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Research Article

A Generalized Homogeneously Weighted Moving Average Monitoring Scheme for Monitoring the Process Mean

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Abstract: This paper introduces a generalized version of the Homogenously Weighted Moving Average (HWMA) monitoring scheme for monitoring the process mean (i.e., HWMA $\overline{X}$ scheme). The proposed scheme is constructed using r $(r \geq 1)$ smoothing parameters λ_i (where $0 < \lambda_i \leq 1$ with $i = 1, \ldots, r$) instead of one, as is the case for the HWMA $\overline{X}$ scheme. Thus, the existing HWMA $\overline{X}$ scheme is a special case of the new Generalized HWMA $\overline{X}$ (GHWMA $\overline{X}$) scheme when $r = 1$. The properties of the new scheme are derived, and the In-Control (IC) and Out-Of-Control (OOC) performances are investigated in terms of the zero-state Average Run Length (ARL), Percentiles of the Run Length (PRL), Standard Deviation of the Run Length (SDRL), and Conditional Expected Delay (CED) criterion. The ability of the proposed scheme to detect small $(0 < \delta \leq 1)$, moderate $(1 < \delta \leq 2)$, large $(2 < \delta \leq 3)$, and small-to-large $(0 < \delta \leq 3)$ shifts are investigated using the Expected ARL ($EARL$) metric, where δ is the magnitude of the shift expressed in standard deviation. The performance of the GHWMA $\overline{X}$ scheme is compared with that of some well-known memory-type monitoring schemes. It is found that the new GHWMA $\overline{X}$ scheme is flexible through the adjustment of the smoothing parameters and outperforms the existing classical schemes when the weights are well-selected. The results reveal that the GHWMA $\overline{X}$ scheme with $r > 1$ has a better overall performance than the HWMA $\overline{X}$ scheme, regardless of the magnitude of the shifts. The HWMA $\overline{X}$ scheme is not consistent and effective in detecting shifts after a change point compared to the GHWMA $\overline{X}$ scheme. This explains the superiority of the GHWMA $\overline{X}$ scheme over the HWMA $\overline{X}$ scheme in terms of the CED. To demonstrate the application and implementation of the new scheme, numerical examples are provided using real-world and simulated data.

***Keywords*:** average run-length, control chart, conditional expected delay, HWMA, generalised HWMA, weight structure

MSC: 62P30, 62-08

1. Introduction

It has been a century since Statistical Process Monitoring (SPM) was introduced in the industry to improve the quality of manufactured products and services rendered to customers. In order to successfully achieve this, seven tools, commonly known as "the magnificent seven", have been put at the disposal of the operators in the industry. These include the histogram, check sheets, Pareto charts, cause and effect, stratification, scatter plots, and monitoring schemes;

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Montgomery [1]. The latter is the most utilized and regarded as more effective than the other six tools by researchers. A monitoring scheme's primary function is to monitor the stability of a process by distinguishing between two types of process variation: chance and assignable causes of variability. These two sources of variability determine the state of the statistical process. Thus, a process that runs only in the presence of natural causes of variation is considered In-Control (IC), whereas a process that runs in the presence of any assignable causes is considered Out-Of-Control (OOC). With that being said, the effectiveness of a monitoring scheme is its ability to quickly detect the presence of assignable causes of variation in a process. This is expressed in terms of the time to signal or the number of samples plotted on the scheme before it signals for the first time; Montgomery [1].

The magnitude of the shift in the process parameters plays a significant role in selecting a suitable monitoring scheme. Therefore, in the literature, it is important that smaller smoothing parameters are recommended for the quick detection of small shifts and larger smoothing parameters for the quick detection of large shifts; see Ugaz et al. [2], Talordphop et al. [3], Riaz et al. [4]. For instance, to detect larger shifts, memoryless monitoring schemes such as Shewhart-type schemes may be suitable; see Roberts [5], Shewhart [6], Page [7], and Sheu and Lin [8]; whereas small and moderate shifts may be detected using memory type schemes such as Cumulative Sum (CUSUM), Exponentially Weighted Moving Average (EWMA), Generally Weighted Moving Average (GWMA) and Homogeneously Weighted Moving Average (HWMA) schemes; see, for example, Page [7], Roberts [5], Mabude et al. [9], Chan et al. [10], Qiu [11], and Abbas [12]. It is important to note that for memory-type schemes, the weight assigned to historical data (old and recent) also plays a crucial role in the performance of the scheme; Knoth et al. [13], Riaz et al. [14], and Malela-Majika et al. [15]. Knoth et al. [13] criticized the weight structure of some new schemes and reported that the CUSUM and EWMA schemes are preferred over the classical extended and modified EWMA, HWMA, GWMA schemes, etc.

The HWMA $\overline{X}$ scheme was introduced by Abbas [12] to monitor shifts in the process mean. It was reported that the HWMA $\overline{X}$ scheme outperforms the EWMA $\overline{X}$ scheme in monitoring tiny shifts in the process mean. Thus, the difference between the two schemes is that in the EWMA $\overline{X}$ scheme, the data used in the computation of the charting statistics are assigned geometrically decreasing weights from the most recent to the oldest. However, in the design of the HWMA $\overline{X}$ scheme, a specific weight is assigned to the current sample, while the remaining weight is evenly distributed among the previous samples. Adegoke et al. [16] adopted a weight structure similar to that of the HWMA $\overline{X}$ scheme proposed by Abbas [12] to develop the HWMA $\overline{X}$ scheme that incorporates auxiliary variables as regression estimators in addition to the process quality characteristic. This enhancement improved the detection capabilities of the classical HWMA $\overline{X}$ scheme by using an efficient and unbiased estimate of the process mean of the variable of interest. Nawaz and Had [17] investigated the performance of the HWMA $\overline{X}$ monitoring scheme using Ranked Set Sampling (RSS) and its variation, namely extreme RSS, median RSS, and neoteric RSS. Abid et al. [18] introduced the Double HWMA (DHWMA) $\overline{X}$ monitoring scheme. Abid et al. [19] proposed a new CUSUM $\overline{X}$ scheme using the HWMA $\overline{X}$ statistic as an input in the CUSUM statistic. Dawod et al. [20] proposed a new linear profile monitoring scheme using the HWMA statistic in a Bayesian framework. Adegoke et al. [21] and Abbas [12] introduced multivariate HWMA $\overline{X}$ schemes for monitoring the mean vector under known and unknown process parameters, respectively. Raza et al. [22] proposed two nonparametric HWMA $\overline{X}$ schemes. For a more detailed account of HWMA-type schemes, readers are referred to the review article by Malela-Majika et al. [15].

The literature has reported that the extended EWMA and HWMA $\overline{X}$ schemes perform better in zero than in steady-state. The steady-state performances of these schemes are known to be worse in most cases. For this reason, there is a need to introduce simple schemes that perform better in the steady-state. This paper introduces a new Generalized HWMA (GHWMA) $\overline{X}$ monitoring scheme when the process parameters are known. The proposed GHWMA $\overline{X}$ scheme is designed using r $(r \geq 1)$ smoothing parameters $(\lambda_i,\ i = 1,\ \ldots,\ r)$. Therefore, it can be noted that the HWMA $\overline{X}$ scheme is a special case of the GHWMA $\overline{X}$ scheme when $r = 1$. The sensitivity of the proposed scheme will be measured using the properties of the run-length distribution in zero-state and the Conditional Expected Delay (CED) criterion. The effect of the number of smoothing parameters (r) on the GHWMA $\overline{X}$ scheme will also be investigated using extensive simulations.

The rest of this paper is organized as follows: Section 2 highlights the mathematical background of the HWMA $\overline{X}$ scheme. Section 3 introduces the proposed GHWMA $\overline{X}$ scheme, assuming the process parameters are known. In addition, the weight structure of the new GHWMA $\overline{X}$ statistic is analyzed. Section 4 investigates the IC robustness and

OOC performance of the GHWMA $\overline{X}$ scheme. In addition, the performance of the proposed scheme is compared to that of the EWMA, CUSUM, HWMA, double EWMA (DEWMA), Hybrid EWMA (HEWMA), and GWMA $\overline{X}$ schemes in terms of the zero-state run-length properties and CED criterion. In Section 5, illustrative examples using real-world and simulated data are provided to demonstrate the application and implementation of the proposed GHWMA $\overline{X}$ scheme. Lastly, concluding remarks, recommendations, and future research work are provided in Section 6.

2. Mathematical background of the HWMA $\overline{X}$ monitoring scheme

Let $X_{tj}\{t = 1,\ 2,\ \ldots,$ and $j = 1,\ 2,\ \ldots,\ n\}$ be a set of samples of independent normal random variables, i.e., X_{tj} follows a $N\left(\mu_0 + \delta\sigma_0,\ \sigma_0^2\right)$, where μ_0 is the IC mean value, σ_0^2 is the IC variance and δ is the magnitude of the shift in standard deviation units. When $\delta = 0$, the process is considered to be IC, which implies X_{tj} follows a $N\left(\mu_0,\ \sigma_0^2\right)$. However, when $\delta \neq 0$ the process is OOC.

Let $\overline{X}_t \left(= \sum_{j=1}^{n} X_{tj}/n\right)$ be the sample mean of the i^{th} sample. The plotting statistic of the HWMA $\overline{X}$ scheme (denoted as H_t) is defined as

$$H_t = \lambda \overline{X}_t + (1-\lambda)\overline{\overline{X}}_{t-1}, \tag{1}$$

with

$$\overline{\overline{X}}_{t-1} = \frac{\sum_{v=1}^{t-1} \overline{X}_v}{t-1},$$

where $\lambda\,(0 < \lambda \leq 1)$ is the smoothing constant and $\overline{\overline{X}}_{t-1}$ is the mean of the previous $t-1$ sample means. The initial value of $\overline{\overline{X}}_{t-1}$ (i.e. $\overline{\overline{X}}_0$) is typically set to be equal to the target mean μ_0.

Abbas [12] showed that equation (1) can also be written as

$$H_t = \lambda \overline{X}_t + \left[\left(\frac{1-\lambda}{t-1}\right)\overline{X}_{t-1} + \left(\frac{1-\lambda}{t-1}\right)\overline{X}_{t-2} + \cdots + \left(\frac{1-\lambda}{t-1}\right)\overline{X}_2 + \left(\frac{1-\lambda}{t-1}\right)\overline{X}_1\right]. \tag{2}$$

From equation (2), it can be seen that the HWMA $\overline{X}$ statistic assigns a weight λ to the current sample and a weight $(1-\lambda)$ is equally distributed among the previous samples. It can be shown that the mean and variance of the plotting statistic in equations (1) or (2) are given by

$$E\left(H_t\right) = \mu_0$$

and

$$Var\left(H_t\right) = \sigma_{H_t}^2 = \begin{cases} \dfrac{\lambda^2\sigma_0^2}{n}, & \text{if } t = 1 \\ \\ \dfrac{\sigma_0^2}{n}\left(\lambda^2 + \dfrac{(1-\lambda)^2}{(t-1)}\right), & \text{if } t > 1 \end{cases} \tag{3}$$

respectively. Therefore, the time-varying lower and upper control limits (i.e., LCL_t and UCL_t) of the HWMA $\overline{X}$ monitoring scheme are defined as

$$LCL_t = \begin{cases} \mu_0 - L\dfrac{\lambda\sigma_0}{\sqrt{n}}, \; t = 1 \\ \\ \mu_0 - L\dfrac{\sigma_0}{\sqrt{n}}\sqrt{\left(\lambda^2 + \dfrac{(1-\lambda)^2}{(t-1)}\right)}, \; t > 1 \end{cases} \tag{4}$$

and

$$UCL_t = \begin{cases} \mu_0 + L\dfrac{\lambda\sigma_0}{\sqrt{n}}, \; t = 1 \\ \\ \mu_0 + L\dfrac{\sigma_0}{\sqrt{n}}\sqrt{\left(\lambda^2 + \dfrac{(1-\lambda)^2}{(t-1)}\right)}, \; t > 1 \end{cases}$$

respectively; where L is the control limits constant that is set to have a pre-specified IC ARL (ARL_0). Thus, the HWMA $\overline{X}$ scheme gives a signal if the plotting statistic in equation (1) plots beyond the control limits defined in equation (4); that is, if $H_t \geq UCL_t$ or $H_t \leq LCL_t$. In case the process has been running for a long time (i.e., $t \to \infty$), the term $\dfrac{(1-\lambda)^2}{t-1} \to 0$. Therefore, the control limits in equation (4) reduce to the following asymptotic control limits

$$\begin{aligned} LCL &= \mu_0 - L\frac{\lambda\sigma_0}{\sqrt{n}}, \text{ and} \\ UCL &= \mu_0 + L\frac{\lambda\sigma_0}{\sqrt{n}}, \end{aligned} \tag{5}$$

and, in this case, the process is OOC if $H_t \geq UCL$ or $H_t \leq LCL$.

3. The proposed GHWMA $\overline{X}$ scheme

3.1 *Mathematical background of the GHWMA $\overline{X}$ scheme*

Let us assume that instead of one smoothing parameter, as was the case in Section 2, now we have r $(r \geq 1)$ smoothing parameters denoted as λ_i $(i = 1, 2, \ldots, r)$, where $0 < \lambda_i \leq 1$ and $0 < \sum_{i=1}^{r} \lambda_i \leq 1$. In addition, the condition "$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$" is imposed to adjust the weight of the latest r samples such that each of them is lager or equal to the weight assigned to past data. This allows to correct the weight structure of the GHWMA $\overline{X}$ scheme. Thus, the GHWMA $\overline{X}$ statistic (denoted by GH_t) is a generalized form of the HWMA $\overline{X}$ statistic and its charting statistic is defined as

$$GH_t = \lambda_1\overline{X}_t + \lambda_2\overline{X}_{t-1} + \lambda_3\overline{X}_{t-2} + \ldots + \lambda_{r-1}\overline{X}_{t-r+2} + \lambda_r\overline{X}_{t-r+1} + \bar{\lambda}\overline{\overline{X}}_{t-r}, \tag{6}$$

where

$$\bar{\lambda} = 1 - \sum_{i=1}^{r} \lambda_i$$

$$\overline{\overline{X}}_{t-r} = \frac{\sum_{k=1}^{t-1} \overline{X}_k}{t-r}$$

and

$$\overline{X}_k = \overline{\overline{X}}_k = \mu_0, \text{ for } k \leq 0.$$

When $t \neq r$, equation (6) can also be written as:

$$\lambda_1 \overline{X}_t + \lambda_2 \overline{X}_{t-1} + \lambda_3 \overline{X}_{t-2} + \ldots + \lambda_{r-1} \overline{X}_{t-r+2} + \lambda_r \overline{X}_{t-r+1}$$
$$+ \frac{\bar{\lambda}}{t-r} \left(\overline{X}_{t-r} + \overline{X}_{t-r-1} + \overline{X}_{t-r-2} + \ldots + \overline{X}_3 + \overline{X}_2 + \overline{X}_1 \right), \tag{7}$$

which can be simplified to

$$GH_t = \sum_{i=1}^{r} \lambda_i \overline{X}_{t-i+1} + \frac{\bar{\lambda}}{t-r} \sum_{k=1}^{t-r} \overline{X}_k. \tag{8}$$

When $t = r$, the equations (6)-(7) become:

$$GH_r = \lambda_1 \overline{X}_r + \lambda_2 \overline{X}_{r-1} + \lambda_3 \overline{X}_{r-2} + \ldots + \lambda_{r-1} \overline{X}_2 + \lambda_r \overline{X}_1 + \bar{\lambda} \mu_0. \tag{9}$$

It is also shown that (see Appendix A),

• When $t > r$,

$$GH_t = \sum_{i=1}^{t} \lambda_i \overline{X}_{t-i+1}, \tag{10}$$

where

$$\lambda_{r+1} = \lambda_{r+2} = \ldots = \lambda_t = \frac{\bar{\lambda}}{t-r}, \; t > r.$$

• When $t \leq r$,

$$GH_t = \sum_{i=1}^{t} \lambda_i \overline{X}_{t-i+1} + \left(\sum_{k=t+1}^{r} \lambda_i + \bar{\lambda} \right) \mu_0, \tag{11}$$

where

$$\lambda_i = \frac{\bar{\lambda}}{t-r}, \; i = r+1, \; r+2, \; \ldots, \; t.$$

Thus, the mean and variance of the GHWMA $\overline{X}$ statistic are given by (see Appendix A)

$$E\left(GH_t\right) = \mu_0$$

and

$$Var\left(GH_t\right) = \sigma^2_{GH_t} = \begin{cases} \dfrac{\sigma_0^2}{n} \sum_{i=1}^{t} \lambda_i^2, & \text{if } t \leq r \\ \dfrac{\sigma_0^2}{n} \left(\sum_{i=1}^{r} \lambda_i^2 + \dfrac{\left(1 - \sum_{i=1}^{r} \lambda_i\right)^2}{(t-r)} \right), & \text{if } t > r \end{cases} \tag{12}$$

respectively. Therefore, the exact or time-varying lower and upper control limits (i.e., LCL_t and UCL_t) of the GHWMA $\overline{X}$ scheme is defined as

$$UCL_t / LCL_t = \mu_0 \pm L\sqrt{Var\left(GH_t\right)}, \tag{13}$$

where L represents the control limits constant that is set to yield an attained IC ARL close or equal to a pre-specified ARL_0. The ARL_0 is generally set to be equal to some high desired values such as 370 and 500. In other words, the scheme signals whenever the plotting statistic defined in equation (6) plots beyond the control limits defined in equation (13); i.e., if $GH_t \geq UCL_t$ or $GH_t \leq LCL_t$. When the process has been running for a long time (i.e., $t \to \infty$), then the term $\dfrac{\left(1 - \sum_{i=1}^{r} \lambda_i\right)^2}{(t-r)} \to 0$. Thus, since $t(> r) \to \infty$, the asymptotic control limits become

$$UCL/LCL = \mu_0 \pm L\frac{\sigma_0}{\sqrt{n}} \sqrt{\sum_{i=1}^{r} \lambda_i^2}. \tag{14}$$

In this case, the process is OOC if $GH_t \geq UCL$ or $GH_t \leq LCL$.

3.2 *The weight function of the GHWMA-type scheme and other popular schemes*

In this subsection, the GHWMA scheme weight structure amongst samples is compared with the EWMA and HWMA schemes' weight structures for the first 20 samples used to compute the 20$^{\text{th}}$ charting statistic. The weight structure of the charting statistic is fundamental in that it enables the shift detection capability of a scheme. Figure 1 displays the weight structure of the EWMA (λ), HWMA (λ) and GHWMA (λ_i) schemes with $r = 2$ and 3 for the GHWMA scheme. The smoothing parameters of the HWMA and EWMA schemes are $\lambda = 0.5,\ 0.2,\ 0.1$ and 0.05. Whereas the combination of the smoothing parameters of the GHWMA scheme for $r = 2$ and 3 are $(0.3,\ 0.3)$, $(0.3,\ 0.1)$, $(0.3,\ 0.3,\ 0.3)$, $(0.3,\ 0.1,\ 0.05)$ and $(0.5,\ 0.2,\ 0.1)$. Note that the 20th sample is designated as the current sample, and the rest (i.e., samples 1 to 19) are regarded as previous (or past) samples. Figure 1 shows that the EWMA scheme weight decreases exponentially as the samples become older. For instance, when $\lambda = 0.1$, the EWMA scheme weights of the current and 19$^{\text{th}}$ previous samples are 0.1 and 0.033 , respectively. Thus, the older the sample, the smaller the weight. The larger the value of λ, the smaller the weight as compared to smaller values of λ. The converse is true for smaller values of λ. The HWMA (0.1) scheme assigns a weight of 0.1 to the current sample, and the rest $(t-1)$ samples are assigned an equal weight of 0.0474. The GHWMA $(0.3,\ 0.3,\ 0.3)$ scheme will assign a weight of 0.3 to the latest three samples, and the rest will be divided equally among the remaining $(t-3)$ samples. However, the GHWMA $(0.3,\ 0.1,\ 0.05)$ scheme will assign the weight of 0.3, 0.1 and 0.05 to the current, the previous and the second previous samples, respectively, and the rest of the weight is divided equally among the remaining $(t-3)$ samples. It can be noticed that the GHWMA scheme with a large value of $r > 1$ will improve the weight structure of the HWMA scheme when its largest smoothing parameter is equal to the smoothing parameter of the HWMA. When $r = 1$, the GHWMA scheme is equivalent to the HWMA scheme.

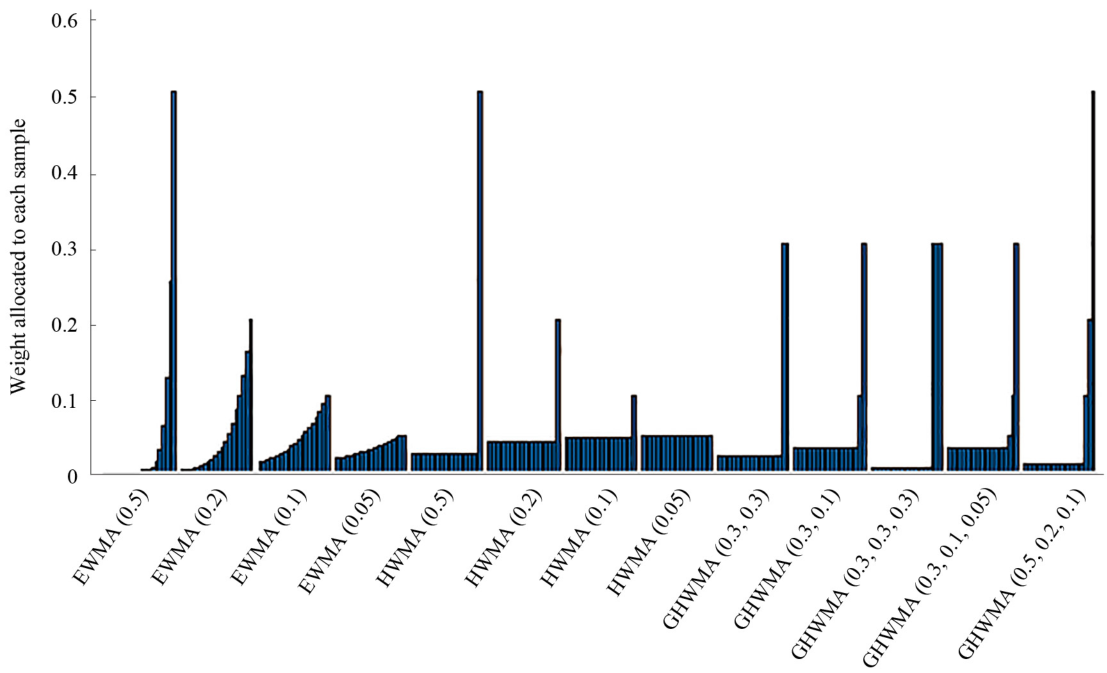


Figure 1. The numerical values of the weights for the EWMA, HWMA and GHWMA charting statistics of the first 20 samples

Note that even though the GHWMA scheme with $r > 1$ is more complex than the HWMA scheme, the main advantage is that it uses its flexibility to improve the weight structure and, therefore, improve its ability to detect shifts in the process parameters, especially when these shifts are not expected at the start-up of the process.

4. Performance analysis of the proposed GHWMA $\overline{X}$ scheme

4.1 *IC Robustness of the GHWMA $\overline{X}$ scheme*

Monitoring schemes are considered IC robust whenever their IC characteristics of the run-length distribution are similar (or significantly close) to the ARL_0 across all continuous distributions. In this paper, we study the IC robustness of the GHWMA $\overline{X}$ scheme by providing the visuals to verify whether the proposed GHWMA $\overline{X}$ scheme is IC robust. This will be done by checking whether, for instance, the attained IC Average Run-Length (ARL) values of the GHWMA $\overline{X}$ scheme under different distributions lie within the range $ARL_0 \pm 0.1\ ARL_0$, where the ARL_0 is considered to be the nominal IC ARL value of 500. The significance and practical value of the IC robustness analysis is to examine if the monitoring scheme will still maintain the same performance properties if a practitioner uses a distribution other than the normal distribution. Therefore, in order to study the IC robustness of the GHWMA $\overline{X}$ scheme, we considered different symmetrical and skewed distributions, namely:

i. Standard normal distribution; i.e., $N(0,\ 1)$,

ii. Student's t distribution with degrees of freedom $v = 5,\ 10$ and 30; i.e., $t(5),\ t(10)$ and $t(30)$,

iii. Gamma distribution with parameters $\omega = 1,\ 3,\ 40$ and $\beta = 1$; i.e., $Gamma(1,\ 1)$, $Gamma(3,\ 1)$ and $Gamma\ (40,\ 1)$,

iv. Weibull distribution with parameters $\tau = 1$ and $\kappa = 1$; i.e., $Wei(1,\ 1)$, and

v. Standard double exponential distribution with $\mu = 0$ and $\beta = 1$, i.e., $DEXP(0,\ 1)$.

Of note in this instance is that for a fair comparison, a further transformation process was implemented using the above-mentioned distributions to ensure that each distribution's mean and standard deviation are 0 and 1, respectively. To investigate the IC robustness behavioural patterns of the proposed GHWMA $\overline{X}$ scheme, the IC characteristics of the run-length distribution computed using Monte Carlo simulations in SAS IML®9.4 is as depicted in Table 1 and Figures 2a-c. From Table 1 and Figures 2a-c, it can be observed that the GHWMA $\overline{X}$ scheme is not IC robust, as the IC characteristics of its run-length distribution are significantly different across the distributions considered in this paper. When comparing HWMA $\overline{X}$ (i.e., the GHWMA $\overline{X}$ with $r = 1$) and GHWMA $\overline{X}(r > 1)$ schemes, it can be observed that regardless of the underlying process distribution, the GHWMA $\overline{X}$ scheme with $r > 1$ is more IC robust than the HWMA $\overline{X}$ scheme, as its attained IC ARL values are much closer to the value nominal $ICARL_0$ was defined as ARL_0 earlier as compared to the HWMA $\overline{X}$ scheme (see Figure 2a). The findings in Table 1 and Figure 2 can be summarized as follows:

• When $r = 1$, the GHWMA $\overline{X}$ scheme with smaller smoothing parameters is more robust than the GHWMA $\overline{X}$ scheme with larger smoothing parameters (see Figure 2b).

• The number of attained IC ARL that are plotted within the range $ARL_0 \pm 0.1ARL_0$ (i.e., between 450 and 550) for the GHWMA $\overline{X}(r = 4)$ scheme is more than that of the HWMA $\overline{X}$ scheme.

• It is, therefore, clear that when increasing the value of r, the scheme becomes more IC robust; hence, the GHWMA $\overline{X}(r > 1)$ scheme can be preferable to the GHWMA $\overline{X}(r = 1)$ scheme in terms of the IC robustness.

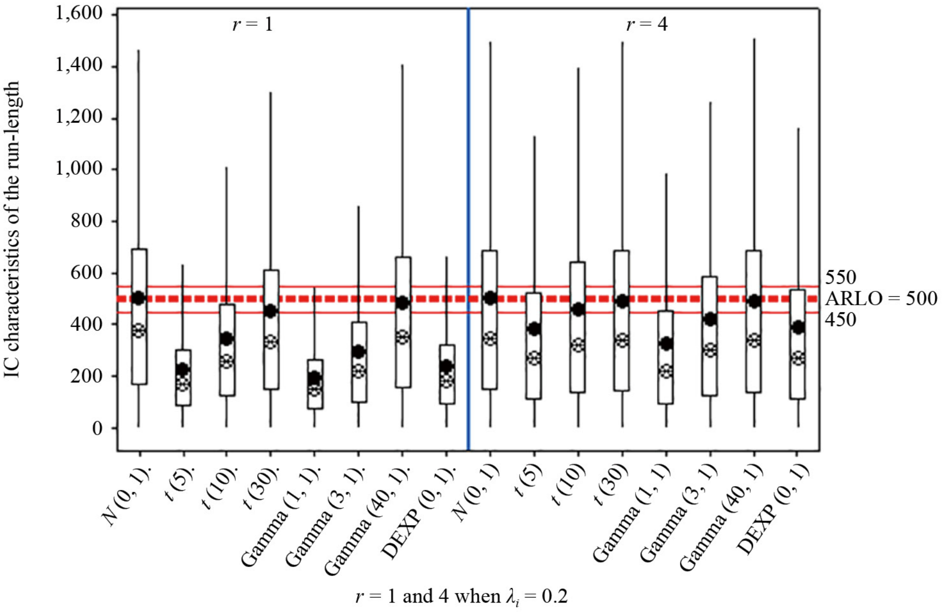
• Both monitoring schemes are IC robust for $t(v)$ distributions when the degrees of freedom $v \geq 30$. Otherwise, they are not IC robust when $v < 30$.

• When $\lambda_1 = \lambda_2 = \ldots = \lambda_r$, smaller smoothing parameters will yield better IC robustness properties as compared to both larger smoothing parameters and a combination of different smoothing parameters (see Figure 2c).

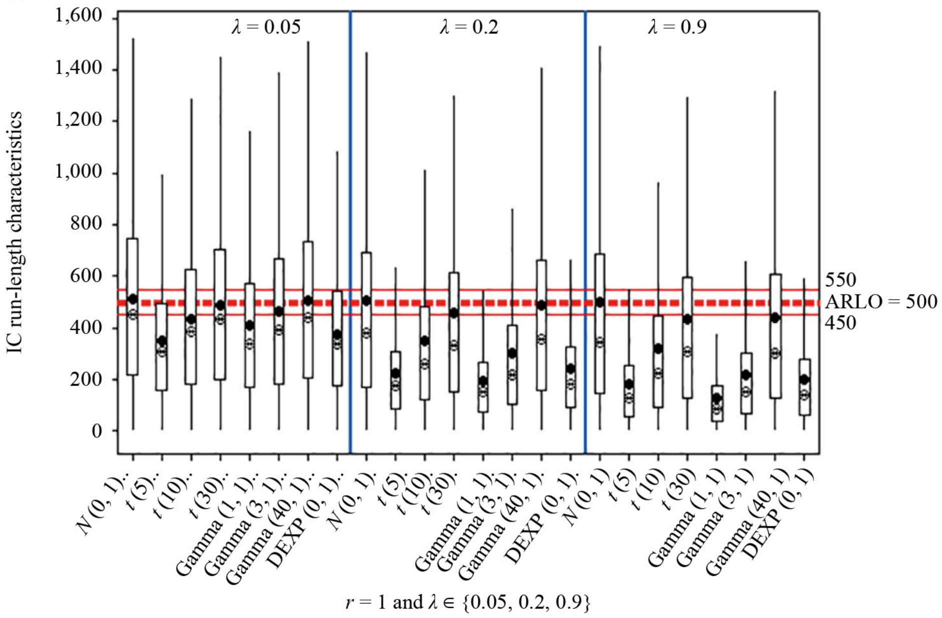
Table 1. IC run-length properties of the GHWMA $\overline{X}$ scheme under different continuous distributions when $n = 5$, $r \in \{1, 4\}$, $\lambda_i \in \{0.05, 0.1, 0.2, 0.3, 0.9\}$ and $ARL_0 = 500$

r				$r = 1$							$r = 4$			
Percentile	ARL_0	$SDRL_0$	P5	P25	P50	P75	P95	ARL_0	$SDRL_0$	P5	P25	P50	P75	P95
Distribution				$(\lambda_1 = 0.05)$					$(\lambda_1 = 0.05,\ \lambda_2 = 0.05,\ \lambda_3 = 0.05,\ \lambda_4 = 0.05)$					
N (0, 1)	500.8	372.6	20.0	206.0	439.0	728.0	1198.0	499.3	462.3	15.0	157.0	377.0	704.0	1,402.0
t (5)	346.9	242.6	13.0	161.0	312.0	494.0	797.0	388.2	341.1	12.0	135.0	303.0	548.0	1,056.0
t (10)	437.5	316.6	16.0	188.0	391.0	631.0	1026.0	459.5	418.0	13.0	149.0	349.0	650.0	1,284.0
t (30)	487.8	358.9	19.0	203.0	431.0	707.0	1161.0	489.4	452.0	14.0	154.0	368.0	693.0	1,385.0
Gamma (1, 1)	405.3	318.2	12.0	166.0	340.0	574.0	1019.0	434.5	411.4	12.0	138.0	320.0	606.0	1,244.0
Gamma (3, 1)	465.9	357.9	17.0	189.0	398.0	668.0	1150.0	480.0	453.8	14.0	150.0	355.0	673.0	1,376.0
Gamma (40, 1)	501.8	373.7	19.0	205.0	437.0	727.0	1205.0	501.1	466.3	15.0	157.0	374.0	706.0	1,413.0
Wei (1, 1)	381.6	292.3	13.0	162.0	329.0	537.0	1001.0	402.1	301.9	13.0	132.0	314.0	591.0	1,199.0
DEXP (0, 1)	369.3	257.7	14.0	168.0	334.0	531.0	842.0	403.3	361.1	11.0	136.0	312.0	570.0	1,116.0
L				2.6112							2.8594			
Distribution				$(\lambda_1 = 0.2)$					$(\lambda_1 = 0.2,\ \lambda_2 = 0.2,\ \lambda_3 = 0.2,\ \lambda_4 = 0.2)$					
N (0, 1)	500.0	471.1	36.0	162.0	363.0	687.0	1434.5	500.2	510.6	19.0	138.0	344.0	693.0	1,514.0
t (5)	219.4	189.3	23.0	83.0	168.0	301.0	592.0	333.1	339.1	12.0	92.0	228.0	463.0	1,005.0
t (10)	351.3	316.1	31.0	124.0	263.0	483.0	976.0	437.9	441.3	16.0	121.0	302.0	615.0	1,316.0
t (30)	453.7	424.3	34.0	151.0	331.0	624.0	1292.0	484.5	493.2	17.0	133.0	334.0	674.0	1,467.0
Gamma (1, 1)	196.3	171.3	20.0	74.0	149.0	267.0	532.0	274.6	279.2	11.0	76.0	118.0	381.0	827.0
Gamma (3, 1)	300.8	274.2	27.0	106.0	221.0	412.0	843.0	385.1	392.6	14.0	106.0	263.0	535.0	1,178.0
Gamma (40, 1)	474.6	443.5	35.0	157.0	346.0	654.0	1356.0	491.3	502.7	18.0	135.0	338.0	682.0	1,481.5
Wei (1, 1)	213.4	185.2	22.0	80.0	161.0	296.0	597.0	290.2	298.1	12.0	81.0	174.0	400.0	992.0
DEXP (0, 1)	241.3	208.0	25.0	91.0	184.0	330.0	652.0	354.1	358.2	12.0	98.0	246.0	495.0	1,062.0
L				3.0599							3.0134			
Distribution				$(\lambda_1 = 0.9)$					$(\lambda_1 = 0.3,\ \lambda_2 = 0.2,\ \lambda_3 = 0.1,\ \lambda_4 = 0.05)$					
N (0, 1)	500.3	498.5	26.0	144.0	348.0	692.0	1,493.0	500.3	501.0	26.0	145.0	347.0	690.0	1,503.0
t (5)	182.1	181.8	10.0	53.0	126.0	251.0	545.0	270.3	269.2	15.0	80.0	188.0	373.0	800.0
t (10)	320.3	319.0	17.0	93.0	222.0	446.0	955.5	401.0	396.3	22.0	118.0	280.0	557.0	1,185.0
t (30)	442.2	441.4	24.0	128.0	308.5	613.0	1,317.0	473.2	472.9	25.0	137.0	328.0	653.0	1,422.0
Gamma (1, 1)	124.6	123.6	7.0	36.0	87.0	172.0	372.0	213.1	209.1	12.0	64.0	150.0	295.0	628.0
Gamma (3, 1)	214.1	212.3	12.0	62.0	149.0	297.0	636.5	328.3	324.6	19.0	97.0	230.0	457.0	965.0
Gamma (40, 1)	447.9	446.8	23.0	131.0	312.0	618.0	1334.0	478.3	476.0	25.0	139.0	333.0	662.0	1431.0
Wei (1, 1)	178.8	144.3	9.0	41.0	98.0	201.0	411.0	239.1	248.0	13.0	72.0	167.0	324.0	712.0
DEXP (0, 1)	200.6	199.6	11.0	59.0	139.0	278.0	596.0	294.4	291.2	17.0	88.0	206.0	406.0	874.0
L				3.0890							3.0365			

(a)


IC characteristics of the run-length
1,600
1,400
1,200
1,000
800
600
400
200
0
r = 1
r = 4
550
ARLO = 500
450
N (0, 1).
t (5).
t (10).
t (30).
Gamma (1, 1).
Gamma (3, 1).
Gamma (40, 1).
DEXP (0, 1).
N (0, 1)
t (5)
t (10)
t (30)
Gamma (1, 1)
Gamma (3, 1)
Gamma (40, 1)
DEXP (0, 1)
r = 1 and 4 when λ_i = 0.2

(b)


IC run-length characteristics
1,600
1,400
1,200
1,000
800
600
400
200
0
λ = 0.05
λ = 0.2
λ = 0.9
550
ARLO = 500
450
N (0, 1).
t (5).
t (10).
t (30).
Gamma (1, 1).
Gamma (3, 1).
Gamma (40, 1).
DEXP (0, 1).
N (0, 1)
t (5)
t (10)
t (30)
Gamma (1, 1)
Gamma (3, 1)
Gamma (40, 1)
DEXP (0, 1)
r = 1 and λ ∈ {0.05, 0.2, 0.9}

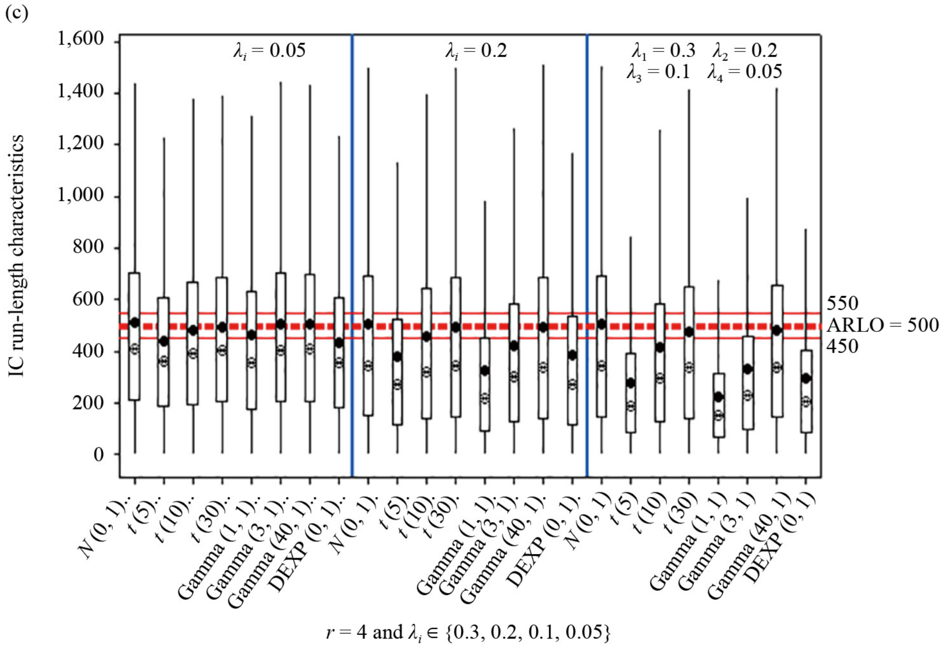


Figure 2. IC robustness of the GHWMA $\overline{X}$ scheme in terms of IC run-length characteristics when $n = 5$ and $ARL_0 = 500$

4.2 *OOC performance of the GHWMA $\overline{\mathbf{X}}$ scheme*

In this subsection, the performance of the GHWMA $\overline{X}$ monitoring scheme is explored in terms of the *ARL* profiles for different shifts.

The selection of the smoothing parameters can significantly affect a scheme's ability to detect different shifts or ranges of shifts. Small smoothing parameters are often selected when one is interested in monitoring small shifts in the process parameters. When the interest is to monitor moderate shifts, the smoothing parameter should be between 0.2 and 0.3. Larger values of λ ($\lambda > 0.3$) are recommended for significant shifts. Several researchers have investigated the effect of smoothing parameters on the performance of the monitoring schemes. Lazariv et al. [23] reported that smaller values of the smoothing parameter lead to an increasing influence of previous observations.

Later on, Ugaz et al. [2] proposed a new adaptive EWMA monitoring scheme using time-varying smoothing parameters through optimization. Talordphop et al. [3] reported that parametric extended EWMA monitoring schemes' control limit constants increase when the smoothing parameter value increases, while that of nonparametric monitoring schemes decreases. Riaz et al. [4] recommended that the smoothing parameters be selected based on the size of the detected shift, where smaller smoothing constants are chosen for minor shifts.

Let vector $\lambda_{opt} = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be the optimal combination of the smoothing parameters. Then, for the given ARL_0 value, the optimal λ_{opt} vector is obtained by minimizing the OOC ARL and the *EARL* values for a specific shift and overall performance, respectively. Thus, the optimization problem can be written as follows:

$$\lambda_{opt} = \underset{0<\lambda_i \leq 1;\ 0<\sum_{i=1}^{r}\lambda_i \leq 1}{\arg\min} (ARL_1, EARL) \tag{15}$$

Subject to:

$$ICARL(\lambda_{opt}) = ARL_0,$$

where ARL_1 is the OOC ARL for a specific shift (δ), $ICARL$ is the attained IC ARL value, and ARL_0 is the nominal IC ARL value. The optimal vector λ_{opt} with optimal values of ARL_1 and/or $EARL$ gives the optimal smoothing parameters of the GHWMA $\overline{X}$ scheme.

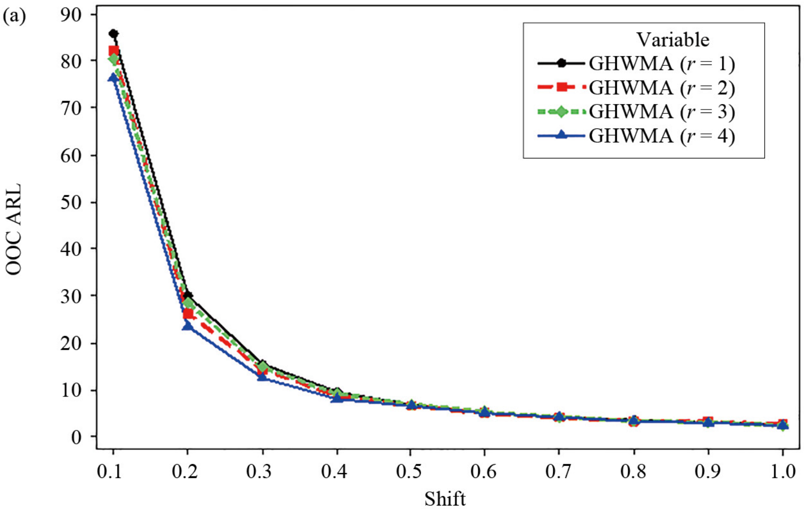
Table 2 displays the ARL profiles of the GHWMA $\overline{X}$ scheme when $n = 5$, $\lambda \in \{0.05,\ 0.2,\ 0.3\}$ with $r \in \{1,\ 2,\ 3\}$ and $ARL_0 = 500$. According to Abbas [12], the smoothing parameter λ is always chosen based on the size of the shifts selected by practitioners. It has been highlighted in the literature that $\lambda = 0.1, 0.5$, and 0.9 are respectively recommended for quick detection of small shifts, moderate to large shifts, and very large shifts. The findings in Table 2 can be summarised as follows:

• It can be observed that for any given value of r, the GHWMA $\overline{X}$ scheme is generally more sensitive for small values of λ. Thus, the performance of the GHWMA $\overline{X}$ scheme deteriorates as the value of λ increases. This implies that the scheme takes less time to give a first OOC signal when the smoothing parameter λ is smaller than when it is larger. For example, consider when the number of smoothing parameters $r = 2$ at a shift of 0.2, the ARL value increases from 26.3 to 30.3 when both λ_1 and λ_2 increase from 0.05 to 0.2, respectively. Likewise, the ARL increases from 30.3 to 40.6 when both λ_1 and λ_2 increase from 0.2 to 0.3.

• The same pattern can be observed for the $SDRL$ and MRL; thus, for the very same conditions, the $SDRL$ increases from 21.1 to 24.2, and the MRL value increases from 25.0 to 29.0 as both λ_1 and λ_2 increase from 0.05 to 0.2. Likewise, these values continue to increase when λ_1 and λ_2 increase from 0.2 to 0.3 (see Table 2).

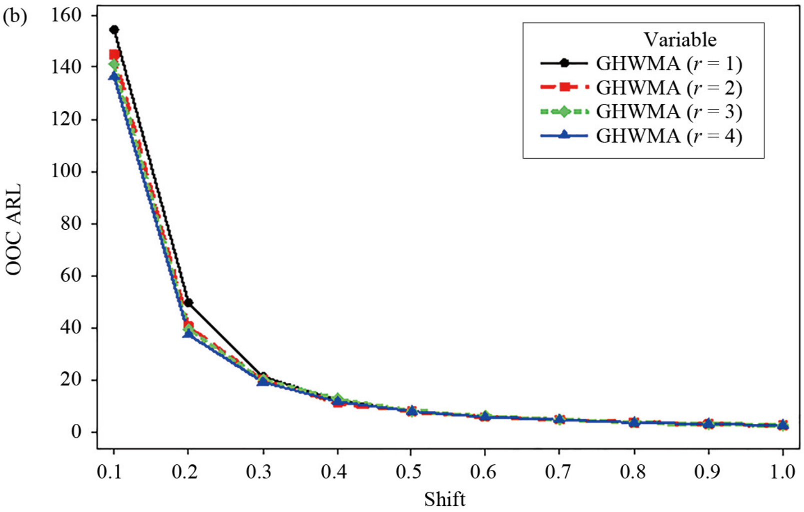
• It can also be observed that as the value of a given smoothing parameter λ increases, the value of the corresponding constant L increases; this implies that the larger the value of the smoothing parameter, the wider the control limits become and will delay the OOC signal. Similar patterns were observed in Table 3; however, the order of different λ values was altered to investigate its effect on the performance of the GHWMA $\overline{X}$ monitoring scheme. The results show a slight difference in the performance since the magnitude of the smoothing parameter affects the weight assigned to past and current data. Table 2 and Figures 3a-d reveal that the sensitivity of the GHWMA $\overline{X}$ scheme increases as r increases when λ_i $(i = 1,\ \ldots,\ r)$ are equal. Thus, if the number of smoothing parameters increases, the performance of the GHWMA $\overline{X}$ scheme increases when the weights are selected carefully. When the values of λ_i are different, for small λ_i values, the GHWMA $\overline{X}$ scheme performs better for larger values of r. However, for large λ_i values, the performance of the GHWMA $\overline{X}$ scheme deteriorates. Thus, choosing the weights carefully when using more smoothing parameters is important. It can also be observed from Figures 4a-d that if r increases, the Expected ARL ($EARL$) values decrease whenever the value of λ is kept constant. In other words, the overall performance of the proposed GHWMA $\overline{X}$ scheme increases as r increases. This is evident from Figure 4. For instance, when $\lambda = 0.05$, the $EARL$ value decreased from 9.1 to 8.5 when r increases from 1 to 2, and thereafter decreased to 8.3 when r increases to 4 (see Figures 4 and Table 4). Note that a decrease in $EARL$ values imply that the scheme's performance has improved.

(a)



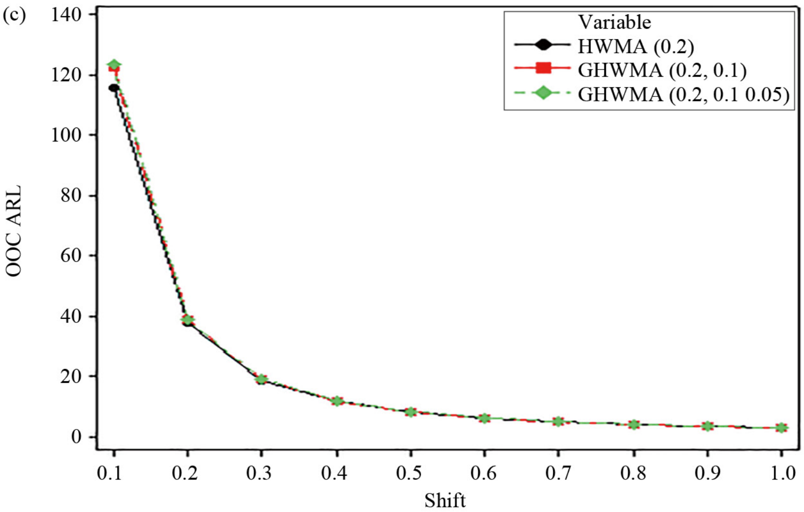
$r \in \{1, 2, 3, 4\}$ for $\lambda_i \in \{0.05\}$

(b)



$r \in \{1, 2, 3\}$ for $\lambda_i \in \{0.3\}$

(c)



$r \in \{1, 2,3\}$ for $\lambda_i \in \{0.05, 0.1, 0.2\}$

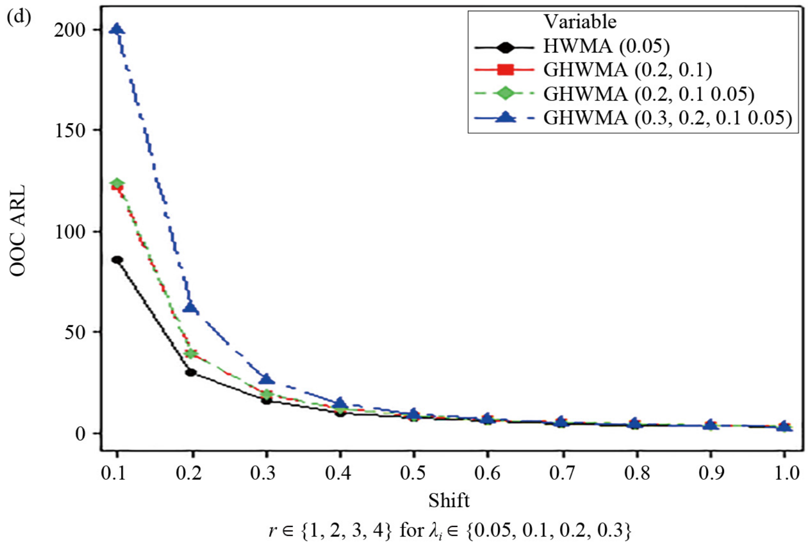
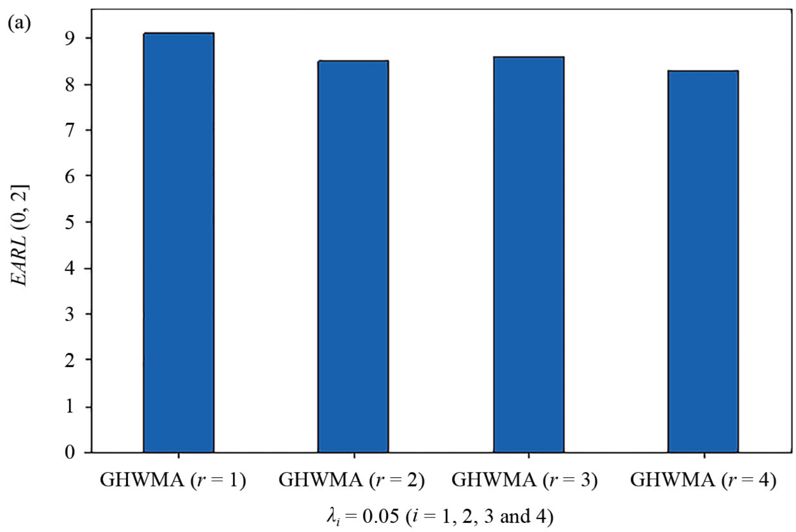


Figure 3. (a)-(b):Comparing different r values when $r \in \{1, 4\}$ for $\lambda \in \{0.05, 0.3\}$; (c)-(d): Comparing different r values when $r \in \{1, 2, 3, 4\}$ for $\lambda \in \{0.05, 0.1, 0.2, 0.3\}$



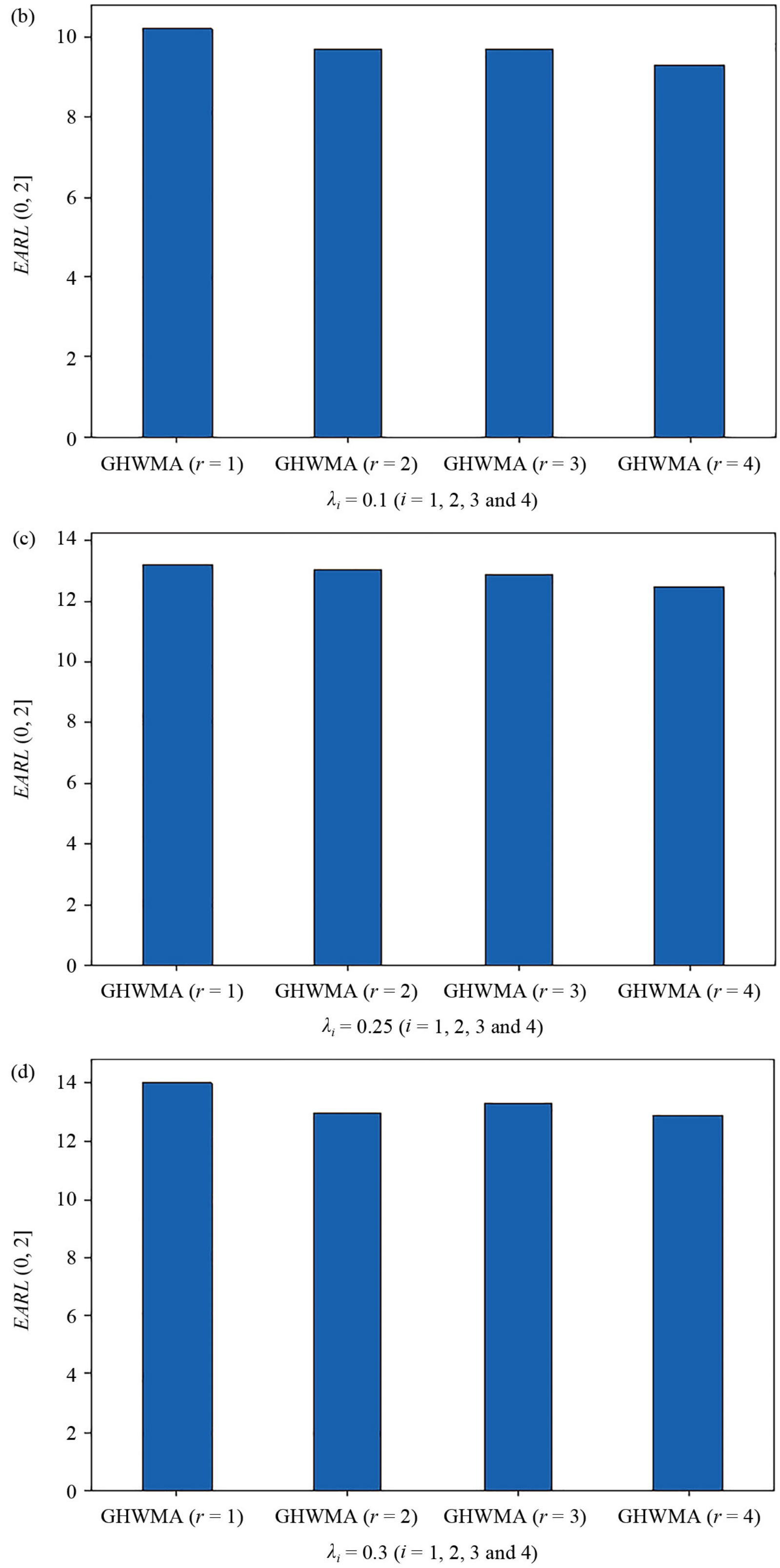


Figure 4. (a)-(d): Comparing different r values when $r \in \{1, 2, 3, 4\}$ for $\lambda_i \in \{0.05, 0.1, 0.2, 0.3\}$

Table 2. OOC *ARL*, *SDRL* and *MRL* profiles of the GHWMA $\overline{X}$ schemes when $n = 5$, $r \in \{1, 2, 3\}$, $\lambda_i \in \{0.05, 0.2, 0.3\}$ and $ARL_0 = 500$

Shift	$r = 1(\lambda_i)$			$r = 2(\lambda_i)$			$r = 3(\lambda_i)$		
	$\lambda_1 = 0.05$	$\lambda_1 = 0.2$	$\lambda_1 = 0.3$	$\lambda_1 = 0.05$, $\lambda_2 = 0.05$	$\lambda_1 = 0.2$, $\lambda_2 = 0.2$	$\lambda_1 = 0.3$, $\lambda_2 = 0.3$	$\lambda_1 = 0.05$, $\lambda_2 = 0.05$, $\lambda_3 = 0.05$	$\lambda_1 = 0.2$, $\lambda_2 = 0.2$, $\lambda_3 = 0.2$	$\lambda_1 = 0.3$, $\lambda_2 = 0.3$, $\lambda_3 = 0.3$
	ARL (*SDRL*, *MRL*)	*ARL* (*SDRL*, *MRL*)	*ARL* (*SDRL*, *MRL*)	*ARL* (*SDRL*, *MRL*)	*ARL* (*SDRL*, *MRL*)	*ARL* (*SDRL*, *MRL*)	*ARL* (*SDRL*, *MRL*)	*ARL* (*SDRL*, *MRL*)	*ARL* (*SDRL*, *MRL*)
0.0	500.8 (372.6, 439.0)	500.0 (471.1, 363.0)	500.7 (491.7, 351.0)	500.2 (373.3, 426.0)	499.4 (488.4, 351.0)	500.6 (499.4, 347.0)	499.6 (390.2, 412)	499.6 (497.1, 348.0)	499.5 (499.8, 347.0)
0.1	85.8 (62.5, 74.0)	116.1 (96.8, 90.0)	154.7 (142.6, 112.0)	82.1 (61.7, 70.0)	111.2 (93.3, 89.0)	145.2 (147.9, 113.0)	80.4 (64.1, 73.0)	115.4 (101.1, 100.0)	141.3 (176.2, 126.0)
0.2	30.0 (20.7, 26.0)	37.8 (26.9, 31.0)	46.9 (38.8, 36.0)	26.3 (21.1, 25.0)	30.3 (24.2, 29.0)	40.6 (40.8, 38.0)	28.6 (20.1, 21.0)	35.6 (30.3, 33.0)	39.4 (33.8, 35.0)
0.3	15.5 (10.2, 14.0)	18.7 (12.0, 16.0)	21.5 (15.7, 18.0)	14.3 (10.1, 13)	15.4 (10.9, 14.0)	20.1 (14.9, 17.0)	14.9 (10.2,15)	18.0 (11.6, 15.0)	20.1 (17.2, 21.0)
0.4	9.6 (6.0, 8.0)	11.5 (6.8, 10.0)	12.27 (8.1, 10.0)	8.8 (6.0, 9.0)	10.0 (6.5, 10.0)	11.1 (7.9, 11.0)	9.2 (6.1, 9.0)	12.2 (7.2, 11.0)	12.6 (10.7, 11.0)
0.5	6.8 (3.9, 6.0)	7.9 (4.4, 7.0)	8.2 (5.0, 7.0)	6.6 (3.7, 6.0)	7.7 (4.1, 6.0)	7.9 (4.7, 7.0)	7.0 (4.1, 7.0)	8.0 (4.8, 7.0)	8.1 (6.4, 8.0)
0.6	5.2 (2.8, 5.0)	5.9 (3.1, 5.0)	6.0 (3.4, 5.0)	5.0 (2.9, 5.0)	5.6 (3.0, 5.0)	5.9 (3.2, 5.0)	5.3 (3.0, 6.0)	6.0 (3.5, 5.0)	6.1 (3.5, 6.0)
0.7	4.2 (2.1, 4.0)	4.7 (2.3, 4.0)	4.6 (2.4, 4.0)	4.1 (2.2, 4.0)	4.6 (2.1, 4.0)	4.7 (2.3, 4.0)	4.3 (2.9, 5.0)	4.5 (2.5, 4.0)	4.8 (2.5, 4.0)
0.8	3.5 (1.8, 3.0)	3.8 (1.8, 4.0)	3.7 (1.9, 3.0)	3.4 (2.2, 4.0)	3.7 (1.9, 3.0)	3.7 (2.0, 3.0)	3.3 (2.5, 3.0)	3.6 (1.9, 3.0)	3.5 (2.1, 3.0)
0.9	3.0 (1.5, 3.0)	3.2 (1.5, 3.0)	3.1 (1.5, 3.0)	3.2 (2.0, 2.0)	3.2 (1.6, 3.0)	3.0 (1.6, 3.0)	3.1 (2.1, 2.0)	3.0 (1.5, 3.0)	3.1 (1.8, 3.0)
1.0	2.6 (1.4, 3.0)	2.8 (1.3, 3.0)	2.7 (1.2, 3.0)	2.7 (1.7, 2.0)	2.7 (1.4, 2.0)	2.6 (1.3, 2.0)	2.5 (1.7, 2.0)	2.5 (1.3, 2.0)	2.6 (1.4, 2.0)
1.1	2.2 (1.2, 3.0)	2.4 (1.1, 3.0)	2.3 (1.1, 2.0)	2.3 (1.5, 2.0)	2.3 (1.2, 2.0)	2.2 (1.1, 2.0)	2.1 (1.3, 2.0)	2.2 (1.1, 2.0)	2.2 (1.1, 2.0)
1.2	2.0 (1.1, 1.0)	2.2 (1.0, 2.0)	2.1 (1.0, 2.0)	1.9 (1.2, 2.0)	2.0 (1.0, 2.0)	1.9 (0.9, 2.0)	1.8 (1.0, 2.0)	1.9 (0.9, 2.0)	1.9 (0.9, 2.0)
1.3	1.8 (1.0, 1.0)	1.9 (0.9, 2.0)	1.8 (0.9, 2.0)	1.7 (1.0, 1.0)	1.8 (0.9, 2.0)	1.7 (0.8, 2.0)	1.6 (0.8, 1.0)	1.6 (0.8, 2.0)	1.7 (0.8, 2.0)
1.4	1.6 (0.9, 1.0)	1.7 (0.9, 1.0)	1.6 (0.8, 1.0)	1.5 (0.8, 1.0)	1.6 (0.7, 1.0)	1.6 (0.7, 1.0)	1.5 (0.7, 1.0)	1.6 (0.7, 1.0)	1.4 (0.7, 1.0)
1.5	1.4 (0.8, 1.0)	1.6 (0.8, 1.0)	1.5 (0.7, 1.0)	1.3 (0.6, 1.0)	1.4 (0.6, 1.0)	1.4 (0.6, 1.0)	1.3 (0.5, 1.0)	1.4 (0.6, 1.0)	1.3 (0.6, 1.0)
1.6	1.3 (0.7, 1.0)	1.4 (0.7, 1.0)	1.4 (0.6, 1.0)	1.2 (0.5, 1.0)	1.3 (0.5, 1.0)	1.3 (0.5, 1.0)	1.2 (0.5, 1.0)	1.3 (0.5, 1.0)	1.2 (0.5, 1.0)
1.7	1.2 (0.6, 1.0)	1.3 (0.6, 1.0)	1.3 (0.5, 1.0)	1.2 (0.4, 1.0)	1.2 (0.5, 1.0)	1.2 (0.5, 1.0)	1.2 (0.4, 1.0)	1.2 (0.4, 1.0)	1.2 (0.4, 1.0)
1.8	1.1 (0.5, 1.0)	1.2 (0.5, 1.0)	1.2 (0.4, 1.0)	1.1 (0.3, 1.0)	1.2 (0.4, 1.0)	1.2 (0.4, 1.0)	1.1 (0.3, 1.0)	1.2 (0.4, 1.0)	1.1 (0.4, 1.0)
1.9	1.1 (0.4, 1.0)	1.1 (0.4, 1.0)	1.1 (0.4, 1.0)	1.1 (0.3, 1.0)	1.1 (0.3, 1.0)	1.1 (0.3, 1.0)	1.1 (0.3, 1.0)	1.1 (0.3, 1.0)	1.1 (0.3, 1.0)
2.0	1.1 (0.3, 1.0)	1.1 (0.3, 1.0)	1.1 (0.3, 1.0)	1.0 (0.2, 1.0)	1.1 (0.3, 1.0)	1.1 (0.3, 1.0)	1.0 (0.2, 1.0)	1.1 (0.3, 1.0)	1.0 (0.3, 1.0)
L	2.6112	3.0599	3.0820	2.7825	3.0634	3.0725	2.8429	3.0410	3.0450

Table 3. OOC profiles of the GHWMA $\overline{X}$ schemes when $n = 5$, $r \in \{2, 3, 4\}$, $\lambda_i \in \{0.05, 0.1, 0.2, 0.25, 0.3, 0.4, 0.5\}$ and $ARL_0 = 500$

Shift	$r = 2\,(\lambda_i)$			$r = 3\,(\lambda_i)$			$r = 4\,(\lambda_i)$		
	$\lambda_1 = 0.2$, $\lambda_2 = 0.1$	$\lambda_1 = 0.1$, $\lambda_2 = 0.05$	$\lambda_1 = 0.2$, $\lambda_2 = 0.05$	$\lambda_1 = 0.3$, $\lambda_2 = 0.2$, $\lambda_3 = 0.1$	$\lambda_1 = 0.2$, $\lambda_2 = 0.1$, $\lambda_3 = 0.05$	$\lambda_1 = 0.25$, $\lambda_2 = 0.1$, $\lambda_3 = 0.05$	$\lambda_1 = 0.3$, $\lambda_2 = 0.2$, $\lambda_3 = 0.1$, $\lambda_4 = 0.05$	$\lambda_1 = 0.4$, $\lambda_2 = 0.3$, $\lambda_3 = 0.2$, $\lambda_4 = 0.05$	$\lambda_1 = 0.5$, $\lambda_2 = 0.25$, $\lambda_3 = 0.1$, $\lambda_4 = 0.05$
	ARL $(SDRL, MRL)$	ARL $(SDRL, MRL)$	ARL $(SDRL, MRL)$	ARL $(SDRL, MRL)$	ARL $(SDRL, MRL)$	ARL $(SDRL, MRL)$	ARL $(SDRL, MRL)$	ARL $(SDRL, MRL)$	ARL $(SDRL, MRL)$
0.0	499.8 (478.0, 357.0)	500.3 (416.9, 394.0)	499.5 (470.9, 362.0)	499.9 (499.1, 348.0)	500.1 (479.9, 357.0)	500.0 (482.1, 356.0)	500.3 (501.0, 347.0)	500.4 (501.6, 346.0)	499.7 (498.1, 348.0)
0.1	110.5 (95.3, 83.0)	86.4 (69.1, 81.0)	100.1 (83.8, 82.0)	119.0 (123.9, 99.0)	107.7 (97.5, 81.0)	110.6 (101.2, 84.0)	114.7 (124.6, 101.0)	171.1 (160.2, 126.0)	199.7 (168.4, 151.0)
0.2	30.3 (28.5, 32.0)	27.5 (20.6, 28.0)	28.9 (25.0, 30.0)	34.9 (29.5, 30.0)	28.9 (26.8, 30.0)	32.0 (30.0, 31.0)	33.8 (37.0, 35.0)	52.4 (108.4, 76.0)	50.6 (114.8, 80.0)
0.3	15.0 (10.4, 12.0)	13.2 (9.7, 11.0)	14.8 (10.9, 11.0)	17.5 (11.3, 13.0)	14.2 (10.3, 11.0)	16.3 (11.4, 12.0)	20.9 (15.6, 16.0)	21.8 (17.4, 17.0)	23.0 (20.2, 19.0)
0.4	9.6 (6.2, 9.0)	9.1 (6.0, 8.0)	9.5 (6.2, 8.0)	10.3 (7.7, 9.0)	9.7 (6.1, 9.0)	10.3 (7.2, 10.0)	12.0 (8.6, 11.0)	14.3 (8.8, 11.0)	15.7 (11.5, 12.0)
0.5	7.0 (4.3, 7.0)	7.2 (4.2, 8.0)	7.2 (4.3, 7.0)	7.5 (4.6, 7.0)	7.2 (4.4, 8.0)	7.7 (4.6, 9.0)	8.1 (4.9, 7.0)	10.5 (6.0, 9.0)	11.5 (6.8, 10.0)
0.6	6.0 (3.1, 6.0)	6.2 (3.1, 6.0)	6.2 (3.0, 6.0)	6.0 (3.6, 5.0)	6.2 (3.2, 6.0)	6.3 (3.2, 7.0)	6.3 (3.9, 6.0)	7.3 (5.4, 6.0)	7.9 (5.9, 6.0)
0.7	4.8 (2.3, 5.0)	5.0 (2.5, 5.0)	5.0 (2.3, 5.0)	4.6 (2.6, 4.0)	4.9 (2.5, 5.0)	5.1 (2.6, 6.0)	4.7 (2.7, 4.0)	5.3 (4.0, 4.0)	5.8 (4.3, 5.0)
0.8	3.9 (1.9, 4.0)	4.1 (2.1, 4.0)	4.1 (2.0, 4.0)	3.7 (1.9, 3.0)	4.0 (2.1, 3.0)	4.0 (2.2, 5.0)	3.8 (2.1, 3.0)	4.0 (2.7, 3.0)	4.3 (2.9, 4.0)
0.9	3.3 (1.7, 3.0)	3.4 (1.9, 4.0)	3.5 (1.7, 4.0)	3.1 (1.6, 3.0)	3.3 (1.8, 3.0)	3.4 (2.0, 5.0)	3.1 (1.7, 3.0)	3.1 (1.9, 3.0)	3.4 (2.1, 3.0)
1.0	2.8 (1.5, 2.0)	2.9 (1.7, 2.0)	3.0 (1.5, 3.0)	2.6 (1.3, 2.0)	2.8 (1.5, 2.0)	3.0 (1.9, 5.0)	2.6 (1.4, 2.0)	2.6 (1.4, 2.0)	2.8 (1.6, 2.0)
1.1	2.4 (1.3, 2.0)	2.4 (1.5, 2.0)	2.5 (1.4, 2.0)	2.2 (1.1, 2.0)	2.3 (1.3, 2.0)	2.4 (1.8, 4.0)	2.2 (1.1, 2.0)	2.2 (1.1, 2.0)	2.3 (1.3, 2.0)
1.2	2.1 (1.1, 2.0)	2.1 (1.3, 2.0)	2.2 (1.2, 2.0)	2.0 (0.9, 2.0)	2.0 (1.0, 2.0)	2.2 (1.7, 2.0)	1.9 (0.9, 2.0)	1.9 (0.9, 2.0)	2.0 (1.0, 2.0)
1.3	1.8 (0.9, 2.0)	1.8 (1.0, 2.0)	1.9 (1.0, 2.0)	1.7 (0.8, 2.0)	1.8 (0.9, 2.0)	1.8 (1.6, 2.0)	1.7 (0.8, 2.0)	1.7 (0.8, 2.0)	1.8 (0.9, 2.0)
1.4	1.6 (0.8, 1))	1.6 (0.9, 1.0)	1.7 (0.9, 1.0)	1.6 (0.7, 1.0)	1.6 (0.7, 1.0)	1.7 (1.5, 1.0)	1.6 (0.7, 1.0)	1.6 (0.7, 1.0)	1.6 (0.7, 1.0)
1.5	1.5 (0.7, 1.0)	1.4 (0.7, 1.0)	1.5 (0.7, 1.0)	1.4 (0.6, 1.0)	1.4 (0.6, 1.0)	1.5 (1.3, 1.0)	1.4 (0.6, 1.0)	1.4 (0.6, 1.0)	1.4 (0.6, 1.0)
1.6	1.3 (0.6, 1.0)	1.3 (0.6, 1.0)	1.4 (0.6, 1.0)	1.3 (0.5, 1.0)	1.3 (0.5, 1.0)	1.3 (1.1, 1.0)	1.3 (0.5, 1.0)	1.3 (0.5, 1.0)	1.3 (0.5, 1.0)
1.7	1.2 (0.5, 1.0)	1.2 (0.5, 1.0)	1.3 (0.5, 1.0)	1.2 (0.5, 1.0)	1.2 (0.5, 1.0)	1.3 (0.9, 1.0)	1.2 (0.4, 1.0)	1.2 (0.4, 1.0)	1.2 (0.5, 1.0)
1.8	1.2 (0.4, 1.0)	1.2 (0.4, 1.0)	1.2 (0.4, 1.0)	1.2 (0.4, 1.0)	1.2 (0.4, 1.0)	1.2 (0.7, 1.0)	1.2 (0.4, 1.0)	1.2 (0.4, 1.0)	1.2 (0.4, 1.0)
1.9	1.1 (0.3, 1.0)	1.1 (0.3, 1.0)	1.1 (0.3, 1.0)	1.1 (0.3, 1.0)	1.1 (0.3, 1.0)	1.1 (0.5, 1.0)	1.1 (0.3, 1.0)	1.1 (0.3, 1.0)	1.1 (0.3, 1.0)
2.0	1.1 (0.3, 1.0)	1.1 (0.3, 1.0)	1.1 (0.3, 1.0)	1.1 (0.3, 1.0)	1.1 (0.3, 1.0)	1.1 (0.4, 1.0)	1.1 (0.3, 1.0)	1.1 (0.3, 1.0)	1.1 (0.3, 1.0)
L	3.0605	2.9600	3.0615	3.0620	3.0564	3.0700	3.0365	3.0420	3.0650

Table 4. Performance comparison with other popular schemes when $r \in \{1, 3\}$ for $\lambda_i \in \{0.05, 0.1, 0.25\}$

Shift	EWMA			CUSUM			GHWMA($r=1$)			GHWMA($r=2$)			GHWMA ($r=3$)			
	$\lambda_1 = 0.05$	$\lambda_1 = 0.1$	$\lambda_1 = 0.25$	$k_c = 0.125$	$k_c = 0.25$	$k_c = 0.5$	$\lambda_1 = 0.05$	$\lambda_1 = 0.1$	$\lambda_1 = 0.25$	$\lambda_1 = 0.05, \lambda_2 = 0.05$	$\lambda_1 = 0.1, \lambda_2 = 0.1$	$\lambda_1 = 0.25, \lambda_2 = 0.05$	$\lambda_1 = 0.05, \lambda_2 = 0.05, \lambda_3 = 0.05$	$\lambda_1 = 0.1, \lambda_2 = 0.1, \lambda_3 = 0.1$	$\lambda_1 = 0.25, \lambda_2 = 0.25, \lambda_3 = 0.25$	$\lambda_1 = 0.25, \lambda_2 = 0.1, \lambda_3 = 0.05$
0.0	504.9	499.7	499.7	500.8	502.4	500.3	500.8	500.9	499.9	500.2	500.5	499.7	499.6	500.3	500.0	500.0
0.1	95.9	124.7	198.5	97.3	115.3	93.2	85.8	95.4	133.6	82.1	96.7	104.5	80.4	96.9	143.2	110.6
0.2	28.8	35.6	59.9	39.7	37.2	34.1	30.0	34.1	41.4	26.3	29.1	29.1	28.6	32.5	38.3	32.0
0.3	14.4	16.6	24.8	24.6	20.5	18.2	15.5	17.9	19.8	14.3	14.9	15.4	14.9	17.3	20.0	16.3
0.4	8.8	10	13.2	17.9	14.1	11.2	9.6	11.2	11.8	8.8	9.7	10.7	9.2	11.2	12.8	10.3
0.5	6.1	6.8	8.3	14	10.7	8.6	6.8	7.8	8.0	6.6	7.2	8.4	7.0	7.7	8.5	7.7
0.6	4.5	5.0	5.9	11.5	8.7	6.1	5.2	5.9	5.9	5.0	6.1	6.6	5.3	5.9	6.1	6.3
0.7	3.5	3.9	4.4	9.8	7.3	4.3	4.2	4.7	4.6	4.1	4.8	5.4	4.3	4.6	4.7	5.1
0.8	2.9	3.1	3.5	8.5	6.3	3.3	3.5	3.9	3.7	3.4	3.8	4.2	3.3	3.8	3.8	4.0
0.9	2.4	2.6	2.9	7.6	5.6	2.5	3.0	3.3	3.1	3.2	3.3	3.6	3.1	3.1	3.1	3.4
1.0	2.1	2.2	2.5	6.8	5.0	2.3	2.6	2.9	2.7	2.7	2.9	3.2	2.5	2.5	2.6	3.0
1.1	1.8	2.0	2.1	6.2	4.5	2.1	2.2	2.5	2.4	2.3	2.4	2.6	2.1	2.1	2.3	2.4
1.2	1.6	1.7	1.9	5.7	4.1	2.0	2.0	2.2	2.1	1.9	2.1	2.3	1.8	1.9	2.0	2.2
1.3	1.5	1.6	1.7	5.3	3.8	1.7	1.8	2.0	1.9	1.7	1.7	1.8	1.6	1.7	1.7	1.8
1.4	1.4	1.4	1.5	4.9	3.6	1.6	1.6	1.8	1.7	1.5	1.5	1.7	1.5	1.5	1.6	1.7
1.5	1.3	1.3	1.4	4.6	3.3	1.5	1.4	1.6	1.5	1.3	1.4	1.4	1.3	1.4	1.4	1.5
1.6	1.2	1.2	1.3	4.4	3.2	1.5	1.3	1.4	1.4	1.2	1.2	1.3	1.2	1.3	1.3	1.3
1.7	1.1	1.2	1.2	4.1	3.0	1.4	1.2	1.3	1.3	1.2	1.2	1.2	1.2	1.2	1.2	1.3
1.8	1.1	1.1	1.2	3.9	2.9	1.3	1.1	1.2	1.2	1.1	1.1	1.1	1.1	1.2	1.2	1.2
1.9	1.1	1.1	1.1	3.7	2.7	1.3	1.1	1.1	1.1	1.1	1.0	1.0	1.1	1.1	1.1	1.1
2.0	1.0	1.1	1.1	3.5	2.6	1.2	1.1	1.1	1.1	1.0	1.0	1.0	1.0	1.1	1.1	1.1
EARL	9.1	11.2	16.9	14.2	13.2	10.0	9.1	10.2	12.5	8.5	9.7	10.3	8.6	10.3	12.9	10.7
CLC	2.6450	2.8242	3.0002	5.8870	3.8780	3.8780	2.6112	2.9377	3.0737	2.7825	2.8133	3.0679	2.8429	3.0021	3.0437	3.0700

Note: CLC = Control Limit Constant

Table 4. (cont.)

Shift	HEWMA ($\lambda_1 = 0.05$)			HEWMA ($\lambda_1 = 0.25$)			DEWMA			GWMA			
	$\lambda_2 = 0.1$	$\lambda_2 = 0.2$	$\lambda_2 = 0.25$	$\lambda_2 = 0.1$	$\lambda_2 = 0.15$	$\lambda_2 = 0.2$	$\lambda_1 = \lambda_2 = 0.05$	$\lambda_1 = \lambda_2 = 0.1$	$\lambda_1 = \lambda_2 = 0.25$	$a = 0.5, q = 0.75$	$a = 1, q = 0.75$	$a = 0.5, q = 0.9$	$a = 1, q = 0.9$
0.0	499.7	500.5	501.2	501.1	502.3	504.6	499.7	501.8	499.7	500.4	499.0	499.7	500.0
0.1	85.3	89.9	99.0	130.1	149.2	161.1	80.9	95.4	167.1	149.4	197.3	100.3	126.7
0.2	29.5	32.8	47.4	42.6	51.8	55.2	34.7	32.5	55.4	49.1	59.6	39.8	38.7
0.3	17.1	21.1	29.3	25.5	27.7	30.1	17.8	20.1	23.2	24.0	24.6	21.2	21.6
0.4	11.3	13.4	22.6	13.4	18.0	21.3	9.8	18.9	14.5	14.3	13.0	13.9	12.5
0.5	8.7	9.7	17.4	11.2	12.3	13.2	7.3	11.4	11.0	9.6	8.2	8.3	8.8
0.6	6.1	7.2	14.1	9.8	10.9	8.1	5.9	7.2	8.1	6.9	5.9	6.1	6.1
0.7	4.3	6.1	12.9	8.7	6.1	5.9	5.0	5.9	6.0	5.2	4.4	4.7	4.9
0.8	3.7	5.0	10.3	8.0	3.9	4.1	4.1	4.2	4.5	4.1	3.5	3.8	3.2
0.9	2.5	4.6	7.2	6.3	2.5	2.9	3.3	2.5	3.2	3.3	2.9	3.1	2.6
1.0	2.3	4.2	5.4	4.3	2.2	2.2	2.6	2.0	2.7	2.8	2.5	2.6	2.3
1.1	1.8	3.7	4.0	3.6	1.9	1.9	2.0	1.9	2.1	2.4	2.1	2.2	2.0
1.2	1.6	3.0	3.2	2.9	1.7	1.7	1.6	1.6	1.7	2.1	1.9	2.0	1.8
1.3	1.4	2.3	2.6	2.2	1.5	1.5	1.4	1.5	1.5	1.8	1.7	1.7	1.6
1.4	1.3	1.9	2.1	1.9	1.4	1.4	1.3	1.4	1.4	1.6	1.5	1.6	1.4
1.5	1.2	1.7	1.4	1.7	1.3	1.3	1.2	1.2	1.3	1.5	1.4	1.4	1.3
1.6	1.1	1.5	1.3	1.6	1.2	1.2	1.2	1.2	1.2	1.3	1.3	1.3	1.2
1.7	1.1	1.3	1.2	1.5	1.1	1.1	1.1	1.1	1.1	1.3	1.2	1.2	1.2
1.8	1.1	1.2	1.1	1.3	1.1	1.1	1.1	1.1	1.1	1.2	1.2	1.2	1.1
1.9	1.0	1.2	1.1	1.2	1.1	1.1	1.0	1.1	1.1	1.1	1.1	1.1	1.1
2.0	1.0	1.1	1.0	1.1	1.0	1.0	1.0	1.0	1.0	1.1	1.1	1.1	1.0
EARL	9.2	10.6	14.2	13.9	14.9	15.9	9.2	10.7	15.5	14.2	16.8	10.9	12.1
CLC	2.2213	2.3492	2.3913	2.5489	2.6399	2.6969	2.0859	2.3689	2.7354	3.063	3.002	2.998	2.825

Note: CLC = Control Limit Constant

4.3 *Performance comparison with other popular schemes*

Recently, many scholars have recommended the use of the CED criterion when assessing the performance of memory-type schemes; see, for example, Knoth [24] and Knoth et al. [13]. The CED is defined as the delay in detecting a change from the first opportunity and not from the time of change itself. The CED, denoted as D_τ, is then defined by

$$D_\tau = E_\tau(N - \tau + 1 \mid N \geq \tau), \ \tau = 1, \ 2, \ \ldots, \tag{16}$$

where N represents the number of samples until an alarm is raised and τ is the change point applied to the following model:

$$\mu = \begin{cases} \mu_0, & t < \tau \\ \mu_1 = \mu_0 + \delta\sigma, & t \geq \tau' \end{cases} \tag{17}$$

where μ_0 and μ_1 represent the IC and OOC process means, respectively. In Section 4.1, it was assumed that $(\mu_0, \ \sigma) = (0, \ 1)$. Therefore, $\mu = 0$ when $t < \tau$. Otherwise, $\mu = \delta$.

Note that when $\tau = 1$, the CED is equivalent to the zero-state *ARL* criterion, and when $\tau \to \infty$, the CED is equivalent to the steady-state *ARL* criterion. Note that it is essential to investigate the performance of a memory-type scheme when $\tau \in [1, \ \infty)$.

4.3.1 *Performance comparison in terms of the zero-state ARL criterion*

In this subsection, the proposed GHWMA $\overline{X}$ scheme is compared with the existing EWMA and CUSUM, DEWMA, HEWMA, and GWMA $\overline{X}$ schemes in terms of their zero-state *ARL* profiles, i.e., when $\tau = 1$. Table 4 compares the OOC performances of the competing schemes when $\lambda_i \in \{0.05, \ 0.1, \ 0.25\}$. This comparison is summarised in Figure 5 when $\lambda_i = 0.05$ and $k = 0.5$.

Thus, from Figure 5 and Table 4, it can be observed that:

• The zero-state *ARL* profile of the CUSUM $\overline{X}$ scheme recorded higher values than the EWMA and GHWMA $\overline{X}$ schemes, regardless of the size of the shift. Thus, the CUSUM $\overline{X}$ scheme takes more time to detect shifts in the process mean as compared to the EWMA and GHWMA $\overline{X}$ schemes.

• The GHWMA $\overline{X}$ scheme with $r = 2$ performs better than the competing schemes for small shifts. The EWMA $\overline{X}$ scheme outperforms the CUSUM and GHWMA $\overline{X}$ schemes for moderate shifts. For large Shifts, the EWMA, CUSUM and GHWMA $\overline{X}$ perform similarly (see Figure 5a).

• The proposed GHWMA $\overline{X}$ scheme is found to be superior to the HEWMA and DEWMA $\overline{X}$ schemes for small shifts (when $\delta < 0.8$) and when $0.8 \leq \delta < 1.2$, the HEWMA $\overline{X}$ scheme performs better than the DEWMA and GHWMA $\overline{X}$ schemes (Figure 5b).

• In terms of the EARL values (i.e., overall performance), the GHWMA $\overline{X}$ scheme outperforms all competing schemes considered in this study.

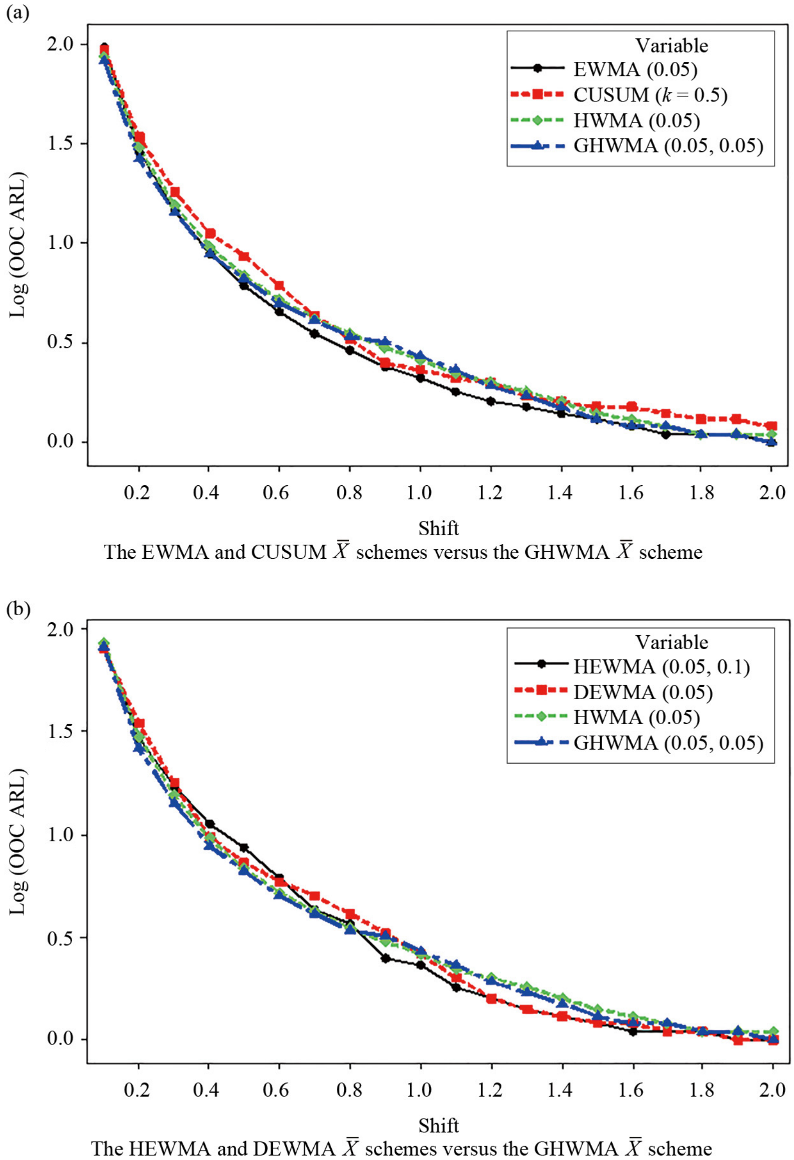
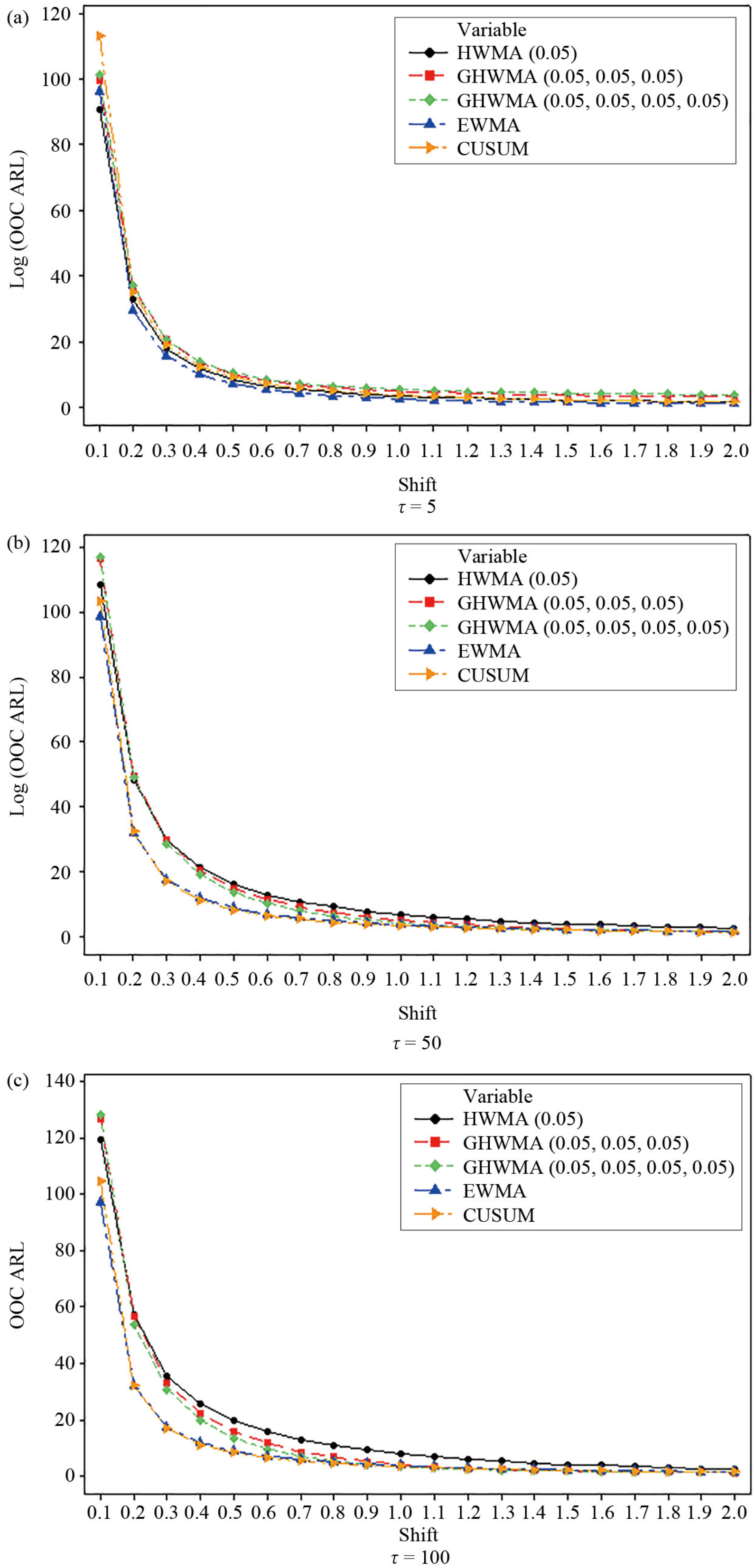


Figure 5. Comparison with other popular monitoring schemes when $\lambda = 0.1$

4.3.2 *Performance comparison in terms of the expected conditional delay criterion*

The GHWMA $\overline{X}$ scheme is compared with the popular EWMA and CUSUM $\overline{X}$ schemes in terms of the CED profiles (denoted as OOC *ARL*) when the smoothing parameters of the EWMA $\overline{X}$ scheme (i.e., λ) and that of the GHWMA $\overline{X}$ scheme (λ_i) are equal to 0.05. Figure 6 shows that for small values of $\tau \leq 10$, the performances of the competing schemes are almost similar for small shifts, and for large shifts, the EWMA $\overline{X}$ scheme performs better, followed by the GHWMA $(r = 1)$ and CUSUM $\overline{X}$ schemes; see, for example, Figure 6a. When $10 < \tau \leq 500$, regardless of the magnitude of the shifts in the process mean, the EWMA and CUSUM $\overline{X}$ schemes outperform the GHWMA $\overline{X}$ schemes, and their performance is almost similar for small shifts, regardless of the number of smoothing parameters (see, Figures 6b, d). When $\tau > 500$, for small and moderate shifts in the process mean, the EWMA and CUSUM $\overline{X}$ schemes outperform the GHWMA $\overline{X}$ scheme. Note that for large shifts, the competing schemes perform similarly (see Figures 6e-f).


(a)
Log (OOC ARL)
Variable
HWMA (0.05)
GHWMA (0.05, 0.05, 0.05)
GHWMA (0.05, 0.05, 0.05, 0.05)
EWMA
CUSUM
Shift
τ = 5
(b)
Log (OOC ARL)
Variable
HWMA (0.05)
GHWMA (0.05, 0.05, 0.05)
GHWMA (0.05, 0.05, 0.05, 0.05)
EWMA
CUSUM
Shift
τ = 50
(c)
OOC ARL
Variable
HWMA (0.05)
GHWMA (0.05, 0.05, 0.05)
GHWMA (0.05, 0.05, 0.05, 0.05)
EWMA
CUSUM
Shift
τ = 100

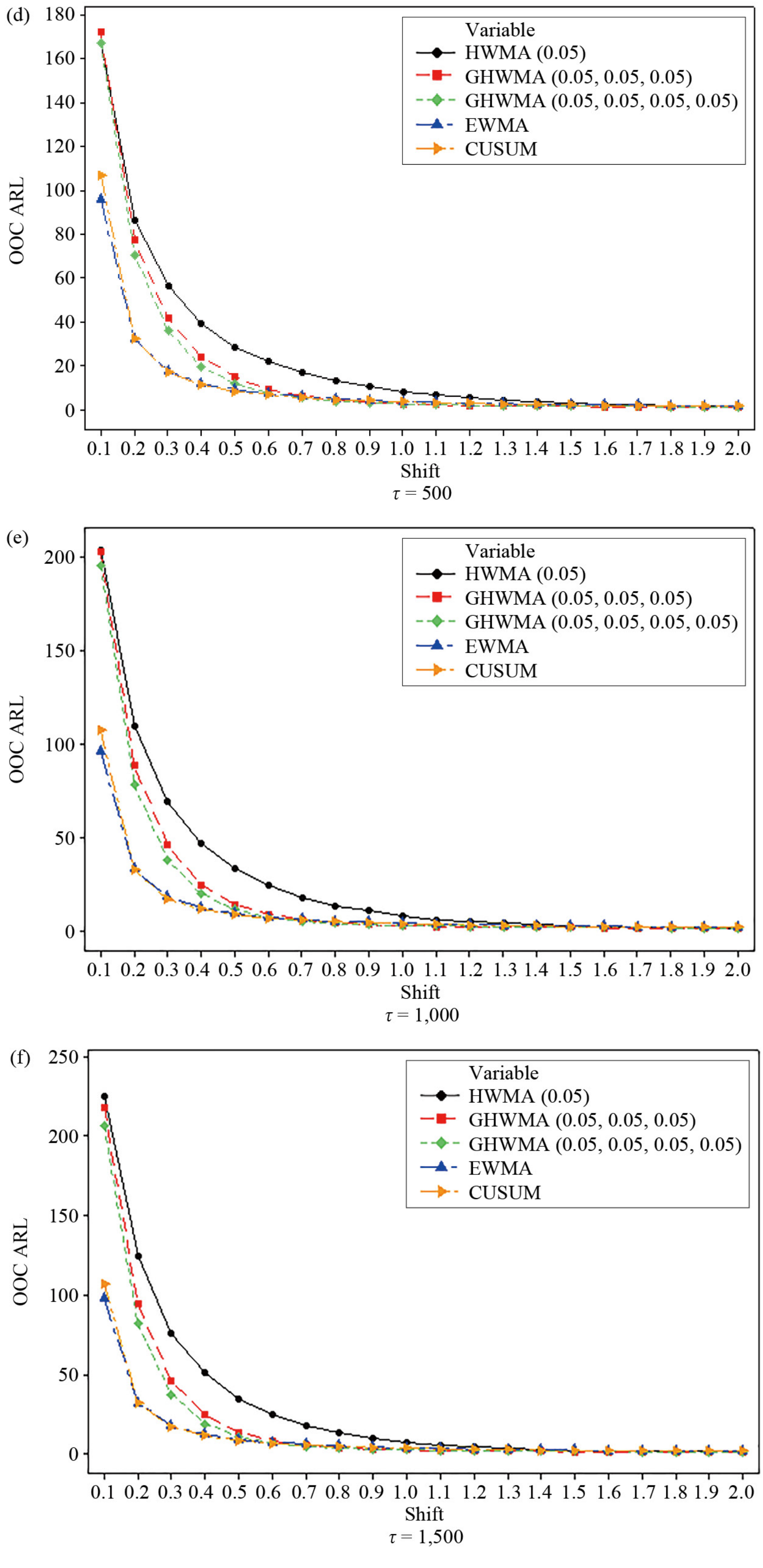


Figure 6. Comparison of the competing schemes in terms of the CED values

Comparing the GHWMA $\overline{X}$ scheme with $r = 1$ (i.e., HWMA $\overline{X}$ scheme) with the GHWMA $\overline{X}$ scheme with $r > 1$, it is observed that the HWMA $\overline{X}$ scheme should be used in processes where shifts are expected to happen at the start-up of the process. In other words, the HWMA $\overline{X}$ scheme is not consistent and effective when shifts are expected to happen after τ as compared to the GHWMA $\overline{X}$ scheme. The flexibility of the GHWMA $\overline{X}$ scheme can allow the choice of optimal process parameters that can improve the performance of the proposed scheme in such cases. This explains the superiority of the GHWMA $\overline{X}$ scheme over the HWMA $\overline{X}$ scheme in terms of the CED for different τ values. In addition, the HWMA $\overline{X}$ scheme is low in monitoring moderate and large shifts and good in detecting small shifts. However, a better selection of process parameters can help the GHWMA $\overline{X}$ scheme to outperform the HWMA $\overline{X}$ scheme in such cases.

4.3.3 *Computational complexity analysis*

In this section, we analyze the computational complexity of the proposed GHWMA $\overline{X}$ scheme in terms of the time it takes to compute the characteristics of the run-length distribution for $r > 1$ as compared to the HWMA $\overline{X}$ scheme. Table 5 presents the results of the computational times (in seconds) and attained *ICARL* values (in parentheses) of the HWMA and GHWMA $\overline{X}$ schemes when $\lambda = \lambda_i = 0.05$ ($i = 1, 2, 3$ and 4) using 10,000, 20,000, 50,000 and 100,000 simulations for a prespecified ARL_0 value of 500. The pseudo codes of the HWMA and GHWMA $\overline{X}$ schemes are given in Appendix B. The results from Table 5 (in terms of the computational time) are summarized in Figure 7. Note that $r = 1$ corresponds to the HWMA $\overline{X}$ scheme. Figure 7 and Table 5 show that the computational time is directly proportional to the number of smoothing parameters. In other words, the larger the value of r, the higher the computational time. For instance, using 10,000 simulations, the HWMA $\overline{X}$ scheme takes 23 seconds to yield the results. However, the GHWMA $\overline{X}$ scheme with $r = 4$ takes 34 seconds to yield the results. Using 100,000 simulations, the HWMA $\overline{X}$ scheme takes 223 seconds to yield the results, while the GHWMA $\overline{X}$ scheme with $r = 4$ takes 299 seconds to yield the results.

The difference between the computational time for different values of r increases as the number of simulations increases. For instance, when we have 10,000 simulations, the percentage difference between the computational times for successive r varies between 7% and 21%. However, when we increase the number of simulations to 100,000, it varies between 7% and 13%. In addition, Table 5 shows that the higher the number of smoothing parameters and/or simulation runs, the closer the *ICARL* is to the prespecified value of the ARL_0.

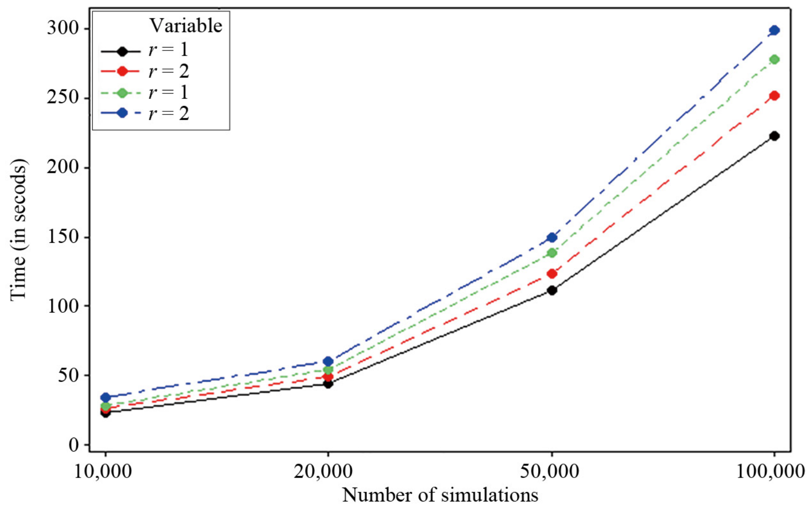


Figure 7. Computational time of the GHWMA $\overline{X}$ scheme for different number of simulations

Table 5. Computational times and attained IC *ARL* values of the GHWMA and HWMA $\bar{X}$ schemes for $\lambda = 0.05$ when $ARL_0 = 500$

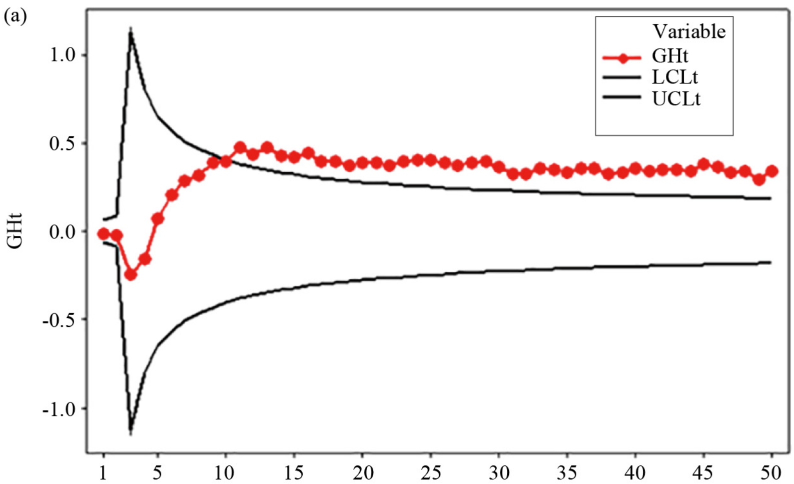
Number of simulations	$r = 1$	$r = 2$	$r = 3$	$r = 4$
10,000	23 (504.9)	26 (503.0)	28 (502.1)	34 (501.5)
20,000	44 (502.0)	49 (499.1)	54 (499.9)	60 (500.0)
50,000	111 (501.5)	123 (499.3)	138 (500.0)	149 (500.0)
100,000	223 (500.6)	252 (500.0)	278 (500.2)	299 (500.0)

5. Illustrative examples of the proposed GHWMA $\overline{X}$ scheme when $r \in \{1, 4\}$

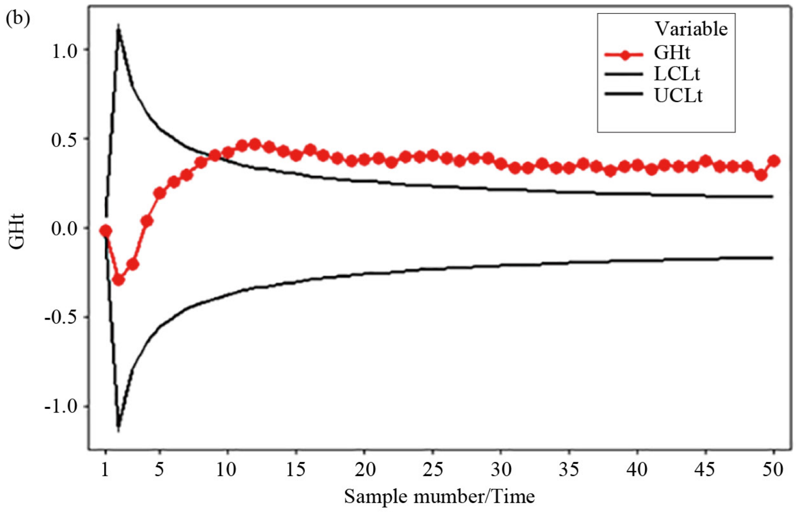
5.1 *Numerical example using simulated data*

To illustrate the implementation of the GHWMA $\overline{X}$ scheme when $r \in \{1, 2, 3, 4\}$ and $\lambda_i = 0.05$, fifty samples of size 5 were simulated using the normal distribution with a small shift of 0.25 standard deviation in the process mean (i.e., $N(0.25, 1)$ distribution). The control limit constants used in the design of the proposed GHWMA $\overline{X}$ scheme are given in Tables 1 and 2. The plots of the charting statistics of the GHWMA $\overline{X}$ scheme when $r = 1, 2, 3$ and 4 are shown in Figure 8, along with those of the EWMA and CUSUM $\overline{X}$ schemes for comparison purposes when $\lambda_i = \lambda = 0.05$ and $k = 0.5$. Figure 8 shows that when $r = 1$ and 2, the proposed GHWMA $\overline{X}$ scheme gives a signal on the 9$^{\text{th}}$ and 10$^{\text{th}}$ samples, respectively. When $r = 3$ and 4, it gives a signal for the first time on the 11$^{\text{th}}$ sample. However, the EWMA $\overline{X}$ scheme does not give a signal, while the CUSUM $\overline{X}$ scheme gives a signal on the 37$^{\text{th}}$ sample (see Figure 8). This example reveals that the proposed GHWMA $\overline{X}$ scheme outperforms the EWMA and CUSUM $\overline{X}$ schemes in detecting small shifts.

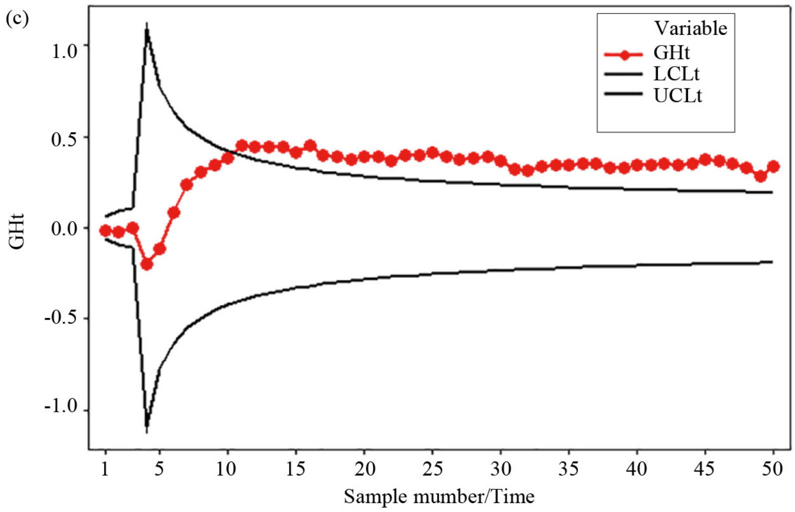
(a)



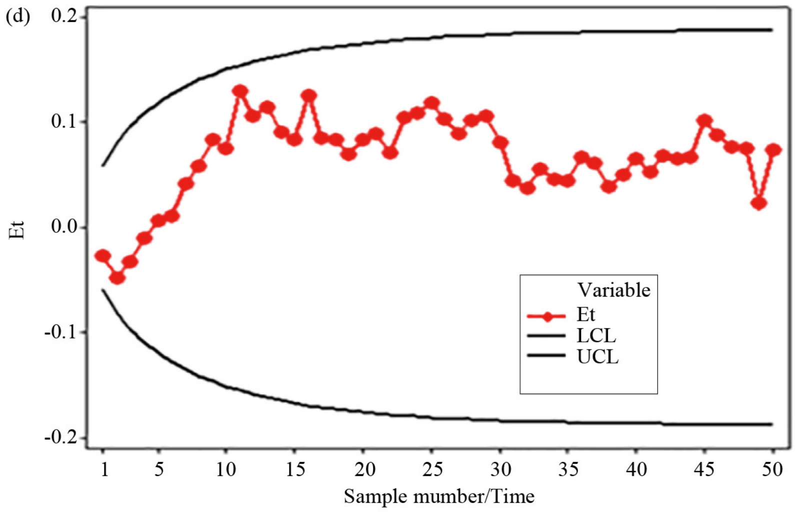
GHWMA $\overline{X}$ scheme when $r = 2$ and $\lambda_i = 0.05$



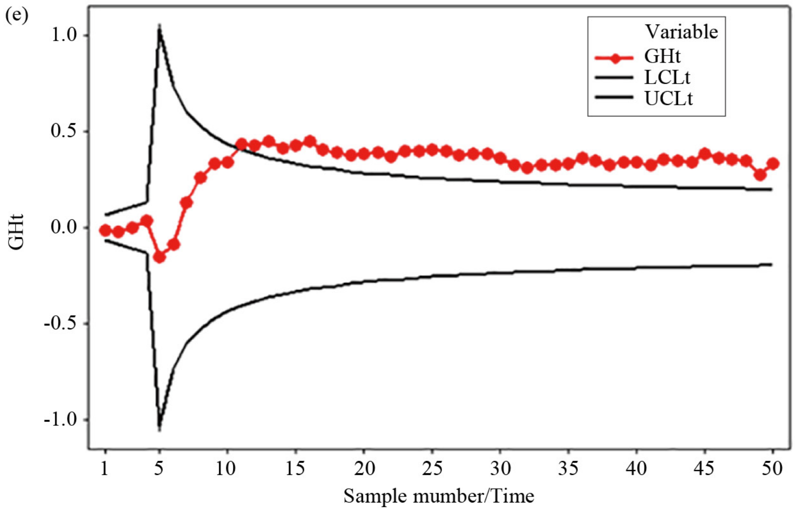
HWMA $\overline{X}$ scheme when $\lambda = 0.05$



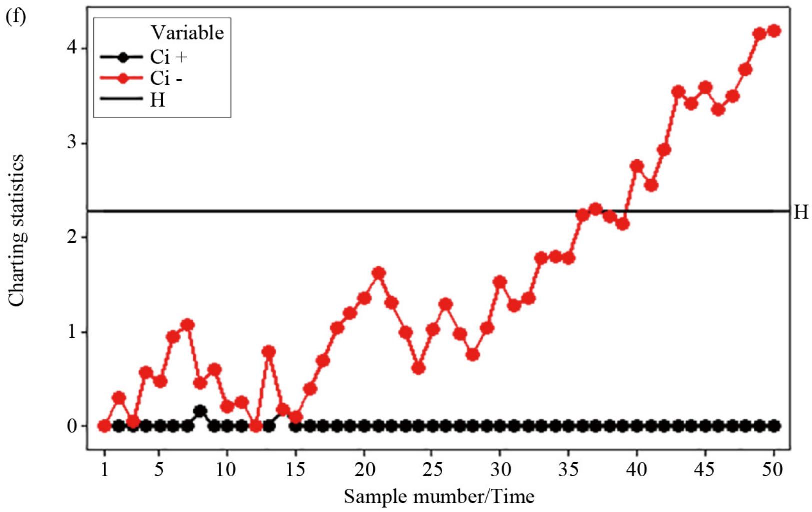
GHWMA $\overline{X}$ scheme when $r = 3$ and $\lambda_i = 0.05$



EWMA $\overline{X}$ scheme when $\lambda = 0.05$



GHWMA $\overline{X}$ scheme when r = 4 and λ_i = 0.05

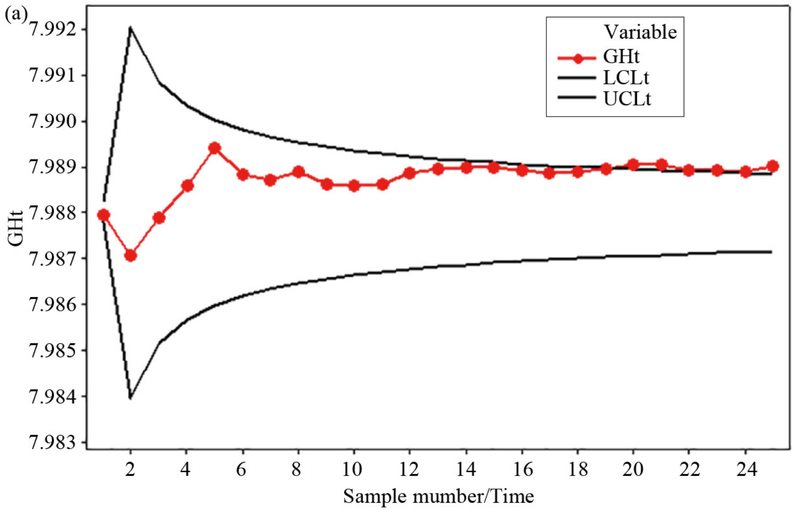


CUSUM $\overline{X}$ scheme when k = 0.5

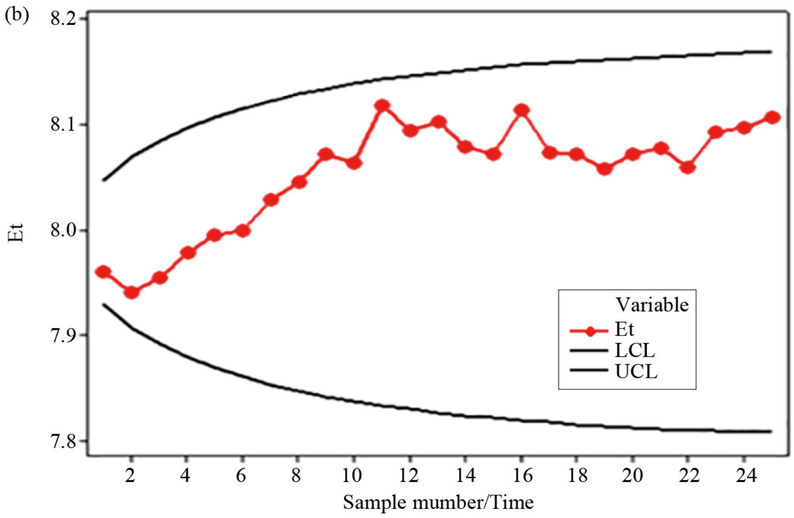
Figure 8. GHWMA, HWMA, EWMA and CUSUM $\overline{X}$ schemes using simulated data when $n = 5$, $\lambda_i = 0.05$, $\lambda = 0.05$, $k = 0.05$ and $ARL_0 = 500$

5.2 *Numerical example using real-world data*

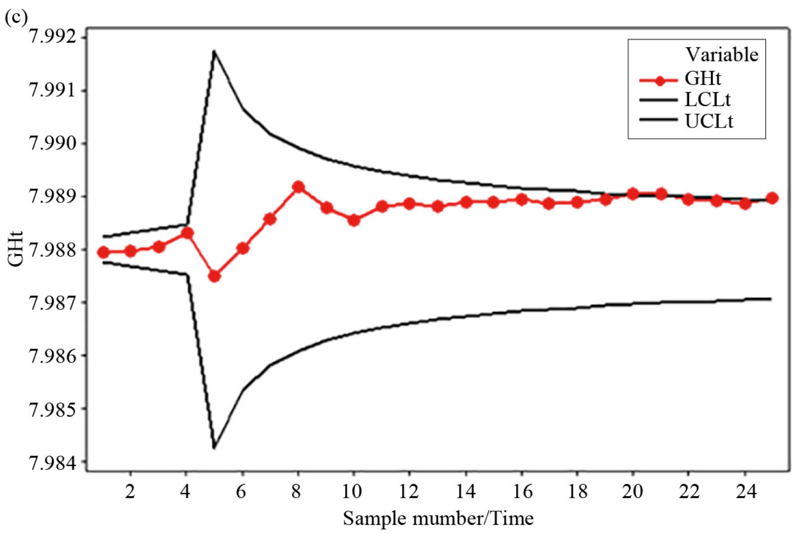
In this subsection, the design and implementation of the proposed GHWMA $\overline{X}$ scheme are illustrated using the dataset from Mahmoud and Aufy [25]. The data represent the shaft diameter and the target mean and standard deviation are given by 7.988 and 0.00363 millimetres (mm), respectively. To assess the production process, measurements of twenty-five samples have been taken, each consisting of five items from the final production stage. For a nominal $ARL_0 = 500$, $n = 5$ and $\lambda_i = 0.05$ for $r = 1$ and 4, it is found that the control limit constants are given by 2.6112 and 2.8594, respectively. Note that these L values are similar to those in Section 4.



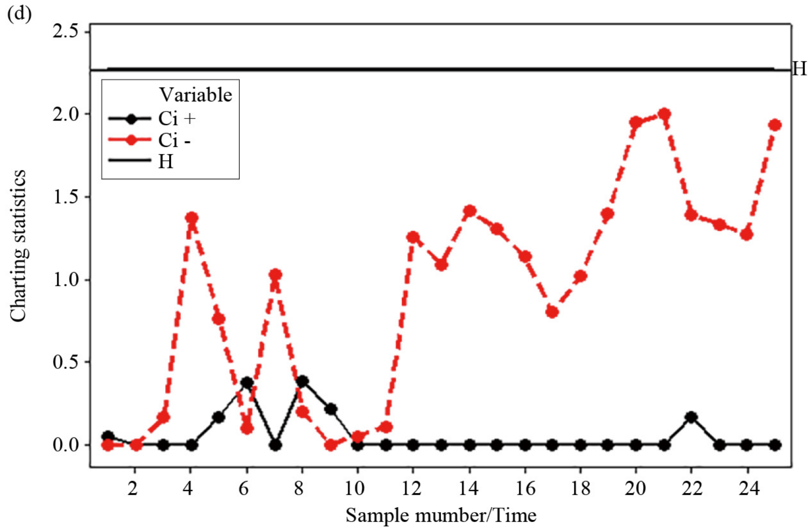
HWMA $\overline{X}$ scheme when r = 1 and λ = 0.05



EWMA $\overline{X}$ scheme when λ = 0.05



GHWMA $\overline{X}$ scheme when r = 4 and λ = 0.05



CUSUM $\overline{X}$ scheme when k = 0.5

Figure 9. GHWMA, EWMA and CUSUM $\overline{X}$ schemes for the shaft diameter of a production process when $n = 5$, $\lambda_i = 0.05$, $\lambda = 0.05$, $k = 0.05$ and $ARL_0 = 500$

The plots of the charting statistics for the GHWMA $\overline{X}$ scheme when $r = 1$ and 4 are shown in Figures 9a and 9c, respectively. The proposed GHWMA $\overline{X}$ scheme is also compared to the EWMA and CUSUM $\overline{X}$ schemes, as was the case in the previous example. The CUSUM $\overline{X}$ scheme is designed using $k = 0.5$ since it was reported by many researchers to be the optimal reference value [26]. The data set is standardized to yield the same control limit $H = 2.2662$. The EWMA $\overline{X}$ is designed using $\lambda = 0.05$. The plots of the EWMA and CUSUM $\overline{X}$ schemes are also shown in Figures 9b and 9d, respectively. In this example, Figure 9 shows that the GHWMA $\overline{X}$ scheme gives a signal on the 20$^{\text{th}}$ sample for both $r = 1$ and 2; see Figures 9a and 9c. However, both the EWMA and CUSUM $\overline{X}$ schemes do not give signals. This proves again the superiority of the proposed GHWMA $\overline{X}$ monitoring scheme.

6. Concluding remarks

This paper introduced the generalized HWMA $\overline{X}$ (GHWMA $\overline{X}$) scheme for monitoring the process mean when the process parameters are known. Unlike the HWMA $\overline{X}$ scheme, the GHWMA $\overline{X}$ scheme uses r ($r \geq 1$) smoothing parameters instead of one, as is the case with the HWMA $\overline{X}$ scheme. Thus, it was shown that the well-known HWMA $\overline{X}$ scheme is a special case of the GHWMA $\overline{X}$ scheme when $r = 1$. In this paper, the IC robustness and OOC performance of the GHWMA $\overline{X}$ were investigated using the characteristics of the run-length distribution and the CED criterion. It was found that generally, the GHWMA $\overline{X}$ is not IC robust; however, when compared with the well-known HWMA $\overline{X}$ (i.e., the GHWMA $\overline{X}$ with $r = 1$), it is more robust. In addition, the performance of the GHWMA $\overline{X}$ scheme increases as r increases. Compared to the EWMA and CUSUM $\overline{X}$ schemes, it was found that the GHWMA $\overline{X}$ scheme performs much better than the EWMA and CUSUM $\overline{X}$ schemes for small shifts in zero-state and also in terms of the overall performance. In terms of the CED values, the GHWMA $\overline{X}$ scheme is outperformed by the EWMA $\overline{X}$ scheme. The GHWMA $\overline{X}$ scheme with $r > 1$ has attractive properties as compared to the HWMA $\overline{X}$ (i.e., GHWMA $\overline{X}$ scheme with $r = 1$) and CUSUM $\overline{X}$ scheme.

It is recommended that practitioners consider this new monitoring scheme since it outperforms its counterparts in many situations, as shown in this paper. We further recommend investigating the performance of the proposed GHWMA $\overline{X}$ scheme using unknown process parameters with measurement errors, since perfect measurements with known process parameters are rare to find in real-life applications. Researchers should also construct other parametric and nonparametric schemes using the newly proposed GHWMA scheme statistic.

Conflict of interest

The authors declare no competing financial interest.

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Appendix A

Derivation of the mean and variance of the GHWMA $\overline{X}$ scheme statistic

Define

$$\overline{\overline{X}}_{t-r} = \frac{1}{t-r}\sum_{k=1}^{t-r}\overline{X}_k, \quad \bar{\lambda} = 1 - \sum_{k=1}^{r}\lambda_k, \quad \lambda_{r+1} = \lambda_{r+2} = \cdots = \lambda_t = \frac{\bar{\lambda}}{t-r}, \quad t > r.$$

And for $t > r$,

$$GH_t = \lambda_1\overline{X}_t + \lambda_2\overline{X}_{t-1} + \lambda_3\overline{X}_{t-2} + \cdots + \lambda_{r-1}\overline{X}_{t-r+2} + \lambda_r\overline{X}_{t-r+1} + \bar{\lambda}\overline{\overline{X}}_{t-r}$$

$$= \sum_{k=t-r+1}^{t}\lambda_{t-k+1}\overline{X}_k + \bar{\lambda}\overline{\overline{X}}_{t-r} = \sum_{k=1}^{t}\lambda_{t-k+1}\overline{X}_k = \sum_{k=1}^{t}\lambda_k\overline{X}_{t-k+1}.$$

It is to be noted that

$$\overline{X}_k = \mu_0, \text{ for } k \leq 0.$$

And for $t \leq r$,

$$GH_t = \sum_{k=1}^{t}\lambda_k\overline{X}_{t-k+1} + \left(\sum_{k=t+1}^{r}\lambda_k + \bar{\lambda}\right)\mu_0.$$

A.1 Derivation of the expectation of GH_t

For $t \leq r$, we have

$$E(GH_t) = E\left(\sum_{k=1}^{t}\lambda_k\overline{X}_k + \left(\sum_{k=t+1}^{r}\lambda_k + \bar{\lambda}\right)\mu_0\right) = \sum_{k=1}^{t}\lambda_k E(\overline{X}_k) + \left(\sum_{k=t+1}^{r}\lambda_k + \bar{\lambda}\right)\mu_0 = \mu_0.$$

And for $t > r$,

$$E(GH_t) = E\left(\sum_{k=1}^{t}\lambda_k\overline{X}_{t-k+1}\right) = \sum_{k=1}^{t}\lambda_k E(\overline{X}_{t-k+1}) = \left(\sum_{k=1}^{t}\lambda_k\right)\mu_0 = \mu_0$$

A.2 Derivation of the variance of GH_t

Here, it is to be noted that $\overline{X}_k,\ k = 1,\ 2,\ \ldots,\ t$ are independently distributed.

For $t \leq r$, we have

$$Var(GH_t) = Var\left(\sum_{k=1}^{t} \lambda_k \overline{X}_k + \left(\sum_{k=t+1}^{r} \lambda_k + \overline{\lambda}\right)\right) = Var\left(\sum_{k=1}^{t} \lambda_k \overline{X}_k\right) = \left(\sum_{k=1}^{t} \lambda_k^2\right) \frac{\sigma_0^2}{n}.$$

For $t > r$, then

$$Var(GH_t) = \sum_{i=1}^{r} \lambda_i^2 Var(\overline{X}_{t-i+1}) + \frac{\bar{\lambda}^2}{(t-r)^2} \sum_{k=1}^{t-r} Var(\overline{X}_k) + 2Cov\left(\sum_{i=1}^{r} \lambda_i \overline{X}_{t-i+1},\ \frac{\bar{\lambda}}{t-r} \sum_{k=1}^{t-r} \overline{X}_k\right)$$

$$= \sum_{i=1}^{r} \lambda_i^2 \frac{\sigma_0^2}{n} + \frac{(t-r)\bar{\lambda}^2}{(t-r)^2} \frac{\sigma_0^2}{n} + 2\lambda_1 \frac{\bar{\lambda}}{t-r} Cov(\overline{X}_t,\ \overline{X}_1) + 2\lambda_2 \frac{\bar{\lambda}}{t-r} Cov(\overline{X}_{t-1},\ \overline{X}_1) + 2\lambda_3 \frac{\bar{\lambda}}{t-r} Cov(\overline{X}_{t-2},\ \overline{X}_1) + \ldots$$

$$+ 2\lambda_r \frac{\bar{\lambda}}{t-r} Cov(\overline{X}_{t-r+1},\ \overline{X}_1) + 2\lambda_1 \frac{\bar{\lambda}}{t-r} Cov(\overline{X}_t,\ \overline{X}_2) + 2\lambda_2 \frac{\bar{\lambda}}{t-r} Cov(\overline{X}_{t-1},\ \overline{X}_2) + 2\lambda_3 \frac{\bar{\lambda}}{t-r} Cov(\overline{X}_{t-2},\ \overline{X}_2)$$

$$+ \cdots + 2\lambda_r \frac{\bar{\lambda}}{t-r} Cov(\overline{X}_{t-r+1},\ \overline{X}_2) + \cdots + 2\lambda_1 \frac{\bar{\lambda}}{t-r} Cov(\overline{X}_t,\ \overline{X}_3) + 2\lambda_2 \frac{\bar{\lambda}}{t-r} Cov(\overline{X}_{t-1},\ \overline{X}_3)$$

$$+ 2\lambda_3 \frac{\bar{\lambda}}{t-r} Cov(\overline{X}_{t-2},\ \overline{X}_3) + \cdots + 2\lambda_r \frac{\bar{\lambda}}{t-r} Cov(\overline{X}_{t-r+1},\ \overline{X}_3) + \cdots + 2\lambda_1 \frac{\bar{\lambda}}{t-r} Cov(\overline{X}_t,\ \overline{X}_{t-r})$$

$$+ 2\lambda_2 \frac{\bar{\lambda}}{t-r} Cov(\overline{X}_{t-1},\ \overline{X}_{t-r}) + 2\lambda_3 \frac{\bar{\lambda}}{t-r} Cov(\overline{X}_{t-2},\ \overline{X}_{t-r}) + \cdots + 2\lambda_r \frac{\bar{\lambda}}{t-r} Cov(\overline{X}_{t-r+1},\ \overline{X}_{t-r}).$$

We know that the $\overline{X}_i$'s are i.i.d. then,

$$Cov(\overline{X}_i,\ \overline{X}_k) = 0 \quad \forall i \neq k.$$

Thus,

$$Var(GH_t) = \left(\sum_{i=1}^{r} \lambda_i^2 + \frac{\bar{\lambda}^2}{t-r}\right)$$

Hence,

$$Var(GH_t) = \left(\sum_{i=1}^{r} \lambda_i^2 + \frac{(1 - \sum_{i=1}^{r} \lambda_i)^2}{t-r}\right) \frac{\sigma_0^2}{n}$$

Appendix B

This appendix presents the pseudo codes of the proposed GHWMA and HWMA $\overline{X}$ schemes to conceptualize the computational complexity of the two charts.

B.1 Pseudo code of the HWMA $\overline{X}$ scheme

1 1.1 Specify the number of simulations (s), test sample size (n), number of smoothing parameters (r), smoothing parameter (λ), etc.
1.2 Initialize the run-length vector ($rlvec$).
1.3 Specify the parameters of the underlying process distribution, μ_0 and σ_0, and the shift in the location parameter (δ), where $0 \leq \delta < \infty$. For the in-control case, $\mu_0 = 0$ and $\sigma_0 = 1$, and for the out-of-control case, $\mu_1 = \delta$ and $\sigma_0 = 1$, where $\delta \neq 0$.

2 Set the control limit constant L to some value satisfying a prespecified value τ' (say, $\tau' = 500$).

3 3.1 For $k = 1$ to s
3.2 Initialize $count$, $signal$ and $xvec$: $count = 0$, $signal = 1$ and $Xvec = \{\}$.
3.3 For $t = 1$ to ∞ until ($signal = 1$)
3.4 Randomly generate a test sample from a distribution such as $X \sim N(\mu_1,\ \sigma_0^2)$; i.e., $X \sim N(\delta,\ 1)$
$count = count + 1$
3.5 # Calculate the control limits:
3.6 if $t = 1$, $UCL_t/LCL_t = \mu_0 \pm L(\lambda^2\sigma_0^2/n)^{1/2}$
3.7 if $t > 1$, $UCL_t/LCL_t = \mu_0 \pm L\left((\lambda^2 + (1-\lambda)^2/(t-1))\sigma_0^2/n\right)^{1/2}$
3.8 # Calculate the charting statistic:
3.9 $\overline{X}_t = \sum_{j=1}^{n} X_j$
3.10 $Xvec[t,] = \overline{X}_t$
3.11 $H_t = \lambda\overline{X}_t + (1-\lambda)\sum_{v=1}^{t-1}\overline{X}_v/(t-1)$
3.12 # Compare the charting statistic to the control limits:
3.13 if $H_t \in (LCL_t,\ UCL_t)$ then $signal = 0$
3.14 else $signal = 1$
3.15 $rlvec[k,\ ] = count$
3.16 end loop t
3.17 end loop k

4 Find the empirical run-length distribution represented by the vector $rlvec$.

5 Use $rlvec$ to compute the characteristics of the run-length distribution (such as the mean, standard deviation and percentiles of run-length distribution).

6 If the attained $ICARL \notin [\tau' - \varepsilon,\ \tau' + \varepsilon]$ then return to Step 2 else go to Step 7.
Note: $\varepsilon = \tau'/100$ was chosen so that $ICARL$ is close to τ'.

7 Record the characteristics of the run-length distribution and all other important information, then stop.

B.2 Pseudo code of the proposed GHWMA $\overline{X}$ scheme

1 1.1 Specify the number of simulations (s), test sample size (n), number of smoothing. parameters (r), smoothing parameters (λ_i), etc.
1.2 Initialize the run-length vector ($rlvec$).
1.3 Specify the parameters of the underlying process distribution, μ_0 and σ_0, and the shift in the location parameter δ, where $0 \leq \delta < \infty$. For the in-control case $\mu_0 = 0$, and $\sigma_0 = 1$, and for the out-of-control case, $\mu_1 = \delta$, and $\sigma_0 = 1$ where $\delta \neq 0$.

2 Set the control limit constant L to some value satisfying a prespecified value τ' (say, $\tau' = 500$).

3 3.1 For $k = 1$ to s:

3.2 Initialize `count`, `signal` and `xvec`: `count` = 0, `signal` = 1 and `Xvec` = {}.

3.3 For $t = 1$ to ∞ `until` (`signal` = 1)

3.4 Randomly generate a test sample from a distribution such as $X \sim \mathcal{N}(\mu_1,\ \sigma_0^2)$; i.e., $X \sim \mathcal{N}(\delta,\ 1)$
`count` = `count` + 1

3.5 # Calculate the control limits:

3.6 if $t = 1$, $UCL_t/LCL_t = \mu_0 \pm L\left(\lambda_1^2 \frac{\sigma_0^2}{n}\right)^{1/2}$

3.7 if $t = 2$, $UCL_t/LCL_t = \mu_0 \pm L\left((\lambda_1^2 + \lambda_2^2)\frac{\sigma_0^2}{n}\right)^{1/2}$

3.8 ...

3.9 if $t = r$, $UCL_t/LCL_t = \mu_0 \pm L\left((\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_r^2)\frac{\sigma_0^2}{n}\right)^{1/2}$

3.10 if $t > r$, $UCL_t/LCL_t = \mu_0 \pm L\left(\left(\sum_{i=1}^{r} \lambda_i^2 + (1 - \sum_{i=1}^{r} \lambda_i)^2/(t-r)\right)\frac{\sigma_0^2}{n}\right)^{1/2}$

3.11 # Calculate the charting statistic:

3.12 $\overline{X}_t = \frac{1}{n}\sum_{j=1}^{n} \frac{X_j}{n}$,

3.13 $\texttt{Xvec}[t,] = \overline{X}_t$

3.14 if $t = 1$, $GH_1 = \lambda_1\overline{X}_1 + \lambda_2\mu_0 + \cdots + \lambda_r\mu_0 + \bar{\lambda}\mu_0$

3.15 if $t = 2$, $GH_2 = \lambda_1\overline{X}_1 + \lambda_2\texttt{Xvec}[t-1,] + \lambda_3\mu_0 + \cdots + \lambda_r\mu_0 + \bar{\lambda}\mu_0$

3.16 ...

3.17 if $t = r$, $GH_r = \lambda_1\overline{X}_1 + \lambda_2\texttt{Xvec}[t-1,] + \lambda_3\texttt{Xvec}[t-2,] + \cdots + \lambda_r\texttt{Xvec}[t-r+1,] + \bar{\lambda}\mu_0$

3.18 if $t > r$, $GH_t = \lambda_1\overline{X}_1 + \lambda_2\texttt{Xvec}[t-1,] + \lambda_3\texttt{Xvec}[t-2,] + \cdots + \lambda_r\texttt{Xvec}[t-r+1,]$
$+\bar{\lambda}\sum_{v=1}^{t-r} \frac{\overline{X}_v}{t-r}$.

3.19 # Compare the charting statistic to the control limits:

3.20 if $GH_t \in (LCL_t,\ UCL_t)$ then `signal` = 0

3.21 else `signal` = 1

3.22 `rlvec` [k,] = `count`

3.23 end loop t

3.24 end loop k

4 Find the empirical run-length distribution represented by the vector *rlvec*.

5 Use *rlvec* to compute the characteristics of the run-length distribution (such as the mean, standard deviation and percentiles of run-length distribution).

6 If the attained $ICARL \notin [\tau' - \varepsilon,\ \tau' + \varepsilon]$ then return to Step 2 else go to Step 7.
Note that ε was chosen to be equal to $\tau'/100$ so that the $ICARL$ is much closer to τ'.

7 Record the characteristics of the run-length distribution and all other important information, then stop.