interval variable $[\underline{s},\ \overline{s}]$ for the variable s and matching interval operators for the real arithmetic operations. The resulting $\mathtt{C}([\underline{s},\ \overline{s}])$ is said to be a natural interval extension of $\mathtt{C}(s)$ specifically, in case where $\mathtt{C}(s)$ is both monotonic and continuous, we obtain

$$\mathtt{C}([\underline{s},\ \overline{s}])=[\min\{\mathtt{C}(\underline{s}),\ \mathtt{C}(\overline{s})\},\ \max\{\mathtt{C}(\underline{s}),\ \mathtt{C}(\overline{s})\}$$

(a) If $\mathtt{C}(s)=e^s$, $s\in\mathbb{R}$ then $\mathtt{C}([\underline{s},\ \overline{s}])=[e^{\underline{s}},\ e^{\overline{s}}]$.

(b) If $\mathtt{C}(s)=\ln s$, $\ln s>0$ then $\mathtt{C}([\underline{s},\ \overline{s}])=[\ln\underline{s},\ \ln\overline{s}]$, for $\underline{s}>0$.

Theorem 2 [28] Suppose $\mathtt{C}=[\underline{\mathtt{C}},\ \overline{\mathtt{C}}]$ is a function with interval valued, where $\mathtt{C}:[\phi_1,\ \phi_2]\to\mathbb{R}_I$. Then the function $\mathtt{C}$ is said to be Riemann integrable at $[\phi_1,\ \phi_2]$, provided $\underline{\mathtt{C}}$ and $\overline{\mathtt{C}}$ be Riemann integrable at $[\phi_1,\ \phi_2]$ and

$$\int_{\phi_1}^{\phi_2} \mathtt{C}(\zeta)d\zeta = \left[\int_{\phi_1}^{\phi_2} \underline{\mathtt{C}}(\zeta)d\zeta,\ \int_{\phi_1}^{\phi_2} \overline{\mathtt{C}}(\zeta)d\zeta\right].$$

An additional general form of numerous types of convex functions is a log-h-convex function. Additionally, they examined the fundamental characteristics of log-h-convex functions and formulated integral inequalities for them.

Theorem 3 Let $\mathtt{C}$ be a log-h-convex function with $h\left(\frac{1}{2}\right) \neq 0$,

$$\mathtt{C}\left(\frac{\phi_1+\phi_2}{2}\right)^{\frac{1}{2h(\frac{1}{2})}} \leq \exp\left[\frac{1}{(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_2} \ln \mathtt{C}(\upsilon)d\upsilon\right]$$

$$\leq \left[\mathtt{C}(\phi_1)\mathtt{C}(\phi_2)\right]^{\int_0^1 h(\zeta)d\zeta}.$$

Liu et al. [29] extended the notion of the log-h-convex function to functions that have interval valued.

Definition 10 Let $\mathtt{C} = [\underline{\mathtt{C}},\ \overline{\mathtt{C}}]$ and $\mathtt{C} \in I\mathbb{R}_{([\phi_1,\ \phi_2])}$ is an interval valued function with $\mathtt{C} : [\phi_1,\ \phi_2] \to \mathbb{R}_I^+$ on $[\phi_1,\ \phi_2]$. A function $\mathtt{C}$ is defined as cr-log-h-convex,

$$\mathtt{C}(\phi_1\mu + (1-\mu)\phi_2) \preceq_{cr} \left[\mathtt{C}(\phi_1)\right]^{h(\mu)} \left[\mathtt{C}(\phi_2)\right]^{h(1-\mu)},$$

where the function $h : [0,\ 1] \to \mathbb{R}^+$ is a nonnegative.

Remark 7 The function $\mathtt{C}$ reduces to log-h-convex when $\underline{\mathtt{C}} = \overline{\mathtt{C}}$.

Theorem 4 [30] For interval valued functions $\mathtt{C},\ \tilde{H} : [\phi_1,\ \phi_2] \to \mathbb{R}_I^+$ where $\tilde{H} = [\underline{H},\ \overline{H}]$ and $\mathtt{C} = [\underline{\mathtt{C}},\ \overline{\mathtt{C}}]$. If $\mathtt{C},\ \tilde{H} \in I\mathbb{R}_{([\phi_1,\ \phi_2])}$, and $\mathtt{C}(\zeta) \preceq_{cr} \tilde{H}(\zeta)$ for all $\zeta \in [\phi_1,\ \phi_2]$, then

$$\int_{\phi_1}^{\phi_2} \mathtt{C}(\zeta)d\zeta \preceq_{cr} \int_{\phi_1}^{\phi_2} \tilde{H}(\zeta)d\zeta.$$

2. Main results

In this section, we concentrate on deriving various inequalities linked to Katugampola fractional integrals and their weighted versions, tailored specifically for the newly introduced class of cr-log-h-convex functions. The results presented here aim to highlight the rich structure and applicability of this generalized convexity, offering new perspectives and extending existing inequalities within the framework of fractional calculus and interval-valued functions.

Here $SX(cr\text{-log-}h,\ [\phi_1^{\xi},\ \phi_2^{\xi}],\ \mathbb{R}_I^+)$ is the collection of cr-log-h-convex functions on $[\phi_1^{\xi},\ \phi_2^{\xi}]$. While $I\mathbb{R}_{[\phi_1^{\xi},\ \phi_2^{\xi}]}$ is collection of interval-valued functions on $[\phi_1^{\xi},\ \phi_2^{\xi}]$, which are Riemann integrable on given interval .

Theorem 5 Let $\mathtt{C} : [\phi_1,\ \phi_2] \to \mathbb{R}_I^+$ be an interval-valued function defined as $\mathtt{C} = [\underline{\mathtt{C}},\ \overline{\mathtt{C}}]$, where

$$\mathtt{C} \in I\mathbb{R}^+_{[\phi_1^{\xi},\ \phi_2^{\xi}]},$$

and let $h:[0,\ 1]\to\mathbb{R}^{+}$ be a function such that $h\left(\frac{1}{2}\right)\neq 0$. If

$$\mathtt{C}\in SX(cr\text{-log-}h,\ [\phi_1^{\xi},\ \phi_2^{\xi}],\ \mathbb{R}_I^{+}),$$

then for any $\alpha\geq 0$, $\xi\geq 0$, the following inequality holds:

$$\mathtt{C}\left(\frac{\phi_1^{\xi}+\phi_2^{\xi}}{2}\right)^{\frac{1}{\alpha\xi h\left(\frac{1}{2}\right)}}\preceq_{cr}\exp\left[\frac{\xi^{\alpha-1}\Gamma(\alpha)}{(\phi_2^{\xi}-\phi_1^{\xi})^{\alpha}}\left({}^{\xi}K^{\alpha}_{\phi_1^{+}}\ln\mathtt{C}(\phi_2^{\xi})+{}^{\xi}K^{\alpha}_{\phi_2^{-}}\ln\mathtt{C}(\phi_1^{\xi})\right)\right]$$

$$\preceq_{cr}\left[\mathtt{C}(\phi_1^{\xi})\cdot\mathtt{C}(\phi_2^{\xi})\right]^{\int_0^1\left(\zeta^{\alpha\xi-1}+(1-\zeta^{\xi})^{\alpha-1}\zeta^{\xi-1}\right)h(\zeta^{\xi})\ d\zeta}.$$

Proof. For an interval valued function $\mathtt{C}\in SX(cr\text{-log-}h,\ \left[\phi_1^{\xi},\ \phi_2^{\xi}\right],\mathbb{R}_I^{+})$ and for $y,\ z\in\left[\phi_1^{\xi},\ \phi_2^{\xi}\right]$ and $\mu=\frac{1}{2}$, we have

$$\mathtt{C}\left(\frac{y+z}{2}\right)\preceq_{cr}[\mathtt{C}(y)\mathtt{C}(z)]^{h\left(\frac{1}{2}\right)}. \tag{6}$$

On utilizing the property of log, we have

$$\frac{1}{h\left(\frac{1}{2}\right)}\ln\mathtt{C}\left(\frac{y+z}{2}\right)\preceq_{cr}\ln\left[\mathtt{C}(y)\right]+\ln\left[\mathtt{C}(z)\right]. \tag{7}$$

On substituting $y=\zeta^{\xi}\phi_1^{\xi}+(1-\zeta^{\xi})\phi_2^{\xi}$ and $z=(1-\zeta^{\xi})\phi_1^{\xi}+\zeta^{\xi}\phi_2^{\xi}$ in (7), we obtain

$$\frac{1}{h\left(\frac{1}{2}\right)}\ln\left[\mathtt{C}\left(\frac{\phi_1^{\xi}+\phi_2^{\xi}}{2}\right)\right]\preceq_{cr}\ln\left[\mathtt{C}(\zeta^{\xi}\phi_1^{\xi}+(1-\zeta^{\xi})\phi_2^{\xi})\right]+\ln\left[\mathtt{C}((1-\zeta^{\xi})\phi_1^{\xi}+\zeta^{\xi}\phi_2^{\xi})\right]. \tag{8}$$

On multiplying (8) with $\zeta^{\alpha\xi-1}$ and integrating w.r.t ζ over the interval $[0,\ 1]$, we obtain

$$\frac{1}{\alpha\xi}\frac{1}{h\left(\frac{1}{2}\right)}\ln\left[\mathtt{C}\left(\frac{\phi_1^{\xi}+\phi_2^{\xi}}{2}\right)\right]$$

$$\preceq_{cr}\int_0^1\zeta^{\alpha\xi-1}\ln\left[\mathtt{C}(\zeta^{\xi}\phi_1^{\xi}+(1-\zeta^{\xi})\phi_2^{\xi})\right]d\zeta+\int_0^1\zeta^{\alpha\xi-1}\ln\left[\mathtt{C}((1-\zeta^{\xi})\phi_1^{\xi}+\zeta^{\xi}\phi_2^{\xi})\right]d\zeta$$

$$= \left[\int_0^1 \zeta^{\alpha\xi-1} \ln\left[\underline{\mathsf{C}}(\zeta^\xi \phi_1^\xi + (1-\zeta^\xi)\phi_2^\xi)\right] d\zeta,\ \int_0^1 \zeta^{\alpha\xi-1} \ln\left[\overline{\mathsf{C}}(\zeta^\xi \phi_1^\xi + (1-\zeta^\xi)\phi_2^\xi)\right] d\zeta\right]$$

$$+ \left[\int_0^1 \zeta^{\alpha\xi-1} \ln\left[\underline{\mathsf{C}}((1-\zeta^\xi)\phi_1^\xi + \zeta^\xi \phi_2^\xi)\right] d\zeta,\ \int_0^1 \zeta^{\alpha\xi-1} \ln\left[\overline{\mathsf{C}}((1-\zeta^\xi)\phi_1^\xi + \zeta^\xi \phi_2^\xi)\right] d\zeta\right].$$

Following an appropriate substitution, we get

$$\frac{1}{\alpha\xi}\frac{1}{h\left(\frac{1}{2}\right)}\ln\left[\mathsf{C}\left(\frac{\phi_1^\xi+\phi_2^\xi}{2}\right)\right]$$

$$= \left[\frac{\xi^{\alpha-1}\Gamma(\alpha)}{(\phi_2^\xi-\phi_1^\xi)^\alpha}\left[\frac{\xi^{1-\alpha}}{\Gamma(\alpha)}\int_{\phi_1}^{\phi_2}(\phi_2^\xi-\upsilon^\xi)^{\alpha-1}\upsilon^{\xi-1}\ln\left[\underline{\mathsf{C}}(\upsilon^\xi)\right]d\upsilon\right],\right.$$

$$\left.\frac{\xi^{\alpha-1}\Gamma(\alpha)}{(\phi_2^\xi-\phi_1^\xi)^\alpha}\left[\frac{\xi^{1-\alpha}}{\Gamma(\alpha)}\int_{\phi_1}^{\phi_2}(\phi_2^\xi-\upsilon^\xi)^{\alpha-1}\upsilon^{\xi-1}\ln\left[\overline{\mathsf{C}}(\upsilon^\xi)\right]d\upsilon\right]\right]$$

$$+ \left[\frac{\xi^{\alpha-1}\Gamma(\alpha)}{(\phi_2^\xi-\phi_1^\xi)^\alpha}\left[\frac{\xi^{1-\alpha}}{\Gamma(\alpha)}\int_{\phi_1}^{\phi_2}(\upsilon^\xi-\phi_1^\xi)^{\alpha-1}\upsilon^{\xi-1}\ln\left[\underline{\mathsf{C}}(\upsilon^\xi)\right]d\upsilon\right],\right.$$

$$\left.\frac{\xi^{\alpha-1}\Gamma(\alpha)}{(\phi_2^\xi-\phi_1^\xi)^\alpha}\left[\frac{\xi^{1-\alpha}}{\Gamma(\alpha)}\int_{\phi_1}^{\phi_2}(\upsilon^\xi-\phi_1^\xi)^{\alpha-1}\upsilon^{\xi-1}\ln\left[\overline{\mathsf{C}}(\upsilon^\xi)\right]d\upsilon\right]\right].$$

Above equation can be written as

$$\frac{1}{\alpha\xi}\frac{1}{h\left(\frac{1}{2}\right)}\ln\left[\mathsf{C}\left(\frac{\phi_1^\xi+\phi_2^\xi}{2}\right)\right]$$

$$\preceq_{cr} \frac{\xi^{\alpha-1}\Gamma(\alpha)}{(\phi_2^\xi-\phi_1^\xi)^\alpha}\left[\frac{\xi^{1-\alpha}}{\Gamma(\alpha)}\int_{\phi_1}^{\phi_2}(\phi_2^\xi-\upsilon^\xi)^{\alpha-1}\upsilon^{\xi-1}\ln\left[\mathsf{C}(\upsilon^\xi)\right]d\upsilon\right]$$

$$+ \frac{\xi^{\alpha-1}\Gamma(\alpha)}{(\phi_2^\xi-\phi_1^\xi)^\alpha}\left[\frac{\xi^{1-\alpha}}{\Gamma(\alpha)}\int_{\phi_1}^{\phi_2}(\upsilon^\xi-\phi_1^\xi)^{\alpha-1}\upsilon^{\xi-1}\ln\left[\mathsf{C}(\upsilon^\xi)\right]d\upsilon\right]$$

$$= \frac{\xi^{\alpha-1}\Gamma(\alpha)}{(\phi_2^\xi-\phi_1^\xi)^\alpha}\left[{}^\xi K_{\phi_1^+}^\alpha \ln\mathsf{C}(\phi_2^\xi) + {}^\xi K_{\phi_2^-}^\alpha \ln\mathsf{C}(\phi_1^\xi)\right]. \tag{9}$$

Similarly, as $\mathtt{C} \in SX(cr\text{-log-}h, \left[\phi_1^\xi,\ \phi_2^\xi\right], \mathbb{R}_I^+)$ we obtain

$$\mathtt{C}(\zeta^\xi \phi_1^\xi + (1-\zeta^\xi)\phi_2^\xi) \preceq_{cr} \left[\mathtt{C}(\phi_1^\xi)\right]^{h\left(\zeta^\xi\right)} \left[\mathtt{C}(\phi_2^\xi)\right]^{h(1-\zeta^\xi)} \tag{10}$$

$$\mathtt{C}((1-\zeta^\xi)\phi_1^\xi + \zeta^\xi \phi_2^\xi) \preceq_{cr} \left[\mathtt{C}(\phi_1^\xi)\right]^{h(1-\zeta^\xi)} \left[\mathtt{C}(\phi_2^\xi)\right]^{h\left(\zeta^\xi\right)}. \tag{11}$$

We can write (10) and (11) as

$$\ln\left[\mathtt{C}((1-\zeta^\xi)\phi_1^\xi + \zeta^\xi \phi_2^\xi)\right] \tag{12}$$

$$\preceq_{cr} h\left(\zeta^\xi\right) \ln\left[\mathtt{C}(\phi_2^\xi)\right] + h\left(1-\zeta^\xi\right) \ln\left[\mathtt{C}(\phi_1^\xi)\right].$$

$$\ln\left[\mathtt{C}((1-\zeta^\xi)\phi_2^\xi + \zeta^\xi \phi_1^\xi)\right] \tag{13}$$

$$\preceq_{cr} h\left(1-\zeta^\xi\right) \ln\left[\mathtt{C}(\phi_2^\xi)\right] + h\left(\zeta^\xi\right) \ln\left[\mathtt{C}(\phi_1^\xi)\right].$$

On adding (12) and (13), we have

$$\ln\mathtt{C}\left(\zeta^\xi \phi_1^\xi + (1-\zeta^\xi)\phi_2^\xi\right) + \ln\mathtt{C}\left((1-\zeta^\xi)\phi_1^\xi + \zeta^\xi \phi_2^\xi\right)$$

$$\preceq_{cr} h\left(\zeta^\xi\right) \ln\left[\mathtt{C}(\phi_1^\xi)\right] + h\left(1-\zeta^\xi\right) \ln\left[\mathtt{C}(\phi_2^\xi)\right] + h\left(1-\zeta^\xi\right) \ln\left[\mathtt{C}(\phi_1^\xi)\right] + h\left(\zeta^\xi\right) \ln\left[\mathtt{C}(\phi_2^\xi)\right]. \tag{14}$$

On multiplying (14) with $\zeta^{\alpha\xi-1}$ and integrating w.r.t ζ over the interval $[0,\ 1]$, we obtain.

$$\left[\int_0^1 \zeta^{\alpha\xi-1} \ln\left[\underline{\mathtt{C}}(\zeta^\xi \phi_1^\xi + (1-\zeta^\xi)\phi_2^\xi)\right] d\zeta,\ \int_0^1 \zeta^{\alpha\xi-1} \ln\left[\overline{\mathtt{C}}(\zeta^\xi \phi_1^\xi + (1-\zeta^\xi)\phi_2^\xi)\right] d\zeta\right]$$

$$+ \left[\int_0^1 \zeta^{\alpha\xi-1} \ln\left[\underline{\mathtt{C}}((1-\zeta^\xi)\phi_1^\xi + \zeta^\xi \phi_2^\xi)\right] d\zeta,\ \int_0^1 \zeta^{\alpha\xi-1} \ln\left[\overline{\mathtt{C}}((1-\zeta^\xi)\phi_1^\xi + \zeta^\xi \phi_2^\xi)\right] d\zeta\right]$$

$$= \left[\ln\mathtt{C}(\phi_1^\xi)\right] \int_0^1 \zeta^{\alpha\xi-1} h\left(\zeta^\xi\right) d\zeta + \left[\ln\mathtt{C}(\phi_2^\xi)\right] \int_0^1 \zeta^{\alpha\xi-1} h\left(1-\zeta^\xi\right) d\zeta$$

$$+ \left[\ln\mathtt{C}(\phi_1^\xi)\right] \int_0^1 \zeta^{\alpha\xi-1} h\left(1-\zeta^\xi\right) d\zeta + \left[\ln\mathtt{C}(\phi_2^\xi)\right] \int_0^1 \zeta^{\alpha\xi-1} h\left(\zeta^\xi\right) d\zeta.$$

After suitable substitution, we have

$$\left[\frac{\xi^{\alpha-1}\Gamma(\alpha)}{(\phi_2^\xi-\phi_1^\xi)^\alpha}\left[\frac{\xi^{1-\alpha}}{\Gamma(\alpha)}\int_{\phi_1}^{\phi_2}(\phi_2^\xi-\upsilon^\xi)^{\alpha-1}\upsilon^{\xi-1}\ln\left[\underline{\mathtt{C}}(\upsilon^\xi)\right]d\upsilon\right],\right.$$

$$\left.\frac{\xi^{\alpha-1}\Gamma(\alpha)}{(\phi_2^\xi-\phi_1^\xi)^\alpha}[\left[\frac{\xi^{1-\alpha}}{\Gamma(\alpha)}\int_{\phi_1}^{\phi_2}(\phi_2^\xi-\upsilon^\xi)^{\alpha-1}\upsilon^{\xi-1}\ln\left[\overline{\mathtt{C}}(\upsilon^\xi)\right]d\upsilon\right]\right]$$

$$+\left[\frac{\xi^{\alpha-1}\Gamma(\alpha)}{(\phi_2^\xi-\phi_1^\xi)^\alpha}\left[\frac{\xi^{1-\alpha}}{\Gamma(\alpha)}\int_{\phi_1}^{\phi_2}(\upsilon^\xi-\phi_1^\xi)^{\alpha-1}\upsilon^{\xi-1}\ln\left[\underline{\mathtt{C}}(\upsilon^\xi)\right]d\upsilon\right],\right.$$

$$\left.\frac{\xi^{\alpha-1}\Gamma(\alpha)}{(\phi_2^\xi-\phi_1^\xi)^\alpha}\left[\frac{\xi^{1-\alpha}}{\Gamma(\alpha)}\int_{\phi_1}^{\phi_2}(\upsilon^\xi-\phi_1^\xi)^{\alpha-1}\upsilon^{\xi-1}\ln\left[\overline{\mathtt{C}}(\upsilon^\xi)\right]d\upsilon\right]\right]$$

$$=\ln\left[\mathtt{C}(\phi_1^\xi)\right]^{\int_0^1\zeta^{\alpha\xi-1}h(\zeta^\xi)d\zeta}+\ln\left[\mathtt{C}(\phi_2^\xi)\right]^{\int_0^1(1-\zeta^\xi)^{\alpha-1}h(\zeta^\xi)\zeta^{\xi-1}d\zeta}$$

$$+\ln\left[\mathtt{C}(\phi_2^\xi)\right]^{\int_0^1\zeta^{\alpha\xi-1}h(\zeta^\xi)d\zeta}+\ln\left[\mathtt{C}(\phi_1^\xi)\right]^{\int_0^1(1-\zeta^\xi)^{\alpha-1}h(\zeta^\xi)\zeta^{\xi-1}d\zeta}.$$

Above equation can be written as

$$\frac{\xi^{\alpha-1}\Gamma(\alpha)}{(\phi_2^\xi-\phi_1^\xi)^\alpha}\left[\frac{\xi^{1-\alpha}}{\Gamma(\alpha)}\int_{\phi_1}^{\phi_2}(\phi_2^\xi-\upsilon^\xi)^{\alpha-1}\upsilon^{\xi-1}\left[\ln\mathtt{C}(\upsilon^\xi)\right]d\upsilon\right]$$

$$+\ \frac{\xi^{\alpha-1}\Gamma(\alpha)}{(\phi_2^\xi-\phi_1^\xi)^\alpha}\left[\frac{\xi^{1-\alpha}}{\Gamma(\alpha)}\int_{\phi_1}^{\phi_2}(\upsilon^\xi-\phi_1^\xi)^{\alpha-1}\upsilon^{\xi-1}\left[\ln\mathtt{C}(\upsilon^\xi)\right]d\upsilon\right]$$

$$\preceq_{cr}\ln\left[\mathtt{C}(\phi_1^\xi)\right]^{\int_0^1\left(\zeta^{\alpha\xi-1}+(1-\zeta^\xi)^{\alpha-1}\zeta^{\xi-1}\right)h(\zeta^\xi)d\zeta}$$

$$+\ln\left[\mathtt{C}(\phi_2^\xi)\right]^{\int_0^1\left(\zeta^{\alpha\xi-1}+(1-\zeta^\xi)^{\alpha-1}\zeta^{\xi-1}\right)h(\zeta^\xi)d\zeta}.$$

On utilizing the definition of Katugampola fractional integrals, we have

$$\exp\left[\frac{\xi^{\alpha-1}\Gamma(\alpha)}{(\phi_2^\xi-\phi_1^\xi)^\alpha}\left[{}^\xi K_{\phi_1^+}^\alpha\ln\left[\mathtt{C}(\phi_2^\xi)\right]\ +\ {}^\xi K_{\phi_2^-}^\alpha\ln\left[\mathtt{C}(\phi_1^\xi)\right]\right]\right]$$

$$\preceq_{cr}\left[\mathtt{C}(\phi_1^\xi)\mathtt{C}(\phi_2^\xi)\right]^{\int_0^1\left(\zeta^{\alpha\xi-1}+(1-\zeta^\xi)^{\alpha-1}\zeta^{\xi-1}\right)h(\zeta^\xi)d\zeta}. \tag{15}$$

On combining (9) and (15), we get the required result. □

Corollary 1 On taking $\underline{\mathtt{C}} = \overline{\mathtt{C}},\ \alpha = 1$ and $\xi = 1$ we get Theorem 3.1 from [31].

Corollary 2 For $\xi = 1$, we have

$$\mathtt{C}\left(\frac{\phi_1+\phi_2}{2}\right)^{\frac{1}{\alpha h(\frac{1}{2})}} \preceq_{cr} \exp\frac{\Gamma(\alpha)}{(\phi_2-\phi_1)^{\alpha}}\left[J^{\alpha}_{\phi_1^+}\ln\left[\mathtt{C}(\phi_2)\right]+J^{\alpha}_{\phi_2^-}\ln\left[\mathtt{C}(\phi_1)\right]\right]$$

$$\preceq_{cr}\left[\mathtt{C}(\phi_1)\mathtt{C}(\phi_2)\right]^{\int_0^1(\zeta^{\alpha-1}\ +\ (1-\zeta)^{\alpha-1})h(\zeta)d\zeta}.$$

Corollary 3 If $h(\zeta) = 1$, we obtain

$$\sqrt{\mathtt{C}\left(\frac{\phi_1^{\xi}+\phi_2^{\xi}}{2}\right)}\preceq_{cr}\exp\left[\frac{\xi^{\alpha}\Gamma(\alpha+1)}{2(\phi_2^{\xi}-\phi_1^{\xi})^{\alpha}}\left({}^{\xi}K^{\alpha}_{\phi_1^+}\ln\left[\mathtt{C}(\phi_2^{\xi})\right]+{}^{\xi}K^{\alpha}_{\phi_2^-}\ln\left[\mathtt{C}(\phi_1^{\xi})\right]\right)\right]$$

$$\preceq_{cr}[\mathtt{C}(\phi_1^{\xi})\mathtt{C}(\phi_2^{\xi})].$$

Theorem 6 For $\alpha = 1,\ \xi = 1$, we obtain

$$\mathtt{C}\left(\frac{\phi_1+\phi_2}{2}\right)^{\frac{1}{2h(\frac{1}{2})}} \preceq_{cr}\exp\left[\frac{1}{(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_2}\left[\ln\mathtt{C}(\zeta)\right]d\zeta\right] \tag{16}$$

$$\preceq_{cr}\left(\mathtt{C}(\phi_1)\mathtt{C}(\phi_2)\right)^{\int_0^1 h(\zeta)d\zeta}.$$

Remark 8 If $h(\zeta) = 1$, $\xi = 1$, $\alpha = 1$, we obtain the following inequalities:

$$\sqrt{\mathtt{C}\left(\frac{\phi_1+\phi_2}{2}\right)}\preceq_{cr}\exp\left(\frac{1}{(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_2}\left[\ln\mathtt{C}(\zeta)\right]d\zeta\right) \tag{17}$$

$$\preceq_{cr}\mathtt{C}(\phi_1)\mathtt{C}(\phi_2).$$

Theorem 7 or an interval valued function $\mathtt{C} : [\phi_1,\ \phi_2] \to \mathbb{R}_I^+$, where $\mathtt{C} = [\underline{\mathtt{C}},\ \overline{\mathtt{C}}]$ and $\mathtt{C} \in I\mathbb{R}_{[\phi_1^{\xi},\ \phi_2^{\xi}]}$ and h be a functionh: $[0,\ 1] \to \mathbb{R}^+$ with $h\left(\frac{1}{2}\right) \neq 0$. If $\tilde{H}$: $\left[\phi_1^{\xi},\ \phi_2^{\xi}\right] \to \mathbb{R}^+$ is function which is symmetric about $\frac{\phi_1^{\xi}+\phi_2^{\xi}}{2}$. If $\mathtt{C} \in SX(cr$-log-$h,\ \left[\phi_1^{\xi},\ \phi_2^{\xi}\right],\ \mathbb{R}_I^+)$ then for any $\alpha \geq 0,\ \xi \geq 0$ we have

$$\frac{1}{2h\left(\frac{1}{2}\right)}\ln\left[\mathtt{C}\left(\frac{\phi_1^{\xi}+\phi_2^{\xi}}{2}\right)\right]\frac{\xi^{\alpha-1}\Gamma(\alpha)}{(\phi_2^{\xi}-\phi_1^{\xi})^{\alpha}}\left[{}^{\xi}K^{\alpha}_{\phi_1^+}\tilde{H}(\phi_2^{\xi})+{}^{\xi}K^{\alpha}_{\phi_2^-}\tilde{H}(\phi_1^{\xi})\right]$$

$$\preceq_{cr}\frac{\xi^{\alpha-1}\Gamma(\alpha)}{(\phi_2^{\xi}-\phi_1^{\xi})^{\alpha}}\left[{}^{\xi}K^{\alpha}_{\phi_1^+}\left[\ln\mathtt{C}(\phi_2^{\xi})\right]\tilde{H}(\phi_2^{\xi})+{}^{\xi}K^{\alpha}_{\phi_2^-}\left[\ln\mathtt{C}(\phi_1^{\xi})\right]\tilde{H}(\phi_1^{\xi})\right]$$

$$\preceq_{cr}\left(\ln\left[\mathtt{C}(\phi_1^{\xi})\mathtt{C}(\phi_2^{\xi})\right]\right)\int_0^1\left[\zeta^{\alpha\xi-1}+(1-\zeta^{\xi})^{\alpha-1}\zeta^{\xi-1}\right]h\left(\zeta^{\xi}\right)\tilde{H}(\zeta^{\xi}\phi_1^{\xi}+(1-\zeta^{\xi})\phi_2^{\xi})d\zeta.$$

Proof. Since $\mathtt{C}\in SX(cr\text{-log-}h,\ [\phi_1^{\xi},\ \phi_2^{\xi}],\mathbb{R}_I^+)$ then, we obtain

$$\mathtt{C}(\zeta^{\xi}\phi_1^{\xi}+(1-\zeta^{\xi})\phi_2^{\xi})\preceq_{cr}\left[\mathtt{C}(\phi_1^{\xi})\right]^{h\left(\zeta^{\xi}\right)}\left[\mathtt{C}(\phi_2^{\xi})\right]^{h\left(1-\zeta^{\xi}\right)}$$

$$\mathtt{C}((1-\zeta^{\xi})\phi_1^{\xi}+\zeta^{\xi}\phi_2^{\xi})\preceq_{cr}\left[\mathtt{C}(\phi_1^{\xi})\right]^{h\left(1-\zeta^{\xi}\right)}\left[\mathtt{C}(\phi_2^{\xi})\right]^{h\left(\zeta^{\xi}\right)}$$

On utilizing the property of log, we have

$$\ln\mathtt{C}(\zeta^{\xi}\phi_1^{\xi}+(1-\zeta^{\xi})\phi_2^{\xi})\preceq_{cr}h\left(\zeta^{\xi}\right)\ln\left[\mathtt{C}(\phi_1^{\xi})\right]+h\left(1-\zeta^{\xi}\right)\ln\left[\mathtt{C}(\phi_2^{\xi})\right].\tag{18}$$

$$\ln\mathtt{C}((1-\zeta^{\xi})\phi_1^{\xi}+\zeta^{\xi}\phi_2^{\xi})\preceq_{cr}\left(1-\zeta^{\xi}\right)\ln\left[\mathtt{C}(\phi_1^{\xi})\right]+h\left(\zeta^{\xi}\right)\ln\left[\mathtt{C}(\phi_2^{\xi})\right].\tag{19}$$

On multiplying (18) with $\zeta^{\alpha\xi-1}\tilde{H}(\zeta^{\xi}\phi_1^{\xi}+(1-\zeta^{\xi})\phi_2^{\xi})$ and integrating over the interval $[0,\ 1]$ w.r.t ζ we get

$$\int_0^1\zeta^{\alpha\xi-1}\left[\ln\mathtt{C}(\zeta^{\xi}\phi_1^{\xi}+(1-\zeta^{\xi})\phi_2^{\xi})\right]\tilde{H}(\zeta^{\xi}\phi_1^{\xi}+(1-\zeta^{\xi})\phi_2^{\xi})d\zeta$$

$$\preceq_{cr}\int_0^1\zeta^{\alpha\xi-1}h\left(\zeta^{\xi}\right)\left[\ln\mathtt{C}(\phi_1^{\xi})\right]\tilde{H}(\zeta^{\xi}\phi_1^{\xi}+(1-\zeta^{\xi})\phi_2^{\xi})d\zeta$$

$$+\int_0^1\zeta^{\alpha\xi-1}h\left(1-\zeta^{\xi}\right)\left[\ln\mathtt{C}(\phi_2^{\xi})\right]\tilde{H}(\zeta^{\xi}\phi_1^{\xi}+(1-\zeta^{\xi})\phi_2^{\xi})d\zeta.\tag{20}$$

On multiplying (19) with $\zeta^{\alpha\xi-1}\tilde{H}((1-\zeta^{\xi})\phi_1^{\xi}+\zeta^{\xi}\phi_2^{\xi})$ and integrating w.r.t ζ over the interval $[0,\ 1]$, we obtain

$$\int_0^1 \zeta^{\alpha\xi-1}\left[\ln\mathsf{C}((1-\zeta^{\xi})\phi_1^{\xi}+\zeta^{\xi}\phi_2^{\xi})\right]\tilde{H}((1-\zeta^{\xi})\phi_1^{\xi}+\zeta^{\xi}\phi_2^{\xi})d\zeta$$

$$\preceq_{cr}\int_0^1 \zeta^{\alpha\xi-1}h\left(1-\zeta^{\xi}\right)\left[\ln\mathsf{C}(\phi_1^{\xi})\right]\tilde{H}((1-\zeta^{\xi})\phi_1^{\xi}+\zeta^{\xi}\phi_2^{\xi})d\zeta$$

$$+\int_0^1 \zeta^{\alpha\xi-1}h\left(\zeta^{\xi}\right)\left[\ln\mathsf{C}(\phi_2^{\xi})\right]\tilde{H}((1-\zeta^{\xi})\phi_1^{\xi}+\zeta^{\xi}\phi_2^{\xi})d\zeta. \tag{21}$$

On combining (20) and (21), we obtain

$$\int_0^1 \zeta^{\alpha\xi-1}\left[\ln\mathsf{C}(\zeta^{\xi}\phi_1^{\xi}+(1-\zeta^{\xi})\phi_2^{\xi})\right]\tilde{H}(\zeta^{\xi}\phi_1^{\xi}+(1-\zeta^{\xi})\phi_2^{\xi})d\zeta$$

$$+\int_0^1 \zeta^{\alpha\xi-1}\left[\ln\mathsf{C}((1-\zeta^{\xi})\phi_1^{\xi}+\zeta^{\xi}\phi_2^{\xi})\right]\tilde{H}((1-\zeta^{\xi})\phi_1^{\xi}+\zeta^{\xi}\phi_2^{\xi})d\zeta$$

$$\preceq_{cr}\left[\ln\mathsf{C}(\phi_1^{\xi})\right]\int_0^1 \zeta^{\alpha\xi-1}h\left(\zeta^{\xi}\right)\tilde{H}(\zeta^{\xi}\phi_1^{\xi}+(1-\zeta^{\xi})\phi_2^{\xi})d\zeta$$

$$+\left[\ln\mathsf{C}(\phi_2^{\xi})\right]\int_0^1 \zeta^{\alpha\xi-1}h\left(1-\zeta^{\xi}\right)\tilde{H}(\zeta^{\xi}\phi_1^{\xi}+(1-\zeta^{\xi})\phi_2^{\xi})d\zeta$$

$$+\left[\ln\mathsf{C}(\phi_1^{\xi})\right]\int_0^1 \zeta^{\alpha\xi-1}h\left(1-\zeta^{\xi}\right)\tilde{H}(\zeta^{\xi}\phi_1^{\xi}+(1-\zeta^{\xi})\phi_2^{\xi})d\zeta$$

$$+\left[\ln\mathsf{C}(\phi_2^{\xi})\right]\int_0^1 \zeta^{\alpha\xi-1}h\left(\zeta^{\xi}\right)\tilde{H}(\zeta^{\xi}\phi_1^{\xi}+(1-\zeta^{\xi})\phi_2^{\xi})d\zeta.$$

After suitable substitution and utilizing that $\tilde{H}$ is symmetric about $\dfrac{\phi_1^{\xi}+\phi_2^{\xi}}{2}$, we obtain

$$\frac{\xi^{\alpha-1}\Gamma(\alpha)}{(\phi_2^{\xi}-\phi_1^{\xi})^{\alpha}}\left[\frac{\xi^{1-\alpha}}{\Gamma(\alpha)}\int_{\phi_1}^{\phi_2}(\phi_2^{\xi}-\upsilon^{\xi})^{\alpha-1}\left[\ln\mathsf{C}(\upsilon^{\xi})\right]\tilde{H}(\upsilon^{\xi})\upsilon^{\xi-1}d\upsilon\right]$$

$$+\frac{\xi^{\alpha-1}\Gamma(\alpha)}{(\phi_2^{\xi}-\phi_1^{\xi})^{\alpha}}\left[\frac{\xi^{1-\alpha}}{\Gamma(\alpha)}\int_{\phi_1}^{\phi_2}(\upsilon^{\xi}-\phi_1^{\xi})^{\alpha-1}\left[\ln\mathsf{C}(\upsilon^{\xi})\right]\tilde{H}(\upsilon^{\xi})\upsilon^{\xi-1}d\upsilon\right]$$

$$\preceq_{cr}\left[\ln\mathsf{C}(\phi_1^{\xi})\right]\int_0^1 \zeta^{\alpha\xi-1}h\left(\zeta^{\xi}\right)\tilde{H}(\zeta^{\xi}\phi_1^{\xi}+(1-\zeta^{\xi})\phi_2^{\xi})d\zeta$$

$$+\left[\ln \mathtt{C}(\phi_2^{\xi})\right]\int_0^1 \zeta^{\alpha\xi-1}h\left(\zeta^{\xi}\right)\tilde{H}(\zeta^{\xi}\phi_1^{\xi}+(1-\zeta^{\xi})\phi_2^{\xi})d\zeta$$

$$+\left[\ln \mathtt{C}(\phi_1^{\xi})\right]\int_0^1 (1-\zeta^{\xi})^{\alpha-1}h\left(\zeta^{\xi}\right)\zeta^{\xi-1}\tilde{H}(\zeta^{\xi}\phi_1^{\xi}+(1-\zeta^{\xi})\phi_2^{\xi})d\zeta$$

$$+\left[\ln \mathtt{C}(\phi_2^{\xi})\right]\int_0^1 (1-\zeta^{\xi})^{\alpha-1}h\left(\zeta^{\xi}\right)\zeta^{\xi-1}\tilde{H}(\zeta^{\xi}\phi_1^{\xi}+(1-\zeta^{\xi})\phi_2^{\xi})d\zeta.$$

On utilizing the definition of Katugampola fractional integrals, we have

$$\frac{\xi^{\alpha-1}\Gamma(\alpha)}{(\phi_2^{\xi}-\phi_1^{\xi})^{\alpha}}\left[{}^{\xi}K^{\alpha}_{\phi_1^+}\left[\ln \mathtt{C}(\phi_2^{\xi})\right]\tilde{H}(\phi_2^{\xi})\;+\;{}^{\xi}K^{\alpha}_{\phi_2^-}\left[\ln \mathtt{C}(\phi_1^{\xi})\right]\tilde{H}(\phi_1^{\xi})\right]$$

$$\preceq_{cr}\left[\ln \mathtt{C}(\phi_1^{\xi})\right]\int_0^1 (\zeta^{\alpha\xi-1}+(1-\zeta^{\xi})^{\alpha-1}\zeta^{\xi-1})h\left(\zeta^{\xi}\right)\tilde{H}(\zeta^{\xi}\phi_1^{\xi}+(1-\zeta^{\xi})\phi_2^{\xi})d\zeta$$

$$+\left[\ln \mathtt{C}(\phi_2^{\xi})\right]\int_0^1 (\zeta^{\alpha\xi-1}+(1-\zeta^{\xi})^{\alpha-1}\zeta^{\xi-1})h\left(\zeta^{\xi}\right)\tilde{H}(\zeta^{\xi}\phi_1^{\xi}+(1-\zeta^{\xi})\phi_2^{\xi})d\zeta$$

$$\preceq_{cr}\ln\left[\mathtt{C}(\phi_1^{\xi})\mathtt{C}(\phi_2^{\xi})\right]\int_0^1 (\zeta^{\alpha\xi-1}+(1-\zeta^{\xi})^{\alpha-1}\zeta^{\xi-1})h\left(\zeta^{\xi}\right)\tilde{H}(\zeta^{\xi}\phi_1^{\xi}+(1-\zeta^{\xi})\phi_2^{\xi})d\zeta. \tag{22}$$

Now, on multiplying (8) with $\zeta^{\alpha\xi-1}\tilde{H}(\zeta^{\xi}\phi_1^{\xi}+(1-\zeta^{\xi})\phi_2^{\xi})$ and integrating over the interval $[0,\ 1]$ w.r.t ζ

$$\frac{1}{h\left(\frac{1}{2}\right)}\ln\left[\mathtt{C}(\frac{\phi_1^{\xi}+\phi_2^{\xi}}{2})\right]\int_0^1 \zeta^{\alpha\xi-1}\tilde{H}(\zeta^{\xi}\phi_1^{\xi}+(1-\zeta^{\xi})\phi_2^{\xi})d\zeta \tag{23}$$

$$\preceq_{cr}\int_0^1 \zeta^{\alpha\xi-1}\tilde{H}(\zeta^{\xi}\phi_1^{\xi}+(1-\zeta^{\xi})\phi_2^{\xi})\ln\left[\mathtt{C}(\zeta^{\xi}\phi_1^{\xi}+(1-\zeta^{\xi})\phi_2^{\xi})\right]d\zeta$$

$$+\int_0^1 \zeta^{\alpha\xi-1}\tilde{H}((1-\zeta^{\xi})\phi_1^{\xi}+\zeta^{\xi}\phi_2^{\xi})\ln\left[\mathtt{C}((1-\zeta^{\xi})\phi_1^{\xi}+\zeta^{\xi}\phi_2^{\xi})\right]d\zeta.$$

Again Multiply (8) by $\zeta^{\alpha\xi-1}\tilde{H}((1-\zeta^{\xi})\phi_1^{\xi}+\zeta^{\xi}\phi_2^{\xi})$ and integrating w.r.t ζ over the interval $[0,\ 1]$, we obtain

$$\frac{1}{h\left(\frac{1}{2}\right)}\ln\left[\mathtt{C}\left(\frac{\phi_1^{\xi}+\phi_2^{\xi}}{2}\right)\right]\int_0^1 \zeta^{\alpha\xi-1}\tilde{H}((1-\zeta^{\xi})\phi_1^{\xi}+\zeta^{\xi}\phi_1^{\xi})d\zeta \tag{24}$$

$$\preceq_{cr}\int_0^1 \zeta^{\alpha\xi-1}\tilde{H}(\zeta^{\xi}\phi_1^{\xi}+(1-\zeta^{\xi})\phi_2^{\xi})\ln\left[\mathtt{C}(\zeta^{\xi}\phi_1^{\xi}+(1-\zeta^{\xi})\phi_2^{\xi})\right]d\zeta$$

$$+\int_0^1 \zeta^{\alpha\xi-1}\tilde{H}((1-\zeta^{\xi})\phi_1^{\xi}+\zeta^{\xi}\phi_2^{\xi})\ln\left[\mathtt{C}((1-\zeta^{\xi})\phi_1^{\xi}+\zeta^{\xi}\phi_2^{\xi})\right]d\zeta.$$

On combining (23) and (24), we have

$$\frac{1}{h\left(\frac{1}{2}\right)}\ln\left[\mathtt{C}\left(\frac{\phi_1^{\xi}+\phi_2^{\xi}}{2}\right)\right]\int_0^1 \zeta^{\alpha\xi-1}\tilde{H}(\zeta^{\xi}\phi_1^{\xi}+(1-\zeta^{\xi})\phi_2^{\xi})d\zeta$$

$$+\frac{1}{h\left(\frac{1}{2}\right)}\ln\left[\mathtt{C}\left(\frac{\phi_1^{\xi}+\phi_2^{\xi}}{2}\right)\right]\int_0^1 \zeta^{\alpha\xi-1}\tilde{H}((1-\zeta^{\xi})\phi_1^{\xi}+\zeta^{\xi}\phi_2^{\xi})d\zeta$$

$$\preceq_{cr} 2\left[\int_0^1 \zeta^{\alpha\xi-1}\tilde{H}(\zeta^{\xi}\phi_1^{\xi}+(1-\zeta^{\xi})\phi_2^{\xi})\ln\left[\mathtt{C}(\zeta^{\xi}\phi_1^{\xi}+(1-\zeta^{\xi})\phi_2^{\xi})\right]d\zeta\right]$$

$$+2\left[\int_0^1 \zeta^{\alpha\xi-1}\tilde{H}((1-\zeta^{\xi})\phi_1^{\xi}+\zeta^{\xi}\phi_2^{\xi})\ln\left[\mathtt{C}((1-\zeta^{\xi})\phi_1^{\xi}+\zeta^{\xi}\phi_2^{\xi})\right]d\zeta\right].$$

With the appropriate substitution, we get

$$\frac{1}{h\left(\frac{1}{2}\right)}\ln\left[\mathtt{C}\left(\frac{\phi_1^{\xi}+\phi_2^{\xi}}{2}\right)\right]\left[\frac{\xi^{\alpha-1}\Gamma(\alpha)}{(\phi_2^{\xi}-\phi_1^{\xi})^{\alpha}}\left[\frac{\xi^{1-\alpha}}{\Gamma(\alpha)}\int_{\phi_1}^{\phi_2}(\phi_2^{\xi}-\upsilon^{\xi})^{\alpha-1}\tilde{H}(\upsilon^{\xi})\upsilon^{\xi-1}d\upsilon\right]\right.$$

$$\left.+\frac{\xi^{\alpha-1}\Gamma(\alpha)}{(\phi_2^{\xi}-\phi_1^{\xi})^{\alpha}}\left[\frac{\xi^{1-\alpha}}{\Gamma(\alpha)}\int_{\phi_1}^{\phi_2}(\upsilon^{\xi}-\phi_1^{\xi})^{\alpha-1}\tilde{H}(\upsilon^{\xi})\upsilon^{\xi-1}d\upsilon\right]\right]$$

$$=2\left[\frac{\xi^{\alpha-1}\Gamma(\alpha)}{(\phi_2^{\xi}-\phi_1^{\xi})^{\alpha}}\left[\frac{\xi^{1-\alpha}}{\Gamma(\alpha)}\int_{\phi_1}^{\phi_2}(\phi_2^{\xi}-\upsilon^{\xi})^{\alpha-1}\left[\ln\underline{\mathtt{C}}(\upsilon^{\xi})\right]\tilde{H}(\upsilon^{\xi})\upsilon^{\xi-1}d\upsilon\right]\right.,$$

$$\left.\frac{\xi^{\alpha-1}\Gamma(\alpha)}{(\phi_2^{\xi}-\phi_1^{\xi})^{\alpha}}\left[\frac{\xi^{1-\alpha}}{\Gamma(\alpha)}\int_{\phi_1}^{\phi_2}(\phi_2^{\xi}-\upsilon^{\xi})^{\alpha-1}\left[\ln\overline{\mathtt{C}}(\upsilon^{\xi})\right]\tilde{H}(\upsilon^{\xi})\upsilon^{\xi-1}d\upsilon\right]\right]$$

$$+2\left[\frac{\xi^{\alpha-1}\Gamma(\alpha)}{(\phi_2^{\xi}-\phi_1^{\xi})^{\alpha}}\left[\frac{\xi^{1-\alpha}}{\Gamma(\alpha)}\int_{\phi_1}^{\phi_2}(\upsilon^{\xi}-\phi_1^{\xi})^{\alpha-1}\left[\ln\underline{\mathsf{C}}(\upsilon^{\xi})\right]\tilde{H}(\upsilon^{\xi})\upsilon^{\xi-1}d\upsilon\right],\right.$$

$$\left.\frac{\xi^{\alpha-1}\Gamma(\alpha)}{(\phi_2^{\xi}-\phi_1^{\xi})^{\alpha}}\left[\frac{\xi^{1-\alpha}}{\Gamma(\alpha)}\int_{\phi_1}^{\phi_2}(\upsilon^{\xi}-\phi_1^{\xi})^{\alpha-1}\left[\ln\overline{\mathsf{C}}(\upsilon^{\xi})\right]\tilde{H}(\upsilon^{\xi})\upsilon^{\xi-1}d\upsilon\right]\right].$$

On utilizing the definition of Katugampola fractional integrals, we have

$$\frac{1}{2h\left(\frac{1}{2}\right)}\ln\left[\mathsf{C}\left(\frac{\phi_1^{\xi}+\phi_2^{\xi}}{2}\right)\right]\frac{\xi^{\alpha-1}\Gamma(\alpha)}{(\phi_2^{\xi}-\phi_1^{\xi})^{\alpha}}\left[{}^{\xi}K_{\phi_1^+}^{\alpha}\tilde{H}(\phi_2^{\xi})+{}^{\xi}K_{\phi_2^-}^{\alpha}\tilde{H}(\phi_1^{\xi})\right] \tag{25}$$

$$\preceq_{cr}\frac{\xi^{\alpha-1}\Gamma(\alpha)}{(\phi_2^{\xi}-\phi_1^{\xi})^{\alpha}}\left[{}^{\xi}K_{\phi_1^+}^{\alpha}\left[\ln\mathsf{C}(\phi_2^{\xi})\right]\tilde{H}(\phi_2^{\xi})+{}^{\xi}K_{\phi_2^-}^{\alpha}\left[\ln\mathsf{C}(\phi_1^{\xi})\right]\tilde{H}(\phi_1^{\xi})\right].$$

On combining (22) and (25), we get the required result. □

Remark 9 For $\xi=1$ and $\alpha=1$ we obtain,

$$\frac{1}{2(\phi_2-\phi_1)h\left(\frac{1}{2}\right)}\ln\left[\mathsf{C}\left(\frac{\phi_1+\phi_2}{2}\right)\right]\int_{\phi_1}^{\phi_2}\tilde{H}(\zeta)d\zeta$$

$$\preceq_{cr}\frac{1}{(\phi_2-\phi_1)}\left[\int_{\phi_1}^{\phi_2}\mathsf{C}(\zeta)\tilde{H}(\zeta)d\zeta\right]$$

$$\preceq_{cr}\left[\mathsf{C}(\phi_1)\mathsf{C}(\phi_2)\right]\int_0^1 h(\zeta)\tilde{H}(\zeta\phi_1+(1-\zeta)\phi_2)d\zeta.$$

Remark 10 For $\xi=1$ we obtain,

$$\frac{1}{2h\left(\frac{1}{2}\right)}\ln\left[\mathsf{C}\left(\frac{\phi_1+\phi_2}{2}\right)\right]\frac{\Gamma(\alpha)}{(\phi_2-\phi_1)^{\alpha}}\left[J_{\phi_1^+}^{\alpha}\tilde{H}(\phi_2)+J_{\phi_2^-}^{\alpha}\tilde{H}(\phi_1)\right]$$

$$\preceq_{cr}\frac{\Gamma(\alpha)}{(\phi_2-\phi_1)^{\alpha}}\left[J_{\phi_1^+}^{\alpha}\left[\ln\mathsf{C}(\phi_2)\right]\tilde{H}(\phi_2)+J_{\phi_2^-}^{\alpha}\left[\ln\mathsf{C}(\phi_1)\right]\tilde{H}(\phi_1)\right]$$

$$\preceq_{cr}\ln\left[\mathsf{C}(\phi_1)\mathsf{C}(\phi_2)\right]\int_0^1(\zeta^{\alpha-1}+(1-\zeta)^{\alpha-1})h(\zeta)\tilde{H}(\zeta\phi_1+(1-\zeta)\phi_2)d\zeta.$$

3. Illustrative examples and graphical validation

This section shows examples in figures created with MATLAB software to confirm the derived inequalities for *cr*-log-*h*-convex functions using Katugampola fractional integrals. These examples highlight the results' practical applicability and significance.

Example 1 Suppose an interval valued function $\mathtt{C}(\upsilon) = \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \to \mathbb{R}_I^+$ defined by $\mathtt{C}(\upsilon) = [e^{-\sin\upsilon}, e^{-\cos\upsilon}]$. If $h(\zeta) = \zeta$ for all $\zeta \in [0, 1]$ then for $\alpha = 2$, $\xi = 1$ we have:

$$\frac{\phi_1 + \phi_2}{2} = \frac{3\pi}{8}, \ (\phi_2^\xi - \phi_1^\xi)^\alpha = \frac{\pi^2}{16}, \ \xi^{\alpha-1} = 1$$

$$\mathtt{C}\left(\frac{\phi_1^\xi + \phi_2^\xi}{2}\right)^{\frac{1}{\alpha\xi h\left(\frac{1}{2}\right)}} = [0.397, 0.682]$$

$$\exp\left[\frac{\xi^{\alpha-1}\Gamma(\alpha)}{(\phi_2^\xi - \phi_1^\xi)^\alpha}\left({}^\xi K_{\phi_1^+}^\alpha \ln\mathtt{C}(\phi_2^\xi) + {}^\xi K_{\phi_2^-}^\alpha \ln\mathtt{C}(\phi_1^\xi)\right)\right] = [0.406, 0.688]$$

$$\left[\mathtt{C}(\phi_1^\xi)\mathtt{C}(\phi_2^\xi)\right]^{\int_0^1 (\zeta^{\alpha\xi-1} + (1-\zeta^\xi)^{\alpha-1}\zeta^{\xi-1})h(\zeta^\xi)d\zeta} = \left([e^{\frac{-1}{\sqrt{2}}}, e^{\frac{-1}{\sqrt{2}}}].[e^{-1}, 1]\right)^{\frac{1}{2}} = [0.426, 0.702].$$

As $[0.397, 0.682] \preceq_{cr} [0.406, 0.688] \preceq_{cr} [0.426, 0.702]$. Thus Theorem 5 is satisfied.

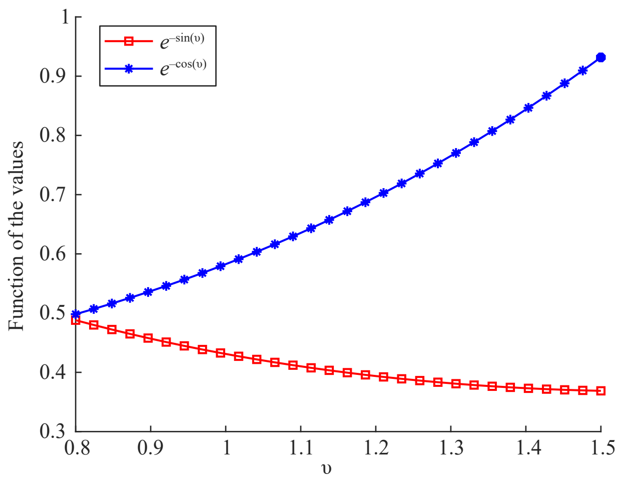

Figure 1. Comparison of $e^{-\sin(\upsilon)}$ and $e^{-\cos(\upsilon)}$

Figure 1 displays the graphs of the bounding functions $e^{-\sin(\upsilon)}$ and $e^{-\cos(\upsilon)}$. The figure illustrates the behavior of the interval-valued function $\mathtt{C}(\upsilon)$ across the interval. The containment of the calculated bounds in Example 1 within this visual range confirms the theoretical inequalities derived in Theorem 5.

Example 2 Suppose an interval valued function $\mathtt{C}(\upsilon) = [1, 3] \to \mathbb{R}_I^+$ defined by $\mathtt{C}(\upsilon) = [e^{3\upsilon}, e^{5\upsilon}]$. Let $h(\zeta) = \zeta$ for all $\zeta \in [0, 1]$ then for $\alpha = 2$, $\xi = 1$ we have

$$\frac{\phi_1+\phi_2}{2}=2,\ (\phi_2^{\xi}-\phi_1^{\xi})^{\alpha}=4,\ \xi^{\alpha-1}=1$$

$$\mathsf{C}\left(\frac{\phi_1^{\xi}+\phi_2^{\xi}}{2}\right)^{\frac{1}{\alpha\xi h(\frac{1}{2})}}=\left[e^6,\ e^{10}\right]$$

$$\exp\left[\frac{\xi^{\alpha-1}\Gamma(\alpha)}{(\phi_2^{\xi}-\phi_1^{\xi})^{\alpha}}\left({}^{\xi}K^{\alpha}_{\phi_1^+}\ln\mathsf{C}(\phi_2^{\xi})\ +\ {}^{\xi}K^{\alpha}_{\phi_2^-}\ln\mathsf{C}(\phi_1^{\xi})\right)\right]=\left[e^6,\ e^{10}\right]$$

$$[\mathsf{C}(\phi_1^{\xi})\mathsf{C}(\phi_2^{\xi})]^{\int_0^1(\zeta^{\alpha\xi-1}\ +\ (1-\zeta^{\xi})^{\alpha-1}\zeta^{\xi-1})h(\zeta^{\xi})d\zeta}=\left([e^3,\ e^5]\cdot[e^9,\ e^{15}]\right)^{\frac{1}{2}}=\left[e^6,\ e^{10}\right]$$

As $\left[e^6,\ e^{10}\right]\preceq_{cr}\left[e^6,\ e^{10}\right]\preceq_{cr}\left[e^6,\ e^{10}\right]$. Thus Theorem 5 is satisfied.

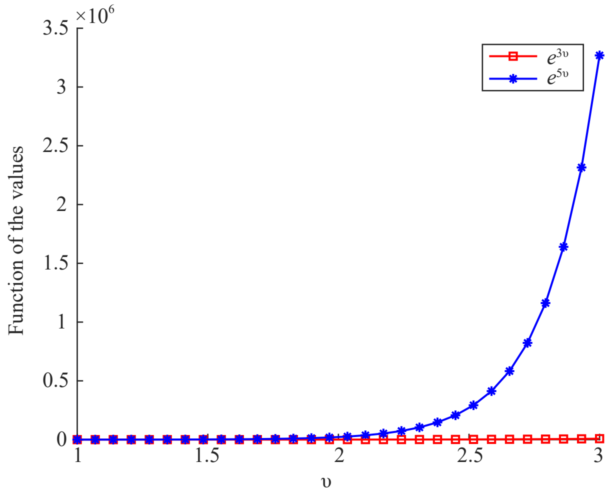

Figure 2. Comparison of $e^{3\upsilon}$ and $e^{5\upsilon}$

Figure 2 illustrates the rapidly increasing behavior of the exponential bounds $e^{3\upsilon}$ and $e^{5\upsilon}$. The plot provides a visual understanding of the tightness and validity of the inequality relations for interval-valued functions under the assumptions of Theorem 5.

Example 3 Suppose an interval valued function $\mathsf{C}(\upsilon)=[1,\ e]\ \to\mathbb{R}_I^+$ defined by $\mathsf{C}(\upsilon)=[\upsilon,\ \upsilon^2]$. If $h(\zeta)=e^{\zeta}-1$ for all $\zeta\in[0,\ 1]$ then for $\alpha=2,\ \xi=1$ we have

$$\frac{\phi_1+\phi_2}{2}=\frac{1+e}{2},\ (\phi_2^{\xi}-\phi_1^{\xi})^{\alpha}=2.9525,\ \xi^{\alpha-1}=1,\ h\left(\frac{1}{2}\right)=1.4142$$

$$\mathtt{C}\left(\frac{\phi_1^{\xi}+\phi_2^{\xi}}{2}\right)^{\frac{1}{\alpha\xi h(\frac{1}{2})}}=[1.613,\ 2.601]$$

$$\exp\left[\frac{\xi^{\alpha-1}\Gamma(\alpha)}{(\phi_2^{\xi}-\phi_1^{\xi})^{\alpha}}\left({}^{\xi}K^{\alpha}_{\phi_1^+}\ln\mathtt{C}(\phi_2^{\xi})\ +\ {}^{\xi}K^{\alpha}_{\phi_2^+}\ln\mathtt{C}(\phi_1^{\xi})\right)\right]=[1.789,\ 3.202]$$

$$[\mathtt{C}(\phi_1^{\xi})\mathtt{C}(\phi_2^{\xi})]^{\int_0^1(\zeta^{\alpha\xi-1}\ +\ (1-\zeta^{\xi})^{\alpha-1}\zeta^{\xi-1})h(\zeta^{\xi})d\zeta}=[2.051,\ 4.206]$$

As $[1.612,\ 2.601]\preceq_{cr}[1.789,\ 3.202]\preceq_{cr}[2.051,\ 4.206]$. Thus Theorem 5 is satisfied.

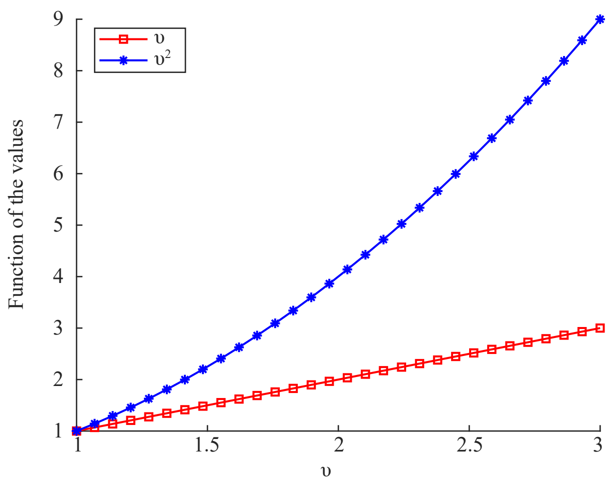

Figure 3. Comparison of υ and υ^2

Figure 3 displays the functions υ^3+1 and υ^3+4, forming the lower and upper bounds of the interval-valued function $\mathtt{C}(\upsilon)$. The polynomial nature of the bounds results in a smooth, increasing interval width over the domain. The numerical values obtained align with the plotted curves, and the containment of the computed expressions within the expected intervals visually confirms the validity of Theorem 5.

4. Conclusion

In this work, we introduced and developed a new class of convex functions *cr*-log-*h*-convex functions within the context of interval-valued functions and the Katugampola fractional integral framework. Our main contribution is the derivation of generalized Hermite-Hadamard-type inequalities for these functions, including their weighted versions. These results significantly extend classical convexity-based inequalities by incorporating the more flexible and encompassing structure of *cr*-log-*h*-convexity. Compared to previous studies that typically focused on real-valued convex or log-convex functions, our approach broadens the applicability to interval-valued functions, which are crucial in modeling uncertainty and imprecision, particularly in fuzzy systems and optimization. Furthermore, the integration of the Katugampola fractional integral, which generalizes both Riemann-Liouville and Hadamard integrals, allows our results to cover a wider spectrum of fractional models. We also discussed several special cases that reduce our general results to known inequalities, thereby confirming the consistency and generality of the proposed framework. To demonstrate the validity and practical usefulness of the findings, three illustrative examples were provided. Overall, this study offers a systematic generalization of classical integral inequalities and contributes a novel toolset for researchers in fractional calculus, convex analysis, fuzzy systems, and interval mathematics. Future work may build upon this framework to explore further generalizations or applications in fields such as economics, engineering, and information theory.

Data availability

No datasets were generated or analysed during the current study.

CRediT authorship contribution statement

Conceptualization, P.O.M. and R.P.A.; data curation, M.A.Y.; funding acquisition, N.C.; investigation, P.O.M. and R.P.A.; methodology, S.M.; project administration, P.O.M., A.A. and N.C.; software, P.O.M. and A.A.; validation, S.M.; writing - original draft, M.A.Y. and A.A.; writing- review and editing, N.C. All authors have read and agreed to the published version of the manuscript.

Acknowledgements

Researchers Supporting Project number (RSP2025R153), King Saud University, Riyadh, Saudi Arabia.

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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