Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

Research Article

Solvability of Caputo-Katugampola Fractional Differential Equation Involving p-Laplacian at Resonance

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Received: 12 May 2025; **Revised:** 11 August 2025; **Accepted:** 13 August 2025

Abstract: In this paper, we investigate the existence of solutions for Caputo-Katugampola fractional differential equations at resonance that involve two different orders and the p-Laplacian operator, by using the theory of the coincidence degree due to Mawhin and improving it due to Ge's theory. The dimension of the kernel for a fractional differential operator is two or one according to the value of type of Katugampola fractional integral. We use inequalities and nonlinear analytic techniques to investigate the existence of the solutions. In the simplest case, we automatically found the solvability of the Caputo-Katugampola equations with p-Laplacian operator. We provide two examples to illustrate our results.

Keywords: coincidence degree theory, Caputo-Katugampola fractional derivative, p-Laplacian operator, resonance

MSC: 34A12, 34A08, 47H10, 47H11

1. Introduction

Without a doubt, fractional differential equations are a potent mathematical example that offers further treatment flexibility with a variety of practical applications. Fractional differential equations with boundary and initial value conditions have been extensively studied in recent years, yielding several useful results [1–3]. They are typically studied using nonlinear analysis techniques such as fixed-point theorems, coincidence degree theory, variation method, upper and lower solutions method, among others [4–6].

Fractional order calculus has been discovered in the last few decades to be present in both theoretical and applied aspects of many different fields of science and engineering, including biological systems, robotics, dielectric polarization, electromagnetic waves, colored noise, heat conduction, memory features, biology, finance, and the illustration of heredity. The existence results for fractional differential evolution inclusions are investigated in [7–10]. The analysis of the optimal control results and solvability concerning fractional differential equations have been discussed in [11–15].

As a result, many scientists are interested in fractional differential equations since they can describe a variety of events, and the p-Laplacian operator is also frequently utilized to simulate a variety of physical and non-physical phenomena (See [16–18] and their references). If the associated homogeneous boundary value problem has a nontrivial solution, the boundary value problem is said to be at resonance. Many experts have recently focused on these problems.

In this research, we work on a type of fractional differential equation that can be expressed in the form $\mathbf{N}\upsilon = \mathbf{M}\upsilon$, where $\mathbf{N}$, $\mathbf{M}$ are two mappings and $\mathbf{N}$ is a linear Fredholm mapping of index zero. Also, it can not be reduced to a fixed

DOI: https://doi.org/10.37256/cm.6520257190

point problem for the operator $\mathbf{N^{-1}M}$ with the resonant problems (See [19–25] and their references). Otherwise, the problem is called non-resonant. There are many contributions published to investigate the existence of solutions to it (See [26–29] and their references).

Wang et al. [19] examined the existence of a solution by implementing a fixed point index theorem for the following resonant problem:

$$(D_{0^+}^{\alpha}\upsilon)(t)+H(t,\,\upsilon(t),\,(D_{0^+}^{\beta}\upsilon)(t))=0,\qquad t\in(0,\,1),$$

with boundary conditions

$$\upsilon(0)=0=\upsilon'(0),\qquad (D_{0^+}^{\beta}\upsilon)(1)=\sum_{i=1}^{m}\eta_i(D_{0^+}^{\beta}\upsilon)(\zeta_i),$$

where $D_{0^+}^{\alpha}$ is Riemann-Liouville derivative, $\alpha\in(2,\,3)$, $\beta\in(0,\,\alpha-2)$ and $\eta_i>0$, $0<\zeta_1<\cdots<\zeta_m<1$ with $\sum_{i=1}^{m}\eta_i\zeta_i^{\alpha-\beta-1}=1$.

Song et al. investigated [20] by executing the coincidence degree theory, the existence of a solution to the following resonant problem:

$$({}^{c}D_{1^-}^{\alpha}(D_{0^+}^{\beta}\upsilon))(t)=H(t,\,\upsilon(t),\,(D_{0^+}^{\beta+1}\upsilon)(t),\,D_{0^+}^{\beta}\upsilon(t)),\qquad t\in(0,\,1),$$

with boundary conditions

$$\upsilon(0)=0=\upsilon'(0),\qquad \upsilon(1)=\int_0^1\upsilon(t)dA(t),$$

where the Caputo fractional derivative ${}^{c}D_{1^-}^{\alpha}$ of order $\alpha\in(1,\,2]$ and the Riemann-Liouville fractional derivative $D_{0^+}^{\beta}$ of order $\beta\in(0,\,1]$.

There are a number of researchers who have contributed to proving the existence of a solution to many equations involving the p-Laplacian operator at resonance.

In [21], Azouzi et al. studied the following problem:

$$(\varphi_p\left((D_{0^+}^{\alpha}\upsilon)(t)\right))'=H(t,\,\upsilon(t),\,(D_{0^+}^{\alpha-1}\upsilon)(t)),\qquad t\in[0,\,1],\quad \alpha\in(1,\,2),$$

with boundary conditions

$$\upsilon(0)=0=(D_{0^+}^{\alpha}\upsilon)(1),\qquad (D_{0^+}^{\alpha-1}\upsilon)(1)=\sum_{i=1}^{m-2}\beta_i(D_{0^+}^{\alpha-1}\upsilon)(\eta_i),$$

where $D^{\alpha}_{0^+}$ is the Riemann-Liouville derivative, $H : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, φ_p is a p-Laplacian operator, $0 < \eta_1 < \cdots < \eta_{m-2} < 1$ and $\beta_i \in \mathbb{R}_+$, where $i = 1, 2, \ldots, m-2$, $m \geq 3$ and $\sum_{i=1}^{m-2} \beta_i = 1$.

Sun et al. investigated [22] the following problem:

$$D^{\beta}_{0^+} \left(\varphi_p \left((D^{\alpha}_{0^+} \upsilon)(t)\right)\right) = \mu(t) H(t,\, \upsilon(t),\, (D^{\alpha-(n-1)}_{0^+} \upsilon)(t),\, (D^{\alpha-(n-2)}_{0^+} \upsilon)(t),\, \cdots,\, (D^{\alpha}_{0^+} \upsilon)(t)),$$

with boundary conditions

$$(D^{\alpha}_{0^+} \upsilon)(0) = (D^{\alpha-2}_{0^+} \upsilon)(0) = (D^{\alpha-3}_{0^+} \upsilon)(0) = \cdots = (D^{\alpha-(n-2)}_{0^+} \upsilon)(0) = \upsilon(0) = 0,$$

$$\Gamma_1(\upsilon) = 0 = \Gamma_2(\upsilon),$$

where $t \in [0, +\infty)$, $\beta \in (0, 1]$, $\alpha \in (n-1, n]$ with $n \geq 3$, φ_p is a p-Laplacian operator and Γ_1, Γ_2 are continuous linear functions with the resonance conditions: $\Gamma_1(t^{\alpha-1}) = \Gamma_2(t^{\alpha-n+1}) = \Gamma_1(t^{\alpha-n+1}) = \Gamma_2(t^{\alpha-1}) = 0$.

Here, we are interested in proving the existence of the solutions (at least one solution) to the Caputo-Katugampola differential equation of two different orders with the p-Laplacian operator at resonance, it can be expressed as

$$\begin{cases} \left({}^{\rho}_{c}D^{\beta}_{0^+} \varphi_p \left({}^{\rho}_{c}D^{\alpha}_{0^+} \upsilon \right) \right)(t) = H(t,\, \upsilon(t),\, ({}^{\rho}_{c}D^{\alpha}_{0^+} \upsilon)(t)), & t \in [0, S], \\ ({}^{\rho}_{c}D^{\alpha}_{0^+} \upsilon)(0) = 0 = ({}^{\rho}_{c}D^{\alpha}_{0^+} \upsilon)(S), & \upsilon'(0) = \upsilon_0, \end{cases} \tag{1}$$

where

- $H : [0, S] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function;
- $1 < \alpha \leq 2$, $0 < \beta \leq 1$ and $\rho > 0$;
- ${}^{\rho}_{c}D^{\alpha}_{0^+}$ and ${}^{\rho}_{c}D^{\beta}_{0^+}$ are the Caputo-Katugampola fractional derivatives of orders α and β respectively and type ρ;
- φ_p with $p > 1$ is a p-Laplacian operator defined by $\varphi_p(x) = |x|^{p-2}x$ which has the inverse φ_k such that $\frac{1}{p} + \frac{1}{k} = 1$.

The n-Laplace equation, the turbulent movement of a gas in a porous medium, and non-Newtonian fluid theory are all studied using equations of the aforementioned kind with ordinary derivatives. Some existence results for such boundary value problems are achieved by applying a novel continuation theorem. For $p = 2$, the differential equation (1) with the boundary condition is linear. However, $\varphi_p(x)$ is not linear with regard to x when $p \neq 2$. Therefore, compared to the linear instance, the debate in this situation is more complicated.

The Katugampola fractional integral combines the Riemann-Liouville fractional integral and the Hadamard fractional integral into a single form. It is also strongly connected to the Erdelyi-Kober operator, which generalizes the Riemann-Liouville fractional integral. The Katugampola fractional derivative is defined using the Katugampola fractional integral, and it, like any other fractional differential operator, allows for the use of real or complex number powers of the integral and differential operators. They combine the qualities of Caputo and Hadamard derivatives and may be scaled to multiple functions, making them ideal for modeling systems with nonlinear and fractal-like tendencies.

The Caputo and Caputo-Hadamard fractional derivatives are both generalized by the Caputo-Katugampola derivative. A smooth transition between these two common fractional derivatives is made possible by the inclusion of a parameter ρ. This implies that you can tailor the derivative to better fit the unique features of the system you are describing by varying the parameter ρ. Memory effects and non-local behaviors, which are frequently found in real-world occurrences, can be modeled by including the parameter ρ.

To explicitly highlight the novelty of the proposed method, we clearly state how our approach differs from and improves upon existing methods in the literature. In this research, we investigate the existence of a solution for a resonant problem that involves Caputo-Katugampola fractional derivatives with two different orders and type $\rho \geq 0$ and a p-Laplacian operator. We use two theorems according to the value of type of fractional derivative ρ. We focus on two cases:

• When $\rho \in (0, 1)$, the index of Fredholm operator is equal to zero. So, we apply the coincidence degree theory due to Mawhin;

• When $\rho \in [1, \infty)$, the index of the Fredholm operator does not finish. So, we apply the Ge-Mawhin continuation theorem.

The primary challenges in this paper, in contrast to earlier research, are as follows:

First: We are aware that the larger the kernel dimension, the more challenging it is to create the projections, according to Mawhin's continuation theorem, especially if there are two probabilities;

Second: Assessing prior bounds is the toughest part;

Third: It is challenging to create an example.

The organization of this article is as follows. We introduce some knowledge, definitions and lemmas to get the main results of ours in section 2. In section 3, we define some operations and offer their properties that are needed to obtain our main results. We study the existence of a solution for our problem and present an example to illustrate the applicability of our results in section 4. At the last, the Conclusion section is introduced to show the final results that were obtained in this paper.

2. Preliminaries

Some basic lemmas, definitions and knowledge regarding fractional calculus theory are offered in this section to help in obtaining our main results. Most of them are taken from [30–36].

2.1 *Spaces*

Throughout this paper, we use the Banach space $\mathcal{Y} = C[0, S]$ with the norm

$$||\upsilon||_{\mathcal{Y}} = \sup\{|\upsilon(\mathrm{y})|, \mathrm{y} \in [0, S]\}.$$

The set of functions

$$\mathcal{X} = \{\upsilon(\mathrm{y}) : \upsilon(\mathrm{y}), ({}^{\rho}_{c}D^{\alpha}_{0^+}\upsilon)(\mathrm{y}) \in \mathcal{Y}, \mathrm{y} \in [0, S]\}$$

with the norm

$$||\upsilon||_{\mathcal{X}} = \max\{||\upsilon||_{\mathcal{Y}}, ||{}^{\rho}_{c}D^{\alpha}_{0^+}\upsilon||_{\mathcal{Y}}\}.$$

It is obvious that $\mathcal{X}$ is a Banach space.

Consider the space $\mathbb{X}^p_c(a, b)$ with $-\infty \leq a < b \leq \infty$, which contains Lebesgue measurable function υ on the closed interval $[a, b]$ with complex-valued and $\|\upsilon\|_{\mathbb{X}^p_c} < \infty$ and

$$\|\upsilon\|_{\mathbb{X}_c^p} = \left(\int_a^b |v^c \upsilon(v)|^p \frac{dv}{v}\right)^{\frac{1}{p}}, \quad (1 \le p < \infty, c \in \mathbb{R}).$$

For $p = \infty$,

$$\|\upsilon\|_{\mathbb{X}_c^p} = ess \sup_{\mathrm{x} \in [a, b]} \mathrm{x}^c |\upsilon(\mathrm{x})|.$$

Take $AC[a, b]$ is the set of all absolutely continuous functions on the closed interval $[a, b]$. Then, for $n \in \mathbb{N}$,

$$AC^n[a, b] = \left\{\upsilon : [a, b] \to \mathbb{C}, \ \left(\mathrm{x}^{1-\rho} \frac{d}{d\mathrm{x}}\right)^{n-1} \upsilon(\mathrm{x}) \in AC[a, b]\right\}$$

with $AC^1[a, b] = AC[a, b]$ [37, 38]. It is worth mentioning that this space will be used to introduce the definition of the Caputo-Katugampola fractional derivative.

2.2 *p-Laplacian operator*

Let $p > 1$. The p-Laplacian operator φ_p is defined as $\varphi_p(x) = |x|^{p-2}x$ which has the inverse φ_k such that $\frac{1}{p} + \frac{1}{k} = 1$. We state below some needed results to establish some important inequalities used to arrive at the main results:

Lemma 2.1 [23] Let φ_p be a p-Laplacian operator, and we have

Case 1: If $1 < p \le 2$, $\mathrm{xy} > 0$ and $|\mathrm{x}|, |\mathrm{y}| \ge \mathrm{n} > 0$. Then,

$$|\varphi_p(\mathrm{x}) - \varphi_p(\mathrm{y})| \le (p-1)\mathrm{n}^{\mathrm{p}-2}|\mathrm{x} - \mathrm{y}|; \tag{2}$$

Case 2: If $p > 2$ and $|\mathrm{x}|, |\mathrm{y}| \le \mathrm{r}$, where r is a constant. Then,

$$|\varphi_p(\mathrm{x}) - \varphi_p(\mathrm{y})| \le (p-1)\mathrm{r}^{\mathrm{p}-2}|\mathrm{x} - \mathrm{y}|. \tag{3}$$

Lemma 2.2 [24] Let φ_p be a p-Laplacian operator. Then for all $\mathrm{x}, \mathrm{y} > 0$, we have

Case 1: When $p \in (1, 2)$. Then,

$$|\varphi_p(\mathrm{x} + \mathrm{y})| \le |\varphi_p(\mathrm{x})| + |\varphi_p(\mathrm{y})|;$$

Case 2: When $p \ge 2$. Then,

$$|\varphi_p(\mathrm{x} + \mathrm{y})| \le 2^{p-2}\left(|\varphi_p(\mathrm{x})| + |\varphi_p(\mathrm{y})|\right).$$

2.3 *Fractional calculus*

Here, we present the key terms and ideas associated with our problem. Additionally, we use these results to ascertain our problem's solution form (1).

Definition 2.1 [30] Let $\rho,\ \gamma \in \mathbb{R}^+$ and the function $\mathrm{h} \in \mathbb{X}_c^p(a,\ b)$. Then, the Katugampola integral is given by

$$({}^{\rho}I_{0+}^{\gamma}\mathrm{h})(\mathrm{x}) = \frac{\rho^{1-\gamma}}{\Gamma(\gamma)} \int_0^{\mathrm{x}} \mathrm{s}^{\rho-1}(\mathrm{x}^{\rho} - \mathrm{s}^{\rho})^{\gamma-1}\mathrm{h}(\mathrm{s})ds.$$

Definition 2.2 [30] The Caputo-Katugampola fractional derivative of order γ and type $\rho \in \mathbb{R}^+$ with $r-1 < \gamma \le r,\ r \in \mathbb{N}$ of a function $\mathrm{h} \in AC^r[a,\ b]$ is given by

$$({}_{c}^{\rho}D_{0+}^{\gamma}\mathrm{h})(\mathrm{x}) = \left({}^{\rho}I_{0+}^{r-\gamma}\left(\mathrm{x}^{1-\rho}\frac{d}{d\mathrm{x}}\right)^{r}\mathrm{h}\right)(\mathrm{x})$$

$$= \frac{\rho^{1+\gamma-r}}{\Gamma(r-\gamma)} \int_0^{\mathrm{x}} \mathrm{s}^{\rho-1}(\mathrm{x}^{\rho} - \mathrm{s}^{\rho})^{r-\gamma-1}\left(\mathrm{s}^{1-\rho}\frac{d}{d\mathrm{s}}\right)^{r}\mathrm{h}(\mathrm{s})ds.$$

Lemma 2.3 [30] Let $r \in \mathbb{N},\ \gamma,\ \delta,\ \rho \in \mathbb{R}^+$ such that $r-1 < \gamma \le r$ and $\gamma \ge \delta$. Then, we have

- $({}_{c}^{\rho}D_{0+}^{\delta}\ {}^{\rho}I_{0+}^{\gamma}\mathrm{h})(\mathrm{x}) = ({}^{\rho}I_{0+}^{\gamma-\delta}\mathrm{h})(\mathrm{x}),\ \mathrm{h} \in \mathbb{X}_c^p(a,\ b)$;
- $({}^{\rho}I_{0+}^{\gamma}\ {}_{c}^{\rho}D_{0+}^{\gamma}\mathrm{h})(\mathrm{x}) = \mathrm{h}(\mathrm{x}) - a_0 - a_1\mathrm{x}^{\rho} - \cdots - a_{r-1}\mathrm{x}^{\rho(\mathrm{r}-1)},\ \mathrm{h} \in AC^r[a,\ b]$,

where a_k with $k = 0,\ 1,\ \ldots,\ r-1$ are constants.

Lemma 2.4 [30] Let $r \in \mathbb{N},\ r-1 < \delta \le r,\ \gamma > 0,\ \rho > 0$ and $t > 0$. Then,

- ${}^{\rho}I_{0+}^{\delta}\left(\frac{\mathrm{x}^{\rho}}{\rho}\right)^{\gamma-1} = \frac{\Gamma(\gamma)}{\Gamma(\gamma+\delta)}\left(\frac{\mathrm{x}^{\rho}}{\rho}\right)^{\gamma+\delta-1}$;
- ${}_{c}^{\rho}D_{0+}^{\delta}\left(\frac{\mathrm{x}^{\rho}}{\rho}\right)^{\gamma-1} = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\delta)}\left(\frac{\mathrm{x}^{\rho}}{\rho}\right)^{\gamma-\delta-1},\ \delta < \gamma$;
- ${}_{c}^{\rho}D_{0+}^{\delta}\left(\frac{\mathrm{x}^{\rho}}{\rho}\right)^{m} = 0,\ m = 0,\ 1,\ 2,\ 3,\ \ldots,\ r-1$.

Lemma 2.5 The following problem and problem (1) are equivalent

$$\begin{cases} ({}_{c}^{\rho}D_{0+}^{\alpha}\upsilon)(t) = \varphi_k\left({}^{\rho}I_{0+}^{\beta}H(t,\ \upsilon(t),\ ({}_{c}^{\rho}D_{0+}^{\alpha}\upsilon)(t))\right), & t \in [0,\ S], \\ ({}_{c}^{\rho}D_{0+}^{\alpha}\upsilon)(0) = 0 = ({}_{c}^{\rho}D_{0+}^{\alpha}\upsilon)(S), & \upsilon'(0) = \upsilon_0. \end{cases} \tag{4}$$

Proof. By Lemma 2.3, problem (1) has a solution

$$\varphi_p\left(({}_{c}^{\rho}D_{0+}^{\alpha}\upsilon)(t)\right) = {}^{\rho}I_{0+}^{\beta}H(t,\ \upsilon(t),\ ({}_{c}^{\rho}D_{0+}^{\alpha}\upsilon)(t)) + a_0,$$

where $a_0 \in \mathbb{R}$, when substituting $t = 0$, we get $a_0 = 0$. Then, we obtain

$$\varphi_p\left(({}_{c}^{\rho}D_{0+}^{\alpha}\upsilon)(t)\right) = {}^{\rho}I_{0+}^{\beta}H(t,\ \upsilon(t),\ ({}_{c}^{\rho}D_{0+}^{\alpha}\upsilon)(t)). \tag{5}$$

Applying φ_k on both sides of (5), we get

$$({}^{\rho}_{c}D^{\alpha}_{0^+}\upsilon)(t) = \varphi_k\left({}^{\rho}I^{\beta}_{0^+}H(t,\,\upsilon(t),\,({}^{\rho}_{c}D^{\alpha}_{0^+}\upsilon)(t))\right).$$

Now, by applying ${}^{\rho}I^{\alpha}_{0^+}$ on both sides of the above formula, we get two constants $b_0,\, b_1 \in \mathbb{R}$

$$\upsilon(t) = {}^{\rho}I^{\alpha}_{0^+}\varphi_k\left({}^{\rho}I^{\beta}_{0^+}H(t,\,\upsilon(t),\,({}^{\rho}_{c}D^{\alpha}_{0^+}\upsilon)(t))\right) + b_0 + b_1 t^{\rho}.$$

Hence,

$$\upsilon'(t) = {}^{\rho}I^{\alpha-1}_{0^+}\varphi_k\left({}^{\rho}I^{\beta}_{0^+}H(t,\,\upsilon(t),\,({}^{\rho}_{c}D^{\alpha}_{0^+}\upsilon)(t))\right) + \rho b_1 t^{\rho-1}.$$

In the case of $\rho > 1$, we can not calculate the values of b_0 and b_1 under our boundary conditions. If $\rho = 1$, we get $b_1 = v_0$ and can not calculate the value of b_0. The previous equation can be rewritten as

$$t^{1-\rho}\upsilon'(t) = t^{1-\rho}\,{}^{\rho}I^{\alpha-1}_{0^+}\varphi_k\left({}^{\rho}I^{\beta}_{0^+}H(t,\,\upsilon(t),\,({}^{\rho}_{c}D^{\alpha}_{0^+}\upsilon)(t))\right) + \rho b_1$$

which implies that $b_1 = 0$ in the case of $0 < \rho < 1$ and also can not calculate the value of b_0. Therefore, $\upsilon(t)$ is a solution of equation (1), provided that it solves equation (4). □

2.4 *Fredholm mapping*

Certain operators that appear in the Fredholm theory of integral equations are known as Fredholm operators named in honor of Erik Ivar Fredholm. A Fredholm operator, defined as a bounded linear operator $\mathbf{N} : \mathcal{X} \to \mathcal{Y}$ between two Banach spaces, has a finite-dimensional kernel and an algebraic cokernel. In additional details:

Definition 2.3 [31] If a linear mapping

$$\mathbf{N} : dom\mathbf{N} \subset \mathcal{X} \to \mathcal{Y},$$

where $\mathcal{X}$ and $\mathcal{Y}$ are real Banach spaces, satisfies the following conditions:

• $ker\mathbf{N}$ and $Im\mathbf{N}$ have finite dimensions;

• $Im\mathbf{N}$ is closed.

Then we call this mapping a Fredholm mapping.

Let a Fredholm mapping $\mathbf{N}$ have index zero and the continuous projectors $\mathbf{F} : \mathcal{X} \to \mathcal{X}$, $\mathbf{G} : \mathcal{Y} \to \mathcal{Y}$ such that $ker\mathbf{N} = Im\mathbf{F}$, $Im\mathbf{N} = ker\mathbf{G}$ and $\mathcal{X} = ker\mathbf{N} \oplus ker\mathbf{F}$, $\mathcal{Y} = Im\mathbf{N} \oplus Im\mathbf{G}$. A continuous mapping $\mathbf{F} : \mathcal{X} \to \mathcal{X}$ is a projector, when $\mathbf{F}^2\upsilon = \mathbf{F}\upsilon$, $\forall \upsilon \in \mathcal{X}$. It follows that the invertible map $\mathbf{N}_\mathbf{F} : dom\mathbf{N} \cap ker\mathbf{F} \to Im\mathbf{N}$ and we denote the inverse of $\mathbf{N}_\mathbf{F}$ by $\mathbf{K}_\mathbf{F}$.

Definition 2.4 [31] The mapping $\mathbf{M}$ is called $\mathbf{N}$-compact on $\overline{\mathbf{\Omega}}$, where $\mathbf{\Omega} \subset \mathcal{X}$ is an open bounded set if it satisfies the following conditions:

• $\mathbf{GM}(\overline{\mathbf{\Omega}})$ is bounded;

• $\mathbf{K}_{\mathbf{F},\,\mathbf{G}}\mathbf{M} = \mathbf{K}_\mathbf{F}(\mathbf{I}-\mathbf{G})\mathbf{M} : \overline{\mathbf{\Omega}} \to \mathcal{X}$ is compact.

Remark 2.1 In the Arzela-Ascoli theorem: $\mathbf{K}_{\mathbf{F},\,\mathbf{G}}\mathbf{M}(\overline{\mathbf{\Omega}})$ is compact $\Leftrightarrow$ it is bounded and equicontinuous.

Remark 2.2 [31] The Fredholm mapping has an integer index

$$IndN = dim(kerN) - codim(ImN),$$

where $codim(ImN) = dim(cokerN) = dim(\mathcal{Y}/ImN)$.

Remark 2.3 [39] Let $U_1, U_2 \subset V$. Then $V = U_1 \oplus U_2$, provided that the following conditions hold:

• For all $v \in V$, there exists $u_1 \in U_1$ and $u_2 \in U_2$ such that $v = u_1 + u_2$;

• $U_1 \cap U_2 = \{0\}$.

Definition 2.5 [24, 25] The mapping $\mathbf{N} : \mathcal{X} \cap dom\mathbf{N} \to \mathcal{Y}$ is a quasi-linear operator if the following conditions hold:

• $ker\mathbf{N}$ is linearly homeomorphic to $\mathbb{R}^r$ ($r < \infty$);

• $Im\mathbf{N} \subset \mathcal{Y}$ is closed.

Definition 2.6 [24, 25] Assume that $\mathcal{X}_1$ is a subspace from the Banach space $\mathcal{X}$. A continuous mapping $\mathbf{F} : \mathcal{X} \to \mathcal{X}_1$ is a semi-projector, if

• $\mathbf{F}^2\upsilon = \mathbf{F}\upsilon$ for all $\upsilon \in \mathcal{X}$;

• $\mathbf{F}(r\upsilon) = r(\mathbf{F}\upsilon)$ for all $\upsilon \in \mathcal{X}$, and $r \in \mathbb{R}$.

2.5 *Fixed point theorems*

The presence of at least one point (referred to as a "fixed point") that a given function maps to itself under specific circumstances is guaranteed by fixed point theorems, which are mathematical conclusions. These theorems are fundamental to many branches of mathematics and can be used in topology, analysis, and even problem-solving in the real world.

Theorem 2.1 (Mawhin's continuation theorem [31]) Let $\mathbf{N}$ be a Fredholm operator with index $\mathbf{0}$, and $\mathbf{M}$ be $\mathbf{N}$-compact on $\overline{\mathbf{\Omega}}$ and suppose it satisfies the following conditions:

• $\mathbf{N}\upsilon \neq \xi\mathbf{M}\upsilon$, $\forall(\upsilon, \xi) \in [(dom\mathbf{N} \setminus ker\mathbf{N}) \cap \partial\mathbf{\Omega}] \times (0, 1)$;

• $\mathbf{M}\upsilon \notin Im\mathbf{N}$, $\forall \upsilon \in (ker\mathbf{N} \cap \partial\mathbf{\Omega})$;

• $deg(\mathbf{JGM}|_{ker\mathbf{N}}, \mathbf{\Omega} \cap ker\mathbf{N}, 0) \neq 0$, where $\mathbf{J} : Im\mathbf{G} \to ker\mathbf{N}$ is any isomorphism and a continuous projection $\mathbf{G} : \mathcal{Y} \to \mathcal{Y}$ as above with $Im\mathbf{N} = ker\mathbf{G}$.

Then, in $dom\mathbf{N} \cap \overline{\mathbf{\Omega}}$ of the equation $\mathbf{N}\upsilon = \mathbf{M}\upsilon$, there is at least one solution.

Theorem 2.2 (Ge-Mawhin's continuation theorem [32]) Let $\mathbf{N}$ be a quasi-linear operator, $\mathbf{M}_\xi = \xi\mathbf{M}$, where $\xi \in [0, 1]$ is $\mathbf{N}$-compact on $\overline{\mathbf{\Omega}}$ and suppose it satisfies the following conditions:

• $\mathbf{N}\upsilon \neq \xi\mathbf{M}\upsilon$, $\forall(\upsilon, \xi) \in [dom\mathbf{N} \cap \partial\mathbf{\Omega}] \times (0, 1)$;

• $deg(\mathbf{JGM}|_{ker\mathbf{N}}, \mathbf{\Omega} \cap ker\mathbf{N}, 0) \neq 0$, where $\mathbf{J} : Im\mathbf{G} \to ker\mathbf{N}$ is homeomorphism with $\mathbf{J}(\theta) = \theta$ (θ origin element), and a semi-projection $\mathbf{G} : \mathcal{Y} \to \mathcal{Y}_1 \subseteq \mathcal{Y}$ as above.

Then, in $dom\mathbf{N} \cap \overline{\mathbf{\Omega}}$ of the equation $\mathbf{N}\upsilon = \mathbf{M}\upsilon$, there is at least one solution.

Remark 2.4 [24, 25] We can use the theory of Mawhin's continuation is due to Ge when the index of the operator $\mathbf{N}$ does not finish, the conditions of this theory are very similar to the theory of the coincidence degree that due to Mawhin, except that the operator $\mathbf{N}$ must be quasi-linear.

Remark 2.5 The last part in the proof of Lemma 2.5 with the previous theorems and last remark tell that:

• When $\rho \in (0, 1)$, the index of Fredholm operator is equal to zero. So, we have shown that by applying the coincidence degree theory due to Mawhin;

• When $\rho \in [1, \infty)$, the index of Fredholm operator does not finish. So, we have applied an improved version of Ge-Mawhin's continuation theorem.

3. Basic constructions

We define the operator $\mathbf{N} : dom\mathbf{N} \subset \mathcal{X} \to \mathcal{Y}$ by

$$\mathbf{N}\upsilon = {}^{\rho}_{c}D^{\alpha}_{0^+}\upsilon,$$

where

$$dom\mathbf{N} = \{\upsilon \in \mathcal{X} : ({}^{\rho}_{c}D^{\alpha}_{0^+}\upsilon)(0) = 0 = ({}^{\rho}_{c}D^{\alpha}_{0^+}\upsilon)(S),\ \upsilon'(0) = \upsilon_0\},$$

and $\mathcal{X}$, $\mathcal{Y}$ are two Banach spaces defined in the previous section. It is obvious that $ker\mathbf{N}$ has a finite dimension, since

$$ker\mathbf{N} = \{a_0 + a_1 t^{\rho} | a_i \in \mathbb{R},\ i = 0,\ 1\} = \mathbb{R}^2.$$

It is clear that $dim(ker\mathbf{N}) = 2$ if $a_i \neq 0,\ i = 0,\ 1$. If one of them is identically zero (not both), then $dim(ker\mathbf{N}) = 1$. This depends on the value of type ρ according to Lemma 2.5 ($a_1 = 0$ if $\rho \in (0,\ 1)$).

Let $y \in \mathcal{Y}$, then there exists $\upsilon \in dom\mathbf{N}$ such that $y = {}^{\rho}_{c}D^{\alpha}_{0^+}\upsilon$. From conditions $({}^{\rho}_{c}D^{\alpha}_{0^+}\upsilon)(0) = 0 = ({}^{\rho}_{c}D^{\alpha}_{0^+}\upsilon)(S)$, we get $y(0) = 0 = y(S)$. So, $Im\mathbf{N} = \{y \in \mathcal{Y} | y(0) = 0 = y(S)\}$ is closed with finite dimension.

We define the operator $\mathbf{G} : \mathcal{Y} \to \mathcal{Y}_1$ by

$$(\mathbf{G}y)(t) = y(S).$$

This operator is a projector mapping since $\mathbf{G}^2 = \mathbf{G}$.

It is clear that $ker\mathbf{G} = \{y \in \mathcal{Y} | y(S) = 0\} = Im\mathbf{N}$. In addition, it is easy to see that

$$0 = \mathbf{G}y - \mathbf{G}y = \mathbf{G}y - \mathbf{G}^2 y = \mathbf{G}(\mathbf{I} - \mathbf{G})y$$

which implies that $(\mathbf{I} - \mathbf{G})y \in ker\mathbf{G}$. By assuming $y \in \mathcal{Y}$, we get

$$y = y - \mathbf{G}y + \mathbf{G}y = (\mathbf{I} - \mathbf{G})y + \mathbf{G}y$$

which leads to

$$\mathcal{Y} = ker\mathbf{G} + Im\mathbf{G} = Im\mathbf{N} + Im\mathbf{G}.$$

By letting $y_0 \in (ker\mathbf{G} \cap Im\mathbf{G})$, we get $y_0 \in ker\mathbf{G}$ and $y_0 \in Im\mathbf{G}$ which are respectively, equivalently $\mathbf{G}y_0 = 0$ and there exists $y_1 \in \mathcal{Y}$ such that

$$y_0 = \mathbf{G}y_1 = \mathbf{G}^2 y_1 = \mathbf{G}y_0 = 0.$$

Hence, $ker\mathbf{G} \cap Im\mathbf{G} = \{0\} = Im\mathbf{N} \cap Im\mathbf{G}$. By Remark 2.3, we have

$$\mathcal{Y} = Im\mathbf{N} \oplus Im\mathbf{G}.$$

We define the operator $\mathbf{F} : \mathcal{X} \to \mathcal{X}_1$ by

$$\mathbf{F}\upsilon = \upsilon(0) + a_1 t^{\rho},$$

where $a_1 \in \mathbb{R}$. It is clear that it is a projector mapping with $ker\mathbf{F} = \{\upsilon \in \mathcal{X} | \upsilon(0) = -a_1 t^{\rho}\}$ and $Im\mathbf{F} = \{a_0 + a_1 t^{\rho} | a_i \in \mathbb{R},\, i = 0,\, 1\} = ker\mathbf{N}$. By Remark 2.3, we have

$$\mathcal{X} = ker\mathbf{N} \oplus ker\mathbf{F}.$$

Hence, by Definition 2.3 and Remark 2.2, we get $\mathbf{N}$ is a Fredholm mapping and $Ind\mathbf{N} = 0$. When $\rho \geq 1$, then $Ind\mathbf{N} \neq 0$. But $\mathbf{N}$ is a quasilinear operator for all $\rho \in \mathbb{R}^+$.

Lemma 3.1 Let the operator $\mathbf{K_F} : Im\mathbf{N} \to (dom\mathbf{N} \cap ker\mathbf{F})$ be defined by

$$\mathbf{K_F}\mathrm{y} = {}^{\rho}I^{\alpha}_{0^+}\mathrm{y},$$

with the norm

$$||\mathbf{K_F}\mathrm{y}||_{\mathcal{X}} \leq \max\{\tau_\alpha,\, 1\}||\mathrm{y}||_{\mathcal{Y}},$$

where $\mathrm{y} \in Im\mathbf{N}$ and

$$\tau_\alpha = \frac{S^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}. \tag{6}$$

Then, $\mathbf{K_F} = \mathbf{N_F^{-1}}$, where $\mathbf{N_F} : (dom\mathbf{N} \cap ker\mathbf{F}) \to Im\mathbf{N}$.

Proof. By Lemma 2.3, for $\mathrm{y} \in Im\mathbf{N}$, we have

$$\mathbf{N_F}(\mathbf{K_F}\mathrm{y}) = \mathbf{N_F}\left({}^{\rho}I^{\alpha}_{0^+}\mathrm{y}\right)$$

$$= {}^{\rho}_{c}D^{\alpha}_{0^+}\, {}^{\rho}I^{\alpha}_{0^+}\mathrm{y} = \mathrm{y}.$$

Furthermore, if $\upsilon \in (dom\mathbf{N} \cap ker\mathbf{F})$, then $\upsilon \in ker\mathbf{F}$. So, $\upsilon(0) = -a_1 t^{\rho}$, we have

$$\mathbf{K_F}\left(\mathbf{N_F}\upsilon\right) = \mathbf{K_F}\left({}^{\rho}_{c}D^{\alpha}_{0^+}\upsilon\right)$$

$$= {}^{\rho}I^{\alpha}_{0^+}\, {}^{\rho}_{c}D^{\alpha}_{0^+}\upsilon$$

$$= \upsilon - \upsilon(0) - a_1 t^{\rho} = \upsilon.$$

This shows that $\mathbf{K_F} = \mathbf{N_F^{-1}}$. From $t \in [0, S]$ and $||\mathrm{y}||_{\mathcal{Y}} = \max\{|\mathrm{y}(t)|, t \in [0, S]\}$, we get

$$|\mathbf{K_F}\mathrm{y}(t)| = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\left|\int_0^t \mathrm{s}^{\rho-1}(t^{\rho} - \mathrm{s}^{\rho})^{\alpha-1}\mathrm{y}(\mathrm{s})d\mathrm{s}\right|$$

$$\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \mathrm{s}^{\rho-1}(t^{\rho} - \mathrm{s}^{\rho})^{\alpha-1}|\mathrm{y}(\mathrm{s})|d\mathrm{s}$$

$$\leq \frac{||\mathrm{y}||_{\mathcal{Y}} S^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}.$$

Therefore, $||\mathbf{K_F}\mathrm{y}||_{\mathcal{X}} \leq \max\{\tau_{\alpha}, 1\}||\mathrm{y}||_{\mathcal{Y}}$. □

Lemma 3.2 Let $\upsilon \in \mathcal{X}$, τ_{α} be defined in (6) and $a_1 \in \mathbb{R}$. Then, we have

$$||(\mathbf{I} - \mathbf{F})\upsilon||_{\mathcal{X}} \leq \max\{\tau_{\alpha}, 1\}||\mathbf{N}\upsilon||_{\mathcal{Y}}$$

and

$$||\mathbf{F}\upsilon||_{\mathcal{X}} \leq |\upsilon(0)| + \tau,$$

where

$$\tau = |a_1|S^{\rho}. \tag{7}$$

Proof. By Lemma 3.1, for $\upsilon \in \mathcal{X}$,

$$||(\mathbf{I} - \mathbf{F})\upsilon||_{\mathcal{X}} = ||\mathbf{K_F}\mathbf{N}(\mathbf{I} - \mathbf{F})\upsilon||_{\mathcal{X}} \leq \max\{\tau_{\alpha}, 1\}||\mathbf{N}(\mathbf{I} - \mathbf{F})\upsilon||_{\mathcal{Y}} \leq \max\{\tau_{\alpha}, 1\}||\mathbf{N}\upsilon||_{\mathcal{Y}}.$$

This comes from the fact that $\mathbf{F}\upsilon \in Im\mathbf{F} = ker\mathbf{N}$, which means $\mathbf{NF}\upsilon = 0$. Also, by the definition of the operator $\mathbf{F}$ and from $t \in [0, S]$, we get

$$|\mathbf{F}\upsilon(t)| = |\upsilon(0) + a_1 t^{\rho}| \quad \text{and} \quad |\mathbf{NF}\upsilon| = 0.$$

Therefore, $||\mathbf{F}\upsilon||_{\mathcal{X}} \leq |\upsilon(0)| + \tau$. □

We define the operator $\mathbf{M} : \mathcal{X} \to \mathcal{Y}$ by

$$\mathbf{M}\upsilon = \varphi_k \left({}^{\rho}I^{\beta}_{0+} H(t,\, \upsilon(t),\, ({}^{\rho}_{c}D^{\alpha}_{0+}\upsilon)(t)) \right).$$

Lemma 3.3 Assume $\mathbf{\Omega}$ is an open bounded subset of $\mathcal{X}$ such that $dom\mathbf{N} \cap \overline{\mathbf{\Omega}} \neq \emptyset$. Then, $\mathbf{M}$ is $\mathbf{N}$-compact on $\overline{\mathbf{\Omega}}$.

Proof. From the continuity of a function H and the definition of an operator $\mathbf{M}$, there exists $c \in \mathbb{R}^+$ such that for each $t \in [0,\, S]$,

$$|H(t,\, \upsilon(t),\, ({}^{\rho}_{c}D^{\alpha}_{0+}\upsilon)(t))| \leq c \tag{8}$$

and $|\mathbf{M}\upsilon(t)| \leq (c\tau_{\beta})^{k-1}$ with $\upsilon \in \overline{\mathbf{\Omega}}$. Moreover, from the definition of an operator $\mathbf{G}$, we get

$$||\mathbf{GM}\upsilon||_{\mathcal{Y}} = |\mathbf{M}\upsilon(S)| \leq (c\tau_{\beta})^{k-1} \triangleq d. \tag{9}$$

So, $\mathbf{GM}(\overline{\mathbf{\Omega}})$ is bounded. Similarly, by the definitions of operators $\mathbf{K_F}$ and $\mathbf{G}$, we get that $\mathbf{K_F}(\mathbf{I} - \mathbf{G})\mathbf{M}(\overline{\mathbf{\Omega}})$ is bounded.

Now, we want to prove that $\mathbf{K_F}(\mathbf{I} - \mathbf{G})\mathbf{M}(\overline{\mathbf{\Omega}})$ is equicontinuous. To do this, take $\upsilon \in \overline{\mathbf{\Omega}}$, $0 \leq t_1 < t_2 \leq S$ with using (9), we get

$$\begin{aligned}
|\mathbf{K_F M}\upsilon(t_2) - \mathbf{K_F M}\upsilon(t_1)| &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \left| \int_0^{t_2} \mathrm{s}^{\rho-1}(t_2^{\rho} - \mathrm{s}^{\rho})^{\alpha-1}\mathbf{M}\upsilon(\mathrm{s})ds - \int_0^{t_1} \mathrm{s}^{\rho-1}(t_1^{\rho} - \mathrm{s}^{\rho})^{\alpha-1}\mathbf{M}\upsilon(\mathrm{s})ds \right| \\
&\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_1} \mathrm{s}^{\rho-1}[(t_2^{\rho} - \mathrm{s}^{\rho})^{\alpha-1} - (t_1^{\rho} - \mathrm{s}^{\rho})^{\alpha-1}]|\mathbf{M}\upsilon(\mathrm{s})|d\mathrm{s} \\
&\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_1}^{t_2} \mathrm{s}^{\rho-1}(t_2^{\rho} - \mathrm{s}^{\rho})^{\alpha-1}|\mathbf{M}\upsilon(\mathrm{s})|d\mathrm{s} \\
&\leq \frac{d\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_1} \mathrm{s}^{\rho-1}\left((t_2^{\rho} - \mathrm{s}^{\rho})^{\alpha-1} - (t_1^{\rho} - \mathrm{s}^{\rho})^{\alpha-1}\right)d\mathrm{s} \\
&\quad + \frac{d\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_1}^{t_2} \mathrm{s}^{\rho-1}(t_2^{\rho} - \mathrm{s}^{\rho})^{\alpha-1}d\mathrm{s} \\
&= \frac{d}{\rho^{\alpha}\Gamma(\alpha+1)} \left(t_2^{\rho\alpha} - t_1^{\rho\alpha}\right)
\end{aligned}$$

which implies that

$$
\begin{aligned}
& |\mathbf{K_F}(\mathbf{I}-\mathbf{G})\mathbf{M}\upsilon(t_2)-\mathbf{K_F}(\mathbf{I}-\mathbf{G})\mathbf{M}\upsilon(t_1)| \\
\leq\ & |\mathbf{K_F}\mathbf{M}\upsilon(t_2)-\mathbf{K_F}\mathbf{M}\upsilon(t_1)|+|\mathbf{K_F}\mathbf{G}\mathbf{M}\upsilon(t_2)-\mathbf{K_F}\mathbf{G}\mathbf{M}\upsilon(t_1)| \\
\leq\ & \frac{2d}{\rho^{\alpha}\Gamma(\alpha+1)}\left(t_2^{\rho\alpha}-t_1^{\rho\alpha}\right).
\end{aligned}
$$

Since $t^{\rho\alpha}$ is uniformly continuous on $[0, S]$, $\mathbf{K_F}(\mathbf{I}-\mathbf{G})\mathbf{M}(\overline{\mathbf{\Omega}})$ is equicontinuous.
Now, we will prove ${}^{\rho}_{c}D^{\alpha}_{0^+}\mathbf{K_F}(\mathbf{I}-\mathbf{G})\mathbf{M}(\overline{\mathbf{\Omega}})$ is equicontinuous.
Case I: When $1<p\leq 2$, we get $k\geq 2$. For any $\upsilon\in\overline{\mathbf{\Omega}}$, $0\leq t_1<t_2\leq S$, from (6), there exists $\tau_\beta\in\mathbb{R}^+$ such that

$$
|\,{}^{\rho}I^{\beta}_{0^+}H(t,\,\upsilon(t),\,({}^{\rho}_{c}D^{\alpha}_{0^+}\upsilon)(t))|=|({}^{\rho}I^{\beta}_{0^+}H)(t)|\leq\tau_\beta.
$$

So, by Lemma 2.3 and Lemma 2.1 (case 2), we get

$$
\begin{aligned}
& \left|{}^{\rho}_{c}D^{\alpha}_{0^+}\mathbf{K_F}(\mathbf{I}-\mathbf{G})\mathbf{M}\upsilon(t_2)-{}^{\rho}_{c}D^{\alpha}_{0^+}\mathbf{K_F}(\mathbf{I}-\mathbf{G})\mathbf{M}\upsilon(t_1)\right| \\
=\ & |(\mathbf{I}-\mathbf{G})\mathbf{M}\upsilon(t_1)-(\mathbf{I}-\mathbf{G})\mathbf{M}\upsilon(t_2)| \\
=\ & \left|\varphi_k\left(({}^{\rho}I^{\beta}_{0^+}H)(t_2)\right)-\varphi_k\left(({}^{\rho}I^{\beta}_{0^+}H)(t_1)\right)\right| \\
\leq\ & \frac{(k-1)\tau_\beta^{k-2}}{\rho^{\beta-1}\Gamma(\beta)}\left|\int_0^{t_2}\mathrm{s}^{\rho-1}(t_2^{\rho}-\mathrm{s}^{\rho})^{\beta-1}H(\mathrm{s})d\mathrm{s}-\int_0^{t_1}\mathrm{s}^{\rho-1}(t_1^{\rho}-\mathrm{s}^{\rho})^{\beta-1}H(\mathrm{s})ds\right| \\
\leq\ & \frac{(k-1)c\tau_\beta^{k-2}}{\rho^{\beta-1}\Gamma(\beta)}\left[\int_0^{t_1}\mathrm{s}^{\rho-1}((t_1^{\rho}-\mathrm{s}^{\rho})^{\beta-1}-(t_2^{\rho}-\mathrm{s}^{\rho})^{\beta-1})d\mathrm{s}+\int_{t_1}^{t_2}\mathrm{s}^{\rho-1}(t_2^{\rho}-\mathrm{s}^{\rho})^{\beta-1}d\mathrm{s}\right] \\
=\ & \frac{(k-1)c\tau_\beta^{k-2}}{\rho^{\beta}\Gamma(\beta+1)}\left[t_1^{\rho\beta}+(t_2^{\rho}-t_1^{\rho})^{\beta}-t_2^{\rho\beta}+(t_2^{\rho}-t_1^{\rho})^{\beta}\right] \\
\leq\ & \frac{2(k-1)c\tau_\beta^{k-2}}{\rho^{\beta}\Gamma(\beta+1)}(t_2^{\rho}-t_1^{\rho})^{\beta}.
\end{aligned}
$$

Since t^{ρ} is uniformly continuous on $[0, S]$, $\mathbf{K_F}(\mathbf{I}-\mathbf{G})\mathbf{M}(\overline{\mathbf{\Omega}})$ is compact for $1<p\leq 2$.
Case II: When $p>2$ $(1<k<2)$, we will prove that $\mathbf{K_F}(\mathbf{I}-\mathbf{G})\mathbf{M}(\overline{\mathbf{\Omega}})$ is pointwise equicontinuous in two steps. Let any $\upsilon\in\overline{\mathbf{\Omega}}$ and $t_1\in[0, S]$. Then, either ${}^{\rho}I^{\beta}_{0^+}H(t_1)=0$ or ${}^{\rho}I^{\beta}_{0^+}H(t_1)\neq 0$.
Step 1: When $({}^{\rho}I^{\beta}_{0^+}H)(t_1)=0$. Then, for all $t>t_1$, we have

$$\left| {}^{\rho}_{c}D^{\alpha}_{0^+}\mathbf{K_F}(\mathbf{I}-\mathbf{G})\mathbf{M}\upsilon(t) - {}^{\rho}_{c}D^{\alpha}_{0^+}\mathbf{K_F}(\mathbf{I}-\mathbf{G})\mathbf{M}\upsilon(t_1)\right|$$

$$= \left|\varphi_k\left({}^{\rho}I^{\beta}_{0^+}H(t)\right)\right|$$

$$= \left|{}^{\rho}I^{\beta}_{0^+}H(t) - {}^{\rho}I^{\beta}_{0^+}H(t_1)\right|^{k-1}$$

$$\leq \left(\frac{c\rho^{1-\beta}}{\Gamma(\beta)}\left[\int_0^{t_1} \mathrm{s}^{\rho-1}\left((t_1^{\rho}-\mathrm{s}^{\rho})^{\beta-1}-(t^{\rho}-\mathrm{s}^{\rho})^{\beta-1}\right)d\mathrm{s} + \int_{t_1}^{t}\mathrm{s}^{\rho-1}(t^{\rho}-\mathrm{s}^{\rho})^{\beta-1}d\mathrm{s}\right]\right)^{k-1}$$

$$\leq \left(\frac{2c}{\rho^{\beta}\Gamma(\beta+1)}\left(t^{\rho}-t_1^{\rho}\right)^{\beta}\right)^{k-1}.$$

Step 2: If ${}^{\rho}I^{\beta}_{0^+}H(t_1)\neq 0$, then, there is $r>0$ such that $\left|{}^{\rho}I^{\beta}_{0^+}H(t_1)\right|\geq r$. Whenever $t>t_1$, we get

$$\left| {}^{\rho}_{c}D^{\alpha}_{0^+}\mathbf{K_F}(\mathbf{I}-\mathbf{G})\mathbf{M}\upsilon(t) - {}^{\rho}_{c}D^{\alpha}_{0^+}\mathbf{K_F}(\mathbf{I}-\mathbf{G})\mathbf{M}\upsilon(t_1)\right|$$

$$\leq \left|\varphi_k\left({}^{\rho}I^{\beta}_{0^+}H(t)\right) - \varphi_k\left({}^{\rho}I^{\beta}_{0^+}H(t_1)\right)\right|$$

$$\leq \frac{(k-1)r^{k-2}\rho^{1-\beta}}{\Gamma(\beta)}\left|\int_0^{t}\mathrm{s}^{\rho-1}(t^{\rho}-\mathrm{s}^{\rho})^{\beta-1}H(\mathrm{s})d\mathrm{s} - \int_0^{t_1}\mathrm{s}^{\rho-1}(t_1^{\rho}-\mathrm{s}^{\rho})^{\beta-1}H(\mathrm{s})d\mathrm{s}\right|$$

$$\leq \frac{(k-1)cr^{k-2}}{\rho^{\beta-1}\Gamma(\beta)}\left[\int_0^{t_1}\mathrm{s}^{\rho-1}((t_1^{\rho}-\mathrm{s}^{\rho})^{\beta-1}-(t^{\rho}-\mathrm{s}^{\rho})^{\beta-1})d\mathrm{s} + \int_{t_1}^{t}\mathrm{s}^{\rho-1}(t^{\rho}-\mathrm{s}^{\rho})^{\beta-1}d\mathrm{s}\right]$$

$$\leq \frac{(k-1)cr^{k-2}}{\rho^{\beta}\Gamma(\beta+1)}\left[t_1^{\rho\beta}+(t^{\rho}-t_1^{\rho})^{\beta}-t^{\rho\beta}+(t^{\rho}-t_1^{\rho})^{\beta}\right]$$

$$\leq \frac{2(k-1)cr^{k-2}}{\rho^{\beta}\Gamma(\beta+1)}|t^{\rho}-t_1^{\rho}|^{\beta}.$$

t^{ρ} and t_1^{ρ} are uniformly continuous on $[0, S]$ and $\mathbf{K_F}(\mathbf{I}-\mathbf{G})\mathbf{M}(\overline{\mathbf{\Omega}})$ is pointwise equicontinuous for $p>2$. Hence, $\mathbf{K_F}(\mathbf{I}-\mathbf{G})\mathbf{M}(\overline{\mathbf{\Omega}})$ is compact for all $p>1$. □

4. Main results

The following hypotheses are assumed for the continuous function $H(t, \upsilon(t), ({}^{\rho}_{c}D^{\alpha}_{0^+}\upsilon)(t))$:
$(\mathfrak{R}_1)$ There exist functions $\psi_i\in\mathcal{Y}$, $i=1, 2, 3$ such that

$$|H(t,\, \upsilon(t),\, ({}_{c}^{\rho}D_{0^+}^{\alpha}\upsilon)(t))| \leq \psi_1(t)+\psi_2(t)|\upsilon(t)|^{p-1}+\psi_3(t)|{}_{c}^{\rho}D_{0^+}^{\alpha}\upsilon(t)|^{p-1}, \quad t\in[0,\, S].$$

Then (1) has a solution at least, if and only if

• $S^{\rho\beta}(\tau_{\alpha}^{p-1}||\psi_2||_{\mathcal{Y}}+||\psi_3||_{\mathcal{Y}}) \neq \rho^{\beta}\Gamma(\beta+1)$, $p\in(1,\,2)$;

• $S^{\rho\beta}(2^{p-2}\tau_{\alpha}^{p-1}||\psi_2||_{\mathcal{Y}}+||\psi_3||_{\mathcal{Y}}) \neq \rho^{\beta}\Gamma(\beta+1)$, $p\geq 2$.

$(\mathfrak{R}_2)$ If $|\upsilon(0)|>\eta_1$, where $\eta_1>0$. Then,

$$\mathbf{GM}\upsilon \neq 0, \quad \text{for all} \quad \upsilon\in(dom\mathbf{N}\setminus ker\mathbf{N}).$$

$(\mathfrak{R}_3)$ For any $b\in\mathbb{R}$, if $|b|>\eta_2$, where $\eta_2>0$, then either

$$b\mathbf{GM}\upsilon<0, \qquad \upsilon\in ker\mathbf{N},$$

or

$$b\mathbf{GM}\upsilon>0, \qquad \upsilon\in ker\mathbf{N}.$$

Lemma 4.1 Let $\xi\in(0,\,1)$. Then, the set

$$\mathbf{\Omega_1}=\{\upsilon\in(dom\mathbf{N}\setminus ker\mathbf{N}):\mathbf{N}\upsilon=\xi\mathbf{M}\upsilon\}$$

is bounded.

Proof. Let $\upsilon\in\mathbf{\Omega_1}$. Then, $\upsilon\in(dom\mathbf{N}\setminus ker\mathbf{N})$. This means $\upsilon\notin ker\mathbf{N}$, which implies that $\mathbf{N}\upsilon=\xi\mathbf{M}\upsilon\neq 0$. So, $\xi\neq 0$ and $\mathbf{M}\upsilon\in Im\mathbf{N}=ker\mathbf{G}$. Hence, $\mathbf{GM}\upsilon=0$. From assumption $(\mathfrak{R}_2)$, there exists $\eta_1>0$ such that $|\upsilon(0)|\leq\eta_1$ and by Lemma 3.2, we get

$$|\upsilon|=|(\mathbf{I}-\mathbf{F})\upsilon+\mathbf{F}\upsilon|\leq\max\{\tau_{\alpha},\,1\}|\mathbf{N}\upsilon|+\eta_1+\tau, \tag{10}$$

where τ_{α} and τ are defined in (6) and (7), respectively. From $(\mathfrak{R}_1)$ and $\mathbf{N}\upsilon=\xi\mathbf{M}\upsilon$, for each $\upsilon\in\mathbf{\Omega_1}$, we have

$${}_{c}^{\rho}D_{0^+}^{\alpha}\upsilon(t)=\xi\varphi_k\left({}^{\rho}I_{0^+}^{\beta}H(t,\,\upsilon(t),\,({}_{c}^{\rho}D_{0^+}^{\alpha}\upsilon)(t))\right)$$

and hence,

$$
\begin{aligned}
\left|\varphi_p\left({}^{\rho}_{c}D^{\alpha}_{0^+}\upsilon(t)\right)\right| &= \xi^{p-1}\left|{}^{\rho}I^{\beta}_{0^+}H(t,\, \upsilon(t),\, ({}^{\rho}_{c}D^{\alpha}_{0^+}\upsilon)(t))\right| \\
&\leq \frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_0^t \mathrm{s}^{\rho-1}(t^{\rho}-\mathrm{s}^{\rho})^{\beta-1}|H(\mathrm{s},\, \upsilon(\mathrm{s}),\, ({}^{\rho}_{c}D^{\alpha}_{0^+}\upsilon)(\mathrm{s}))|d\mathrm{s} \\
&\leq \frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_0^t \mathrm{s}^{\rho-1}(t^{\rho}-\mathrm{s}^{\rho})^{\beta-1}\left(||\psi_1||_{\mathcal{Y}}+||\psi_2||_{\mathcal{Y}}|\upsilon|^{p-1}+||\psi_3||_{\mathcal{Y}}|{}^{\rho}_{c}D^{\alpha}_{0^+}\upsilon|^{p-1}\right)d\mathrm{s} \\
&\leq \frac{S^{\rho\beta}}{\rho^{\beta}\Gamma(\beta+1)}\left(||\psi_1||_{\mathcal{Y}}+||\psi_2||_{\mathcal{Y}}|\upsilon|^{p-1}+||\psi_3||_{\mathcal{Y}}|\mathbf{N}\upsilon|^{p-1}\right).
\end{aligned}
$$

From $\left|\varphi_p\left({}^{\rho}_{c}D^{\alpha}_{0^+}\upsilon(t)\right)\right| = |{}^{\rho}_{c}D^{\alpha}_{0^+}\upsilon(t)|^{p-1}$ and (10), we have

$$
|{}^{\rho}_{c}D^{\alpha}_{0^+}\upsilon(t)|^{p-1} \leq \frac{S^{\rho\beta}}{\rho^{\beta}\Gamma(\beta+1)}\left(||\psi_1||_{\mathcal{Y}}+||\psi_2||_{\mathcal{Y}}(\tau_{\alpha}|\mathbf{N}\upsilon|+\eta_1+\tau)^{p-1}+||\psi_3||_{\mathcal{Y}}|\mathbf{N}\upsilon|^{p-1}\right)
$$

which leads to

$$
|\mathbf{N}\upsilon(t)|^{p-1} \leq \frac{S^{\rho\beta}}{\rho^{\beta}\Gamma(\beta+1)}\left(||\psi_1||_{\mathcal{Y}}+||\psi_2||_{\mathcal{Y}}\left(\varphi_p(\tau_{\alpha}|\mathbf{N}\upsilon|+\eta_1+\tau)\right)+||\psi_3||_{\mathcal{Y}}|\mathbf{N}\upsilon|^{p-1}\right).
$$

By Lemma 2.2, we have two cases

Case I: When $p \in (1,\, 2)$, by the relation (2) and the fact that τ_{α}, $|\mathbf{N}\upsilon|$, and $\eta_1+\tau$ are nonnegative, we have

$$
|\mathbf{N}\upsilon(t)|^{p-1} \leq \frac{S^{\rho\beta}}{\rho^{\beta}\Gamma(\beta+1)}\left(||\psi_1||_{\mathcal{Y}}+||\psi_2||_{\mathcal{Y}}\left(\tau_{\alpha}^{p-1}|\mathbf{N}\upsilon|^{p-1}+(\eta_1+\tau)^{p-1}\right)+||\psi_3||_{\mathcal{Y}}|\mathbf{N}\upsilon|^{p-1}\right).
$$

By solving the previous inequality, noting that the reciprocal of $p-1$ is $k-1$, and using assumption $(\mathfrak{R}_1)$, we get

$$
|\mathbf{N}\upsilon| \leq \left|\frac{S^{\rho\beta}(||\psi_1||_{\mathcal{Y}}+(\eta_1+\tau)^{p-1}||\psi_2||_{\mathcal{Y}})}{\rho^{\beta}\Gamma(\beta+1)-S^{\rho\beta}(\tau_{\alpha}^{p-1}||\psi_2||_{\mathcal{Y}}+||\psi_3||_{\mathcal{Y}})}\right|^{k-1} \triangleq \mathcal{K}_1.
$$

Case II: When $p \geq 2$, in the same way, we get

$$
|\mathbf{N}\upsilon| \leq \left|\frac{S^{\rho\beta}(||\psi_1||_{\mathcal{Y}}+2^{p-2}(\eta_1+\tau)^{p-1}||\psi_2||_{\mathcal{Y}})}{\rho^{\beta}\Gamma(\beta+1)-S^{\rho\beta}(2^{p-2}\tau_{\alpha}^{p-1}||\psi_2||_{\mathcal{Y}}+||\psi_3||_{\mathcal{Y}})}\right|^{k-1} \triangleq \mathcal{K}_2.
$$

This means that there exists a constant $\mathcal{K}_i > 0$, where $i = 1,\, 2$ such that

$$|\mathbf{N}\upsilon| \leq \max_{i=1,\,2}\{\mathcal{K}_i\}. \tag{11}$$

Hence, by (10) and (11), $\mathbf{\Omega_1}$ is bounded. □

Lemma 4.2 The set $\mathbf{\Omega_2} = \{\upsilon \in ker\mathbf{N} : \mathbf{M}\upsilon \in Im\mathbf{N}\}$ is bounded.

Proof. Let $\upsilon \in \mathbf{\Omega_2}$. Then, we have $\upsilon \in ker\mathbf{N} = Im\mathbf{F}$. So, $\upsilon = a_0 + a_1 t^\rho$, where $a_0,\, a_1 \in \mathbb{R}$. Since $\mathbf{M}\upsilon \in Im\mathbf{N} = ker\mathbf{G}$, $\mathbf{GM}\upsilon = 0$. By Lemma 3.2 and assumption $(\mathfrak{R}_3)$, there exists a positive constant η_2 such that $|a_0| \leq \eta_2$; then we have $|\upsilon| \leq |a_0| + \tau \leq \eta_2 + \tau$, where τ is defined in (7). Therefore, $\mathbf{\Omega_2}$ is bounded. □

Lemma 4.3 Let $\mathbf{J} : Im\mathbf{G} \to ker\mathbf{N}$ be a linear isomorphism operator defined by

$$\mathbf{J}(r) = r(a_0 + a_1 t^\rho), \qquad r \in \mathbb{R},$$

where $a_0,\, a_1 \in \mathbb{R}$ and $t \in [0,\, S]$. Then, the set

$$\mathbf{\Omega_3} = \{\upsilon \in ker\mathbf{N} : \pm\xi\upsilon + (1-\xi)\mathbf{JGM}\upsilon = 0,\ \xi \in [0,\, 1]\}$$

is bounded.

Proof. Assuming that $\upsilon \in \mathbf{\Omega_3}$ it follows that $\upsilon \in ker\mathbf{N} = Im\mathbf{F}$ and $\upsilon = r(a_0 + a_1 t^\rho)$. So,

$$\mp\xi r(a_0 + a_1 t^\rho) = (1-\xi)(a_0 + a_1 t^\rho)\mathbf{GM}(r(a_0 + a_1 t^\rho)).$$

Then, we have three cases:

Case 1: If $\xi = 0$, then we have $\mathbf{GM}\upsilon = 0$. By Lemma 3.2 and assumption $(\mathfrak{R}_3)$, there exists $\eta_2 > 0$ such that $|ra_0| \leq \eta_2$; then we have $|\upsilon| \leq \eta_2 + \tau$, where τ is defined in (7). So, $\mathbf{\Omega_3}$ is bounded if $\xi = 0$;

Case 2: If $\xi = 1$, we get $\upsilon = 0$. So, $\mathbf{\Omega_3}$ is bounded if $\xi = 1$;

Case 3: If $\xi \in (0,\, 1)$, then we have $\upsilon = \dfrac{\pm(\xi-1)(a_0 + a_1 t^\rho)\mathbf{GM}\upsilon(t)}{\xi}$. From $t \in [0,\, S]$ and the equation (9) with $\upsilon \in \mathbf{\Omega_3}$, we have

$$|\upsilon(t)| \leq \frac{(1-\xi)}{\xi}(|a_0| + |a_1|t^\rho)||\mathbf{GM}\upsilon||_{\mathcal{Y}} \leq \frac{(|a_0| + |a_1|S^\rho)d}{\xi}.$$

So, $\mathbf{\Omega_3}$ is bounded if $\xi \in (0,\, 1)$.

From the precious analysis, $\mathbf{\Omega_3}$ is bounded for all $\xi \in [0,\, 1]$. □

Theorem 4.1 Suppose that the hypotheses $(\mathfrak{R}_1)$-$(\mathfrak{R}_3)$ hold; then problem (1) has at least one solution in $dom\mathbf{N} \cap \overline{\mathbf{\Omega}}$.

Proof. Assume that $\mathbf{\Omega}$ is an open bounded subset of $\mathcal{X}$ and $\mathbf{\Omega}_i$, $i = 1,\, 2,\, 3$ are defined as in the previous three lemmas such that $\cup_{i=1}^{3}\overline{\mathbf{\Omega_i}} \subset \mathbf{\Omega}$. From Lemma 3.3, $\mathbf{M}$ is $\mathbf{N}$-compact on $\overline{\mathbf{\Omega}}$. By Lemmas 4.1 and 4.2, we get

- $\mathbf{N}\upsilon \neq \xi\mathbf{M}\upsilon$, $\forall(\upsilon,\, \xi) \in [(dom\mathbf{N} \setminus ker\mathbf{N}) \cap \partial\mathbf{\Omega}] \times (0,\, 1)$;
- $\mathbf{M}\upsilon \notin Im\mathbf{N}$, $\forall\upsilon \in (ker\mathbf{N} \cap \partial\mathbf{\Omega})$;
- Let $\mathbf{H}(\upsilon,\, \xi) = \pm\xi\upsilon + (1-\xi)\mathbf{JGM}\upsilon$, by Lemma 4.3, we have

$$\mathbf{H}(\upsilon, \xi) \neq 0, \qquad \forall \upsilon \in (ker\mathbf{N} \cap \partial \mathbf{\Omega}).$$

Then, by the homotopy property of degree, we get

$$\begin{aligned} deg(\mathbf{JGM}|_{ker\mathbf{N}}, \mathbf{\Omega} \cap ker\mathbf{N}, 0) &= deg(\mathbf{H}(., 0), \mathbf{\Omega} \cap ker\mathbf{N}, 0) \\ &= deg(\mathbf{H}(., 1), \mathbf{\Omega} \cap ker\mathbf{N}, 0) \\ &= deg(\pm \mathbf{I}, \mathbf{\Omega} \cap ker\mathbf{N}, 0) \\ &\neq 0. \end{aligned}$$

Hence, by Theorem 2.1 and Remark 2.4, we can conclude that $\mathbf{N}\upsilon = \mathbf{M}\upsilon$ has one solution at least in $dom\mathbf{N} \cap \overline{\mathbf{\Omega}}$, $\forall \rho \in \mathbb{R}^+$. □

5. Illustrative examples

Now, we will present two examples to clarify our main results.

Example 5.1 Consider the fractional p-Laplacian equation for $t \in [0, 1]$:

$$\begin{cases} {}^{2}_{c}D^{\frac{1}{2}}_{0^+} \varphi_{\frac{5}{2}} \left({}^{2}_{c}D^{\frac{3}{2}}_{0^+} \upsilon(t) \right) = 1 + t^2 + t\upsilon^{\frac{3}{2}}(t) - \sqrt{2t} \left({}^{2}_{c}D^{\frac{3}{2}}_{0^+} \upsilon(t) \right)^{\frac{3}{2}}, \\ {}^{2}_{c}D^{\frac{3}{2}}_{0^+} \upsilon(0) = 0 = {}^{2}_{c}D^{\frac{3}{2}}_{0^+} \upsilon(1), \qquad \upsilon'(0) = \upsilon_0. \end{cases} \tag{12}$$

By comparing equations (1) and (12), we get $\alpha = \dfrac{3}{2}$, $\beta = \dfrac{1}{2}$, $\rho = 2$, $S = 1$ and $p = \dfrac{5}{2}$; then $k = \dfrac{5}{3}$ and $\max\{\tau_\alpha, 1\} = \max\left\{\dfrac{2}{3\sqrt{2\pi}}, 1\right\} = 1$. It is clear that $H : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function with

$$|H(t, \upsilon(t), ({}^{\rho}_{c}D^{\alpha}_{0^+} \upsilon)(t))| \leq 1 + t^2 + t|\upsilon(t)|^{\frac{3}{2}} + \sqrt{2t} \left| {}^{2}_{c}D^{\frac{3}{2}}_{0^+} \upsilon(t) \right|^{\frac{3}{2}}.$$

So, according to assumption $(\mathfrak{R}_1)$, $\psi_1(t) = 1 + t^2$, $\psi_2(t) = t$ and $\psi_3(t) = \sqrt{2t}$. Hence,

$$S^{\rho\beta}(2^{p-2}\tau_\alpha^{p-1}||\psi_2||_{\mathcal{Y}} + ||\psi_3||_{\mathcal{Y}}) - \rho^\beta \Gamma(\beta + 1) = \sqrt{2}\left(2 - \Gamma\left(\frac{3}{2}\right)\right) \neq 0,$$

which holds the rest of assumption $(\mathfrak{R}_1)$.

Since $t \in [0, 1]$, we get

$$|H(t,\, \upsilon(t),\, ({}^{\rho}_{c}D^{\alpha}_{0^+}\upsilon)(t))| \leq 2+||\upsilon||^{\frac{3}{2}}_{\mathcal{X}}+\sqrt{2}\left|\left|{}^{2}_{c}D^{\frac{3}{2}}_{0^+}\upsilon\right|\right|^{\frac{3}{2}}_{\mathcal{X}}.$$

Let $\upsilon \in ker\mathbf{N}$. Then $\upsilon(t)=a_0+a_1t^2$, where $a_i \in \mathbb{R}$, $i=0,\, 1$ and ${}^{2}_{c}D^{\frac{3}{2}}_{0^+}\upsilon(t)=0$. So,

$$||\upsilon||_{\mathcal{X}}=\max\left\{||\upsilon||_{\mathcal{Y}},\, \left|\left|{}^{2}_{c}D^{\frac{3}{2}}_{0^+}\upsilon\right|\right|_{\mathcal{Y}}\right\} \leq |a_0|+|a_1|,$$

and

$$\left|\left|{}^{2}_{c}D^{\frac{3}{2}}_{0^+}\upsilon\right|\right|_{\mathcal{Y}}=\max\left\{\left|\left|{}^{2}_{c}D^{\frac{3}{2}}_{0^+}\upsilon\right|\right|_{\mathcal{Y}},\, ||{}^{2}_{c}D^{3}_{0^+}\upsilon||_{\mathcal{Y}}\right\}=0.$$

Hence,

$$|H(t,\, \upsilon(t),\, ({}^{\rho}_{c}D^{\alpha}_{0^+}\upsilon)(t))| \leq 2+(|a_0|+|a_1|)^{\frac{3}{2}},$$

which implies that

$$\begin{aligned}
\mathbf{GM}\upsilon(t) &= \mathbf{G}\left(\varphi_{\frac{5}{3}}\left({}^{2}I^{\frac{1}{2}}_{0^+}H(t,\, \upsilon(t),\, ({}^{\rho}_{c}D^{\alpha}_{0^+}\upsilon)(t))\right)\right)\\
&\leq \mathbf{G}\left(\left(\sqrt{\frac{2}{\pi}}(2+(|a_0|+|a_1|)^{\frac{3}{2}})\int_0^t \mathrm{s}(t^2-\mathrm{s}^2)^{-\frac{1}{2}}d\mathrm{s}\right)^{\frac{2}{3}}\right)\\
&= \mathbf{G}\left(\left(\sqrt{\frac{2}{\pi}}(2+(|a_0|+|a_1|)^{\frac{3}{2}})t\right)^{\frac{2}{3}}\right)\\
&= \left(\sqrt{\frac{2}{\pi}}(2+(|a_0|+|a_1|)^{\frac{3}{2}})\right)^{\frac{2}{3}},
\end{aligned}$$

which satisfies assumption $(\mathfrak{R}_2)$. To hold assumption $(\mathfrak{R}_3)$, we have two cases:

Case 1: If $\mathbf{GM}\upsilon=0$, then we can choose $b=0$;

Case 2: If $\mathbf{GM}\upsilon \neq 0$, we can choose $b=-1$ and $\eta_2=\dfrac{1}{3}$, then we get either $b\mathbf{GM}\upsilon<0$ or $b\mathbf{GM}\upsilon<0$ depending on the sign of $\mathbf{GM}\upsilon$.

Hence, assumptions $(\mathfrak{R}_1)$-$(\mathfrak{R}_3)$ hold. Therefore, equation (12) has a solution (at least one).

Chai in [40] proved the existence of positive solutions for the equation

$$D_{0^+}^{\frac{1}{2}}\left(\varphi_{\frac{3}{2}}\left(D_{0^+}^{\frac{3}{2}}\upsilon(t)\right)\right)+H(t,\,\upsilon(t))=0,\quad t\in(0,\,1),$$

where $H(t,\,\upsilon(t))$ is a continuous function defined by

$$H(t,\,\upsilon(t))=\begin{cases}(\sqrt{\upsilon}+7)\sin\left(\dfrac{\pi}{90}(32t+13)\right), & \text{for}\quad \upsilon\geq 1,\\[2ex] 8\upsilon^2\sin\left(\dfrac{\pi}{90}(32t+13)\right), & \text{for}\quad \upsilon\in[0,\,1).\end{cases}$$

Now, we will study this equation with the conditions of this paper in the following example.

Example 5.2 The fractional p-Laplacian equation:

$$\begin{cases}D_{0^+}^{\frac{1}{2}}\left(\varphi_{\frac{3}{2}}\left(D_{0^+}^{\frac{3}{2}}\upsilon(t)\right)\right)+H(t,\,\upsilon(t),\,(D_{0^+}^{\alpha}\upsilon)(t))=0, & t\in(0,\,1),\\[2ex] D_{0^+}^{\frac{3}{2}}\upsilon(0)=0=D_{0^+}^{\frac{3}{2}}\upsilon(1), & \upsilon'(0)=\upsilon_0,\end{cases}\tag{13}$$

where H is defined as above. By comparing equations (1) and (13), we get $\alpha=\frac{3}{2}$, $\beta=\frac{1}{2}$, $\rho=1=S$ and $p=\frac{3}{2}$; then $k=3$ and $\max\{\tau_\alpha,\,1\}=\max\left\{\frac{4}{3\sqrt{\pi}},\,1\right\}=1$.

Case 1: If $\upsilon\geq 1$. Then,

$$|H(t)|\leq(|\upsilon|^{\frac{1}{2}}+7)\sin\left(\frac{\pi}{90}(32t+13)\right)$$

which implies that $\psi_1(t)=7\sin\left(\frac{\pi}{90}(32t+13)\right)$, $\psi_2(t)=\sin\left(\frac{\pi}{90}(32t+13)\right)$ and $\psi_3(t)=0$. Hence,

$$S^{\rho\beta}(\tau_\alpha^{p-1}||\psi_2||_{\mathcal{Y}}+||\psi_3||_{\mathcal{Y}})-\rho^{\beta}\Gamma(\beta+1)=1-\Gamma\left(\frac{3}{2}\right)\neq 0.$$

Case 2: If $\upsilon\in[0,\,1)$. Then,

$$|H(t)|\leq 8\sin\left(\frac{\pi}{90}(32t+13)\right)$$

which implies that $\psi_1(t)=8\sin\left(\frac{\pi}{90}(32t+13)\right)$ and $\psi_2(t)=\psi_3(t)=0$. Hence,

$$S^{\rho\beta}(\tau_\alpha^{p-1}||\psi_2||_{\mathcal{Y}}+||\psi_3||_{\mathcal{Y}})-\rho^{\beta}\Gamma(\beta+1)=\Gamma\left(\frac{3}{2}\right)\neq 0,$$

which holds assumption $(\mathfrak{R}_1)$.

Let $\upsilon \in ker\mathbf{N}$, $\upsilon(t) = a_0 + a_1 t$, where $a_i \in \mathbb{R}$, $i = 0, 1$ and $(a_0 + a_1) \geq 0$. Since $t \in (0, 1)$, then we get

$$\left|H(t, \upsilon(t), ({}^{\rho}_{c}D^{\alpha}_{0^+}\upsilon)(t))\right| = \begin{cases} \sqrt{a_0 + a_1} + 7 & \text{for} \quad \upsilon \geq 1, \\ \\ 8 & \text{for} \quad \upsilon \in [0, 1). \end{cases}$$

So,

$$\mathbf{GM}\upsilon(t) = \mathbf{G}\left(\varphi_3\left(-I^{\frac{1}{2}}_{0^+} H(t, \upsilon(t), ({}^{\rho}_{c}D^{\alpha}_{0^+}\upsilon)(t))\right)\right).$$

Case 1: If $\upsilon \geq 1$, then

$$\mathbf{GM}\upsilon(t) \leq \mathbf{G}\left(\left(\frac{\sqrt{a_0 + a_1} + 7}{\sqrt{\pi}} \int_0^t (t - \mathrm{s})^{-\frac{1}{2}} d\mathrm{s}\right)^2\right)$$

$$= \mathbf{G}\left(\left(\frac{2\sqrt{a_0 + a_1} + 14)}{\sqrt{\pi}} \sqrt{t}\right)^2\right)$$

$$= \left(\frac{2\sqrt{a_0 + a_1} + 14)}{\sqrt{\pi}}\right)^2.$$

Case 2: If $\upsilon \in [0, 1)$, then

$$\mathbf{GM}\upsilon(t) \leq \mathbf{G}\left(\left(\frac{8}{\sqrt{\pi}} \int_0^t (t - \mathrm{s})^{-\frac{1}{2}} d\mathrm{s}\right)^2\right)$$

$$= \mathbf{G}\left(\frac{256}{\pi} t\right)$$

$$= \frac{256}{\pi}.$$

The above cases satisfy assumption $(\mathfrak{R}_2)$. For all $\upsilon \in ker\mathbf{N}$, we have two cases:

• If $\mathbf{GM}\upsilon = 0$, then we can choose $b = 0$;

• If $\mathbf{GM}\upsilon \neq 0$, we can choose $b = -0.80763$ and $\eta_2 = 0.033109$; then we get either $b\mathbf{GM}\upsilon < 0$ or $b\mathbf{GM}\upsilon < 0$ depending on the sign of $\mathbf{GM}\upsilon$.

Hence, assumptions $(\mathfrak{R}_1)$-$(\mathfrak{R}_3)$ are hold. Therefore, equation (13) has a solution (at least one).

6. Conclusion

In this research, we have investigated the existence of a solution for a resonant problem that involves Caputo-Katugampola fractional derivatives with two different orders, type $\rho \geq 0$ and a p-Laplacian operator.

We have used two theorems according to the value of type of fractional derivative. We arrived at two cases:

• When $\rho \in (0, 1)$, the index of Fredholm operator is equal to zero. So, we have applied the coincidence degree theory due to Mawhin;

• When $\rho \in [1, \infty)$, the index of Fredholm operator does not finish. So, we have applied an improved version of Ge-Mawhin continuation theorem.

We presented the existence criteria and assumption. Finally, we have provided two examples to illustrate our results. As a continuation of this work, future research will insert the control function and investigate its impact on the solution.

Funding

This Project was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, Saudi Arabia under grant no. (IPP: 60-130-2025). The authors, therefore, acknowledge with thanks DSR for technical and financial support.

Conflict of interest

The authors declare no competing financial interest.

References

[1] Kilbas AA, Srivastava HM, Trujillo JJ. *Theory and Applications of Fractional Differential Equations, Vol. 204*. Elsevier; 2006.

[2] Baghani H, Salem A. Solvability and stability of a class of fractional Langevin differential equations. *Bulletin of the Mexican Mathematical Society*. 2024; 30: 46. Available from: https://doi.org/10.1007/s40590-024-00618-3.

[3] Salem A, Almaghamsi L. Solvability of sequential fractional differential equation at resonance. *Mathematics*. 2023; 11(4): 1044. Available from: https://doi.org/10.3390/math11041044.

[4] Podlubny I. *Fractional Differential Equations*. San Diego: Academic Press; 1999.

[5] Miller KS, Ross B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*. New York: John-Wily and Sons, Inc.; 1993.

[6] Salem A, Abdullah S. Controllability results to non-instantaneous impulsive with infinite delay for generalized fractional differential equations. *Alexandria Engineering Journal*. 2023; 70: 525-533. Available from: https://doi.org/10.1016/j.aej.2023.03.004.

[7] Raja MM, Vijayakumar V, Veluvolu KC. Higher-order caputo fractional integrodifferential inclusions of Volterra-Fredholm type with impulses and infinite delay: Existence results. *Journal of Applied Mathematics and Computing*. 2025; 71: 4849-4874. Available from: https://doi.org/10.1007/s12190-025-02412-4.

[8] Raja MM, Vijayakumar V, Shukla A, Nisar KS, Rezapour S. Investigating existence results for fractional evolution inclusions with order $r \in (1, 2)$ in Banach space. *International Journal of Nonlinear Sciences and Numerical Simulation*. 2023; 24(6): 2047-2060. Available from: https://doi.org/10.1515/ijnsns-2021-0368.

[9] Baghani H. An analytical improvement of a study of nonlinear Langevin equation involving two fractional orders in different intervals. *Journal of Fixed Point Theory and Applications*. 2019; 21: 95. Available from: https://doi.org/10.1007/s11784-019-0734-7.

[10] Salem A, Al-Dosari A. Hybrid differential inclusion involving two multi-valued operators with nonlocal multi-valued integral condition. *Fractal and Fractional*. 2022; 6: 109. Available from: https://doi.org/10.3390/fractalfract6020109.

[11] Raja MM, Vijayakumar V, Veluvolu KC. Improved order in Hilfer fractional differential systems: Solvability and optimal control problem for hemivariational inequalities. *Chaos, Solitons & Fractals*. 2024; 188: 115558. Available from: https://doi.org/10.1016/j.chaos.2024.115558.

[12] Raja MM, Vijayakumar V, Veluvolu KC, Shukla A, Nisar KS. Existence and optimal control results for Caputo fractional delay Clark's subdifferential inclusions of order $r \in (1, 2)$ with sectorial operators. *Optimal Control Applications and Methods*. 2024; 45(4): 1832-1850. Available from: https://doi.org/10.1002/oca.3125.

[13] Raja MM, Vijayakumar V, Shukla A, Nisar KS, Albalawi W, Abdel-Aty A-H. A new discussion concerning to exact controllability for fractional mixed Volterra-Fredholm integrodifferential equations of order with impulses. *AIMS Mathematics*. 2023; 8(5): 10802-10821. Available from: https://doi.org/10.3934/math.2023548.

[14] Salem A, Alharbi KN. Controllability for fractional evolution equations with infinite time-delay and non-local conditions in compact and noncompact cases. *Axioms*. 2023; 12(3): 264. Available from: https://doi.org/10.3390/axioms12030264.

[15] Baghani H, Nieto JJ. Some new properties of the Mittag-Leffler functions and their applications to solvability and stability of a class of fractional Langevin differential equations. *Qualitative Theory of Dynamical Systems*. 2024; 23: 18. Available from: https://doi.org/10.1007/s12346-023-00870-4.

[16] Vua H, Van Hoa N. Hyers-Ulam stability of fractional Integro-differential equation with a positive constant coefficient involving the generalized caputo fractional derivative. *Filomat*. 2022; 36(18): 6299-6316. Available from: https://doi.org/10.2298/FIL2218299V.

[17] Elmoataz A, Toutain M, Tenbrinck D. On the p-Laplacian and ∞-Laplacian on graphs with applications in image and data processing. *SIAM Journal on Imaging Sciences*. 2025; 8(4): 2412-2451. Available from: https://doi.org/10.1137/15M1022793.

[18] Salem A, Almaghamsi L, Alzahrani F. An infinite system of fractional order with p-Laplacian operator in a tempered sequence space via measure of noncompactness technique. *Fractal and Fractional*. 2021; 5(4): 182. Available from: https://doi.org/10.3390/fractalfract5040182.

[19] Wang Y, Wang H. Triple positive solutions for fractional differential equation boundary value problems at resonance. *Applied Mathematics Letters*. 2020; 106: 106376. Available from: https://doi.org/10.1016/j.aml.2020.106376.

[20] Song S, Cui Y. Existence of solutions for integral boundary value problems of mixed fractional differential equations under resonance. *Boundary Value Problems*. 2020; 2020: 23. Available from: https://doi.org/10.1186/s13661-020-01332-5.

[21] Azouzi M, Guedda L. Existence result for nonlocal boundary value problem of fractional order at resonance with p-Laplacian operator. *Azerbaijan Journal of Mathematics*. 2023; 13(1): 2218-6816.

[22] Sun B, Zhang S, Jiang W. Solvability of fractional functional boundary-value problems with p-Laplacian operator on a half-line at resonance. *Journal of Applied Analysis & Computation*. 2023; 13(1): 11-33. Available from: https://doi.org/10.11948/20210123.

[23] Hu L, Zhang S. On existence results for nonlinear fractional differential equations involving the p-Laplacian at resonance. *Mediterranean Journal of Mathematics*. 2016; 13: 955-966. Available from: https://doi.org/10.1007/s00009-015-0544-0.

[24] Pang H, Ge W, Tian M. Solvability of nonlocal boundary value problems for ordinary differential equation of higher order with a p-Laplacian. *Computers and Mathematics with Applications*. 2008; 56: 127-142. Available from: https://doi.org/10.1016/j.camwa.2007.11.039.

[25] Shen T, Liu W, Shen X. Solvability of fractional boundary value problem with p-Laplacian operator at resonance. *Advances in Difference Equations*. 2013; 2013: 295. Available from: https://doi.org/10.1186/1687-1847-2013-295.

[26] Salem A. Existence results of solutions for anti-periodic fractional Langevin equation. *Journal of Applied Analysis and Computation*. 2020; 10(6): 2557-2574. Available from: https://doi.org/10.11948/20190419.

[27] Baghani H, Alzabut J, Farokhi-Ostad J, Nieto JJ. Existence and uniqueness of solutions for a coupled system of sequential fractional differential equations with initial conditions. *Journal of Pseudo-Differential Operators and Applications*. 2020; 11: 1731-1741. Available from: https://doi.org/10.1007/s11868-020-00359-7.

[28] Almaghamsi L, Salem A. Fractional Langevin equations with infinite-point boundary condition: Application to Fractional harmonic oscillator. *Journal of Applied Analysis and Computation*. 2023; 13(6): 3504-3523. Available from: https://doi.org/10.11948/20230124.

[29] Kenef E, Merzoug I, Guezane-Lakoud A. Existence, uniqueness and Ulam stability results for a mixed-type fractional differential equations with p-Laplacian operator. *Arabian Journal of Mathematics*. 2023; 12: 633-645. Available from: https://doi.org/10.1007/s40065-023-00436-x.

[30] Salem A, Almaghamsi L, Alzahrani F. An infinite system of fractional order with p-Laplacian operator in a tempered sequence space via measure of noncompactness technique. *Fractal and Fractional*. 2021; 5(4): 182. Available from: https://doi.org/10.3390/fractalfract5040182.

[31] Mawhin J. *Topological Degree Methods in Nonlinear Boundary Value Problems*. Providence, RI, USA: American Mathematical Society; 1979.

[32] Ge W, Ren J. An extension of Mawhin's continuation theorem and its application to boundary value problems with a p-Laplacain. *Nonlinear Analysis: Theory, Methods & Applications*. 2004; 58(3-4): 477-488. Available from: https://doi.org/10.1016/j.na.2004.01.007.

[33] Katugampola U. New approach to a generalized fractional integral. *Applied Mathematics and Computation*. 2011; 218: 860-865. Available from: https://doi.org/10.1016/j.amc.2011.03.062.

[34] Katugampola U. New approach to a generalized fractional derivatives. *Bulletin of Mathematical Analysis and Application*. 2014; 6: 1-15.

[35] O'Regan D, Cho YJ, Chen YQ. *Topological Degree Theory and Application*. Boca Raton, FL, USA: Chapman and Hall/CRC; 2006.

[36] Rezapour S, Souid MS, Etemad S, Bouazza Z, Ntouyas SK, Asawasamrit S, et al. Mawhin's continuation technique for a nonlinear BVP of variable order at resonance via piecewise constant functions. *Fractal and Fractional*. 2021; 5: 216. Available from: https://doi.org/10.3390/fractalfract5040216.

[37] Jarad F, Abdeljawadb T, Baleanu D. On the generalized fractional derivatives and their Caputo modification. *Journal of Nonlinear Sciences and Applications*. 2017; 10: 2607-2619. Available from: http://dx.doi.org/10.22436/jnsa.010.05.27.

[38] Salem A, Malaikah H, Kamel ES. An infinite system of fractional Sturm-Liouville operator with measure of noncompactness technique in Banach space. *Mathematics*. 2023; 11(6): 1444. Available from: https://doi.org/10.3390/math11061444.

[39] Lankham I, Nachtergaele B, Schilling A. *Linear Algebra-As an Introduction to Abstract Mathematics*. University of California; 2007.

[40] Chai G. Positive solutions for boundary value problem of fractional differential equation with p-Laplacian operator. *Boundary Value Problems*. 2012; 2012: 18. Available from: https://doi.org/10.1186/1687-2770-2012-18.