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Research Article

On the P-Adic Valuations of Stirling Numbers of the Second Kind

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Abstract: In this paper, we introduced certain formulas for p-adic valuations of Stirling numbers of the second kind $S(n, k)$ denoted by $v_p(S(n, k))$ for an odd prime p and positive integers k such that $n \geq k$. We have obtained the formulas, $v_p(S(n, n - a))$ for $a = 1, 2, 3$ and $v_p(S(cp^n, cp^k))$ for $1 \leq c \leq p - 1$ and primality test of positive integer n. We have presented the results of $v_p(S(p^2, kp))$ for $2 \leq k \leq p - 1$, $2 < p < 100$ and a table of $v_p(S(p, k))$. We have posed the following conjectures from our analysis:

1. Let $p \neq 7$ be an odd prime and k be an even integer such that $0 < k < p - 1$. Then

$$v_p(S(p^2, kp)) - v_p(S(p^2, p(k+1)) = 3.$$

2. If k be an integer such that $1 < k < p - 1$, then the p-adic valuations satisfy

$$v_p(S(p^2, kp)) = \begin{cases} 5 \text{ or } 6, & \text{if } k \text{ is even} \\ 2 \text{ or } 3, & \text{if } k \text{ is odd} \end{cases}$$

for any prime $p > 7$.

3. For any primes p and positive integer k such that $2 \leq k \leq p - 1$, then

$$v_p(S(p, k)) \leq 2.$$

Keywords: p-adic valuations, stirling numbers of the second kind, congruence, primes, minimum period

MSC: 05A18,11A51,11B73, 11E95

1. Introduction

Stirling numbers of the first and second kinds were introduced by James Stirling [1]. These numbers have been found to be of great utility in various branches of Mathematics such as combinatorics, number theory, calculus of finite differences, theory of algorithms, etc. The p-adic valuations of Stirling numbers of the second kind appear frequently

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in algebraic topology by Davis [2] to obtain new results related to James numbers, v_1-periodic homotopy groups and exponents of $SU(n)$. More details of Stirling numbers of the second kind may be seen on Comtet [3] and Graham et al. [4].

Stirling numbers of the second kind are more interesting than the first kind by their intrinsic nature. There are many interesting results of 2-adic valuations of Stirling numbers of the second kind in the open literature. Recently, Wannemacker's proof [5] of Lengyel's conjecture [6], results of $v_2(k!S(c - 2n + u, k))$ for $c > 0$ by Lengyel [7], the proof of Wannemacker's conjecture by Hong [8], the works of Amdeberhan et al. [9] and Zhao et al. [10] are other notable results of 2-adic valuation. Gessel and Lengyel [11] proved that for an arbitrary prime p and $n = a(p - 1)p^q$, $1 \leq k \leq n$

$$v_p(k!S(n,k)) = \left\lfloor \frac{k-1}{p-1} + \tau(k) \right\rfloor,$$

where a and q are positive integers such that $(a, p) = 1$, q is sufficiently large, $\frac{k}{p}$ is an odd integer and $\tau(p)$ is a non-negative integer.

Strauss [12] and Pan [13] discussed the problems of 3-adic valuations and 2-adic valuations of certain sums of binomial coefficients respectively. Sun [14] also presented the results of p-adic valuations for multinomial coefficients. Friedland [15] used 2-adic valuations of certain ratios of factorials to prove a conjecture of Falikman-Friedland-Lowery on the parity of degrees of projective varieties of $n \times n$ complex symmetric matrices of rank at most k. Some more results of p-adic valuations are also given in Gouvea [16], Koblitz [17] and Adelberg [18].

This paper consists of some interesting results about p-adic valuations for a few class of Stirling numbers of the second kind $S(n, k)$. This number $v_p(S(n, k))$, where either n or k is related to p, has been obtained independently for some values of p, n and k. The values of $v_p(S(n, k))$ are computed by using GP/PARI software and they are presented in Table 1.

2. Materials and methods

Definition 2.1 Let p be a prime. For any non-zero integer a, the p-adic valuation of a, denoted by $v_p(a)$, is defined as the exponent of the highest power of p dividing a.

It may be noted that $v_p(0) = \infty$ and $v_p(a)$ for a non-zero integer a, is a non-negative integer.

So, $v_3(25) = 0$, $v_5(25) = 2$.

Note that, for any prime p, $v_p(\pm 1) = 0$. For a given prime p and any two integers a and b, we have

$$v_p(a+b) \geq min\{v_p(a), v_p(b)\}, \quad v_p(ab) = v_p(a) + v_p(b).$$

The p-adic valuation v_p can further be extended to the field of rational numbers, $r = \frac{a}{b}$, a, $b \in \mathbb{Z}$ and $b \neq 0$ as

$$v_p(r) = v_p(a) - v_p(b).$$

Definition 2.2 Given two non-negative integers n and k, not both zero, the Stirling number of the second kind $S(n, k)$ is defined as the number of ways one can partition a set with n elements into exactly k non-empty subsets.

Example 2.1 All partitions of the set {1, 2, 3, 4} into 2 non-empty subsets are {1}, {2, 3, 4}; {2}, {1, 3, 4}; {3}, {1, 2, 4}; {4}, {1, 2, 3}; {1, 2}, {3, 4}; {1, 3}, {2, 4} and {1, 4}, {2, 3}. Hence, $S(4, 2) = 7$.

By convention, we set $S(0, 0) = 1$ and $S(0, k) = 0$ for $k \geq 1$. Thus, $S(n, k)$ is the number of ways of distributing n distinct balls into k indistinguishable boxes (the order of the boxes does not count) such that no box is empty.

It is clear that $S(n, k) = 0$ if $1 \leq n < k$ and $S(n, n) = 1$ for all $n \geq 0$.

We use the following properties to prove the results of $v_p(S(n, k))$:

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}i^n, \tag{1}$$

which gives

$$S(n,2)=2^{n-1}-1,\ S(n,1)=1,\ S(n,0)=0. \tag{2}$$

It is easy to derive the following specific identities of $S(n, k)$ using the results of ([19] p. 115-116).

$$S(n,n-1)=\binom{n}{2}\ \ if\ \ n\geq 2, \tag{3}$$

$$S(n,n-2)=\binom{n}{3}+3\binom{n}{4}\ \ if\ \ n\geq 4, \tag{4}$$

$$S(n,n-3)=\binom{n}{4}+10\binom{n}{5}+15\binom{n}{6}\ \ if\ \ n\geq 6. \tag{5}$$

3. Results

In this section, we present some basic results of the p-adic valuations of Stirling numbers starting with $S(n, n-1)$ for $n > 1$.

Proposition 3.1 For any positive integer $n > 1$ and an odd prime p

$$v_p\left(S\left(n,n-1\right)\right)=v_p\left(n\right)+v_p\left(n-1\right).$$

Proof. Using the identity (3), we have

$$S(n,n-1)=\binom{n}{2}=\frac{n(n-1)}{2}.$$

The multiplicative property of $v_p(a)$ implies that

$$v_p(S(n, n-1)) = v_p(n) + v_p(n-1) - v_p(2)$$

$$= v_p(n) + v_p(n-1)$$

as $v_p(2) = 0$, p being odd.

Applying Kummer's theorem [20] to the binomial coefficient $\binom{n}{2} = S(n, n-1)$, the above result can be put in the following form

$$v_p(S(n,n-1))=\frac{s_p(n-2)-s_p(n)+2}{p-1}, \tag{6}$$

where $s_p(n)$ denotes the sum of the p-adic digits of n.

Corollary 3.1 Let p be an odd prime. For any positive integer n and c such that $gcd(p, c) = 1$,

$$v_p(S(cp^n, cp^n-1))=n.$$

Proof. By the proposition, we have

$$v_p(S(cp^n, cp^n - 1)) = v_p(cp^n) + v_p(cp^n - 1).$$

Since $v_p(cp^n - 1) = 0$ and using the multiplicative property of $v_p(a)$, we can obtain

$$\begin{aligned} v_p(S(cp^n, cp^n - 1)) &= v_p(cp^n) \\ &= n + v_p(c). \end{aligned}$$

As $gcd(p, c) = 1$, it is clear that $v_p(c) = 0$. This completes the proof.

Proposition 3.2 For any positive integer $n \geq 2$ and an odd prime p,

$$v_p(S(n, n-2)) = \begin{cases} v_p(n) + v_p(n-1) + v_p(n-2) + v_p(3n-5), & \text{if } p > 3, \\ v_3(n) + v_3(n-1) + v_3(n-2) - 1, & \text{if } p = 3. \end{cases}$$

These results can be proved in the similar manner.

Corollary 3.2 For any positive integer n and an odd prime p,

$$v_p(S(cp^n, cp^n - 2)) = \begin{cases} n, & \text{if } p > 5, \\ n+1, & \text{if } p = 5 \text{ and } n > 1, \\ n-1, & \text{if } p = 3, \end{cases}$$

if c is a positive integer not divisible by p.

Proposition 3.3 Let p be an odd prime. For any positive integer $n \geq 6$,

$$v_p(S(n, n-3)) = \begin{cases} v_p(n) + v_p(n-1) + 2v_p(n-2) + 2v_p(n-3), & \text{if } p \geq 5, \\ v_p(n) + v_p(n-1) + 2v_p(n-2) + 2v_p(n-3) - 1, & \text{if } p = 3. \end{cases}$$

Proof. Using the identity (5), we have

$$S(n, n-3) = \binom{n}{4} + 10\binom{n}{5} + 15\binom{n}{6}, \quad \text{if } n \geq 6.$$

It can also be expressed as

$$\begin{aligned} S(n, n-3) &= \binom{n}{4}\left[\frac{n^2 - 5n + 6}{2}\right] \\ &= \binom{n}{4}\left[\frac{(n-2)(n-3)}{2}\right] \\ &= \left[\frac{n(n-1)(n-2)^2(n-3)^2}{2^4 \cdot 3}\right] \end{aligned}$$

The multiplicative property of $v_p(-)$ implies that

$$v_p\left(S(n, n-3)\right) = v_p(n) + v_p(n-1) + 2v_p(n-2) + 2v_p(n-3) - v_p(3)$$

as $v_p(2) = 0$ and p being odd.

Using Kummer's theorem [20] to $\binom{n}{4}$, we get the following result,

$$v_p(S(n, n-3)) = \frac{s_p(n-4) - s_p(n) + s_p(4)}{p-1} + v_p(n-2) + v_p(n-3). \tag{7}$$

where $s_p(n)$ denotes the sum of the p-adic digits of n. This completes the proof.

Corollary 3.3 For any positive integer n and odd prime p, the following result holds

$$v_p(S(cp^n, cp^n - 3)) = \begin{cases} n, & \text{if} \quad p > 3, \\ n+1, & \text{if} \quad p = 3, \end{cases}$$

if p does not divides c (provided $cp^n \neq 3$ if $p = 3$).

Proof. By the proposition, we have

$$v_p(S(cp^n, cp^n - 3)) = v_p(cp^n) + v_p(cp^n - 1) + v_p(cp^n - 2) + 2v_p(cp^n - 3) - v_p(3).$$

Since $v_p(cp^n - 1) = v_p(cp^n - 2) = v_p(cp^n - 3) = v_p(3) = 0$ if $p \geq 5$, we get

$$\begin{aligned} v_p(\mathrm{S}(cp^n, cp^n - 3)) &= v_p(cp^n) \\ &= n + v_p(c). \end{aligned}$$

As $gcd(p, c) = 1$, it is clear that $v_p(c) = 0$.

For the case $p = 3$, $2v_3(c3^n - 3) - v_3(3) = 1$ and $v_3(c3^n - 1) = v_3(c3^n - 2) = 0$ and hence

$$\begin{aligned} v_3(S(c3^n, c3^n - 3)) &= v_p(c3^n) + 1 \\ &= n + 1 \end{aligned}$$

This completes the proof.

Now, we give an alternate proof of the primality of integer n by divisibility of $S(n, k)$ given by Deamio and Touset [21]. The proof of corollary 2 in their paper is not correct if we take $n = 4$ and $p = 2$, then $S(4, 3) = 6 \not\equiv 1 \bmod 2$ and $2 \mid S(4, 3)$. We tackled this problem, in this paper, more simpler manner. This problem with an alternate solution also appears in Pólya et al. [22].

Theorem 3.1 If p is an odd prime, then $p \mid S(n, k)$ if $s_p(k) > s_p(n)$.

The above theorem is an immediate consequence of ([18], Lemma 2.1) which states that

$$v_p(S(n, k)) \geq \frac{s_p(k) - s_p(n)}{p-1}. \tag{8}$$

Replacing n by an odd prime p in the above theorem, we get the following results.

Corollary 3.4 If p is an odd prime, then $p \mid S(p, k)$ if $2 \leq k \leq p - 1$.

The problem in the above Corollary 3.4 appears in Graham et al. [4] and proof was given by Demaio and Touset [21].

Theorem 3.2 A positive integer n is a prime if and only if $n \mid S(n, k)$ for all $2 \leq k \leq n - 1$.

Proof. The generating function of $S(n, k)$ in terms of falling powers is given by

$$x^n = \sum_{k=o}^{n} S(n, k)\{x\}_k \tag{9}$$

for any non-negative integer n.

If n is a positive integer such that $n \mid S(n, k)$ for all $2 \le k \le n - 1$, put $x = n$ in Equation (9)

$$n^n = \sum_{k=o}^{n} S(n,k)\{n\}_k$$

$$= \{n\}_n + \{n\}_1 + \sum_{k=2}^{n-1} S(n,k)\{n\}_k$$

$$= n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 + n + \sum_{k=2}^{n-1} n(n-1)\cdots(n-(k-1))S(n,k).$$

It follows that

$$n^{n-1} = (n-1)(n-2)\cdots 3\cdot 2\cdot 1 + 1 + \sum_{k=2}^{n-1}(n-1)(n-2)\cdots(n-(k-1))S(n,k)$$

Since $n \mid S(n, k)$ for all $2 \le k \le n - 1$, we get

$$0 \equiv (n-1)! + 1 \mod n$$

or

$$(n-1)! \equiv -1 \mod n.$$

Hence, n is prime.

The converse follows from Corollary 3.4.

Lemma 3.1 If p is a prime, then

$$v_p\left(\binom{p-1}{i} - (-1)^i\right) \ge 1 \text{ or } v_p\left(\binom{p-1}{i}\right) = 0.$$

Proof. For $i = 0$, the case is trivial.

We assume that $i > 0$. The binomial coefficient $\binom{p-1}{i}$ is given by

$$\binom{p-1}{i} = \frac{(p-1)!}{(p-1-i)!i!}.$$

Therefore,

$$i!\binom{p-1}{i} = (p-1)(p-2)...(p-i+2)(p-i+1)(p-i)$$

$$\equiv (-1)(-2)...(-i) \mod p$$

$$\equiv (-1)^i i! \mod p.$$

Since $0 < i < p$, $gcd(p, i) = 1$. Then,

$$\binom{p-1}{i} \equiv (-1)^i \mod p.$$

Theorem 3.3 Let p be an odd prime. For any positive integer $n \geq p$,

$$v_p(S(n, p)) = 0$$

if and only if $(p - 1) \mid (n - 1)$.

Proof. Using the above Lemma 3.1, we have

$$p!S(n, p) = \sum_{i=1}^{p} \binom{p}{i} (-1)^{p-i} i^n$$

$$\equiv \sum_{i=1}^{p} \binom{p}{i} (-1)^{p-i} i^n \mod p.$$

Since $\binom{p}{i} = \binom{p-1}{i-1} \frac{p}{i}$, we get

$$(p-1)!S(n, p) \equiv \sum_{i=1}^{p-1} (-1)^{i-1} (-1)^{p-i} i^{n-1}.$$

Using Wilson's theorem, the preceding congruence reduces to

$$S(n, p) \equiv \sum_{i=1}^{p-1} i^{n-1} \mod p,$$

as p is odd.

Now, we use the following well-known results

$$\sum_{i=1}^{p-1} i^{n-1} \equiv \begin{cases} 0 \mod p, & \text{if } (p-1) \nmid (n-1) \\ -1 \mod p, & \text{if } (p-1) \mid (n-1). \end{cases}$$

Hence, the theorem follows.

Theorem 3.4 Let p be an odd prime and c be a positive integer such that $1 \leq c \leq p - 1$. Then, for positive integers n and k such that $k \leq n$,

$$v_p(S(cp^n, cp^k)) = 0.$$

Proof. The theorem is a special case of ([18], Th. 2.2).

We have

$$cp^n - cp^k = c(p^n - p^k) = c(p-1) \sum_{j=0}^{n-k-1} p^{j+k}$$

which implies that $cp^n - cp^k$ is divisible by $p - 1$. We also have $1 \leq c \leq p - 1$ and $1 \leq cp^k \leq cp^n$.

It follows that $S(cp^n, cp^k)$ is a minimum zero case and hence we have

$$v_p(S(cp^n, cp^k)) = \frac{s_p(cp^k) - s_p(cp^n)}{p-1} = 0, \tag{10}$$

since $s_p(cp^n) = s_p(cp^k) = s_p(c) = c$.

Theorem 3.5 Let p be an odd prime, then

$$v_p(S(p^n, 2p)) \geq n$$

for every integer $n \geq 2$.

Proof. Using identity (1)

$$(2p)!S(p^n, 2p) = \sum_{i=0}^{2p} \binom{2p}{i} (-1)^{2p-i} i^{p^n}$$

which can also be written as

$$(2p)!S(p^n, 2p) = \sum_{i=0}^{2p} \binom{2p}{2p-i} (-1)^i (2p-i)^{p^n}$$

$$= \sum_{i=0}^{2p} \binom{2p}{i} (-1)^{2p-i} (2p-i)^{p^n}.$$

Since $\binom{m}{i} = \binom{m}{m-i}$ for every integers $0 \leq i \leq m$ and $2p - i \equiv i \bmod 2$, we have

$$2(2p)!S(p^n, 2p) = \sum_{i=0}^{2p} \binom{2p}{i} (-1)^{2p-i} (i^{p^n} + (2p-i)^{p^n}). \tag{11}$$

If $p \nmid i$ for $0 \leq i \leq 2p$, then

$$2p - i \equiv -i \bmod p,$$

which also yields the congruence

$$(2p-i)^{p^n} \equiv -(i)^{p^n} \bmod p^{n+1}.$$

It follows that

$$\binom{2p}{i} (-1)^{2p-i} ((2p-i)^{p^n} + (i)^{p^n}) \equiv 0 \bmod p^{n+2}, \text{ since } p \mid \binom{2p}{i}. \tag{12}$$

Thus, each terms of the right hand side of (11) is divisible by p^{n+2} and hence

$$(2p)!S(p^2, 2p) \equiv 0 \bmod p^{n+2}$$

Therefore

$$v_p(2(2p)!S(p^2, 2p)) \geq n+2$$

$$v_p(S(p^2, 2p)) \geq n$$

Hence, the theorem follows.

Theorem 3.6 Let p be a prime and n and k be two positive integers with $k \le p - 1$, then there exists a positive integer m in $1 \le m < p - 1$ such that

$$S(n,k) \equiv \begin{cases} S(m,k) \bmod p, & \text{if } n \not\equiv 0 \bmod (p-1), \\ (p-1-k)! \bmod p, & \text{if } n \equiv 0 \bmod (p-1). \end{cases}$$

Proof. By division algorithm, we have

$$n = (p-1)q + m$$

where q is the quotient and m is the remainder such that $0 \le m < p - 1$.

Now

$$k!S(n,k) = \sum_{i=1}^{k} \binom{k}{i} (-1)^{k-i} i^n$$

$$= \sum_{i=1}^{k} \binom{k}{i} (-1)^{k-i} i^{(p-1)q+m}$$

$$\equiv \sum_{i=1}^{k} \binom{k}{i} (-1)^{k-i} i^m \bmod p$$

since $i^{p-1} \equiv 1 \bmod p$ for $1 \le i \le k \le p - 1$ by Fermat's little theorem.

If $m \ne 0$, we have

$$k!S(n, k) \equiv k!S(m, k) \bmod p.$$

Since k is less than p, it follows that $p \nmid k!$ which results

$$S(n, k) \equiv S(m, k) \bmod p.$$

for every n such that $n \not\equiv 0 \bmod p - 1$.

Next, if $m = 0$, we have

$$k!S(n,k) \equiv \sum_{i=1}^{k} \binom{k}{i} (-1)^{k-i} \bmod p$$

$$\equiv \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} - (-1)^k \bmod p$$

$$\equiv (-1)^{k+1} \bmod p,$$

We also know that

$$\binom{p-1}{k} \equiv (-1)^k \bmod p \text{ or}$$

$$\frac{(p-1)!}{(p-1-k)!k!} \equiv (-1)^k \mod p \text{ or}$$

$$\frac{1}{k!} \equiv (-1)^{k+1}(p-1-k)! \mod p$$

which implies that

$$S(n, k) \equiv (p - 1 - k)! \mod p,$$

which completes the proof.

From the above theorem, we see that if $1 \le m < k$

$$S(n, k) \equiv 0 \mod p \text{ since } S(m, k) = 0.$$

However, the case for $m = k$ results

$$S(n, k) \equiv 1 \mod p.$$

We can write the following results

Corollary 3.5 Let p be an odd prime and k be a positive integer less than p, then

$$S(n,k) \equiv \begin{cases} 1 \mod p, & if \;\; n \equiv k \mod (p-1), \\ 0 \mod p, & if \;\; n \equiv i \mod (p-1) \;\; for \;\; 1 \le i \le k-1. \end{cases}$$

If we applied the above theorem and corollary to the special cases for $k = p - 1$, $p - 2$ and $p - 3$, we get

$$S(n, p-1) \equiv \begin{cases} 1 \mod p, & if \;\; n \equiv 0 \mod (p-1), \\ 0 \mod p, & \text{otherwise.} \end{cases}$$

$$S(n, p-2) \equiv \begin{cases} 1 \mod p, & if \;\; n \equiv 0, p-2 \mod (p-1), \\ 0 \mod p, & \text{otherwise.} \end{cases}$$

$$S(n, p-3) \equiv \begin{cases} 2 \mod p, & if \;\; n \equiv 0 \mod (p-1), \\ 3 \mod p, & if \;\; n \equiv p-2 \mod (p-1), \\ 1 \mod p, & if \;\; n \equiv p-3 \mod (p-1), \\ 0 \mod p, & if \;\; \text{otherwise.} \end{cases}$$

assuming $p \neq 3$ for the last two cases.

4. Discussions

We have computed $v_p(S(p^2, kp))$ for primes $3 \le p \le 100$ and $2 \le k \le p - 1$ using PARI/GP software.

Table 1. (p, k) such that $v_p(S(p, k)) = 2$ for $3 \le p \le 1000$ and $2 \le k \le p-1$

(p, k)	(p, k)	(p, k)	(p, k)	(p, k)	(p, k)	(p, k)
(5, 3)	(167, 7)	(307, 12)	(463, 340)	(653, 429)	(857, 592)	(947, 204)
(13, 5)	(167, 103)	(307, 146)	(467, 278)	(659, 457)	(859, 300)	(947, 478)
(19, 14)	(173, 52)	(317, 188)	(499, 63)	(661, 417)	(859, 357)	(953, 391)
(29, 14)	(181, 166)	(331, 20)	(499, 320)	(677, 367)	(859, 558)	(977, 476)
(31, 16)	(193, 23)	(337, 261)	(509, 324)	(683, 271)	(863, 712)	(991, 953)
(41, 13)	(193, 45)	(353, 162)	(521, 169)	(683, 401)	(877, 77)	(997, 786)
(42, 12)	(197, 85)	(359, 96)	(521, 180)	(691, 468)	(877, 204)	
(47, 12)	(211, 62)	(359, 316)	(521, 479)	(709, 330)	(877, 542)	
(53, 5)	(211, 159)	(373, 230)	(523, 343)	(709, 371)	(881, 63)	
(53, 41)	(223, 61)	(379, 253)	(523, 483)	(709, 669)	(881, 72)	
(53, 45)	(227, 187)	(383, 323)	(569, 123)	(733, 47)	(881, 408)	
(59, 35)	(229, 25)	(397, 27)	(569, 348)	(743, 23)	(881, 625)	
(73, 8)	(233, 7)	(397, 78)	(569, 363)	(751, 744)	(887, 149)	
(79, 14)	(239, 134)	(401, 198)	(577, 119)	(761, 54)	(887, 208)	
(89, 32)	(239, 219)	(409, 45)	(577, 434)	(773, 143)	(887, 443)	
(89, 34)	(241, 15)	(409, 80)	(593, 498)	(773, 262)	(907, 611)	
(107, 16)	(251, 233)	(419, 133)	(601, 303)	(787, 228)	(911, 560)	
(127, 8)	(251, 247)	(419, 256)	(601, 515)	(797, 290)	(919, 163)	
(139, 28)	(257, 131)	(419, 310)	(607, 173)	(809, 119)	(929, 347)	
(149, 5)	(269, 98)	(431, 25)	(607, 242)	(811, 733)	(929, 469)	
(151, 50)	(271, 211)	(431, 112)	(607, 518)	(821, 533)	(929, 801)	
(151, 58)	(283, 91)	(431, 116)	(617, 209)	(827, 257)	(937, 528)	
(157, 45)	(283, 201)	(433, 91)	(647, 117)	(827, 765)	(941, 342)	
(163, 101)	(293, 76)	(439, 308)	(647, 309)	(839, 50)	(947, 85)	
(163, 127)	(293, 162)	(461, 341)	(653, 369)	(839, 744)	(947, 116)	

The obtained values of $v_p(S(p^2, kp))$ for different values of (p, k) are

$$v_p(S(p^2,kp)) = \begin{cases} 7, & if \;\; (p,k)=(7,4) \\ 6, & if \;\; (p,k)=(37,4),(59,14),(67,8) \\ 3, & if \;\; k=p-1 \text{ and } (p,k)=(37,5),(59,15),(67,9) \\ 5, & if \;\; k \text{ is even and } (p,k)\neq(7,4),(37,4),(59,14),(67,8) \\ 2, & if \;\; k \text{ is odd and } (p,k)\neq(37,5),(59,15),(67,9). \end{cases} \tag{13}$$

We also provide in Table 1, the pairs of p and k where $v_p(S(p, k)) = 2$ for $3 \le p \le 1000$ and $2 \le k \le p-1$. It should be noted that $v_p(S(p, k)) = 1$ for all the remaining pairs (p, k).

After a closed examinations of the output, we have observed that the arrays of $v_p(S(p^2, kp))$ follow certain patterns which interpret as conjectures.

1. Let $p > 7$ be an odd prime and k be an even integer such that $0 < k < p-1$. Then

$$v_p(S(p^2,kp)) - v_p(S(p^2, p(k+1))) = 3.$$

2. If k be an integer such that $1 < k < p-1$, then the p-adic valuations satisfy

$$v_p(S(p^2, kp) = \begin{cases} 5 \text{ or } 6, & if\ k \text{ is even} \\ 2 \text{ or } 3, & if\ k \text{ is odd} \end{cases}$$

for any prime $p > 7$.

3. For any odd prime p and a positive integer k such that $2 \leq k \leq p - 1$,

$$v_p(S(p, k)) \leq 2.$$

5. Conclusions

This paper deals with some results of p-adic valuations of Stirling number of the second kind, $S(n, k)$ for odd prime p. We have derived the formulas for $v_p(S(n, n-1))$, $v_p(S(cp^n, cp^n - 1))$, $v_p(S(n, n-2))$, $v_p(S(p^n, p^n - 2))$, $v_p(S(n, n-3))$ and $v_p(S(p^n, p^n - 3))$. It has been shown the primality test of n using divisibility of n to $S(n, k)$, $1 < k < n$. We have obtained the results that $v_p(S(n, p))$ depends on the divisibility of $n - 1$ by $p - 1$ and $v_p(S(cp^n, cp^k)) = 0$ for every integer $n \geq k \geq 1$ and $p - 1 \geq c \geq 1$. We also posed three conjectures after analyzing Table 1 and computational results of (13).

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