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Research Article

An Averaging Limit Theorem for Impulsive Delay Stochastic Fractional Differential Equations

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Abstract: In this article, we present an averaging limit theorem for impulsive delay Caputo fractional stochastic differential equations. In contrast to the present literature, a new technique is adopted to overcome the difficulties hired by the impulsive term based on impulsive-type Grönwall inequality. As a result, it is proved that the solution of the non-impulsive averaged delay Caputo fractional stochastic differential equations converges to that of the standard impulsive delay Caputo fractional stochastic differential equations in L^q-sense. Finally, an example is constructed to enhance the analytical result.

Keywords: Averaging limit theorem, Fractional Stochastic Differential Equations (FSDEs) with delay, impulsive effects, L^q approximation, Caputo fractional derivative

MSC: 34C29, 34A08, 60H10, 60J65

1. Introduction

The aim of this article is to investigate the fractional averaging method for the following Impulsive Delay Caputo Fractional Stochastic Differential Equations (IDCFSDEs):

$$\begin{cases} {}^{C}\mathscr{D}^{\nu}_{0^+}\Re(\rho) = \xi(\rho, \Re(\rho), \Re(\rho-\iota)) + \zeta(\rho, \Re(\rho), \Re(\rho-\iota))\frac{dw(\rho)}{d\rho}, \quad \rho \in [0, b], \quad \rho \neq \rho_k, \\ \Re(\rho_k^+) = \Re(\rho_k^-) + I_k(\Re(\rho_k^-)), \quad k = 1, 2, ..., m, \\ \Re(\rho) = \vartheta(\rho), \quad \rho \in [-\iota, 0], \end{cases} \tag{1}$$

where ${}^{C}\mathscr{D}^{\nu}_{0^+}$ is the Caputo fractional derivative of order $\nu \in (\frac{1}{2}, 1)$, $\xi : [0, b] \times \mathbb{R}^l \times \mathbb{R}^l \longrightarrow \mathbb{R}^l$, $\zeta : [0, b] \times \mathbb{R}^l \times \mathbb{R}^l \longrightarrow \mathbb{R}^{l\times d}$ are continuous functions. $\mathbb{R}^l$ is the Euclidean space with norm $|.|$ and $w(.)$ refers to the d-dimensional Brownian motion

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defined on a complete probability space $(\Omega, \Im, \mathbb{P})$. Let $0 = \rho_0 < \rho_1 < \rho_2 < \ldots < \rho_m < \rho_{m+1} = b$ and I_k: $\mathbb{R}^l \longrightarrow \mathbb{R}^l$ are the impulsive functions which characterize the jump of solutions at impulsive points ρ_k. $\Re(\rho_k^+)$ and $\Re(\rho_k^-)$ represent the right and left limits of $\Re(\rho)$ at time ρ_k, respectively. $\mathbb{E}$ denotes the mathematical expectation and $\vartheta(\rho) \in C([-\imath, 0]; \mathbb{R}^l)$, where $\vartheta_0 = \vartheta(0)$.

The theory of fractional differential equations has been emerged as a beneficial tool in modeling many practical, signal, physical, medical, biological, control theory, engineering and image processing problems, etc. [1–8]. Particularly, Fractional order differential systems are generalizations of classical integer order systems and are widely used to describe phenomena in fluid dynamics, finance and physics [9]. Bedi et al. [10] established the existence and approximate controllability of Hilfer fractional evolution equations with almost sectorial operators. Bhat et al. [11] introduced a comparative analysis of nonlinear Urysohn functional integral equations via Nyström method. Khalil et al. [12] provided a new definition of fractional derivative. Kumar and Pandey [13] proved the existence of mild solution of Atangan-Baleanu fractional differential equations with delay. For more details on this field, one can refer to relevant monographs [14, 15].

Noise or random fluctuations are unavoidable in nature as well as in man-made systems, therefore it seems important to consider stochastic fractional systems rather than deterministic fractional systems. The fractional differential systems which involve noise term in the mathematical model of a given phenomenon are called Fractional Stochastic Differential Equations (FSDEs). In recent years, the existence and uniqueness of solutions to FSDEs in both finite and infinite dimensions have attracted more interest in different fields due to the applications in describing various phenomenon in population dynamics, physics, electrical engineering, ecology, medicine, biology, and other fields of science and engineering. Recently, the existence and uniqueness of solutions for different FSDEs models are investigated. For example, Abouagwa and Li [16] introduced the approximation properties for solutions to Itô-Doob FSDEs with non-Lipschitz coefficients; Abouagwa et al. [17] provided the existence, uniqueness and averaging principle for mixed neutral Caputo fractional stochastic evolution equations with infinite delay; Makhlouf et al. [18] introduced the existence, uniqueness and averaging principle for Hadamard Itô-Doob stochastic delay fractional integral equations; Tian and Luo [19] derived the existence and finite-time stability results for impulsive Caputo-type FSDEs with time delays; Zou et al. [20] proved the existence and averaging principle for FSDEs with impulses.

The theory of stochastic averaging principle is important technique for studying the applications of SDEs in many interesting areas. It seeks to obtain an equivalent averaged stochastic system to replace the original one, which reduces the complexity and numerical computations in practical applications. The first systematic study of mean-square averaging principle was reported by Khasminskii [21], then numerous studies of stochastic averaging were obtained. Abouagwa and Li [22] introduced the averaging principle for neutral SDEs with variable delay driven by G-Brownian motion. Xu et al. [23, 24] studied the averaging principle for stochastic dynamical systems driven by fractional Brownian motion under Lipschitz and non-Lipschitz conditions. Recently, researchers extended the method of averaging principle to FSDEs. Ahmed and Zhu [25] derived the averaging principle for Hilfer FSDEs with delay and Poisson jumps; Luo et al. [26] established the averaging principle for FSDEs with time-delays; Luo et al. [27] provided a novel averaging principle for Hilfer-type fractional system under non-Lipschitz condition. For more details on averaging principle for different types of FSDEs with different perturbations, we refer to [28–34] and their cited references.

Recently, impulsive effects play a crucial role in practical situations, which cause abrupt and short changes to the system state. Impulsive effects are unavoidable and even not ignorable sometimes. Therefore, researchers started to describe the existence and averaging method for impulsive SDEs and FSDEs more accurately. For example, Tian and Luo [19] derived the existence and finite-time stability results for impulsive Caputo FSDEs with delays; Abouagwa et al. [35] proved the existence and uniqueness for a class of mixed neutral Caputo FSDEs with impulses and variable delay; Abouagwa et al. [36] studied the averaging principle for multivalued impulsive SDEs driven by G-Brownian noise; Khalaf et al. [37] proved the periodic averaging method for non-Lipschitz impulsive stochastic dynamical system driven by fractional Brownian motion; Zou et al. [20] introduced the existence and averaging method of FSDEs with impulses. For more developments on averaging method for different types of FSDEs with impulses, we mention [38, 39] and their references.

Due to the non-local property of the Caputo fractional derivatives in time, Caputo fractional differential systems are an important model in various areas of science and engineering. Referring to the definition of Caputo fractional derivatives, the memory effect or nonlocal property is presented by a convolution integral with a power-law memory kernel, which makes the Caputo fractional differential systems an excellent tool in complex systems. Inspired by the above discussions and due to the lack of existing results on averaging principle for impulsive Caputo-type FSDEs with delay, we will extend the fractional stochastic averaging method to IDCFSDEs (1). The advantages and major contributions of this article are highlighted as following.

• The convergence in q-th moment between the solution of the non-impulsive DCFSDEs and that of the standard IDCFSDEs will be proved, which is more technical and gives the parameter q a greater degree of freedom to possesses a good robustness.

• Compared with the existing results in [20, 36–39], a new technique to overcome the difficulties hired by the impulsive term $\mathbb{E}(\sup_{0\leqslant\rho\leqslant\nu}|\sum_{k=1}^{n} I_k(\Re_\varepsilon(\rho_k^-)) - \frac{1}{\Gamma(\nu)}\int_0^\rho(\rho-\tau)^{\nu-1}\bar{I}(\Re_\varepsilon^*(\tau))d\tau|^q)$ is applied, which enriches the relevant literature on the averaging theory for other types of FSDEs with impulsive. It should be mentioned that the limitation when seeking the boundedness of expectation in [20, 36–39] due to the impulsive term has been broken by using Lemma 1.

• Moreover, the results are still new even when the parameter $q\equiv 2$ and generalize theorem 3.1 in [32], which is a particular case of our results when $q\equiv 2$, $\Re(\rho-\iota)\equiv 0$ and $I_k(0)\equiv 0$ $(k=1, 2, ..., m)$.

The remaining of this paper proceeds as follows. We introduce some preliminary definitions, assumptions and lemma in Section 2. Section 3 is aimed to derive the main fractional averaging limit theorem to IDCFSDEs. An illustrative example is carried out to enhance our theoretical findings in Section 4 and the paper is concluded in Section 5.

2. Preparations

In this section, we introduce some preliminary definitions, assumptions and lemma needed to establish our main results

Definition 1 ([15]) For any $\nu>0$ and a function $\gamma(\rho)$: $[0, \infty)\longrightarrow\mathbb{R}$, the Riemann-Liouville fractional integral operator is defined as

$$I_{0^+}^{\nu}\gamma(\rho)=\frac{1}{\Gamma(\nu)}\int_0^\rho(\rho-\iota)^{\nu-1}\gamma(\iota)d\iota,\quad \rho>0, \tag{2}$$

where $\Gamma(.)$ is the Euler's Gamma function defined by $\Gamma(\nu)=\int_0^\infty s^{\nu-1}e^{-s}ds$.

Definition 2 ([15]) For any $\nu>0$, the Caputo fractional derivative for a function $\gamma(\rho)$: $[0, \infty)\longrightarrow\mathbb{R}$ is given as

$${}^{C}\mathscr{D}_{0^+}^{\nu}\gamma(\rho)=\frac{1}{\Gamma(\jmath-\nu)}\int_0^\rho(\rho-\iota)^{\jmath-\nu-1}\gamma^{(\jmath)}(\iota)d\iota, \tag{3}$$

where $\jmath-1<\nu<\jmath, \jmath\in\mathbb{N}$. In particular, for $0<\nu<1$, we have

$$I_{0^+}^{\nu}\,{}^{C}\mathscr{D}_{0^+}^{\nu}\gamma(\rho)=\gamma(\rho)-\gamma(0). \tag{4}$$

To prove the averaging limit theorem for Eq. (1), the following assumptions are needed.

Assumption 1 Assume for all $\rho\in[0, b]$ and for any $\phi_1, \phi_2, \varphi_1, \varphi_2\in\mathbb{R}^l$,

$$|\xi(\rho, \phi_1, \varphi_1) - \xi(\rho, \phi_2, \varphi_2)| \vee |\zeta(\rho, \phi_1, \varphi_1) - \zeta(\rho, \phi_2, \varphi_2)| \leqslant C_1(|\phi_1 - \phi_2| + |\varphi_1 - \varphi_2|), \tag{5}$$

and

$$|\xi(\rho, \phi, \varphi)| \vee |\zeta(\rho, \phi, \varphi)| \leqslant C_2(1 + |\phi| + |\varphi|), \tag{6}$$

where $C_1 > 0$ and $C_2 > 0$ are two constants.

Assumption 2 Assume for any $\phi, \varphi \in \mathbb{R}^l$, there exist $c_k \in \mathbb{R}^+ (k = 1, 2, ...)$ such that

$$|I_k(\varphi) - I_k(\phi)| \leqslant c_k|\varphi - \phi|, \quad |I_k(\varphi)| \leqslant \widehat{n},$$

where $\widehat{n} \in \mathbb{R}^+$ is a constant.

Remark 1 Using the method given in [19], the existence and uniqueness theorem of solution for Eq. (1) can be derived in L^q-sense by adopting Assumptions 1 and 2, which is a generalization of theorem 3.1 in [19] (when $q \equiv 2$).

We close this section by borrowing the following impulsive-type Grönwall inequality from [19], which is important to demonstrate the main results.

Lemma 1 ([19]) Assume for any positive and piecewise continuous function $\eta(\rho)$, we have

$$\eta(\rho) \leqslant \beta + \int_{\rho_0}^{\rho} \mathscr{V}(\iota)\eta(\iota)d\iota + \sum_{\rho_0 < \iota_i < \rho} \sigma_i \eta(\iota_i), \quad \rho \geqslant \rho_0, \tag{7}$$

where $\beta \geqslant 0$, $\sigma \geqslant 0$, $\mathscr{V}(\iota) > 0$, and ι_i denote the impulsive points of $\eta(\rho)$. Then the following inequality is true

$$\eta(\rho) \leqslant \beta \prod_{\rho_0 < \iota_i < \rho} (1 + \sigma_i) \exp\left[\int_{\rho_0}^{\rho} \mathscr{V}(\iota)d\iota\right]. \tag{8}$$

3. Averaging limit theorem

The objective of the this section is to present the main averaging limit theorem for IDCFSDEs. Assume the standard form of Eq. (1) is given by

$$\begin{cases} {}^C\mathscr{D}_{0+}^{\nu}\Re_\varepsilon(\rho) = \varepsilon\xi(\rho, \Re_\varepsilon(\rho), \Re_\varepsilon(\rho - \iota)) + \sqrt{\varepsilon}\zeta(\rho, \Re_\varepsilon(\rho), \Re_\varepsilon(\rho - \iota))\frac{dw(\rho)}{d\rho}, \quad t \in [0, \rho], \ \rho \neq \rho_k, \\ \Re_\varepsilon(\rho_k^+) = \Re_\varepsilon(\rho_k^-) + \varepsilon I_k(\Re_\varepsilon(\rho_k^-)), \quad k = 1, 2, ..., m, \\ \Re_\varepsilon(\rho) = \vartheta(\rho), \quad \rho \in [-\iota, 0], \end{cases} \tag{9}$$

where $\varepsilon \in (0, \varepsilon^*]$ is a small positive parameter with a fixed constant ε^*. Moreover, the coefficients ξ and ζ satisfy the Assumptions 1 and 2. According to [19] and the properties of fractional calculus, the equivalent integral form of Eq. (9) can be written as:

$$\Re_\varepsilon(\rho) = \begin{cases} \vartheta(\rho), & \rho \in [-\iota, 0], \\ \vartheta_0 + \frac{\varepsilon}{\Gamma(\nu)} \int_0^\rho (\rho-\tau)^{\nu-1} \xi(\tau, \Re_\varepsilon(\tau), \Re_\varepsilon(\tau-\iota))d\tau \\ \quad + \frac{\sqrt{\varepsilon}}{\Gamma(\nu)} \int_0^\rho (\rho-\tau)^{\nu-1} \zeta(\tau, \Re_\varepsilon(\tau), \Re_\varepsilon(\tau-\iota))dw(\tau), & \rho \in (0, \rho_1], \\ \vartheta_0 + \frac{\varepsilon}{\Gamma(\nu)} \int_0^\rho (\rho-\tau)^{\nu-1} \xi(\tau, \Re_\varepsilon(\tau), \Re_\varepsilon(\tau-\iota))d\tau \\ \quad + \frac{\sqrt{\varepsilon}}{\Gamma(\nu)} \int_0^\rho (\rho-\tau)^{\nu-1} \zeta(\tau, \Re_\varepsilon(\tau), \Re_\varepsilon(\tau-\iota))dw(\tau) \\ \quad + \varepsilon \sum_{k=1}^n I_k(\Re_\varepsilon(\rho_k^-)), & \rho \in (\rho_n, \rho_{n+1}],\ n = 1, 2, .., m. \end{cases} \tag{10}$$

Next, we assume the following averaged form which corresponds to the standard Eq. (10):

$$\Re^*_\varepsilon(\rho) = \begin{cases} \vartheta(\rho), & \rho \in [-\iota, 0], \\ \vartheta_0 + \frac{\varepsilon}{\Gamma(\nu)} \int_0^\rho (\rho-\tau)^{\nu-1} \bar{\xi}(\Re^*_\varepsilon(\tau), \Re^*_\varepsilon(\tau-\iota))d\tau \\ \quad + \frac{\sqrt{\varepsilon}}{\Gamma(\nu)} \int_0^\rho (\rho-\tau)^{\nu-1} \bar{\zeta}(\Re^*_\varepsilon(\tau), \Re^*_\varepsilon(\tau-\iota))dw(\tau), & \rho \in (0, \rho_1], \\ \vartheta_0 + \frac{\varepsilon}{\Gamma(\nu)} \int_0^\rho (\rho-\tau)^{\nu-1} \bar{\xi}(\Re^*_\varepsilon(\tau), \Re^*_\varepsilon(\tau-\iota))d\tau \\ \quad + \frac{\sqrt{\varepsilon}}{\Gamma(\nu)} \int_0^\rho (\rho-\tau)^{\nu-1} \bar{\zeta}(\Re^*_\varepsilon(\tau), \Re^*_\varepsilon(\tau-\iota))dw(\tau) \\ \quad + \frac{\varepsilon}{\Gamma(\nu)} \int_0^\rho (\rho-\tau)^{\nu-1} \bar{I}(\Re^*_\varepsilon(\tau))d\tau, & \rho \in (\rho_n, \rho_{n+1}],\ n = 1, 2, .., m, \end{cases} \tag{11}$$

where $\bar{\xi}\colon \mathbb{R}^l \longrightarrow \mathbb{R}^l$, $\bar{\zeta}\colon \mathbb{R}^l \longrightarrow \mathbb{R}^{l\times d}$ and $\bar{I}\colon \mathbb{R}^l \longrightarrow \mathbb{R}^l$ are measurable functions, which satisfy Assumptions 1, 2 and the following averaging assumption:

Assumption 3 For all $b_1 \in [0, b]$, ϕ, $\varphi \in \mathbb{R}^l$ and $q \geqslant 2$, there exists a positive bounded function $\lambda(b_1)$, such that (i)

$$\frac{1}{b_1}\int_0^{b_1} |\xi(\rho, \phi, \psi) - \bar{\xi}(\phi, \psi)|^q d\rho \vee \frac{1}{b_1}\int_0^{b_1} |\zeta(\rho, \phi, \psi) - \bar{\zeta}(\phi, \psi)|^q d\rho \leqslant \lambda(b_1)(1+|\phi|^q+|\psi|^q),$$

(ii)

$$\bar{I}(\phi) \leqslant \frac{1}{b_1}\sum_{k=1}^m I_k(\phi),$$

where $\lim_{b_1 \longrightarrow \infty} \lambda(b_1) = 0$.

In order to study the L^q-convergence of the solution process $\Re_\varepsilon$ of Eq. (10) towards the solution process $\Re^*_\varepsilon$ of the averaged system (11) as $\varepsilon \longrightarrow 0$, we prove the following lemma.

Lemma 2 Under Assumptions 1 and 3, for $\forall\, b_1 \in [0, b]$, the function $\bar{\zeta}$ satisfies the following growth condition

$$|\bar{\zeta}(\phi,\, \psi)|^q \leqslant C_3(1+|\phi|+|\psi|)^q, \tag{12}$$

where $C_3 = 2^{q-1}\lambda(b_1)+2^{q-1}C_2^q$.

Proof. Applying Jensen's inequality and Assumptions 1 and 3, we have

$$\begin{aligned} |\bar{\zeta}(\phi,\, \psi)|^q &\leqslant \frac{2^{q-1}}{b_1}\int_0^{b_1}|\bar{\zeta}(\phi,\, \psi)-\zeta(\rho,\, \phi,\, \psi)|^q d\rho + \frac{2^{q-1}}{b_1}\int_0^{b_1}|\zeta(\rho,\, \phi,\, \psi)|^q d\rho \\ &\leqslant 2^{q-1}\lambda(b_1)(1+|\phi|^q+|\psi|^q)+2^{q-1}C_2^q(1+|\phi|+|\psi|)^q \\ &\leqslant (2^{q-1}\lambda(b_1)+2^{q-1}C_2^q)(1+|\phi|+|\psi|)^q. \end{aligned} \tag{13}$$

□

Theorem 1 Assume Assumptions 1-3 are true. Then, for arbitrary small parameter $\theta > 0$ and $q \in [2,\, 2(1-\nu)^{-1})$, $\exists\, \ell > 0$, $\varepsilon_1 \in (0,\, \varepsilon^*]$ and $\alpha \in (0,\, 1)$ such that

$$\mathbb{E}\left(\sup_{\rho\in[-\iota,\, \ell\varepsilon^{-\alpha}]}|\Re_\varepsilon(\rho)-\Re_\varepsilon^*(\rho)|^q\right) \leqslant \theta, \tag{14}$$

for $\forall\varepsilon \in (0,\, \varepsilon_1]$.

Proof. On account of Eqs. (10)-(11), it follows for $\forall v \in [0,\, b]$ that

$$\begin{aligned} &\mathbb{E}\left(\sup_{0\leqslant\rho\leqslant v}|\Re_\varepsilon(\rho)-\Re_\varepsilon^*(\rho)|^q\right) \\ &\leqslant \frac{3^{q-1}\varepsilon^q}{(\Gamma(\nu))^q}\mathbb{E}\left(\sup_{0\leqslant\rho\leqslant v}\left|\int_0^\rho(\rho-\tau)^{\nu-1}[\xi(\tau,\, \Re_\varepsilon(\tau),\, \Re_\varepsilon(\tau-\iota))-\bar{\xi}(\Re_\varepsilon^*(\tau),\, \Re_\varepsilon^*(\tau-\iota))]d\tau\right|^q\right) \\ &\quad+\frac{3^{q-1}\varepsilon^{\frac{q}{2}}}{(\Gamma(\nu))^q}\mathbb{E}\left(\sup_{0\leqslant\rho\leqslant v}\left|\int_0^\rho(\rho-\tau)^{\nu-1}[\zeta(\tau,\, \Re_\varepsilon(\tau),\, \Re_\varepsilon(\tau-\iota))-\bar{\zeta}(\Re_\varepsilon^*(\tau),\, \Re_\varepsilon^*(\tau-\iota))]dw(\tau)\right|^q\right) \\ &\quad+3^{q-1}\varepsilon^q\mathbb{E}\left(\sup_{0\leqslant\rho\leqslant v}\left|\sum_{k=1}^n I_k(\Re_\varepsilon(\rho_k^-))-\frac{1}{\Gamma(\nu)}\int_0^\rho(\rho-\tau)^{\nu-1}\bar{I}(\Re_\varepsilon^*(\tau))d\tau\right|^q\right) \\ &:=\mathscr{J}_1+\mathscr{J}_2+\mathscr{J}_3. \end{aligned} \tag{15}$$

Employing Jensen's inequality, we have

$$
\begin{aligned}
\mathscr{J}_1 &\leqslant \frac{6^{q-1}\varepsilon^q}{(\Gamma(\nu))^q}\mathbb{E}\left(\sup_{0\leqslant\rho\leqslant\nu}\left|\int_0^\rho(\rho-\tau)^{\nu-1}[\xi(\tau,\Re_\varepsilon(\tau),\Re_\varepsilon(\tau-\iota))-\xi(\tau,\Re^*_\varepsilon(\tau),\Re^*_\varepsilon(\tau-\iota))]d\tau\right|^q\right)\\
&\quad+\frac{6^{q-1}\varepsilon^q}{(\Gamma(\nu))^q}\mathbb{E}\left(\sup_{0\leqslant\rho\leqslant\nu}\left|\int_0^\rho(\rho-\tau)^{\nu-1}[\xi(\tau,\Re^*_\varepsilon(\tau),\Re^*_\varepsilon(\tau-\iota))-\bar{\xi}(\Re^*_\varepsilon(\tau),\Re^*_\varepsilon(\tau-\iota))]d\tau\right|^q\right)\\
&:=\mathscr{J}_{11}+\mathscr{J}_{12}.
\end{aligned}
\tag{16}
$$

Then, by using Hölder's inequality and Assumption 1, we conclude

$$
\begin{aligned}
\mathscr{J}_{11} &\leqslant \frac{6^{q-1}\varepsilon^q}{(\Gamma(\nu))^q}\mathbb{E}\left(\sup_{0\leqslant\rho\leqslant\nu}\left|\int_0^\rho(\rho-\tau)^{\nu-1}[\xi(\tau,\Re_\varepsilon(\tau),\Re_\varepsilon(\tau-\iota))-\xi(\tau,\Re^*_\varepsilon(\tau),\Re^*_\varepsilon(\tau-\iota))]d\tau\right|^q\right)\\
&\leqslant \frac{(6v)^{q-1}\varepsilon^q}{(\Gamma(\nu))^q}\int_0^v(v-\tau)^{q(\nu-1)}\mathbb{E}|\xi(\tau,\Re_\varepsilon(\tau),\Re_\varepsilon(\tau-\iota))-\xi(\tau,\Re^*_\varepsilon(\tau),\Re^*_\varepsilon(\tau-\iota))|^q d\tau\\
&\leqslant \frac{(6v)^{q-1}(C_1\varepsilon)^q}{(\Gamma(\nu))^q}\int_0^v(v-\tau)^{q(\nu-1)}\mathbb{E}\Big(|\Re_\varepsilon(\tau)-\Re^*_\varepsilon(\tau)|+|\Re_\varepsilon(\tau-\iota)-\Re^*_\varepsilon(\tau-\iota)|\Big)^q d\tau\\
&\leqslant \Pi_{11}\varepsilon^q v^{q-1}\left[\int_0^v(v-\tau)^{q(\nu-1)}\mathbb{E}\left(\sup_{0\leqslant\tau_1\leqslant\tau}|\Re_\varepsilon(\tau_1)-\Re^*_\varepsilon(\tau_1)|^q\right)d\tau\right.\\
&\quad\left.+\int_0^v(v-\tau)^{q(\nu-1)}\mathbb{E}\left(\sup_{0\leqslant\tau_1\leqslant\tau}|\Re_\varepsilon(\tau_1-\iota)-\Re^*_\varepsilon(\tau_1-\iota)|^q\right)d\tau\right],
\end{aligned}
\tag{17}
$$

where $\Pi_{11}=\frac{6^{q-1}C_1^q}{(\Gamma(\nu))^q}$.

By calling Assumption 3 and similar to $\mathscr{J}_{11}$, we get

$$
\begin{aligned}
\mathscr{J}_{12} &\leqslant \frac{6^{q-1}\varepsilon^q}{(\Gamma(\nu))^q}\mathbb{E}\left(\sup_{0\leqslant\rho\leqslant\nu}\left(\left|\int_0^\rho(\rho-\tau)^{\nu-1}[\xi(\tau,\Re^*_\varepsilon(\tau),\Re^*_\varepsilon(\tau-\iota))-\bar{\xi}(\Re^*_\varepsilon(\tau),\Re^*_\varepsilon(\tau-\iota))]d\tau\right|^2\right)^{\frac{q}{2}}\right)\\
&\leqslant \frac{6^{q-1}\varepsilon^q v^{q\nu-\frac{q}{2}}}{(\Gamma(\nu))^q(2\nu-1)^{\frac{q}{2}}}\mathbb{E}\left(\sup_{0\leqslant\rho\leqslant v}\left(\int_0^\rho|\xi(\tau,\Re^*_\varepsilon(\tau),\Re^*_\varepsilon(\tau-\iota))-\bar{\xi}(\Re^*_\varepsilon(\tau),\Re^*_\varepsilon(\tau-\iota))|^2d\tau\right)^{\frac{q}{2}}\right)\\
&\leqslant \frac{6^{q-1}\varepsilon^q v^{q\nu-1}}{(\Gamma(\nu))^q(2\nu-1)^{\frac{q}{2}}}\mathbb{E}\left(\sup_{0\leqslant\rho\leqslant v}\int_0^\rho|\xi(\tau,\Re^*_\varepsilon(\tau),\Re^*_\varepsilon(\tau-\iota))-\bar{\xi}(\Re^*_\varepsilon(\tau),\Re^*_\varepsilon(\tau-\iota))|^q d\tau\right)\\
&\leqslant \Pi_{12}\varepsilon^q v^{q\nu},
\end{aligned}
\tag{18}
$$

where $\Pi_{12} = \frac{6^{q-1}}{(\Gamma(\nu))^q(2\nu-1)^{\frac{q}{2}}}(\sup_{0\leqslant\rho\leqslant v}\lambda(\rho))\mathbb{E}(1+\sup_{0\leqslant\rho\leqslant v}|\Re_\varepsilon^*(\rho)|^q+\sup_{0\leqslant\rho\leqslant v}|\Re_\varepsilon^*(\rho-\iota)|^q)$.

Again, due to Jensen's inequality, we arrive at

$$\begin{aligned}\mathscr{J}_2 &\leqslant \frac{6^{q-1}\varepsilon^{\frac{q}{2}}}{(\Gamma(\nu))^q}\mathbb{E}\left(\sup_{0\leqslant\rho\leqslant v}\left|\int_0^\rho(\rho-\tau)^{\nu-1}[\zeta(\tau,\Re_\varepsilon(\tau),\Re_\varepsilon(\tau-\iota))-\zeta(\tau,\Re_\varepsilon^*(\tau),\Re_\varepsilon^*(\tau-\iota))]dw(\tau)\right|^q\right)\\ &+\frac{6^{q-1}\varepsilon^{\frac{q}{2}}}{(\Gamma(\nu))^q}\mathbb{E}\left(\sup_{0\leqslant\rho\leqslant v}\left|\int_0^\rho(\rho-\tau)^{\nu-1}[\zeta(\tau,\Re_\varepsilon^*(\tau),\Re_\varepsilon^*(\tau-\iota))-\bar{\zeta}(\Re_\varepsilon^*(\tau),\Re_\varepsilon^*(\tau-\iota))]dw(\tau)\right|^q\right)\\ &:=\mathscr{J}_{21}+\mathscr{J}_{22}.\end{aligned}\tag{19}$$

Using Burkholder-Davis-Gundy, Hölder and Doob's martingale inequalities, it concludes

$$\begin{aligned}\mathscr{J}_{21} &\leqslant \frac{6^{q-1}\varepsilon^{\frac{q}{2}}C_q}{(\Gamma(\nu))^q}\mathbb{E}\left(\int_0^v(v-\tau)^{2(\nu-1)}|\zeta(\tau,\Re_\varepsilon(\tau),\Re_\varepsilon(\tau-\iota))-\zeta(\tau,\Re_\varepsilon^*(\tau),\Re_\varepsilon^*(\tau-\iota))|^2d\tau\right)^{\frac{q}{2}}\\ &\leqslant \frac{6^{q-1}\varepsilon^{\frac{q}{2}}C_q}{(\Gamma(\nu))^q}v^{\frac{q}{2}-1}\mathbb{E}\left(\int_0^v(v-\tau)^{q(\nu-1)}|\zeta(\tau,\Re_\varepsilon(\tau),\Re_\varepsilon(\tau-\iota))-\zeta(\tau,\Re_\varepsilon^*(\tau),\Re_\varepsilon^*(\tau-\iota))|^qd\tau\right)\\ &\leqslant \frac{(12)^{q-1}\varepsilon^{\frac{q}{2}}C_1^qC_q}{(\Gamma(\nu))^q}v^{\frac{q}{2}-1}\left[\int_0^v(v-\tau)^{q(\nu-1)}\mathbb{E}\left(\sup_{0\leqslant\tau_1\leqslant\tau}|Y_\varepsilon(\tau_1)-Y_\varepsilon^*(\tau_1)|^q\right)d\tau\right.\\ &\left.+\int_0^v(v-\tau)^{q(\nu-1)}\mathbb{E}\left(\sup_{0\leqslant\tau_1\leqslant\tau}|\Re_\varepsilon(\tau_1-\iota)-\Re_\varepsilon^*(\tau_1-\iota)|^q\right)d\tau\right]\\ &\leqslant \Pi_{21}\varepsilon^{\frac{q}{2}}v^{\frac{q}{2}-1}\left[\int_0^v(v-\tau)^{q(\nu-1)}\mathbb{E}\left(\sup_{0\leqslant\tau_1\leqslant\tau}|\Re_\varepsilon(\tau_1)-\Re_\varepsilon^*(\tau_1)|^q\right)d\tau\right.\\ &\left.+\int_0^v(v-\tau)^{q(\nu-1)}\mathbb{E}\left(\sup_{0\leqslant\tau_1\leqslant\tau}|\Re_\varepsilon(\tau_1-\iota)-\Re_\varepsilon^*(\tau_1-\iota)|^q\right)d\tau\right],\end{aligned}\tag{20}$$

where $\Pi_{21} = \frac{(12)^{q-1}C_1^qC_q}{(\Gamma(\nu))^q}$.

Taking Assumption 1 and Lemma 2 into account and similar to $\mathscr{J}_{21}$, one may get

$$\begin{aligned}\mathscr{J}_{22} &\leqslant \frac{6^{q-1}\varepsilon^{\frac{q}{2}}C_q}{(\Gamma(\nu))^q}\mathbb{E}\left(\int_0^v(v-\tau)^{2(\nu-1)}|\zeta(\tau,\Re_\varepsilon^*(\tau),\Re_\varepsilon^*(\tau-\iota))-\bar{\zeta}(\Re_\varepsilon^*(\tau),\Re_\varepsilon^*(\tau-\iota))|^2d\tau\right)^{\frac{q}{2}}\\ &\leqslant \frac{6^{q-1}\varepsilon^{\frac{q}{2}}C_q}{(\Gamma(\nu))^q}\left(\frac{q-2}{q\nu-2}\right)^{\frac{q-2}{2}}v^{\frac{q\nu-2}{2}}\end{aligned}$$

$$\times\mathbb{E}\left(\int_0^v (v-\tau)^{\frac{q(\nu-1)}{2}}|\zeta(\tau,\Re_\varepsilon^*(\tau),\Re_\varepsilon^*(\tau-\iota))-\bar{\zeta}(\Re_\varepsilon^*(\tau),\Re_\varepsilon^*(\tau-\iota))|^q d\tau\right)$$

$$\leqslant \frac{(12)^{q-1}\varepsilon^{\frac{q}{2}}C_q}{(\Gamma(\nu))^q}\left(\frac{q-2}{q\nu-2}\right)^{\frac{q-2}{2}} v^{\frac{q\nu-2}{2}} \tag{21}$$

$$\times\mathbb{E}\left(\int_0^v (v-\tau)^{\frac{q(\nu-1)}{2}}(|\zeta(\tau,\Re_\varepsilon^*(\tau),\Re_\varepsilon^*(\tau-\iota))|^q+|\bar{\zeta}(\Re_\varepsilon^*(\tau),\Re_\varepsilon^*(\tau-\iota))|^q)d\tau\right)$$

$$\leqslant \Pi_{22}\varepsilon^{\frac{q}{2}} v^{\frac{q(2\nu-1)}{2}},$$

where $\Pi_{22}=\frac{2(36)^{q-1}C_q(C_3+C_2^q)}{(q(\nu-1)+2)(\Gamma(\nu))^q}\left(\frac{q-2}{q\nu-2}\right)^{\frac{q-2}{2}}\mathbb{E}(1+\sup_{0\leqslant\rho\leqslant v}|\Re_\varepsilon^*(\rho)|^q+\sup_{0\leqslant\rho\leqslant v}|\Re_\varepsilon^*(\rho-\iota)|^q)$.

Now, the plus and minus technique and Jensen's inequality give

$$\mathscr{J}_3\leqslant 6^{q-1}\varepsilon^q\mathbb{E}\left(\sup_{0\leqslant\rho\leqslant v}\left|\sum_{k=1}^n I_k(\Re_\varepsilon(\rho_k^-))-\sum_{k=1}^n I_k(\Re_\varepsilon^*(\rho_k^-))\right|^q\right)$$

$$+6^{q-1}\varepsilon^q\mathbb{E}\left(\sup_{0\leqslant\rho\leqslant v}\left|\sum_{k=1}^n I_k(\Re_\varepsilon^*(\rho_k^-))-\frac{1}{\Gamma(\nu)}\int_0^\rho(\rho-\tau)^{\nu-1}\bar{I}(\Re_\varepsilon^*(\tau))d\tau\right|^q\right) \tag{22}$$

$$:=\mathscr{J}_{31}+\mathscr{J}_{32}.$$

Using Jensen's inequality, Hölder's inequality and Assumption 2, one may conclude

$$\mathscr{J}_{31}\leqslant 6^{q-1}\varepsilon^q\mathbb{E}\left(\sup_{0\leqslant\rho\leqslant v}\left(\sum_{k=1}^n|I_k(\Re_\varepsilon(\rho_k^-))-I_k(\Re_\varepsilon^*(\rho_k^-))|\right)^q\right)$$

$$\leqslant 6^{q-1}\varepsilon^q\mathbb{E}\left(\sup_{0\leqslant\rho\leqslant v}\left(\sum_{k=1}^n c_k|\Re_\varepsilon(\rho_k^-)-\Re_\varepsilon^*(\rho_k^-)|\right)^q\right) \tag{23}$$

$$\leqslant (6m)^{q-1}\varepsilon^q\sum_{k=1}^m(c_k)^q\mathbb{E}\left(\sup_{0\leqslant\rho\leqslant v}|\Re_\varepsilon(\rho_k^-)-\Re_\varepsilon^*(\rho_k^-)|^q\right),$$

and

$$
\begin{aligned}
\mathscr{J}_{32} &\leqslant (12)^{q-1}\varepsilon^q \mathbb{E}\left(\sup_{0\leqslant\rho\leqslant v}\left|\sum_{k=1}^{n} I_k(\Re_\varepsilon^*(\rho_k^-))\right|^q\right) \\
&\quad + \frac{(12)^{q-1}\varepsilon^q}{(\Gamma(\nu))^q}\mathbb{E}\left(\sup_{0\leqslant\rho\leqslant v}\left|\int_0^\rho (\rho-\tau)^{\nu-1}\bar{I}(\Re_\varepsilon^*(\tau))d\tau\right|^q\right) \\
&\leqslant (12m)^{q-1}\varepsilon^q \sum_{k=1}^{m} |I_k(\Re_\varepsilon^*(\rho_k^-))|^q \\
&\quad + \frac{(12)^{q-1}\varepsilon^q}{(\Gamma(\nu))^q}\left(\frac{q-1}{q\nu-1}\right)^{q-1} v^{q\nu-1}\int_0^v \mathbb{E}\left|\frac{1}{b_1}\sum_{k=1}^{m} I_k(\Re_\varepsilon^*(\tau))\right|^q d\tau \\
&\leqslant (12)^{q-1}(m\varepsilon)^q(\widehat{n})^q + \frac{(12)^{q-1}(m\widehat{n}\varepsilon)^q}{b_1^q(\Gamma(\nu))^q}\left(\frac{q-1}{q\nu-1}\right)^{q-1} v^{q\nu} \\
&\leqslant \Pi_{32}\varepsilon^q,
\end{aligned}
\tag{24}
$$

where $\Pi_{32} = (12)^{q-1}(m\widehat{n})^q + \frac{(12)^{q-1}(m\widehat{n})^q}{b_1^q(\Gamma(\nu))^q}\left(\frac{q-1}{q\nu-1}\right)^{q-1} v^{q\nu}$.

Taking (15), (17), (18), (20), (21), (23), (24) into account, we obtain

$$
\begin{aligned}
\mathbb{E}\left(\sup_{0\leqslant\rho\leqslant v}|\Re_\varepsilon(\rho)-\Re_\varepsilon^*(\rho)|^q\right) &\leqslant \Pi_{12}\varepsilon^q v^{q\nu} + \Pi_{22}\varepsilon^{\frac{q}{2}} v^{\frac{q(2\nu-1)}{2}} + \Pi_{32}\varepsilon^q + \left(\Pi_{11}\varepsilon^q v^{q-1} + \Pi_{21}\varepsilon^{\frac{q}{2}} v^{\frac{q}{2}-1}\right) \\
&\quad \times \left(\int_0^v (v-\tau)^{q(\nu-1)}\mathbb{E}\left(\sup_{0\leqslant\tau_1\leqslant\tau}|\Re_\varepsilon(\tau_1)-\Re_\varepsilon^*(\tau_1)|^q\right)d\tau\right. \\
&\quad \left. + \int_0^v (v-\tau)^{q(\nu-1)}\mathbb{E}\left(\sup_{0\leqslant\tau_1\leqslant\tau}|\Re_\varepsilon(\tau_1-\iota)-\Re_\varepsilon^*(\tau_1-\iota)|^q\right)d\tau\right) \\
&\quad + (6m)^{q-1}\varepsilon^q\sum_{k=1}^{m}(c_k)^q\mathbb{E}\left(\sup_{0\leqslant\rho\leqslant v}|\Re_\varepsilon(\rho_k^-)-\Re_\varepsilon^*(\rho_k^-)|^q\right).
\end{aligned}
\tag{25}
$$

Letting $\Phi(v) = \mathbb{E}\left(\sup_{0\leqslant\rho\leqslant v}|\Re_\varepsilon(\rho)-\Re_\varepsilon^*(\rho)|^q\right)$ and $\mathbb{E}\left(\sup_{-\iota\leqslant\rho\leqslant 0}|\Re_\varepsilon(\rho)-\Re_\varepsilon^*(\rho)|^q\right) = 0$, then we obtain

$$
\mathbb{E}\left(\sup_{0\leqslant\tau_1\leqslant\tau}|\Re_\varepsilon(\tau_1-\iota)-\Re_\varepsilon^*(\tau_1-\iota)|^q\right) = \Phi(\tau-\iota). \tag{26}
$$

It concludes from Eq. (25) that

$$
\begin{aligned}
\Phi(v) \leqslant\ & \Pi_{12}\varepsilon^q v^{q\nu} + \Pi_{22}\varepsilon^{\frac{q}{2}} v^{\frac{q(2\nu-1)}{2}} + \Pi_{32}\varepsilon^q + \left(\Pi_{11}\varepsilon^q v^{q-1} + \Pi_{21}\varepsilon^{\frac{q}{2}} v^{\frac{q}{2}-1}\right) \\
& \times \left(\int_0^v (v-\tau)^{q(\nu-1)}\Phi(\tau)d\tau + \int_0^v (v-\tau)^{q(\nu-1)}\Phi(\tau-\iota)d\tau\right) \\
& + (6m)^{q-1}\varepsilon^q \sum_{k=1}^{m} (c_k)^q \Phi(v_k^-).
\end{aligned} \tag{27}
$$

Letting $\Lambda(v) = \sup_{\rho\in[-\iota,\, v]} \Phi(\rho)$, $\forall v \in [0,\, b]$, then $\Phi(\tau) \leqslant \Lambda(\tau)$, and $\Phi(\tau-\iota) \leqslant \Lambda(\tau)$. Therefore, we have from Eq. (27)

$$
\begin{aligned}
\Phi(v) \leq\ & \Pi_{12}\varepsilon^q v^{q\nu} + \Pi_{22}\varepsilon^{\frac{q}{2}} v^{\frac{q(2\nu-1)}{2}} + \Pi_{32}\varepsilon^q + 2\left(\Pi_{11}\varepsilon^q v^{q-1} + \Pi_{21}\varepsilon^{\frac{q}{2}} v^{\frac{q}{2}-1}\right) \\
& \times \int_0^v (v-\tau)^{q(\nu-1)}\Lambda(\tau)d\tau + (6m)^{q-1}\varepsilon^q \sum_{k=1}^{m} (c_k)^q \Lambda(v_k^-).
\end{aligned} \tag{28}
$$

For $\forall\rho \in [0,\, v]$, one may get

$$
\begin{aligned}
\Phi(\rho) \leqslant\ & \Pi_{12}\varepsilon^q \rho^{q\nu} + \Pi_{22}\varepsilon^{\frac{q}{2}} \rho^{\frac{q(2\nu-1)}{2}} + \Pi_{32}\varepsilon^q + 2\left(\Pi_{11}\varepsilon^q \rho^{q-1} + \Pi_{21}\varepsilon^{\frac{q}{2}} \rho^{\frac{q}{2}-1}\right) \\
& \times \int_0^\rho (\rho-\tau)^{q(\nu-1)}\Lambda(\tau)d\tau + (6m)^{q-1}\varepsilon^q \sum_{k=1}^{m} (c_k)^q \Lambda(\rho_k^-) \\
\leqslant\ & \Pi_{12}\varepsilon^q v^{q\nu} + \Pi_{22}\varepsilon^{\frac{q}{2}} v^{\frac{q(2\nu-1)}{2}} + \Pi_{32}\varepsilon^q + 2\left(\Pi_{11}\varepsilon^q v^{q-1} + \Pi_{21}\varepsilon^{\frac{q}{2}} v^{\frac{q}{2}-1}\right) \\
& \times \int_0^v (v-\tau)^{q(\nu-1)}\Lambda(\tau)d\tau + (6m)^{q-1}\varepsilon^q \sum_{k=1}^{m} (c_k)^q \Lambda(v_k^-).
\end{aligned} \tag{29}
$$

Then, we can obtain

$$
\begin{aligned}
\Lambda(v) = & \sup_{\rho\in[-\iota,\, v]} \Phi(\rho) \leqslant \max\left\{\sup_{\rho\in[-\iota,\, 0]} \Phi(\rho),\ \sup_{\rho\in[0,\, v]} \Phi(\rho)\right\} \\
\leqslant\ & \Pi_{12}\varepsilon^q v^{q\nu} + \Pi_{22}\varepsilon^{\frac{q}{2}} v^{\frac{q(2\nu-1)}{2}} + \Pi_{32}\varepsilon^q + 2\left(\Pi_{11}\varepsilon^q v^{q-1} + \Pi_{21}\varepsilon^{\frac{q}{2}} v^{\frac{q}{2}-1}\right)
\end{aligned}
$$

$$\times \int_0^v (v-\tau)^{q(\nu-1)} \Lambda(\tau) d\tau + (6m)^{q-1} \varepsilon^q \sum_{k=1}^{m} (c_k)^q \Lambda(v_k^-). \tag{30}$$

Thanks to Lemma 1, we can obtain

$$\begin{aligned} \Lambda(v) \leqslant & \left(\Pi_{12}\varepsilon^q v^{q\nu} + \Pi_{22}\varepsilon^{\frac{q}{2}} v^{\frac{q(2\nu-1)}{2}} + \Pi_{32}\varepsilon^q\right) \prod_{k=1}^{m} \left(1 + (6m)^{q-1} (\varepsilon c_k)^q\right) \\ & \times \exp\left\{\frac{2(\Pi_{11}\varepsilon^q v^{q\nu} + \Pi_{21}\varepsilon^{\frac{q}{2}} v^{q(1-\frac{1}{2})})}{q(\nu-1)+1}\right\}. \end{aligned} \tag{31}$$

Therefore,

$$\begin{aligned} \mathbb{E}\left(\sup_{0 \leqslant \rho \leqslant v} |\Re_\varepsilon(t) - \Re_\varepsilon^*(\rho)|^q\right) \leqslant & [\Pi_{12}\varepsilon^q v^{q\nu} + \Pi_{22}\varepsilon^{\frac{q}{2}} v^{\frac{q(2\nu-1)}{2}} + \Pi_{32}\varepsilon^q] \prod_{k=1}^{m} (1 + (6m)^{q-1} (\varepsilon c_k)^q) \\ & \times \exp\left\{\frac{2(\Pi_{11}\varepsilon^q v^{q\nu} + \Pi_{21}\varepsilon^{\frac{q}{2}} v^{q(1-\frac{1}{2})})}{q(\nu-1)+1}\right\}. \end{aligned} \tag{32}$$

Taking $\ell > 0$ and $\alpha \in (0,\, 1)$ such that for $\forall \rho \in [0,\, \ell\varepsilon^{-\alpha}] \subseteq [0,\, b]$, it is obtained that

$$\mathbb{E}\left(\sup_{0 \leqslant \rho \leqslant \ell\varepsilon^{-\alpha}} |\Re_\varepsilon(\rho) - \Re_\varepsilon^*(\rho)|^q\right) \leqslant \mathscr{Q}\varepsilon^{1-\alpha}, \tag{33}$$

where

$$\begin{aligned} \mathscr{Q} = & [\Pi_{12}\ell^{q\nu}\varepsilon^{q+\alpha(1-q\nu)-1} + \Pi_{22}\ell^{\frac{q(2\nu-1)}{2}}\varepsilon^{\frac{q}{2}(1+\alpha)+\alpha(1-q\nu)-1} + \Pi_{32}\varepsilon^{q+\alpha-1}] \\ & \times \prod_{k=1}^{m} (1 + (6m)^{q-1} (\varepsilon c_k)^q) \exp\left\{\frac{2(\Pi_{11}\ell^{q\nu}\varepsilon^{q(1-\alpha\nu)} + \Pi_{21}\ell^{q(1-\frac{1}{2})}\varepsilon^{q(1-\frac{\alpha}{2})})}{q(\nu-1)+1}\right\} \\ & := const. \end{aligned} \tag{34}$$

Hence, for a given $\theta > 0$, there exists $\varepsilon_1 \in (0,\, \varepsilon^*]$ such that for $\forall \varepsilon \in (0,\, \varepsilon_1]$ and $\rho \in [-\iota,\, \ell\varepsilon^{-\alpha}]$,

$$\mathbb{E}\left(\sup_{\rho \in [-\iota,\, \ell\varepsilon^{-\alpha}]} |\Re_\varepsilon(\rho) - \Re_\varepsilon^*(\rho)|^q\right) \leqslant \theta. \tag{35}$$

□

Remark 2 If $\Re(\rho-\iota)\equiv 0$ in Eq. (1), the proof process in this article is still applicable and the results are new.

Remark 3 Letting $\Re(\rho-\iota)\equiv 0$ and $I_k(.)\equiv 0$ $(k=1, 2, ..., m)$ in Eq. (1), then Theorem 1 will be consistent with Theorem 3.1 in [32] when $q=2$. Therefore, Theorem 1 in this article extends the results obtained in [32].

4. Example

Assume the following IDCFSDEs driven by Brownian motion:

$$\begin{cases} {}^{C}\mathscr{D}_{0^+}^{\frac{4}{5}}\Re_\varepsilon(\rho)=\varepsilon(\Re_\varepsilon(\rho)\cos^2\rho-\rho\Re_\varepsilon(\rho)\sin(\rho-\pi))+\sqrt{\varepsilon}(\Re_\varepsilon(\rho)\sin^2\rho-\Re_\varepsilon(\rho)\cos(\rho-\pi))\frac{dw(\rho)}{d\rho},\ \rho\in[0, b], \\ \Re_\varepsilon(\rho_k^+)=\Re_\varepsilon(\rho_k^-)+\varepsilon k^3\arctan(\Re_\varepsilon(\rho_k^-)),\quad \rho=\rho_k,\ k=1, 2, ..., m, \\ \Re_\varepsilon(\rho)=\vartheta(\rho),\quad \rho\in[-\iota, 0], \end{cases} \tag{36}$$

where $\nu=\frac{4}{5}$, $\xi(\rho, \Re_\varepsilon(\rho), \Re_\varepsilon(\rho-\iota))=\Re_\varepsilon(\rho)\cos^2\rho-\rho\Re_\varepsilon(\rho)\sin(\rho-\pi)$, $\zeta(\rho, \Re_\varepsilon(\rho), \Re_\varepsilon(\rho-\iota))=\Re_\varepsilon(\rho)\sin^2\rho-\Re_\varepsilon(\rho)\cos(\rho-\pi)$ and $I_k(\Re_\varepsilon(\rho_k^-))=k^3\arctan(\Re_\varepsilon(\rho_k^-))$.

Now, we check that the Assumptions 1-2 are satisfied. For any $(\Re_\varepsilon, \Re_\varepsilon^*)\in\mathbb{R}^l\times\mathbb{R}^l$, we have

$$\begin{aligned} &|\xi(\rho, \Re_\varepsilon(\rho), \Re_\varepsilon(\rho-\iota))-\xi(\rho, \Re_\varepsilon^*(\rho), \Re_\varepsilon^*(\rho-\iota))| \\ =&|\Re_\varepsilon\cos^2\rho-\rho\Re_\varepsilon\sin(\rho-\pi)-\Re_\varepsilon^*\cos^2\rho+\rho\Re_\varepsilon^*\sin(\rho-\pi)| \\ \leqslant&|(\Re_\varepsilon-\Re_\varepsilon^*)(\cos^2\rho+\rho\sin\rho)| \\ \leqslant&(1+\rho)|\Re_\varepsilon-\Re_\varepsilon^*|, \end{aligned} \tag{37}$$

$$\begin{aligned} &|\zeta(\rho, \Re_\varepsilon(\rho), \Re_\varepsilon(\rho-\iota))-\zeta(\rho, \Re_\varepsilon^*(\rho), \Re_\varepsilon^*(\rho-\iota))| \\ =&|\Re_\varepsilon\sin^2\rho-\Re_\varepsilon\cos(\rho-\pi)-\Re_\varepsilon^*\sin^2\rho+\Re_\varepsilon^*\cos(\rho-\pi)| \\ \leqslant&|(\Re_\varepsilon-\Re_\varepsilon^*)(\sin^2\rho+\cos(\rho))| \\ \leqslant&2|\Re_\varepsilon-\Re_\varepsilon^*|. \end{aligned} \tag{38}$$

Applying the mean value theorem for any constant $c>0$, we have

$$\begin{aligned} |I_k(\Re_\varepsilon(\rho_k^-))-I_k(\Re_\varepsilon^*(\rho_k^-))|&=k^3|\arctan(\Re_\varepsilon(\rho_k^-))-\arctan(\Re_\varepsilon^*(\rho_k^-))| \\ &=k^3|\arctan'(c)||\Re_\varepsilon(\rho_k^-)-\Re_\varepsilon^*(\rho_k^-)| \end{aligned}$$

$$
\begin{aligned}
&= k^3 \left| \frac{1}{1+c^2} \right| |\mathfrak{R}_\varepsilon(\rho_k^-) - \mathfrak{R}_\varepsilon^*(\rho_k^-)| \\
&\leqslant k^3 |\mathfrak{R}_\varepsilon(\rho_k^-) - \mathfrak{R}_\varepsilon^*(\rho_k^-)|,
\end{aligned} \tag{39}
$$

where $\arctan'(c)$ is the derivative of $\arctan(c)$. Let

$$
\begin{aligned}
|\xi(\rho, \mathfrak{R}_\varepsilon(\rho), \mathfrak{R}_\varepsilon(\rho-\imath))| &= |\mathfrak{R}_\varepsilon \cos^2\rho - \rho \mathfrak{R}_\varepsilon \sin(\rho - \pi)| \\
&\leqslant |\mathfrak{R}_\varepsilon \cos^2\rho + \rho \mathfrak{R}_\varepsilon \sin\rho| \\
&\leqslant (1+\rho)|\mathfrak{R}_\varepsilon|,
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
|\zeta(\rho, \mathfrak{R}_\varepsilon(\rho), \mathfrak{R}_\varepsilon(\rho-\imath))| &= |\mathfrak{R}_\varepsilon \sin^2\rho - \mathfrak{R}_\varepsilon \cos(\rho - \pi)| \\
&\leqslant |\mathfrak{R}_\varepsilon \sin^2\rho + \mathfrak{R}_\varepsilon \cos\rho| \\
&\leqslant 2|\mathfrak{R}_\varepsilon|,
\end{aligned} \tag{41}
$$

$$
|I_k(\mathfrak{R}_\varepsilon(\rho_k^-))| = |k^3 \arctan(\mathfrak{R}_\varepsilon(\rho_k^-))| = k^3 |\arctan(\mathfrak{R}_\varepsilon(\rho_k^-))| \leqslant \frac{k^3\pi}{2}. \tag{42}
$$

Therefore, Assumptions 1-2 are satisfied with $C_1 = C_2 = 2 \vee (1+\rho)$, $\widehat{n} = \frac{k^3\pi}{2}$ and Eq. (36) has a unique solution $\mathfrak{R}_\varepsilon$. Define

$$
\bar{\xi}(\mathfrak{R}_\varepsilon^*(\rho), \mathfrak{R}_\varepsilon^*(\rho-\imath)) = \frac{1}{\pi}\int_0^\pi \xi(\rho, \mathfrak{R}_\varepsilon^*(\rho), \mathfrak{R}_\varepsilon^*(\rho-\imath)) d\rho = \frac{3\mathfrak{R}_\varepsilon^*}{2}, \tag{43}
$$

$$
\bar{\zeta}(\mathfrak{R}_\varepsilon^*(\rho), \mathfrak{R}_\varepsilon^*(\rho-\imath)) = \frac{1}{\pi}\int_0^\pi \zeta(\rho, \mathfrak{R}_\varepsilon^*(\rho), \mathfrak{R}_\varepsilon^*(\rho-\imath)) d\rho = \frac{\mathfrak{R}_\varepsilon^*}{2}, \tag{44}
$$

$$
\bar{I}(\mathfrak{R}_\varepsilon^*(\rho)) = \frac{1}{\pi}\sum_{k=1}^{m} I_k(\mathfrak{R}_\varepsilon^*(\rho_k^-)) = \frac{1}{\pi}\sum_{k=1}^{m} k^3 \arctan(\mathfrak{R}_\varepsilon^*(\rho_k^-)) = \frac{m^2(m+1)^2}{4\pi} \arctan(\mathfrak{R}_\varepsilon^*(\rho_k^-)). \tag{45}
$$

Hence, we conclude the following averaged DCFSDEs without impulsive:

$$
\begin{cases}
{}^C\mathscr{D}_{0^+}^{\frac{4}{5}} \mathfrak{R}_\varepsilon^*(\rho) = \frac{3\varepsilon}{2}\mathfrak{R}_\varepsilon^*(\rho) + \frac{\sqrt{\varepsilon}}{2}\mathfrak{R}_\varepsilon^*(\rho)\frac{dw(\rho)}{d\rho} + \frac{m^2(m+1)^2}{4\pi}\arctan(\mathfrak{R}_\varepsilon^*(\rho_k^-)), \quad \rho \in [0, b], \\
\mathfrak{R}_\varepsilon^*(\rho) = \vartheta(\rho), \quad \rho \in [-\imath, 0].
\end{cases} \tag{46}
$$

Now, we check that the Assumption 3 is satisfied. We have

$$\frac{1}{b_1}\int_0^{b_1} |\xi(\rho, \mathfrak{R}_\varepsilon(\rho), \mathfrak{R}_\varepsilon(\rho-\iota)) - \bar{\xi}(\mathfrak{R}_\varepsilon(\rho), \mathfrak{R}_\varepsilon(\rho-\iota))|^q d\rho$$

$$= \frac{1}{b_1}\int_0^{b_1} |\mathfrak{R}_\varepsilon \cos^2\rho - \rho\mathfrak{R}_\varepsilon \sin(\rho-\pi) - \frac{3\mathfrak{R}_\varepsilon}{2}|^q d\rho$$

$$\leqslant \frac{1}{b_1}\int_0^{b_1} (\rho^q + (\frac{1}{2})^q)|\mathfrak{R}_\varepsilon|^q d\rho$$

$$\leqslant \frac{(2b_1)^q + q + 1}{2(q+1)}(1 + |\mathfrak{R}_\varepsilon|^q), \tag{47}$$

$$\frac{1}{b_1}\int_0^{b_1} |\zeta(\rho, \mathfrak{R}_\varepsilon(\rho), \mathfrak{R}_\varepsilon(\rho-\iota)) - \bar{\zeta}(\mathfrak{R}_\varepsilon(\rho), \mathfrak{R}_\varepsilon(\rho-\iota))|^q d\rho$$

$$= \frac{1}{b_1}\int_0^{b_1} |\mathfrak{R}_\varepsilon \sin^2\rho - \mathfrak{R}_\varepsilon \cos(\rho-\pi) - \frac{\mathfrak{R}_\varepsilon}{2}|^q d\rho$$

$$\leqslant \frac{1}{b_1}\int_0^{b_1} |\frac{3}{2}\mathfrak{R}_\varepsilon|^q d\rho$$

$$\leqslant \frac{3}{2}(1 + |\mathfrak{R}_\varepsilon|^q). \tag{48}$$

Therefore, Assumption 3 is satisfied with $\lambda(b_1) = \frac{3}{2} \vee (\frac{(2b_1)^q+q+1}{2(q+1)})$, $b_1 > 0$.

According to the above discussions, we can get that the assumptions of Theorem 1 are satisfied. Therefore, the standard solution $\mathfrak{R}_\varepsilon$ of Eq. (36) approximates the averaged solution $\mathfrak{R}_\varepsilon^*$ of Eq. (46) in L^q-sense as $\varepsilon \longrightarrow 0$.

5. Conclusion

In this paper, we present a fractional stochastic averaging limit theorem for IDCFSDEs. It is proved that the solution of the averaged DCFSDEs without impulsive approximates that of the standard IDCFSDEs in L^q-sense. Additionally, different from the previous studies in [20, 36–39], a new technique is applied to overcome the difficulty hired by the impulsive term based on impulsive-type Grönwall inequality, which enriches the relevant literature on the averaging theory for FSDEs with impulsive. It should be mentioned that according to Remark 3, the obtained results generalize the results obtained in [32]. The proof process adopted in this article can be applied for developing the averaging method for other types of impulsive FSDEs. For future research, the averaging principle for FSDEs in the sense of a newly defined OBC-fractional derivative, which is introduced recently by Zaid Odibat and Dumitru Baleanu.

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Conflict of interest

The author declares no competing financial interest.

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