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Research Article

Fuzzy Covariance Control for Interval Type-2 Fuzzy Stochastic Systems with Individual State Variance Constraints

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Abstract: A fuzzy covariance control strategy is proposed for a class of uncertain nonlinear stochastic systems by means of the Interval Type-2 (IT2) Takagi-Sugeno Fuzzy Stochastic Model (TSFSM). By leveraging the IT2 TSFSM, the dynamics of nonlinear systems subject to uncertainties and stochastic behaviors can be characterized in a more comprehensive manner. In line with the IT2 membership function design of the TSFSM, the fuzzy controller is developed using the state feedback technique. Subsequently, the state covariance matrices and the variance constraints are constructed for each subsystem of the IT2 TSFSM. By reformulating the closed-loop model into parallel and cross-terms, the number of fuzzy rules required for the analysis can be effectively reduced. This approach also allows for the integration of covariance control theory into the IT2 fuzzy controller design. Using Lyapunov theory, stability criteria involving upper and lower-bound parameters and state covariance matrices are derived. Unlike typical Type-1 (T1) fuzzy control approaches, the upper and lower-bound membership functions complicate the integration of covariance control theory, which is also overcome in this research. Additionally, the stability and variance constraints are further achieved by applying an upper-bound matrix. The stability conditions are also converted into a Linear Matrix Inequality (LMI) problem. Eventually, a comparative simulation is conducted on the nonlinear ship steering system to demonstrate the validity and applicability of the proposed IT2 fuzzy covariance control strategy.

Keywords: interval type-2 fuzzy systems, takagi-sugeno fuzzy model, stochastic systems, covariance control, state variance constraints

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Abbreviation

TSFM	Takagi-Sugeno Fuzzy Model
TSFSM	Takagi-Sugeno Fuzzy Stochastic Model
T1	Type-1
T2	Type-2

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GT2	General Type-2
IT2	Interval Type-2
PDC	Parallel Distributed Compensation
LMI	Linear Matrix Inequality
RMS	Root-Mean-Square

1. Introduction

Modern control systems have become increasingly complex due to the rapid advancement of technology. Nowadays, many scholars have discussed stabilization and controller design issues related to nonlinear systems such as analog circuit systems, power systems, and double-inverted pendulums [1–3]. In an effort to address the control challenges of complex nonlinear systems, a substantial number of researchers have focused on fuzzy control theory. Fuzzy concepts and fuzzy sets were introduced by Zadeh in 1965 [4] and served as a solid foundation for the development of fuzzy control methodologies. By applying fuzzy sets, the Takagi-Sugeno Fuzzy Model (TSFM) has been developed and widely extended to effectively describe nonlinear system behavior [5]. By choosing appropriate operating points, a nonlinear system can be locally linearized into a number of subsystems represented as IF-THEN rules. Thus, each subsystem of the TSFM can describe the local dynamic within a specific operating range. As a result, the extended application of linear control theory, which has been extensively developed, is allowed to control nonlinear systems. For each subsystem, the Parallel Distributed Compensation (PDC) concept is utilized to establish a series of linear controllers [6]. Then, the overall fuzzy model and controller are respectively obtained by defuzzification using membership functions. To perform the stability analysis, the stability conditions of the closed-loop TSFM are derived according to Lyapunov theory and formulated into the Linear Matrix Inequality (LMI) problem [6, 7]. Eventually, feasible solutions for the fuzzy controller design problem can be obtained using the LMI toolbox [8]. To date, the TSFM, typically known as the Type-1 (T1) TSFM, has been extensively utilized in various practical control applications such as inverted pendulum cart systems [9], double inverted pendulum systems [10], unmanned marine vehicles [11] and ship steering systems [12].

Although these T1 TSFMs with crisp membership grades provide a good representation of nonlinear systems, they struggle to comprehensively capture the uncertainties. The uncertainties may result from the structure of control systems, which is damaged during long-time operation, and from the imprecise modeling. The impact of these uncertainties can seriously lead to a degradation in control performance. To better deal with uncertainty problems, the Type-2 (T2) TSFM provides an efficient solution [13]. In contrast to the T1 TSFM, the T2 membership functions in the T2 TSFM contain the primary and secondary membership grades, which reflect the difference in membership grades caused by uncertain factors [14]. The uncertainties are captured within upper and lower-bound membership functions from the primary membership function [15]. Therefore, the T2 TSFM is capable of involving uncertainties in characterizing nonlinear systems. According to the difference in secondary membership grade, the T2 TSFM is classified into the General T2 (GT2) and Interval T2 (IT2) TSFMs. In the GT2 TSFM [16], the GT2 membership functions are constructed by assigning secondary membership grades whose values, ranging from 0 to 1, are determined in response to uncertainty effects. This advantage enables the GT2 membership functions to more effectively capture the uncertain dynamics. Nevertheless, the computational burden significantly increases because the membership functions change from two dimensions to three dimensions. To overcome the problem, the IT2 membership functions have been proposed as a simplified form where all secondary membership grades are set to the maximum value of 1 [17]. Leveraging this setting, the IT2 membership function can efficiently reduce computational requirements while maintaining a certain degree of capability to describe the uncertainty effects. Since computational costs remain a primary concern from the perspective of practical applications, IT2 fuzzy logic systems are widely applied to the control of uncertain nonlinear systems [18, 19].

With the application of the IT2 TSFM, many researchers have put effort into discussing the control problem of uncertain nonlinear systems [18–25]. Nevertheless, the application of IT2 membership functions increases the difficulty in directly converting the output of the IT2 TSFM into crisp values by the defuzzification process. A more efficient approach solves the problem by first converting IT2 membership functions to T1 membership functions through a type-reduction mechanism before defuzzification. Incorporating the type-reduction process can effectively simplify the complexity to obtain a crisp value usable by the control system [20]. Therefore, the development of type-reduction mechanisms also becomes a critical issue for deriving the overall IT2 TSFM. In [21], scholars have proposed a systematic approach that integrates both type-reduction and defuzzification methods for the IT2 TSFM. Moreover, when the upper and lower membership function bounds coincide, the IT2 TSFM reduces to the T1 TSFM. Extending the concept of PDC, researchers have also developed the IT2 fuzzy controller design process and conducted simulations in various practical control applications [22–25]. However, these studies still lack exploration of countermeasures against stochastic behaviors. In particular, the highly complex working environment of the unpredictable ocean causes the ship steering control system to be severely and continuously affected by stochastic disturbances. This research aims to design an IT2 fuzzy controller for nonlinear stochastic systems via the IT2 Takagi-Sugeno Fuzzy Stochastic Model (TSFSM).

Under actual environmental conditions, external disturbances that significantly affect system stability and control performance must be considered in controller design. In general, most external disturbances are unpredictable, which poses significant challenges to controller design. From the energy perspective, the covariance control theory has been developed to enhance system performance in the presence of stochastic behaviors [26]. Based on the covariance theory, the stability of stochastic systems can be ensured by satisfying a Lyapunov equation constructed using state covariance matrices, which represent the state energy. In addition, the energy of each state affected by stochastic processes is suppressed by restricting the individual state variance within specific Root-Mean-Square (RMS) bounds. For perturbed stochastic systems, a design approach based on an upper-bound matrix is employed to analyze and ensure satisfaction of these variance constraints [27]. Extending the concept of linear covariance control theory, the T1 TSFM-based fuzzy covariance control approach has been proposed for nonlinear stochastic systems [28]. It is worth noting that the upper-bound matrix can also be applied in the Lyapunov stability conditions to substitute the state covariance matrix of all fuzzy subsystems. At present, the variance-constrained T1 fuzzy control method has been adopted for practical nonlinear systems, such as ship steering and synchronous generator systems, whose dynamics are significantly affected by stochastic disturbances [29, 30]. Nonetheless, uncertainties are inevitable in these practical systems and cannot be adequately handled by the T1 fuzzy control method.

Given the aforementioned problems, an IT2 fuzzy controller design strategy subject to variance constraints is proposed for uncertain nonlinear stochastic systems by combining IT2 fuzzy control and covariance control theories. In contrast to the typical T1 fuzzy control method, the IT2 TSFM is constructed from both upper and lower-bound models, which introduces challenges to this combination. To solve the controller design problem, the design procedure is outlined as follows. First, the nonlinear system under the effect of uncertainties and stochastic behaviors is represented by the IT2 TSFSM. Subsequently, the IT2 fuzzy controller is developed consistent with the membership function designed for the IT2 T-SFM. For the closed-loop fuzzy model, the state covariance matrices are defined for the state energy of each fuzzy subsystem, and the RMS-based variance constraints are imposed on these matrices. By combining Lyapunov stability theory, the IT2 fuzzy control approach, and covariance control theory, a stability analysis method is proposed for the IT2 fuzzy controller design, based on the covariance matrices as well as the parameters of the upper and lower-bound fuzzy models. Then, an upper-bound matrix serves as a unified positive definite matrix for stability analysis using Lyapunov theory, which further guarantees stability and satisfies the individual state variance constraints. Finally, these stability conditions are derived into the LMI problem and solved via the convex optimization algorithm [31]. To demonstrate the effectiveness of the proposed IT2 fuzzy covariance control method, a simulation of the ship steering system is conducted to compare the proposed method with the T1 fuzzy covariance control method [28] and the typical IT2 fuzzy control method [32].

The structure of this research is outlined as follows. In Section 2, the IT2 TSFSM and the corresponding IT2 fuzzy controller are presented. The definition of state covariance matrix and RMS variance constraints are introduced. In Section 3, the stability criteria of the closed-loop IT2 TSFM are proposed with Lyapunov stability theory and covariance control theory. In Section 4, the comparison and simulation results for a ship steering system with uncertainties and stochastic disturbances are presented. In Section 5, the conclusions are given for this research.

2. System descriptions and problem statements

In this section, the IT2 T-S fuzzy stochastic model is presented to characterize a class of nonlinear systems with uncertainties and stochastic behaviors. The IT2 fuzzy controller is developed based on the upper and lower-bound membership functions. Regarding the energy of the system states, the definitions of the state covariance matrix and variance constraints are also provided. Based on the IT2 fuzzy sets, the IT2 TSFSM is first given as follows.

Fuzzy Model Rule i :

$$\begin{aligned} &\text{IF } z_1(t) \text{ is } M_1^i \text{ and } \ldots \text{ and } z_k(t) \text{ is } M_k^i \text{ THEN} \\ &\dot{x}(t) = \mathbf{A}_i x(t) + \mathbf{B}_i u(t) + \mathbf{D}_i v(t) \end{aligned} \tag{1}$$

where $\mathbf{A}_i \in \Re^{n\times n}$, $\mathbf{B}_i \in \Re^{n\times m}$, $\mathbf{D}_i \in \Re^{n\times v}$ are the constant matrices, $z_k(t)$ are the premise variables with $\alpha = 1, 2, \cdots, k$ denting the index of the variables, M_k^i is the fuzzy set with $i = 1, 2, \cdots, p$ denoting the fuzzy rule numbers, $x(t) \in \Re^n$ is the state vector, $v(t) \in \Re^v$ is an external disturbance vector considered as a zero-mean white noise. Since white noise possesses autocovariance properties, the definition of the corresponding covariance matrix is presented as follows.

$$E[v(\tau)v^T(\gamma)] = \mathbf{V}\delta(\tau-\gamma) \tag{2}$$

where $\mathbf{V}$ denotes the intensity of the white noise, $\delta(\bullet)$ denotes the Dirac delta function of $\bullet$, which describes the autocovariance function of white noise. In order to analyze the autocovariance of white noise, τ and γ denote different time instants.

Assuming that the initial states are uncorrelated with the external disturbance input, the following relationship is established.

$$E[x(0)v^T(t)] = E[x(0)]E[v^T(t)] = 0 \quad \forall t > 0. \tag{3}$$

For the IT2 TSFM (1), the firing strength corresponding to each rule is defined using interval sets as follows.

$$W_i(z_\alpha(t)) = [\underline{w}_i(z_\alpha(t)),\ \overline{w}_i(z_\alpha(t))] \tag{4}$$

where $\underline{w}_i(z(t)) = \prod_{\alpha=1}^{k} \underline{\mu}_{M_\alpha^i}(z_\alpha(t))$ and $\overline{w}_i(z(t)) = \prod_{\alpha=1}^{k} \overline{\mu}_{M_\alpha^i}(z_\alpha(t))$ indicate the lower membership and upper membership grades, $\underline{\mu}_{M_\alpha^i}(z_\alpha(t))$ and $\overline{\mu}_{M_\alpha^i}(z_\alpha(t))$ indicate the lower and upper-bound membership functions. Accordingly, the following relationships hold based on the definition of IT2 membership functions.

$$\overline{w}_i(z(t)) \geq \underline{w}_i(z(t)) \geq 0 \quad \forall i \tag{5}$$

$$\overline{\mu}_{M_\alpha^i}(z_\alpha(t)) \geq \underline{\mu}_{M_\alpha^i}(z_\alpha(t)) \geq 0 \quad \forall i,\ \alpha. \tag{6}$$

From (4)-(6), the defuzzified IT2 TSFSM for (1) is inferred as follows.

$$\dot{x}(t) = m\sum_{i=1}^{p}\overline{h}_i(z(t))\left\{\mathbf{A}_i x(t) + \mathbf{B}_i u(t) + \mathbf{D}_i v(t)\right\}$$

$$+ n\sum_{i=1}^{p}\underline{h}_i(z(t))\left\{\mathbf{A}_i x(t) + \mathbf{B}_i u(t) + \mathbf{D}_i v(t)\right\} \tag{7}$$

where m and n represent the given parameters of the upper and lower-bound models, whose associated membership functions satisfy the following properties.

$$\overline{h}_i(z(t)) = \frac{\overline{w}_i(z(t))}{\sum_{i=1}^{p}\overline{w}_i(z(t))} \geq 0, \quad \underline{h}_i(z(t)) = \frac{\underline{w}_i(z(t))}{\sum_{i=1}^{p}\underline{w}_i(z(t))} \geq 0 \quad \forall i. \tag{8}$$

Corresponding to the IT2 TSFSM (1), the IT2 fuzzy controller is constructed through the definitions of IT2 membership functions given in (4)-(6) as follows.

Fuzzy Controller Rule i :

IF $z_1(t)$ is M_1^i and $\ldots$ and $z_k(t)$ is M_k^i THEN

$$u(t) = \mathbf{F}_i x(t) \tag{9}$$

where $\mathbf{F}_i$ denotes the feedback gain matrix. Similar to the procedure of the TSFM (7), the IT2 fuzzy controller inferred from (9) is presented as follows.

$$u(t) = m\sum_{i=1}^{p}\overline{h}_i(z(t))\mathbf{F}_i x(t) + n\sum_{i=1}^{p}\underline{h}_i(z(t))\mathbf{F}_i x(t). \tag{10}$$

Substituting the control input (10) into the IT2 TSFSM (7), the overall closed-loop IT2 TSFSM is obtained below.

$$\dot{x}(t) = \sum_{i,\ j,\ l,\ q=1}^{p} h_{ijlq}(z(t))\{mA_i x(t) + m^2 B_i F_l x(t) + mnB_i F_q x(t)$$

$$+nA_j x(t) + nmB_j F_l x(t) + n^2 B_j F_q x(t)\} + \sum_{i=1}^{p}\overline{h}_i(z(t))mD_i v(t) + \sum_{i=1}^{p}\underline{h}_i(z(t))nD_i v(t) \tag{11}$$

where

$$h_{ijlq}(z(t)) = \frac{\overline{w}_i(z(t))\underline{w}_j(z(t))\overline{w}_l(z(t))\underline{w}_q(z(t))}{\sum_{i,\ j,\ l,\ q=1}^{p}\overline{w}_i(z(t))\underline{w}_j(z(t))\overline{w}_l(z(t))\underline{w}_q(z(t))} \tag{12}$$

and $i,\ j,\ l,\ q = 1,\ 2,\ \ldots,\ q$ represent the fuzzy rule numbers.

However, the fuzzy rule numbers increase rapidly due to the presence of four indices in the IT2 T-SFSM defined by (11)-(12). This results in conservative stability analysis under the variance constraint. For this reason, the closed-loop IT2 TSFSM given by (11)-(12) is further reformulated as follows.

$$\dot{x}(t) = \sum_{i=1}^{p} h_{iiii}(z(t))\{\mathbf{G}_{iiii}x(t) + (m+n)\mathbf{D}_i v(t))\} + 2\sum_{i<j}^{p} h_{ijjj}(z(t))\mathbf{G}_{ijjj}x(t)$$

$$+2\sum_{j<l}^{p} h_{ijll}(z(t))\mathbf{G}_{ijll}x(t) + 2\sum_{l<q}^{p} h_{ijlq}(z(t))\mathbf{G}_{ijlq}x(t) \tag{13}$$

where

$$\mathbf{G}_{iiii} = (m+n)\mathbf{A}_i + (m^2+2mn+n^2)\mathbf{B}_i\mathbf{F}_i$$

$$\mathbf{G}_{ijjj} = \left(\frac{m}{2}+\frac{n}{2}\right)(\mathbf{A}_i+\mathbf{A}_j) + \left(\frac{m^2}{2}+\frac{mn}{2}\right)(\mathbf{B}_i\mathbf{F}_j+\mathbf{B}_j\mathbf{F}_i) + \left(\frac{nm}{2}+\frac{n^2}{2}\right)(\mathbf{B}_i\mathbf{F}_j+\mathbf{B}_j\mathbf{F}_i)$$

$$\mathbf{G}_{ijll} = m\mathbf{A}_i + \frac{n}{2}(\mathbf{A}_j+\mathbf{A}_l) + \left(\frac{m^2}{2}+\frac{mn}{2}\right)\mathbf{B}_i(\mathbf{F}_j+\mathbf{F}_l) + \left(\frac{nm}{2}+\frac{n^2}{2}\right)(\mathbf{B}_j\mathbf{F}_l+\mathbf{B}_l\mathbf{F}_j)$$

$$\mathbf{G}_{ijlq} = m\mathbf{A}_i + n\mathbf{A}_j + \left(\frac{m^2}{2}+\frac{mn}{2}\right)\mathbf{B}_i(\mathbf{F}_l+\mathbf{F}_q) + \left(\frac{nm}{2}+\frac{n^2}{2}\right)\mathbf{B}_j(\mathbf{F}_l+\mathbf{F}_q).$$

To enhance the response of the closed-loop IT2 TSFSM (13) under stochastic disturbances $v(t)$, the state covariance matrices and individual state variance constraints are introduced with reference to covariance control theory as follows.

Definition 1 From the perspective of the energy concept, the covariance matrices $\mathbf{X}_i$ are defined based on $x(t)$ of each fuzzy subsystem as follows.

$$\mathbf{X}_i = \lim_{t\to\infty} E[x(t)x^T(t)] \quad \forall i \tag{14}$$

where $E[\bullet]$ denotes the expectation of $\bullet$.

For the state covariance matrix (14), the individual state variance constraint is defined as follows to bound the energy of the system states in the presence of stochastic disturbances.

$$[\tilde{\mathbf{X}}_i]_{dd} = [\mathbf{X}_i + \Delta\mathbf{X}_i]_{dd} \le \sigma_d^2 \quad d = 1,\ 2,\ \ldots,\ n \tag{15}$$

where $[\bullet]_{dd}$ denotes the d-th diagonal element of matrix $[\bullet]$, σ_d denotes the individual RMS state variance constraint and $\Delta\mathbf{X}_i$ denotes the uncertain covariance matrix resulting from the uncertain state dynamics.

To facilitate the derivation in the stability analysis involving the combination of IT2 fuzzy control, Lyapunov theory, and covariance control theory, the following definition and lemma are introduced.

Definition 2 Let the interval of integration be $[a,\ b]$. Then, the Dirac delta function satisfies the following equality.

$$\int_a^b f(\sigma)\delta(\sigma-\rho)d\sigma = f(\rho). \tag{16}$$

Accordingly, the Dirac delta function exhibits a sampling property in the sense that it takes effect only at $\sigma=\rho$ and becomes zero elsewhere.

Lemma 1 [33] Given $F(t,\ a(t),\ b(t)) = \int_{a(t)}^{b(t)} f(t,\ \chi)d\chi$, the differentiation with respect to t is performed as follows.

$$\frac{d}{dt}F(t,\ a(t),\ b(t)) = \frac{\partial F}{\partial b}\frac{db}{dt} - \frac{\partial F}{\partial a}\frac{da}{dt} + \frac{\partial F}{\partial t} = f(t,\ b(t))\frac{db}{dt} - f(t,\ a(t))\frac{da}{dt} + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(t,\ \chi)d\chi. \tag{17}$$

Grounded in the above definitions, lemma, and the closed-loop IT2 TSFSM (13), the stability criteria for nonlinear systems subject to stochastic behaviors are established in the next section. Moreover, an IT2 fuzzy controller design scheme based on the upper-bound covariance matrix is proposed to further guarantee stability and individual state variance constraints.

3. Interval type-2 fuzzy controller design subject to variance constraints

Through the combination of IT2 fuzzy control, covariance control, and Lyapunov theories, an IT2 fuzzy controller design strategy is proposed in this section. This means that the parameters m, n of the upper and lower-bound fuzzy models, as well as the individual variance constraint (15), are all integrated into the stability analysis process. Therefore, the impact of uncertainties and stochastic behaviors can be effectively suppressed. However, the upper and lower-bound parameters complicate the problem of combining covariance control and Lyapunov theories. By extending the linear covariance control theory, the following stability criterion is proposed to overcome this challenge based on the covariance matrices (14) and by virtue of the expression of the IT2 TSFM (13).

Theorem 1 If there exist the positive definite matrices $\tilde{\mathbf{X}}_i$ and the feedback gains $\mathbf{F}_i$ such that the following conditions are satisfied, then the closed-loop IT2 TSFSM (13) is mean-square stable.

$$\mathbf{G}_{iiii}\tilde{\mathbf{X}}_i + \tilde{\mathbf{X}}_i\mathbf{G}_{iiii}^T + (m^2+2mn+n^2)\mathbf{D}_i\mathbf{V}\mathbf{D}_i^T = 0 \quad \forall i \tag{18}$$

$$\mathbf{G}_{ijjj}\tilde{\mathbf{X}}_i + \tilde{\mathbf{X}}_i\mathbf{G}_{ijjj}^T < 0 \quad \forall i < j \tag{19}$$

$$\mathbf{G}_{ijll}\tilde{\mathbf{X}}_i + \tilde{\mathbf{X}}_i\mathbf{G}_{ijll}^T < 0 \quad \forall i,\ j < l \tag{20}$$

$$\mathbf{G}_{ijlq}\tilde{\mathbf{X}}_i + \tilde{\mathbf{X}}_i\mathbf{G}_{ijlq}^T < 0 \quad \forall i,\ j,\ l < q. \tag{21}$$

Proof. Since the approach extends the linear covariance control theory, the analysis procedure that combines covariance control and Lyapunov theories is first developed for the parallel terms in the closed-loop IT2 TSFSM (13). Accordingly, upon presenting the linear subsystem within the parallel terms as follows, the Lyapunov equation associated with the state covariance matrix is subsequently derived.

$$\dot{x}(t) = \mathbf{G}_{iiii}x(t) + m\mathbf{D}_i v(t) + n\mathbf{D}_i v(t) \quad \text{for} \quad i = 1,\ 2,\ \ldots,\ p. \tag{22}$$

For the linear differential equation (22), the solution can be expressed as follows.

$$x(t) = e^{\mathbf{G}_{iiii}t}x(0) + \int_0^t e^{\mathbf{G}_{iiii}(t-\tau)}(m\mathbf{D}_i + n\mathbf{D}_i)v(\tau)d\tau. \tag{23}$$

To analyze the effect of white noise on the closed-loop model (22) over the time interval $[t_1,\ t_2]$, the following expression for the state covariance function is used in the Lyapunov stability analysis.

$$\mathbf{R}_x(t_1,\ t_2) = E[x(t_1)x^T(t_2)]. \tag{24}$$

Then, the state covariance function is further derived as follows by substituting (23) into (24).

$$\begin{aligned}
\mathbf{R}_x(t_1,\ t_2) = E\Bigg[&\left(e^{G_{iiii}t_1}x(0) + \int_0^{t_1} e^{G_{iiii}(t_1-\tau)}(mD_i + nD_i)v(\tau)d\tau\right) \\
&\times \left(x^T(0)e^{G_{iiii}^T t_2} + \int_0^{t_2} v^T(\tau)(mD_i + nD_i)^T e^{G_{iiii}^T(t_2-\tau)}d\tau\right)\Bigg] \\
= e^{\mathbf{G}_{iiii}t_1}&E[x(0)x^T(0)]e^{\mathbf{G}_{iiii}^T t_2} + \int_0^{t_2} e^{\mathbf{G}_{iiii}t_1}E[x(0)v^T(\gamma)](m\mathbf{D}_i + n\mathbf{D}_i)^T e^{\mathbf{G}_{iiii}^T(t_2-\gamma)}d\gamma \\
&+ \int_0^{t_1} e^{\mathbf{G}_{iiii}(t_1-\tau)}(m\mathbf{D}_i + n\mathbf{D}_i)E[v(\tau)x^T(0)]^T e^{\mathbf{G}_{iiii}^T t_2}d\tau \\
&+ \int_0^{t_1}\int_0^{t_2} e^{\mathbf{G}_{iiii}(t_1-\tau)}(m\mathbf{D}_i + n\mathbf{D}_i)E[v(\tau)v^T(\gamma)](m\mathbf{D}_i + n\mathbf{D}_i)^T e^{\mathbf{G}_{iiii}^T(t_2-\gamma)}d\tau d\gamma.
\end{aligned} \tag{25}$$

According to the properties (2)-(3), (25) can be transformed as follows.

$$\mathbf{R}_x(t_1,\ t_2) = e^{\mathbf{G}_{iiii}t_1}\tilde{\mathbf{X}}_i(0)e^{\mathbf{G}_{iiii}^T t_2} + \int_0^{t_1}\int_0^{t_2} e^{\mathbf{G}_{iiii}(t_1-\tau)}(m\mathbf{D}_i + n\mathbf{D}_i)\mathbf{V}\delta(\tau-\gamma)(m\mathbf{D}_i + n\mathbf{D}_i)^T e^{\mathbf{G}_{iiii}^T(t_2-\gamma)}d\tau d\gamma. \tag{26}$$

By assuming $t_2 > t_1$ and referring to Definition 2, the following expression is derived from (26).

$$\mathbf{R}_x(t_1,\ t_2) = e^{\mathbf{G}_{iiii}t_1}\tilde{\mathbf{X}}_i(0)e^{\mathbf{G}_{iiii}^T t_2} + \int_0^{t_1} e^{\mathbf{G}_{iiii}(t_1-\tau)}(m\mathbf{D}_i + n\mathbf{D}_i)\mathbf{V}(m\mathbf{D}_i + n\mathbf{D}_i)^T e^{\mathbf{G}_{iiii}^T(t_2-\tau)}d\tau. \tag{27}$$

Analogously, the corresponding expression can be derived for the case $t_1 > t_2$ as follows.

$$\mathbf{R}_x(t_1,\ t_2) = e^{\mathbf{G}_{iiii}t_1}\tilde{\mathbf{X}}_i(0)e^{\mathbf{G}_{iiii}^T t_2} + \int_0^{t_2} e^{\mathbf{G}_{iiii}(t_1-\gamma)}(m\mathbf{D}_i + n\mathbf{D}_i)\mathbf{V}(m\mathbf{D}_i + n\mathbf{D}_i)^T e^{\mathbf{G}_{iiii}^T(t_2-\gamma)}d\gamma. \tag{28}$$

The covariance function valid for all time instants within the interval $[t_1,\ t_2]$ is obtained by integrating the expressions in (27)-(28).

$$\mathbf{R}_x(t_1,\ t_2) = e^{\mathbf{G}_{iiii}t_1}\tilde{\mathbf{X}}_i(0)e^{\mathbf{G}_{iiii}^T t_2} + \int_0^{min(t_1,\ t_2)} e^{\mathbf{G}_{iiii}(t_1-\gamma)}(m\mathbf{D}_i + n\mathbf{D}_i)\mathbf{V}(m\mathbf{D}_i + n\mathbf{D}_i)^T e^{\mathbf{G}_{iiii}^T(t_2-\gamma)}d\gamma. \tag{29}$$

It is noted that the delta function activates only at the time instant $\tau = \gamma$ and remains zero at all other times. This implies that such an instance can only occur at an earlier time. Therefore, the minimization expression in (29) can be applied to the construction of the covariance function.

However, the expression in (29) also leads to computational difficulties and often causes redundant information that is unnecessary. For this reason, a simplified expression, in which $t_1 = t_2 = t$ is set, is provided as follows.

$$\tilde{\mathbf{X}}_i(t) = \mathbf{R}_x(t,\ t) = e^{\mathbf{G}_{iiii}t}\tilde{\mathbf{X}}_i(0)e^{\mathbf{G}_{iiii}^T t} + \int_0^{t} e^{\mathbf{G}_{iiii}(t-\gamma)}(m\mathbf{D}_i + n\mathbf{D}_i)\mathbf{V}(m\mathbf{D}_i + n\mathbf{D}_i)^T e^{\mathbf{G}_{iiii}^T(t-\gamma)}d\gamma \tag{30}$$

for $i = 1,\ 2,\ ...,\ p$.

Taking the time derivative of the state covariance matrix (30) for the subsystem yields the following result.

$$\begin{aligned}\dot{\tilde{\mathbf{X}}}(t) &= \mathbf{G}_{iiii}e^{\mathbf{G}_{iiii}t}\tilde{\mathbf{X}}_i(0)e^{\mathbf{G}_{iiii}^T t} + e^{\mathbf{G}_{iiii}t}\tilde{\mathbf{X}}_i(0)e^{\mathbf{G}_{iiii}^T t}\mathbf{G}_{iiii}^T \\ &+ \frac{d}{dt}(\int_0^t e^{\mathbf{G}_{iiii}(t-\gamma)}(m\mathbf{D}_i + n\mathbf{D}_i)V(m\mathbf{D}_i + n\mathbf{D}_i)^T e^{\mathbf{G}_{iiii}^T(t-\gamma)}d\gamma).\end{aligned} \tag{31}$$

Then, the time derivative of the integral is computed using the Leibniz integral rule presented in Lemma 1, and the following result is obtained.

$$\begin{aligned}\dot{\tilde{\mathbf{X}}}(t) &= \mathbf{G}_{iiii}e^{\mathbf{G}_{iiii}t}\tilde{\mathbf{X}}_i(0)e^{\mathbf{G}_{iiii}^T t} + e^{\mathbf{G}_{iiii}t}\tilde{\mathbf{X}}_i(0)e^{\mathbf{G}_{iiii}^T t}\mathbf{G}_{iiii}^T + (m\mathbf{D}_i + n\mathbf{D}_i)\mathbf{V}(m\mathbf{D}_i + n\mathbf{D}_i)^T \\ &+ \int_0^t \frac{d}{dt}(\mathbf{G}_{iiii}e^{\mathbf{G}_{iiii}(t-\gamma)}(m\mathbf{D}_i + n\mathbf{D}_i)\mathbf{V}(m\mathbf{D}_i + n\mathbf{D}_i)^T e^{\mathbf{G}_{iiii}^T(t-\gamma)})d\gamma.\end{aligned} \tag{32}$$

Applying the chain rule to the last term, (30) can be derived as follows.

$$\dot{\tilde{\mathbf{X}}}(t) = \mathbf{G}_{iiii}e^{\mathbf{G}_{iiii}t}\tilde{\mathbf{X}}_i(0)e^{\mathbf{G}_{iiii}^T t} + e^{\mathbf{G}_{iiii}t}\tilde{\mathbf{X}}_i(0)e^{\mathbf{G}_{iiii}^T t}\mathbf{G}_{iiii}^T + (m\mathbf{D}_i + n\mathbf{D}_i)\mathbf{V}(m\mathbf{D}_i + n\mathbf{D}_i)^T$$

$$+ \int_0^t \mathbf{G}_{iiii}e^{\mathbf{G}_{iiii}(t-\gamma)}(m\mathbf{D}_i + n\mathbf{D}_i)\mathbf{V}(m\mathbf{D}_i + n\mathbf{D}_i)^T e^{\mathbf{G}_{iiii}^T(t-\gamma)}$$

$$+ e^{\mathbf{G}_{iiii}(t-\gamma)}(m\mathbf{D}_i + n\mathbf{D}_i)\mathbf{V}(m\mathbf{D}_i + n\mathbf{D}_i)^T e^{\mathbf{G}_{iiii}^T(t-\gamma)}\mathbf{G}_{iiii}^T d\gamma. \tag{33}$$

Rearranging (33), one can obtain

$$\dot{\tilde{\mathbf{X}}}(t) = \mathbf{G}_{iiii}e^{\mathbf{G}_{iiii}t}\tilde{\mathbf{X}}_i(0)e^{\mathbf{G}_{iiii}^T t} + \int_0^t \mathbf{G}_{iiii}e^{\mathbf{G}_{iiii}(t-\gamma)}(m\mathbf{D}_i + n\mathbf{D}_i)\mathbf{V}(m\mathbf{D}_i + n\mathbf{D}_i)^T e^{\mathbf{G}_{iiii}^T(t-\gamma)}d\gamma$$

$$+ e^{\mathbf{G}_{iiii}t}\tilde{\mathbf{X}}_i(0)e^{\mathbf{G}_{iiii}^T t}\mathbf{G}_{iiii}^T + \int_0^t e^{\mathbf{G}_{iiii}(t-\gamma)}(m\mathbf{D}_i + n\mathbf{D}_i)\mathbf{V}(m\mathbf{D}_i + n\mathbf{D}_i)^T e^{\mathbf{G}_{iiii}^T(t-\gamma)}\mathbf{G}_{iiii}^T d\gamma$$

$$+ (m\mathbf{D}_i + n\mathbf{D}_i)\mathbf{V}(m\mathbf{D}_i + n\mathbf{D}_i)^T. \tag{34}$$

Extracting the matrices $\mathbf{G}_{iiii}$ and $\mathbf{G}_{iiii}^T$ from the integral in (34), with reference to the definition of the state covariance matrices (30), leads to the following form.

$$\dot{\tilde{\mathbf{X}}}(t) = \mathbf{G}_{iiii}\tilde{\mathbf{X}}_i(t) + \tilde{\mathbf{X}}_i(t)\mathbf{G}_{iiii}^T + (m\mathbf{D}_i + n\mathbf{D}_i)\mathbf{V}(m\mathbf{D}_i + n\mathbf{D}_i)^T. \tag{35}$$

Considering that the goal of stability analysis is to ensure the system reaches a steady state, the limit of (35) is taken as follows.

$$\lim_{t\to\infty}\dot{\tilde{\mathbf{X}}}(t) = \mathbf{G}_{iiii}\lim_{t\to\infty}\tilde{\mathbf{X}}_i(t) + \lim_{t\to\infty}\tilde{\mathbf{X}}_i(t)\mathbf{G}_{iiii}^T + (m\mathbf{D}_i + n\mathbf{D}_i)\mathbf{V}(m\mathbf{D}_i + n\mathbf{D}_i)^T. \tag{36}$$

It is noted that the derivative of the state covariance matrices becomes zero at steady state, based on their definition, provided that system stability is ensured. Therefore, the following equation is derived from (36), by referring to the definitions of the state covariance matrices in (14), (20), and (30).

$$\mathbf{G}_{iiii}\tilde{\mathbf{X}}_i + \tilde{\mathbf{X}}_i\mathbf{G}_{iiii}^T + (m\mathbf{D}_i + n\mathbf{D}_i)\mathbf{V}(m\mathbf{D}_i + n\mathbf{D}_i)^T = 0. \tag{37}$$

Rearranging (37), the Lyapunov equation is derived in the following form.

$$\mathbf{G}_{iiii}\tilde{\mathbf{X}}_i + \tilde{\mathbf{X}}_i\mathbf{G}_{iiii}^T + (m^2 + 2mn + n^2)\mathbf{D}_i\mathbf{V}\mathbf{D}_i^T = 0. \tag{38}$$

It can be readily verified that the Lyapunov equation (38) is satisfied when condition (18) is fulfilled under the IT2 fuzzy controller design approach proposed in Theorem 2. Moreover, the following relationship can also be derived from (38).

$$\mathbf{G}_{iiii}\tilde{\mathbf{X}}_i + \tilde{\mathbf{X}}_i\mathbf{G}_{iiii}^T = -(m^2+2mn+n^2)\mathbf{D}_i\mathbf{V}\mathbf{D}_i^T < 0. \tag{39}$$

Based on the closed-loop system (22) and the typical Lyapunov function $V(x(t))=x^T(t)\mathbf{P}x(t)$, the time derivative also can be derived without considering the disturbances as follows.

$$\dot{V}(x(t)) = x^T(t)(\mathbf{G}_{iiii}^T\mathbf{P} + \mathbf{P}\mathbf{G}_{iiii})x(t). \tag{40}$$

It is evident that the result $\dot{V}(x(t)) < 0$ follows from the relationship in (39) and the substitution of $\mathbf{P}^{-1}$ with $\tilde{\mathbf{X}}_i$. Therefore, the mean-square stability of the parallel term in the IT2 T-SFM (13) is verified. In addition, the cross-terms of the subsystem in IT2 T-SFM (13) also achieves mean-square stability according to Lyapunov theory by satisfying conditions (19)-(21) with the substitution of $\mathbf{P}^{-1}$ with $\tilde{\mathbf{X}}_i$. From the above analysis process, the IT2 T-SFSM (13) is concluded to be mean-square stable based on the definition (14) and Lyapunov theory if there exists $\tilde{\mathbf{X}}_i > 0$ such that the conditions (18)-(21) are satisfied by Theorem 2.

However, calculating the exact value of $\tilde{\mathbf{X}}_i$ for each fuzzy rule in the design approach of Theorem 1 is challenging under the uncertainty. Furthermore, the stability conditions derived in Theorem 1 cannot be solved using the LMI technique due to the presence of the equation condition (18) and the matrices $\tilde{\mathbf{X}}_i$. Therefore, an IT2 fuzzy controller design approach is developed in the next section to solve the mean-square stability problem caused by $\tilde{\mathbf{X}}_i$ and to ensure the satisfaction of individual variance constraint (15).

Theorem 2 If there exist a positive definite matrix $\overline{\mathbf{X}}$ and the feedback gains $\mathbf{F}_i$ such that the following stability conditions are satisfied, then the closed-loop IT2 TSFSM (13) is mean-square stable and the individual state variance constraints in (15) are satisfied.

$$\overline{\mathbf{X}} < diag[\sigma_1^2, \cdots, \sigma_m^2] \tag{41}$$

$$\mathbf{G}_{iiii}\overline{\mathbf{X}} + \overline{\mathbf{X}}\mathbf{G}_{iiii}^T + (m^2+2mn+n^2)\mathbf{D}_i\mathbf{V}\mathbf{D}_i^T < 0 \quad \forall i \tag{42}$$

$$\mathbf{G}_{ijjj}\overline{\mathbf{X}} + \overline{\mathbf{X}}\mathbf{G}_{ijjj}^T < 0 \quad \forall i < j \tag{43}$$

$$\mathbf{G}_{ijll}\overline{\mathbf{X}} + \overline{\mathbf{X}}\mathbf{G}_{ijll}^T < 0 \forall i,\ j < l \tag{44}$$

$$\mathbf{G}_{ijlq}\overline{\mathbf{X}} + \overline{\mathbf{X}}\mathbf{G}_{ijlq}^T < 0 \quad \forall i,\ j,\ l < q \tag{45}$$

where $[\eta_1 \ \ldots \ \eta_n]$ denotes a diagonal matrix with corresponding diagonal elements $\eta_1 \ \ldots \ \eta_n$.

Proof. In Theorem 1, the mean-square stability of the closed-loop IT2 T-SFM (13) is ensured by finding positive definite matrices $\tilde{\mathbf{X}}_i$ and feedback gains $\mathbf{F}_i$ to satisfy the sufficient conditions (18)-(21) in the design process. However, the presence of uncertainties in the state dynamics hinders the determination of the exact values of $\tilde{\mathbf{X}}_i$. Moreover, the Lyapunov-based stability analysis for T-SFM typically requires a common positive definite matrix, which is not satisfied in Theorem 1. By introducing an upper-bound matrix, these analytical difficulties can be overcome simultaneously.

Therefore, the analysis process with the application of common upper-bound matrix $\overline{\mathbf{X}}$ is presented based on Theorem 1 as follows. Given the condition for the parallel term in the closed-loop IT2 TSFSM (13), the following inequality is obtained by subtracting (42) from (18).

$$\mathbf{G}_{iiii}(\overline{\mathbf{X}}-\tilde{\mathbf{X}}_i)+(\overline{\mathbf{X}}-\tilde{\mathbf{X}}_i)\mathbf{G}_{iiii}^T<0 \quad \forall i. \tag{46}$$

From Theorem 1, the matrices $\mathbf{G}_{iiii}$ are proved to be stable matrices if the stability of the parallel term in the closed-loop IT2 TSFSM (13) is ensured by condition (18). By following a similar analysis process as in (18) and (39)-(40), the mean-square stability of the parallel terms in the IT2 TSFSM (13) is also derived under condition (42) in Theorem 2 with the substitution of $\mathbf{P}^{-1}$ with $\overline{\mathbf{X}}$. From the inequality (46), the fact $\overline{\mathbf{X}}-\tilde{\mathbf{X}}_i>0$ can also be derived because of the stable matrices $\mathbf{G}_{iiii}$.

Consequently, if the positive definite matrix $\overline{\mathbf{X}}$ can be found to satisfy the conditions (43)-(45), the matrices $\mathbf{G}_{ijjj}$, $\mathbf{G}_{ijll}$, $\mathbf{G}_{ijlq}$ are proved to be stable matrices. Based on the fact that $\overline{\mathbf{X}}-\mathbf{X}_i>0$, the following relationship can be obtained by satisfying condition (41) in Theorem 2.

$$[\tilde{\mathbf{X}}_i]_{dd}\leq[\overline{\mathbf{X}}]_{dd}<\sigma_d^2. \tag{47}$$

It is worth noting that the positive definite matrix $\overline{\mathbf{X}}$ can serve as an upper-bound matrix for all the state covariance matrices. Furthermore, according to relationship (47), the individual variance constraint (15) can also be guaranteed by satisfying condition (41) in Theorem 2. From the above analysis, the closed-loop IT2 TSFSM (13) is concluded to be mean-square stable provided that the conditions (42)-(45) are satisfied by the IT2 fuzzy controller design approach in Theorem 2. Additionally, the individual variance constraint (15) on the state covariance matrices is ensured by the condition (41).

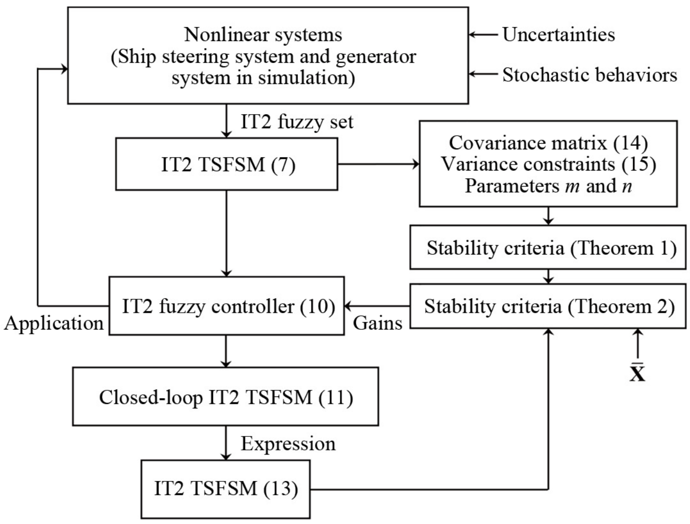

Figure 1. Flowchart of the proposed IT2 fuzzy controller design method

In summary, employing the IT2 fuzzy controller design approach of Theorem 2 allows the upper-bound matrix to efficiently replace the state covariance matrices in the stability conditions. This advantage not only transfers the stability condition from equation (18) to inequality (42), but also provides the upper-bound matrix as a common positive definite matrix consistent with the typical Lyapunov theory for TSFMs. Leveraging the covariance control theory, the individual variance constraint (15) can be directly imposed to suppress the state energy characterized by the covariance matrix under stochastic disturbances. Moreover, the parameters m and n are also combined into the stability conditions so that the designed IT2 fuzzy controller can more effectively handle uncertainties. To clarify the proposed IT2 fuzzy controller design process subject to variance constraints, a step-by-step flowchart is presented in Figure 1.

In the next section, the simulations of a nonlinear ship steering system and a nonlinear generator system are conducted. The comparison results with existing fuzzy control methods are then presented to verify the effectiveness and feasibility of the proposed IT2 fuzzy controller design approach.

4. Simulations of a ship steering system and a generator system

The IT2 fuzzy control approach designed in Theorem 2 is applied respectively to a ship steering system and a generator system in two simulation examples in this section under the consideration of the nonlinearities, uncertainties, and stochastic disturbances. To demonstrate the contribution of the proposed Theorem 2, the T1 fuzzy covariance control method [28] and the IT2 fuzzy control method [32] are considered for comparison. Note that the pole placement constraint in the T1 control method of [28] is not considered in the simulation, as transient performance is of lesser concern in this research. The IT2 fuzzy control method in [32] was developed without considering stochastic disturbances. First, the simulation results of the ship steering system are presented as follows.

Example 1 In the first example, a nonlinear ship steering system is selected to present the simulation results. It is worth noting that a ship operates in a complex ocean environment and is consistently affected by stochastic disturbances, including ocean winds, currents, and waves. Under disturbance effects, structural damage to the ship is inevitably caused over long-term operation. Moreover, modeling errors may also arise due to the complex dynamic behaviors of a ship navigating in the ocean. These effects, which cause the constructed mathematical model to be imprecise, can be regarded as uncertainties in the ship steering system.

Therefore, considering the effects of uncertainties and stochastic disturbances, the dynamics of the ship's steering system governed by surge, sway, and yaw motions, can be characterized as follows.

$$\dot{\eta} = J(\eta)\upsilon \tag{48}$$

$$\dot{\upsilon} = \Gamma_1\eta + \Gamma_2\upsilon + \Gamma_3\tau + \Gamma_4 v \tag{49}$$

where $J(\eta)$ is the rotational matrix related to the yaw angle, Γ_1 is the matrix related to the earth-fixed positions and yaw angle in ship steering systems, Γ_2 is the state matrix related to the body-fixed velocities, Γ_3 is the matrix related to control force and moment and Γ_4 is the matrix related to zero-mean white noise. Moreover, the earth fixed positions and yaw angle ψ of the ship is indicated as the vector $\eta = [x \;\; y \;\; \psi]^T$, the earth fixed velocity and yaw angular velocity, and the surge, sway and yaw motions of the ship steering system are indicated as the vector $\upsilon = [\upsilon_x \;\; \upsilon_y \;\; \upsilon_\psi]^T$, while the control forces and moment are indicated as the vector $\tau = [\tau_1 \;\; \tau_2 \;\; \tau_3]^T$. Note that the disturbances are expressed as the stochastic behaviors v. The detailed information about the state dynamic equation (48)-(49) can be found in [28].

$$\dot{x}_1(t) = \cos(x_3(t))x_4(t) - \sin(x_3(t))x_5(t) \tag{50}$$

$$\dot{x}_2(t) = \sin(x_3(t))x_4(t) + \cos(x_3(t))x_5(t) \tag{51}$$

$$\dot{x}_3(t) = x_6(t) \tag{52}$$

$$\dot{x}_4(t) = -0.0358x_1(t) - 0.0797x_4(t) + 0.9215u_1(t) + v_1(t) \tag{53}$$

$$\dot{x}_5(t) = -0.0208x_2(t) - 0.0818x_5(t) - 0.01224x_6(t) + 0.7805u_2(t) + 7.4562u_3(t) + v_1(t) \tag{54}$$

$$\dot{x}_6(t) = -0.0208x_2(t) - 0.0818x_5(t) - 0.01224x_6(t) + 0.7805u_2(t) + 7.4562u_3(t) + v_1(t) \tag{55}$$

where $(x_1(t),\ x_2(t),\ x_3(t),\ x_4(t),\ x_5(t),\ x_6(t)) = (x,\ y,\ \psi,\ \upsilon_X,\ \upsilon_Y,\ \upsilon_\psi)$ are the system states, $(u_1(t),\ u_2(t),\ u_3(t)) = (\tau_1,\ \tau_2,\ \tau_3)$ are the control forces and moments and $v_1(t) = v(t)$ is the stochastic disturbance.

To construct the IT2 TSFSM and achieve linearization, the yaw angle $x_3(t) = \psi$ is selected as the premise variable due to nonlinearities in the ship steering system (50)-(55). Considering the general operating conditions, the operating domain of the yaw angle is set as $x_3(t) \in \left(-\frac{\pi}{2},\ \frac{\pi}{2}\right)$. Therefore, the IT2 membership function of the TSFSM is designed as shown in Figure 2 based on three operating points within this domain.It is worth noting that the proposed IT2 fuzzy covariance control method can achieve satisfactory control performance by employing typical triangular membership functions. However, many researchers have continued to explore further performance enhancement by developing design strategies and adjustment methods for IT2 membership functions. For instance, a data-driven optimization modeling approach was proposed in [34] for constructing the IT2 TSFM of thermal comfort and energy consumption systems. Additionally, an optimization-based design of IT2 fuzzy sets was presented in [35] using learning-based techniques. These optimization concepts are expected to offer promising directions for the future development of IT2 fuzzy control methods.

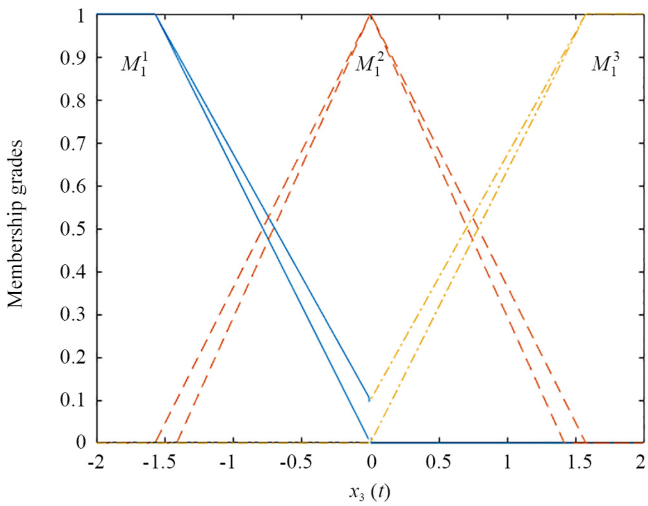

Figure 2. IT2 Membership function of ship steering system

Based on the membership function in Figure 2, the IT2 TSFSM for the ship steering system is established as follows.

Model Rule 1: IF $x_3(t)$ is M_1^1

$$\text{THEN } \dot{x}(t) = \mathbf{A}_1 x(t) + \mathbf{B}_1 u(t) + \mathbf{D}_1 v(t) \tag{56}$$

Model Rule 2: IF $x_3(t)$ is M_1^2

$$\text{THEN } \dot{x}(t) = \mathbf{A}_2 x(t) + \mathbf{B}_2 u(t) + \mathbf{D}_2 v(t) \tag{57}$$

Model Rule 3: IF $x_3(t)$ is M_1^3

$$\text{THEN } \dot{x}(t) = \mathbf{A}_3 x(t) + \mathbf{B}_3 u(t) + \mathbf{D}_3 v(t) \tag{58}$$

where the model matrices are described as follows.

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 0 & 0.0349 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0.0349 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -0.0358 & 0 & 0 & -0.0797 & 0 & 0 \\ 0 & -0.0208 & 0 & 0 & -0.0818 & -0.1224 \\ 0 & -0.0394 & 0 & 0 & -0.2254 & -0.2468 \end{bmatrix}$$

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & -0.0349 & 0 \\ 0 & 0 & 0 & 0.0349 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -0.0358 & 0 & 0 & -0.0797 & 0 & 0 \\ 0 & -0.0208 & 0 & 0 & -0.0818 & -0.1224 \\ 0 & -0.0394 & 0 & 0 & -0.2254 & -0.2468 \end{bmatrix}$$

$$\mathbf{A}_3 = \begin{bmatrix} 0 & 0 & 0 & 0.0349 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0.0349 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -0.0358 & 0 & 0 & -0.0797 & 0 & 0 \\ 0 & -0.0208 & 0 & 0 & -0.0818 & -0.1224 \\ 0 & -0.0394 & 0 & 0 & -0.2254 & -0.2468 \end{bmatrix}$$

$$\mathbf{B}_1=\mathbf{B}_2=\mathbf{B}_3=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\\0.9215&0&0\\0&0.7802&1.4811\\0&1.4811&7.4562\end{bmatrix},\text{ and }\mathbf{D}_1=\mathbf{D}_2=\mathbf{D}_3=\begin{bmatrix}0\\0\\0\\1\\1\\1\end{bmatrix}. \tag{59}$$

It is worth noting that the uncertainty effects are efficiently covered through the expression of IT2 fuzzy sets in the premise part of the TSFSM (56)-(58). To execute the simulation, the initial condition is set as $x(0)=[10\ \ 10\ \ 0^{\circ}\ \ 0\ \ 0\ \ 0]^T$ and the intensity of zero-mean white noises $v(t)$ is set as $\mathbf{V}=1$. Moreover, the following variance constraints are designed to suppress the effects of stochastic disturbances on each state.

$$\sigma_1^2=4,\ \sigma_2^2=4,\ \sigma_3^2=1.5,\ \sigma_4^2=2,\ \sigma_5^2=2,\ \sigma_6^2=0.5. \tag{60}$$

Then, the following feasible solutions can be obtained by solving the design problem of Theorem 2 under the setting of the variance constraints (59) and the parameters $m=0.5$, $n=0.5$ via the LMI toolbox in MATLAB.

$$\overline{\mathbf{X}}=\begin{bmatrix}2.2159&-0.0000&0.0000&-1.2944&-0.0004&-0.0000\\-0.0000&2.2159&-0.0000&0.0004&-1.2944&-0.0000\\0.0000&-0.0000&0.7500&0.0000&-0.0000&-0.3676\\-1.2944&0.0004&0.0000&0.8872&-0.0000&-0.0000\\-0.0004&-1.2944&-0.0000&-0.0000&0.8872&0.0000\\0.0000&-0.0000&-0.3676&-0.0000&0.0000&0.2500\end{bmatrix} \tag{61}$$

$$\mathbf{M}_1=\begin{bmatrix}-0.0754&0.4472&-0.0001&-17.7116&-0.0517&0.0094\\-0.8448&-0.1128&0.0458&-0.0285&-33.5186&6.6736\\0.1671&-0.0042&-0.0541&-0.0062&6.6509&-3.4127\end{bmatrix} \tag{62}$$

$$\mathbf{M}_2=\begin{bmatrix}-0.1553&-0.0072&-0.0000&-17.4700&-0.0535&0.0098\\0.0115&-0.2849&0.0458&-0.0294&-33.0894&6.5880\\-0.0024&0.0291&-0.0541&-0.0063&6.5711&-3.3957\end{bmatrix} \tag{63}$$

$$\mathbf{M}_3=\begin{bmatrix}-0.0634&-0.4424&0.0000&-18.2135&-0.0455&0.0082\\0.8374&-0.1310&0.0474&-0.0254&-34.4725&6.8662\\-0.1656&0.0021&-0.0557&-0.0052&6.8416&-3.5116\end{bmatrix}. \tag{64}$$

Since the control gains for each rule are defined as $\mathbf{F}_i=\mathbf{M}_i\overline{\mathbf{X}}^{-1}$, the following gains for the IT2 fuzzy controller are derived as follows.

$$\mathbf{F}_1 = \begin{bmatrix} -79.1626 & 1.1628 & 0.0662 & -135.4625 & 1.6008 & 0.1372 \\ -2.7581 & -149.7247 & 47.0731 & -3.9782 & -256.2311 & 95.9170 \\ 0.4927 & 29.6279 & -24.2183 & 0.6966 & 50.7241 & -49.2639 \end{bmatrix} \tag{65}$$

$$\mathbf{F}_2 = \begin{bmatrix} -78.3615 & -0.2337 & 0.0685 & -134.0286 & -0.4383 & 0.1418 \\ -0.1461 & -148.3375 & 46.4724 & -0.1691 & -253.7222 & 94.6913 \\ -0.0254 & 29.3740 & -24.0990 & -0.0593 & 50.2634 & -49.0205 \end{bmatrix} \tag{66}$$

$$\mathbf{F}_3 = \begin{bmatrix} -81.3631 & -1.5259 & 0.0580 & -139.2373 & -2.3161 & 0.1203 \\ 2.3922 & -154.0314 & 48.4331 & 3.5418 & -263.5873 & 98.6872 \\ -0.5185 & 30.4841 & -24.9206 & -0.7780 & 52.1879 & -50.6922 \end{bmatrix}. \tag{67}$$

To demonstrate the effectiveness and superiority of the proposed fuzzy control method designed with the IT2 membership function and variance constraints, the T1 fuzzy covariance control method [28], and the typical IT2 fuzzy control method [32] are also employed in the simulation for comparison. First, the gains for the T1 fuzzy controller are obtained by applying the design process in [28], without consideration of the pole placement constraint.

$$\mathbf{F}_1 = \begin{bmatrix} -0.4202 & 0.4462 & 1.4335 & -1.3407 & -0.3118 & 2.7072 \\ -1.3935 & -0.7142 & 4.9051 & -1.1837 & -2.2089 & 9.1571 \\ 0.2260 & 0.1045 & -3.1413 & 0.0703 & 0.3308 & -5.7938 \end{bmatrix} \tag{68}$$

$$\mathbf{F}_2 = \begin{bmatrix} -1.0543 & -0.2319 & 1.3413 & -1.6989 & -0.6896 & 2.5331 \\ -0.0704 & -1.9258 & 4.7572 & -0.3463 & -2.9186 & 8.8779 \\ -0.0334 & 0.3483 & -3.0573 & -0.0849 & 0.4818 & -5.6352 \end{bmatrix} \tag{69}$$

$$\mathbf{F}_3 = \begin{bmatrix} -0.4202 & -0.8940 & 1.4420 & -1.3412 & -1.1385 & 2.7233 \\ 1.1477 & -0.7141 & 4.9077 & 0.3842 & -2.2097 & 9.1620 \\ -0.2788 & 0.1045 & -3.1418 & -0.2413 & 0.3310 & -5.7948 \end{bmatrix}. \tag{70}$$

Note that the T1 fuzzy controller proposed in [28] is implemented using the TSFM presented in (56)-(58), where M_1^1, M_1^2, and M_1^3 are regarded as type-1 triangular fuzzy sets. In addition, the typical IT2 fuzzy control method in [32] is also considered in this simulation for comparison. However, the controller design method in [32] fails to address the disturbance issue, let alone stochastic behaviors. To verify the effectiveness in satisfying the variance constraints, the following control gains are obtained by applying the design method from [32] without considering covariance control theory.

$$\mathbf{F}_1 = \begin{bmatrix} -44.2494 & -0.4942 & 0.0024 & -342.5121 & -3.8976 & 0.0057 \\ 1.0992 & -96.5701 & 0.2324 & 8.6488 & -747.1652 & -0.2748 \\ -0.0.2715 & 23.3294 & -0.2361 & -2.1309 & 180.4844 & -0.0471 \end{bmatrix} \tag{71}$$

$$\mathbf{F}_2 = \begin{bmatrix} -43.6010 & -0.8359 & 0.0196 & -337.4840 & -6.4820 & 0.0473 \\ 2.9397 & -30.1625 & 2.0219 & 22.7514 & -233.4245 & 4.0455 \\ -1.0285 & -11.2488 & -0.5916 & -7.9518 & -87.0107 & -0.9053 \end{bmatrix} \tag{72}$$

$$\mathbf{F}_3 = \begin{bmatrix} -44.2561 & 0.3499 & 0.0111 & -342.5644 & 2.7491 & 0.0268 \\ 0.1039 & -92.3595 & 0.3483 & 0.7273 & -714.5932 & 0.0051 \\ -0.2725 & 21.1072 & -0.2591 & -2.0935 & 163.2939 & -0.1027 \end{bmatrix}. \tag{73}$$

Based on the simulation results presented in Figures 3-9, the proposed IT2 fuzzy covariance control method demonstrates superior control performance under the simultaneous effects of uncertainties and stochastic variations. From Figures 3-5, one can observe that the ship steering systems are particularly vulnerable to the influence of external disturbances by applying the typical IT2 fuzzy control method [32]. The primary reason for the large fluctuations is that external disturbance is not being considered in the controller design process. In other words, this may be the main reason for the system being unstable. Moreover, the T1 fuzzy covariance control method [28], which takes external disturbance into consideration, has better performance than the IT2 fuzzy control method [32]. Even though the T1 fuzzy control method [28] has resistance to external disturbance, it is inevitable to consider the system modeling error due to long-term system operation.

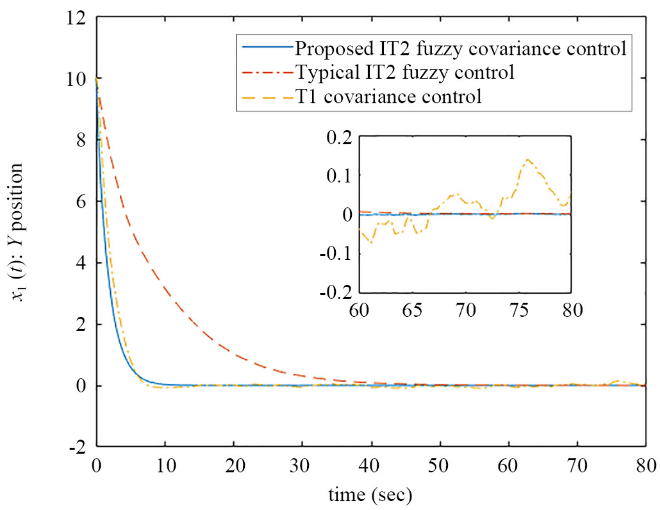

Figure 3. Comparison of $x_1(t)$ responses for the ship steering system

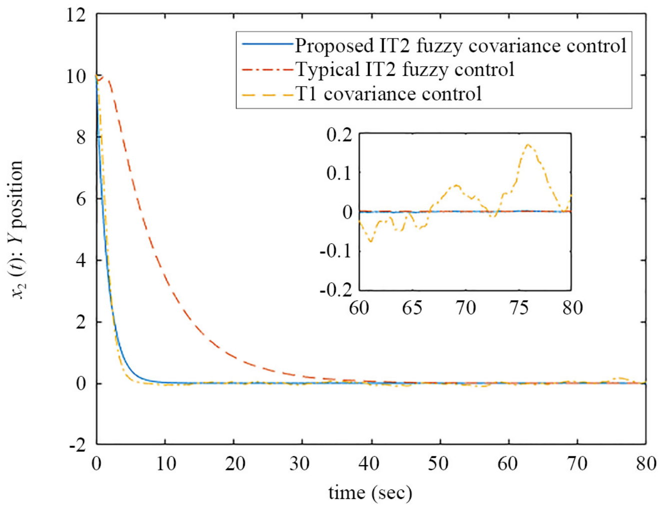

Figure 4. Comparison of $x_2(t)$ responses for the ship steering system

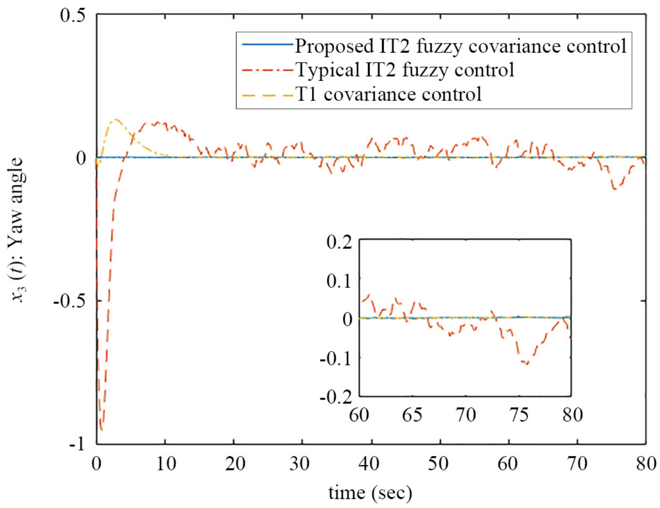

Figure 5. Comparison of $x_3(t)$ responses for the ship steering system

From Figures 6-8, it is observed that the T1 fuzzy covariance control method [28] exhibits a larger variance phenomenon than the proposed IT2 fuzzy covariance control method. Referring to Figure 9, it is obvious that the trajectory of the ship controlled by the proposed IT2 fuzzy covariance control method exhibits enhanced performance relative to T1 fuzzy covariance control method [28] and IT2 fuzzy control method [32]. Because the factors of uncertainties and stochastic disturbances are considered in the proposed IT2 fuzzy covariance control strategy, it can be verified that the designed IT2 fuzzy controller provides a better simulation result to reach the stability and navigational safety of the ship steering system.

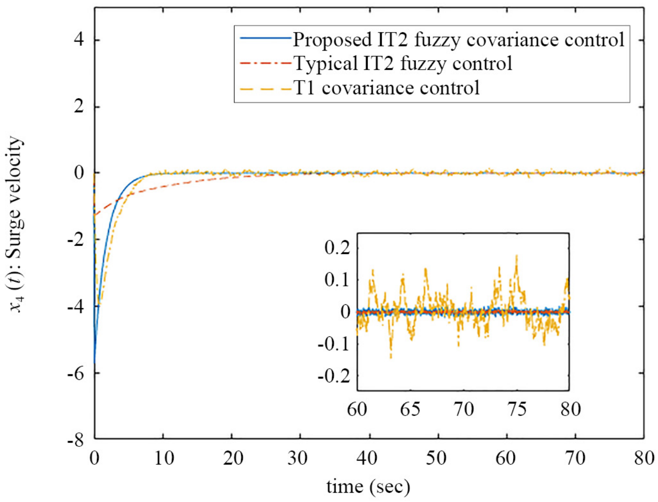

Figure 6. Comparison of $x_4(t)$ responses for the ship steering system

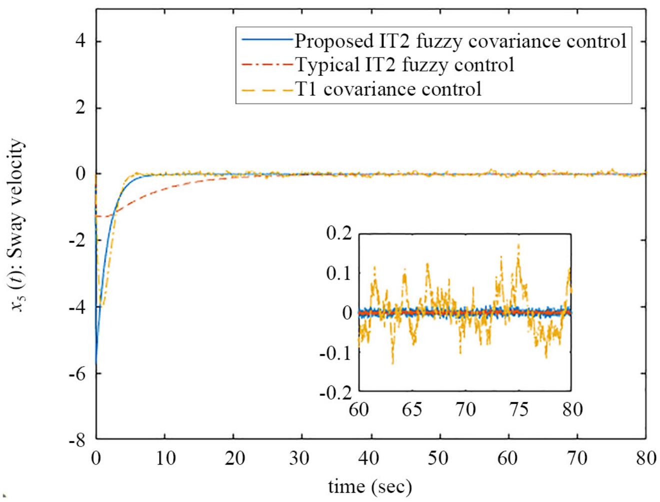

Figure 7. Comparison of $x_5(t)$ responses for the ship steering system

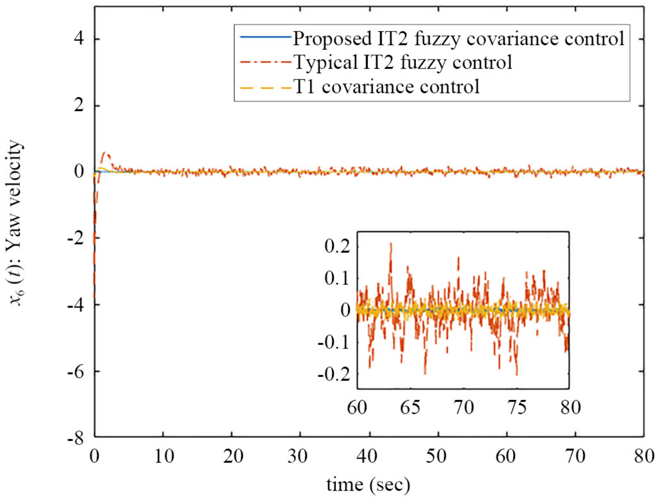

Figure 8. Comparison of $x_6(t)$ responses for the ship steering system

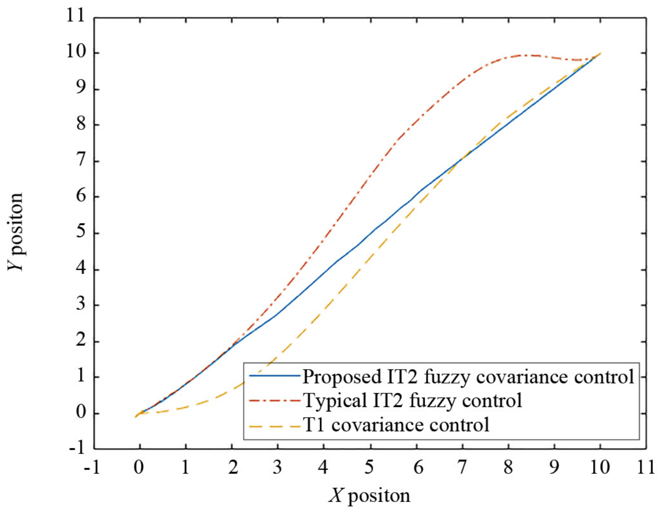

Figure 9. Comparison of the ship trajectories

Example 2 To verify the applicability of the proposed IT2 fuzzy covariance control method across diverse engineering fields, a simulation of a nonlinear synchronous generator system is conducted in the second example. It is worth noting that a generator is significantly affected by stochastic disturbances, which often arise from load variations and grid fluctuations. Moreover, structural damage is also caused by common factors such as wear and tear.

For this reason, discussing the uncertainty and stochastic disturbance problems is also valuable for the nonlinear generator system. Following the research [36], the nonlinear dynamic equations of a generator system are introduced as follows.

$$\dot{x}_1(t) = 2\pi f_0 x_2(t) + 0.1v(t) \tag{74}$$

$$\dot{x}_2(t) = -\frac{D}{H}x_2(t) + \frac{\omega_0}{H}\left(P_m - \frac{V_s}{x'_{d\Sigma}}(x_3(t)+E'_{q0}) + \sin\left(x_1(t)+\frac{\pi}{3}\right) + \frac{(x_d - x'_d)V_s^2}{x_{d\Sigma}x'_{d\Sigma}}\sin\left(x_1(t)+\frac{\pi}{3}\right) + \cos\left(x_1(t)+\frac{\pi}{3}\right)\right) \tag{75}$$

$$\dot{x}_3(t) = -\frac{x_{d\Sigma}}{T_{D0}x'_{d\Sigma}}(x_3(t)+E'_{q0}) + \frac{(x_d - x'_d)}{T_{D0}x'_{d\Sigma}}V_s + \cos(x_1(t)+\frac{\pi}{3}) + \frac{k_a x_{ad}}{T_{D0}R_f}(u_0+u_t) \tag{76}$$

where $x_1(t)$, $x_2(t)$ and $x_3(t)$ are the angular position of the rotor of the generator, the angular velocity and the electromotive force, $u(t)$ is the control input, $v(t)$ is the external disturbance which is regarded as the stochastic behavior in this simulation. For the generator system (73)-(75), the parameters are considered as $f_0 = 50Hz$, $D = 0.8$, $H = 8s$, $\omega_0 = 1p.u.$, $P_m = 0.79p.u$, $V_s = 1p.u$, $x'_{d\Sigma} = 1.18p.u.$, $x_{d\Sigma} = 2.3108p.u.$, $E'_{q0} = 1.2723p.u.$, $x_d = 1.5p.u$, $T_{D0} = 3s$, $k_A = 10$, $R_f = 0.0045p.u.$, $u_0 = 7.2942\times10^{-4}p.u.$ and $x_{ad} = 1.3p.u.$.

Then, the operating domain $x_1(t) \in \left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$ is selected as the premise variable for the nonlinearities in generator system (73)-(75). To achieve linearization of the subsystem in the following IT2 TSFSM, the operating points are selected as $\left(-\frac{\pi}{6}, 0, \frac{\pi}{6}\right)$ and the IT2 membership function is designed as shown in Figure 10.

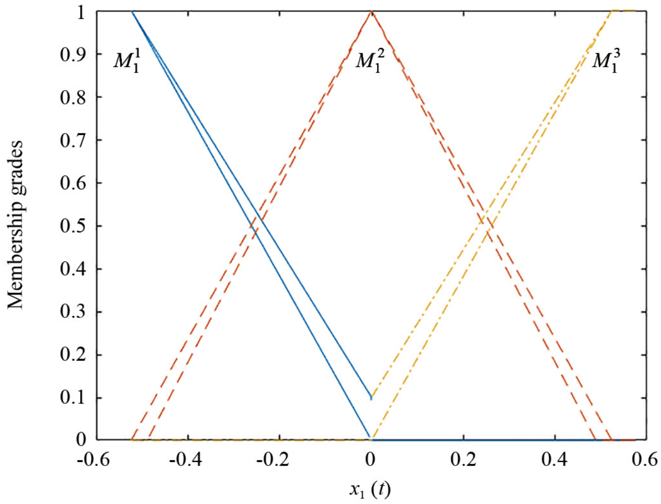

Figure 10. IT2 Membership function of generator system

Model Rule 1: IF $x_1(t)$ is M_1^1

$$\text{THEN } \dot{x}(t) = \mathbf{A}_1 x(t) + \mathbf{B}_1 u(t) + \mathbf{D}_1 v(t) \tag{77}$$

Model Rule 2: IF $x_1(t)$ is M_1^2

$$\text{THEN } \dot{x}(t) = \mathbf{A}_2 x(t) + \mathbf{B}_2 u(t) + \mathbf{D}_2 v(t) \tag{78}$$

Model Rule 3: IF $x_1(t)$ is M_1^3

$$\text{THEN } \dot{x}(t) = \mathbf{A}_3 x(t) + \mathbf{B}_3 u(t) + \mathbf{D}_3 v(t) \tag{79}$$

where the model matrices are described as

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 314.2 & 0 \\ -0.1002 & -0.1 & -0.0563 \\ -0.2519 & 0 & -0.6937 \end{bmatrix}$$

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 314.2 & 0 \\ -0.1009 & -0.1 & -0.0975 \\ -0.2519 & 0 & -0.6937 \end{bmatrix}$$

$$\mathbf{A}_3 = \begin{bmatrix} 0 & 314.2 & 0 \\ -0.085 & -0.1 & -0.1126 \\ -0.3441 & 0 & -0.6937 \end{bmatrix}$$

$$\mathbf{B}_1 = \mathbf{B}_2 = \mathbf{B}_3 = \begin{bmatrix} 0 \\ 0 \\ 962.963 \end{bmatrix}, \text{ and } \mathbf{D}_1 = \mathbf{D}_2 = \mathbf{D}_3 = \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix}. \tag{80}$$

For the simulation of generator system, the initial condition is set as $x(0) = [\ 10^\circ \quad 0 \quad 0\ ]^T$ and the intensity of stochastic behavior is set as $\mathbf{V} = 1$. Based on the IT2 TSFSM (76)-(78), the following control gains are obtained by solving the LMI problem in Theorem 2 using MATLAB, where the parameters of both the individual variance constraint and the upper and lower-bound models are set to $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1$, $m = 0.5$ and $n = 0.5$.

$$\mathbf{F}_1 = \begin{bmatrix} 0.2068 & 32.2030 & -0.6212 \end{bmatrix} \tag{81}$$

$$\mathbf{F}_2 = \begin{bmatrix} 0.2203 & 34.2805 & -0.6603 \end{bmatrix} \tag{82}$$

$$\mathbf{F}_3 = \begin{bmatrix} 0.2171 & 33.7746 & -0.6503 \end{bmatrix}. \tag{83}$$

The differences between the proposed control approach, the T1 fuzzy covariance method in [28], and the typical IT2 fuzzy method in [32] have been explained in Example 1. Applying the design process in [28], the control gains are obtained as follows for the T1 fuzzy controller.

$$\mathbf{F}_1 = \begin{bmatrix} 5.3094 & 934.9676 & -20.5843 \end{bmatrix} \tag{84}$$

$$\mathbf{F}_2 = \begin{bmatrix} 5.3120 & 935.4139 & -20.5915 \end{bmatrix} \tag{85}$$

$$\mathbf{F}_3 = \begin{bmatrix} 5.3129 & 935.5769 & -20.5941 \end{bmatrix}. \tag{86}$$

In addition, the following control gains are also obtained by applying the design process in [32] for the IT2 fuzzy controller.

$$\mathbf{F}_1 = \begin{bmatrix} 1.6580 & 232.0340 & -4.8381 \end{bmatrix} \tag{87}$$

$$\mathbf{F}_2 = \begin{bmatrix} 1.6617 & 232.5552 & -4.8381 \end{bmatrix} \tag{88}$$

$$\mathbf{F}_3 = \begin{bmatrix} 1.6626 & 232.6717 & -4.8403 \end{bmatrix}. \tag{89}$$

Applying the IT2 fuzzy controller with the gains (79)-(81), (85)-(87) and the T1 fuzzy controller with the gains (82)-(84) respectively, the comparison results are presented in Figures 11-13. From 11-13, it can be observed that although the effect of stochastic disturbances is not prominent, the fluctuation controlled by the proposed method is smaller than that of the other two methods during convergence. This phenomenon implies that the proposed IT2 fuzzy controller can achieve smoother responses through the integration of upper and lower-bound parameters with the variance constraint in the controller design process. The enhanced robustness of the control responses can efficiently reduce the damage caused by substantial changes in the nonlinear generator system. Owing to the neglect of stochastic behaviors, the IT2 fuzzy controller designed in [32] cannot suppress stochastic effects. In addition, the application of IT2 fuzzy sets provides a better approach to solving the uncertainty problem than the T1 fuzzy controller designed in [28]. Thus, the proposed IT2 fuzzy covariance controller also offers better performance in controlling the nonlinear generator system.

In conclusion, the proposed fuzzy controller design method, which combines parameters m and n from the upper and lower-bound models in the IT2 TSFSM and the individual variance constraints (15), can offer superior control performance for different engineering applications. Nevertheless, the number of fuzzy rules may increase significantly due to the introduction of four indices, even when using the IT2 TSFSM formulation in (13). To present the proposed design method more clearly, Table 1 summarizes the computational time required to solve the LMI problem of Theorem 2 using MATLAB.

For this reason, two cases of IT2 TSFSM, respectively taken from Example 1 and Example 2 and each constructed with three fuzzy rules, are selected to calculate the computation time for solving the LMI problem in Theorem 2, as presented in Table 1. Additionally, the IT2 TSFSMs, initially constructed with three fuzzy rules for the nonlinear ship steering and generator systems, were extended to five-rule cases respectively. This means that the operating points of the nonlinear ship steering system are selected as $x_3(t) = (-90°,\ -45°,\ 0°,\ 45°,\ 90°)$, whereas those of the nonlinear generator system are selected as $x_1(t) = (-60°,\ -30°,\ 0°,\ 30°,\ 60°)$ to construct the IT2 TSFSM.

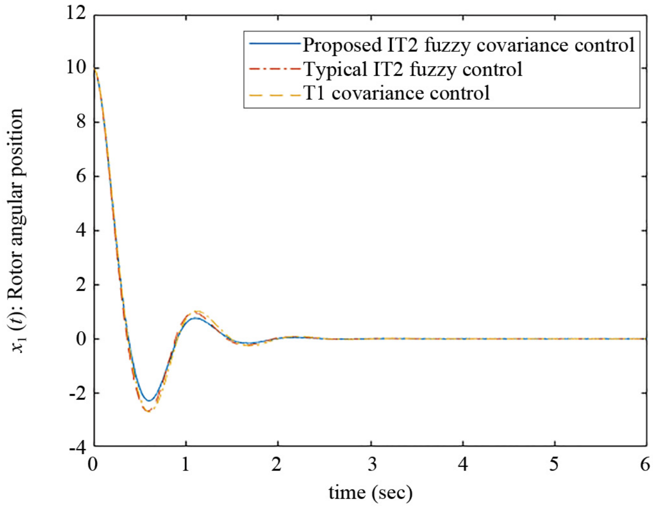

Figure 11. Comparison of $x_1(t)$ responses for the generator system

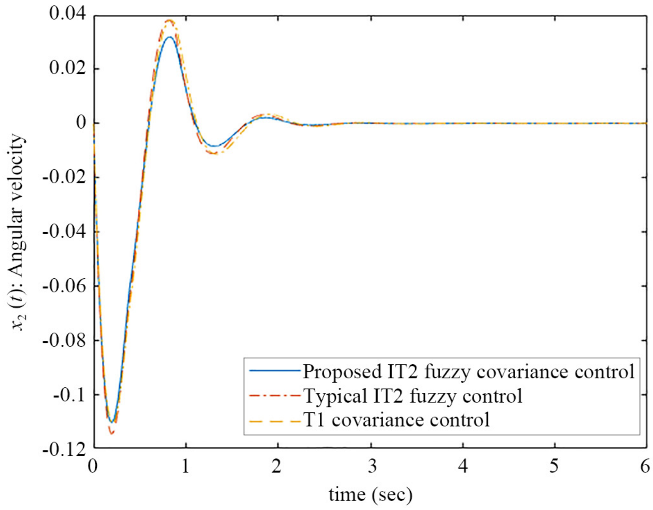

Figure 12. Comparison of $x_2(t)$ responses for the generator system

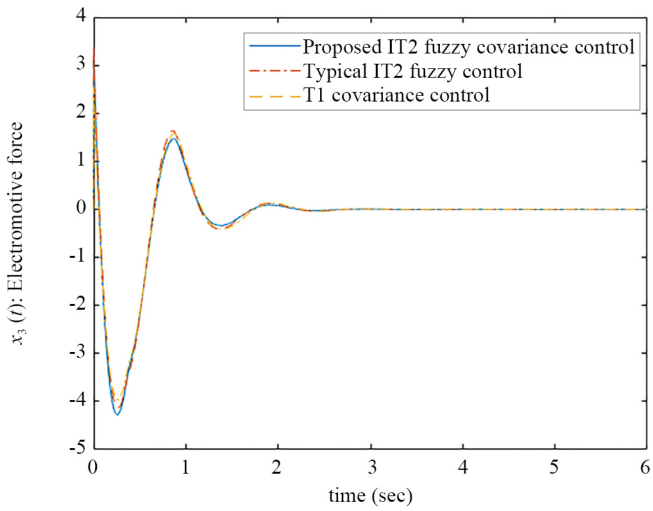

Figure 13. Comparison of $x_3(t)$ responses for the generator system

From Table 1, it is evident that the computational time for solving the LMI increases as the number of fuzzy rules increases. Moreover, an increase in system order, where Example 1 is a sixth-order system and Example 2 is a third-order system, also leads to higher computational time. For the IT2 fuzzy covariance control method proposed in Theorem 2, although the IT2 TSFSM expression (13) effectively reduces the number of fuzzy rules in the stability analysis, the four indices of fuzzy rules still lead to increased computational requirements. In the future, it is valuable to combine efficient methods such as the model reduction method [37] and distributed computing method [38] associated with fuzzy control theory into the proposed design method to overcome this challenge. By applying model reduction, the number of fuzzy rules required for the IT2 TSFSM can be significantly reduced. Additionally, distributed computing optimally distributes LMI computation tasks and contributes to reducing conservativeness.

Table 1. Computational times for different cases

Cases	Numbers of rules	Nonlinear systems	Computational times (seconds)
1	3	Generator system	0.10036
2		Ship steering system	0.127347
3	5	Generator system	0.983203
4		Ship steering system	2.221073

5. Conclusion

This paper presented the IT2 fuzzy covariance control scheme for dealing with uncertainties and stochastic behavior for nonlinear stochastic systems. Compared to the T1 TSFM, the uncertainties can be better captured by the upper and lower-bound membership functions in the IT2 TSFM. Extending the conception of PDC, the IT2 fuzzy controller has been designed with respect to the IT2 TSFM to tolerate system uncertainties including modeling errors. Considering the external disturbance effect in practical circumstances, the covariance control theory was first combined with the IT2 fuzzy controller design process in this paper. Via the derivation of Lyapunov stability conditions and the application of an upper-bound state covariance matrix, the stability of the nonlinear stochastic systems can be achieved for satisfying specified individual state variance constraints. Based on the simulation results, the proposed IT2 fuzzy covariance control method applied to a nonlinear ship steering system and a nonlinear generator system in two respective examples both achieve superior control performance compared to existing fuzzy control methods. This implies that the ship steering system enables more suitable navigation in complex ocean environments. Moreover, the oscillation amplitude of the generator system can also be mitigated to reduce the damage caused by the drastic change. In the future, important control methods such as adaptive control and sliding mode control can be combined into the proposed IT2 fuzzy covariance control strategy. Moreover, the model reduction method and distributed computing method can also be considered to reduce the computational burden in stability analysis based on LMI. To enhance flexibility in designing and applying IT2 fuzzy controllers, GT2 fuzzy sets may be adopted to construct both the TSFM and the fuzzy controller to solve the uncertainty problem.

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Conflict of interest

The authors declare no competing financial interest.

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