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Research Article

Shallow Water Waves with Dispersion Triplet by the Complete Discriminant Approach

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Abstract: This work investigates a class of nonlinear evolution equations that model shallow water wave dynamics and systematically recovers a rich variety of exact wave solutions, including solitary waves, shock waves, singular solitary waves, plane waves, and cnoidal waves. The methodology employed is the complete discriminant approach, a powerful analytical technique that leverages the structure of the polynomial nonlinearities within the governing equations. This approach enables the derivation of a complete spectrum of traveling wave solutions by classifying the roots of the associated algebraic equations based on the signs and multiplicities of the discriminant. By performing a rigorous case-by-case analysis, the study identifies the precise parametric conditions under which each wave type emerges, offering insight into the transition between different nonlinear wave phenomena. The analysis highlights how variations in physical parameters such as wave speed, dispersion coefficients, and nonlinearity strength govern the existence and shape of the obtained waveforms. The considered models and their solutions have broad relevance to coastal engineering, oceanography, and geophysical fluid dynamics, where understanding wave propagation, wave breaking, and pattern formation in shallow water environments is critical. The findings not only recover known wave structures in a unified framework but also reveal novel analytical forms under specific parametric regimes. This comprehensive treatment contributes to the theoretical understanding of shallow water wave dynamics and offers potential for further applications in numerical modeling and experimental validation in real-world shallow water systems.

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1. Introduction

Solitons are self-reinforcing solitary wave packets that maintain their shape while propagating at constant velocity. Since their first observation in shallow water by John Scott Russell in 1834, solitons have become a cornerstone in the study of nonlinear wave phenomena. Their remarkable stability and particle-like behavior stem from a delicate balance between nonlinearity and dispersion. Solitons play a fundamental role in a variety of physical systems, including fluid dynamics, plasma physics, optical fibers, atmospheric sciences, and even biological systems [1–8].

In the context of shallow water wave dynamics, numerous mathematical models have been proposed to describe nonlinear wave propagation [9–16]. Among these, the Korteweg-de Vries (KdV) equation [3] and the Boussinesq equation [1, 8] are two of the most prominent models. These equations account for weak nonlinearity and weak dispersion and have been extensively used to model long waves in shallow channels and coastal environments.

The KdV equation is the simplest integrable equation that supports solitary wave solutions and conservation laws. It has been generalized in several directions. One such generalization is the modified KdV (mKdV) equation [4], which introduces a cubic nonlinearity instead of the quadratic term found in KdV. Another is Gardner's equation [2], which combines both quadratic and cubic nonlinear terms, allowing for a broader class of wave interactions and solution structures.

These three forms that are KdV, mKdV, and Gardner typically feature a single dispersive term. However, recent developments have explored additional dispersion mechanisms. For instance, the Benjamin-Bona-Mahoney (BBM) equation incorporates a temporal dispersion term distinct from the purely spatial one in KdV [5]. Extending this idea, researchers have introduced third-order dispersion into these models to study richer wave dynamics.

Several analytical techniques have been employed to construct exact solutions to these equations [7, 9, 10]. These include the inverse scattering transform, Hirota's bilinear method, Bäcklund transformations, and more recently, computational symbolic approaches such as the tanh-sech method, sine-cosine method, and exp-function methods. However, these methods often produce only a subset of the complete solution space.

In this work, we adopt the complete discriminant approach [6], a robust and systematic method that enables the classification and recovery of a wide spectrum of exact traveling wave solutions. Starting from the traveling wave reduction, the method transforms the original partial differential equations into algebraic forms whose discriminant structures dictate the nature of the solutions. This method facilitates the identification of solitary waves, shock waves, singular solitary waves, plane waves, and cnoidal waves under well-defined parametric constraints.

Unlike the traveling wave hypothesis alone, which may overlook certain classes of solutions, the complete discriminant approach ensures that no admissible solutions are missed, making it especially powerful for studying the full landscape of nonlinear wave structures.

The primary objective of this paper is to revisit the KdV, mKdV, and Gardner equations augmented with higher-order dispersion and to apply the complete discriminant method to retrieve all admissible waveforms. The detailed analysis, solution structures, and existence conditions for the recovered waves are presented in the subsequent sections.

1.1 *Governing model*

The broader class of equations encompassing the generalized Korteweg-de Vries (KdV) equation and related models can be unified within the following structural framework [2]:

$$q_t + F(q)q_x + b_1 q_{xxx} + b_2 q_{xxt} + b_3 q_{xtt} = 0, \tag{1}$$

where x and t denote spatial and temporal coordinates, respectively, and $q(x, t)$ describes the wave profile. The first term captures the temporal evolution of the wave. The nonlinear component $(F(q)q_x)$ accounts for nonlinear effects; for the classical KdV equation, this reduces to a single nonlinear term, while additional nonlinear interactions (represented by $F(q)$) characterize more complex models like the Gardner Equation (GE). The three dispersion terms carry the coefficients b_j for $j = 1, 2, 3$, from triple spatial dispersion, spatio-temporal dispersion, and dual-temporal-spatial dispersion effects, respectively. Notably, the classic KdV and GE equations omit the latter two dispersion terms ($b_2 = b_3 = 0$), which has exhausted the journals.

2. Mathematical preliminaries

To explore solutions of Eq. (1) under the travelling wave assumption, researchers typically adopt the following ansatz:

$$q(\xi) = q(x, t) = g(x - vt). \tag{2}$$

In this context, the function g corresponds to the solitary wave profile, where v represents the wave's propagation speed. By adopting this traveling wave ansatz, Eq. (1) is consequently reduced to a specific Ordinary Differential Equation (ODE):

$$vg' - F(g)g' - \left(b_1 - b_2 v + b_3 v^2\right) g''' = 0, \tag{3}$$

which integrates to

$$c_1 + vg - \int F(g)dg - \left(b_1 - b_2 v + b_3 v^2\right) g'' = 0, \tag{4}$$

where c_1 is an integration constant. Multiplying both sides of Eq. (4) by the derivative g and integrating yields

$$c_2 + 2c_1 g + vg^2 - 2\iint^{g} F(u)dudg - \left(b_1 - b_2 v + b_3 v^2\right) \left(g'\right)^2 = 0. \tag{5}$$

3. Application to shallow water wave equations

The Complete Discriminant Method (CDM) is a systematic and powerful analytical approach for deriving exact solutions to nonlinear partial differential equations [6], particularly those admitting traveling wave reductions such as the KdV, mKdV, and Gardner equations. By applying a traveling wave transformation, the original PDE is reduced to an Ordinary Differential Equation (ODE), often expressed as a polynomial in the dependent variable or its derivative. The CDM focuses on analyzing the discriminant of this polynomial equation to determine the nature and multiplicity of its roots. This analysis enables the classification of all possible solution structures, including solitary waves, shock waves, periodic (cnoidal) waves, and kink-type or plane wave solutions. One of the key strengths of the CDM lies in its completeness: it exhaustively explores the solution space based on the algebraic properties of the reduced equation, offering a unified and general framework that can be applied to a broad class of nonlinear systems. However, its effectiveness is primarily limited to equations reducible to polynomial ODEs, and the method may become computationally intensive for higher-order equations with complex discriminant forms. In this study, the CDM was employed successfully to recover and classify diverse nonlinear wave solutions for the KdV, mKdV, and Gardner equations, demonstrating its robustness and versatility in revealing both known and novel wave phenomena.

3.1 *KdV equation*

Regarding this model, for any real-valued constant a, we have $F(q) = aq$. As derived from (1), this results in [2]

$$q_t + aqq_x + b_1 q_{xxx} + b_2 q_{xxt} + b_3 q_{xtt} = 0, \tag{6}$$

which would imply that (5) is transformed into

$$3c_2 + 6c_1 g + 3vg^2 - ag^3 - 3\left(b_1 - b_2 v + b_3 v^2\right)\left(g'\right)^2 = 0. \tag{7}$$

With the help of the notations

$$p = \left[-\frac{a}{3\left(b_1 - b_2 v + b_3 v^2\right)}\right]^{\frac{1}{3}} g, \quad \xi_1 = \left[-\frac{a}{3\left(b_1 - b_2 v + b_3 v^2\right)}\right]^{\frac{1}{3}} \xi, \tag{8}$$

Eq. (7) becomes

$$\left(p_{\xi_1}\right)^2 = p^3 + e_2 p^2 + e_1 p + e_0, \tag{9}$$

where

$$\begin{aligned} e_2 &= -3va^{-\frac{2}{3}}\left[3\left(-b_1 + b_2 v - b_3 v^2\right)\right]^{-\frac{1}{3}}, \\ e_1 &= -12c_1 a^{-\frac{1}{3}}\left[3\left(-b_1 + b_2 v - b_3 v^2\right)\right]^{-\frac{2}{3}}, \\ e_0 &= c_2\left(b_1 - b_2 v + b_3 v^2\right)^{-1}. \end{aligned} \tag{10}$$

We rewrite Eq. (9) as

$$\pm(\xi_1 - \xi_0) = \int \frac{dp}{\sqrt{F(p)}}, \tag{11}$$

where

$$F(p) = p^3 + e_2 p^2 + e_1 p + e_0. \tag{12}$$

We consider the discriminant system

$$\Delta = -27\left(\frac{2e_2^3}{27}+e_0-\frac{e_1e_2}{3}\right)^2-4\left(e_1-\frac{e_2^2}{3}\right)^3, \quad D_1 = e_1-\frac{e_2^2}{3}. \tag{13}$$

Case 1: $\Delta = 0$, $D_1 < 0$, and $F(p) = (p-\gamma_1)^2(p-\gamma_2)$. When $p > \gamma_2$, if $\gamma_1 > \gamma_2$, the shock wave and singular solitary wave emerge as:

$$q_1 = \left[-\frac{a}{3\left(b_1-b_2v+b_3v^2\right)}\right]^{-\frac{1}{3}}\left[(\gamma_1-\gamma_2)\tanh^2\left[\frac{\sqrt{\gamma_1-\gamma_2}}{2}\left[\left\{-\frac{a}{3\left(b_1-b_2v+b_3v^2\right)}\right\}^{\frac{1}{3}}\xi-\xi_0\right]\right]+\gamma_2\right], \tag{14}$$

and

$$q_2 = \left[-\frac{a}{3\left(b_1-b_2v+b_3v^2\right)}\right]^{-\frac{1}{3}}\left[(\gamma_1-\gamma_2)\coth^2\left[\frac{\sqrt{\gamma_1-\gamma_2}}{2}\left[\left\{-\frac{a}{3\left(b_1-b_2v+b_3v^2\right)}\right\}^{\frac{1}{3}}\xi-\xi_0\right]\right]+\gamma_2\right], \tag{15}$$

respectively. If $\gamma_1 < \gamma_2$, the singular periodic wave evolves as

$$q_3 = \left[-\frac{a}{3\left(b_1-b_2v+b_3v^2\right)}\right]^{-\frac{1}{3}}\left[(\gamma_1-\gamma_2)\tan^2\left[\frac{\sqrt{\gamma_2-\gamma_1}}{2}\left[\left\{-\frac{a}{3\left(b_1-b_2v+b_3v^2\right)}\right\}^{\frac{1}{3}}\xi-\xi_0\right]\right]+\gamma_2\right], \tag{16}$$

where γ_1 and γ_2 are real numbers.

Case 2: $\Delta = 0$, $D_1 = 0$, and $F(p) = (p-\gamma_1)^3$. The plane wave evolves as

$$q_4(x, t) = 4\left[\left\{-\frac{a}{3\left(b_1-b_2v+b_3v^2\right)}\right\}^{\frac{1}{3}}\xi-\xi_0\right]^{-2}+\gamma_1, \tag{17}$$

where γ_1 is a real number.

Case 3: $\Delta > 0$, $D_1 < 0$, and $F(p) = (p-\gamma_1)(p-\gamma_2)(p-\gamma_3)$. When $\gamma_1 < \gamma_2 < \gamma_3$, if $\gamma_1 < p < \gamma_2$, the snoidal wave evolves as

$$\begin{aligned} q_5(x, t) = &\left[-\frac{a}{3\left(b_1-b_2v+b_3v^2\right)}\right]^{-\frac{1}{3}} \\ &\left[\gamma_1+(\gamma_2-\gamma_1)\,\mathrm{sn}^2\left[\frac{\sqrt{\gamma_3-\gamma_1}}{2}\left\{\left\{-\frac{a}{3\left(b_1-b_2v+b_3v^2\right)}\right\}^{\frac{1}{3}}s-s_0\right], m\right\}\right], \end{aligned} \tag{18}$$

and if $p > \gamma_3$, the combo snoidal-cnoidal wave evolves as

$$q_6(x,t)=\left\{-\frac{a}{3\left(b_1-b_2v+b_3v^2\right)}\right\}^{-\frac{1}{3}}\left[\frac{\gamma_3-\gamma_2\,\mathrm{sn}^2\left[\frac{\sqrt{\gamma_3-\gamma_1}}{2}\left\{\left\{-\frac{a}{3\left(b_1-b_2v+b_3v^2\right)}\right\}^{\frac{1}{3}}\xi-\xi_0\right\},m\right]}{\mathrm{cn}^2\left[\frac{\sqrt{\gamma_3-\gamma_1}}{2}\left\{\left\{-\frac{a}{3\left(b_1-b_2v+b_3v^2\right)}\right\}^{\frac{1}{3}}\xi-\xi_0\right\},m\right]}\right], \tag{19}$$

where γ_1, γ_2, and γ_3 are real numbers, and $m^2=\dfrac{\gamma_2-\gamma_1}{\gamma_3-\gamma_1}$.

Case 4: $\Delta<0$, and $F(p)=(p-\gamma_1)(p^2+lp+j)$, $l^2-4j<0$. The cnoidal wave evolves as

$$q_7(x,t)=\left\{-\frac{a}{3\left(b_1-b_2v+b_3v^2\right)}\right\}^{-\frac{1}{3}}$$
$$\left[\gamma_1-\sqrt{\gamma_1^2+l\gamma_1+j}+\frac{2\sqrt{\gamma_1^2+l\gamma_1+j}}{1+\mathrm{cn}\left[\left(\gamma_1^2+l\gamma_1+j\right)^{\frac{1}{4}}\left\{\left\{-\frac{a}{3\left(b_1-b_2v+b_3v^2\right)}\right\}^{\frac{1}{3}}\xi-\xi_0\right\},m\right]}\right], \tag{20}$$

where γ_1, l, and j are real numbers, and $m^2=\dfrac{1}{2}\left(1-\dfrac{\gamma_1+\frac{l}{2}}{\sqrt{\gamma_1^2+l\gamma_1+j}}\right)$.

3.2 *mKdV equation*

For any real-valued constant a, we have $F(q)=aq^2$. As shown in (1), this gives [2]

$$q_t+aq^2q_x+b_1q_{xxx}+b_2q_{xxt}+b_3q_{xtt}=0, \tag{21}$$

which means that (5) transforms into

$$6c_2+12c_1g+6vg^2-ag^4-6\left(b_1-b_2v+b_3v^2\right)\left(g'\right)^2=0. \tag{22}$$

Setting

$$h=\left[-\frac{a}{6\left(b_1-b_2v+b_3v^2\right)}\right]^{\frac{1}{4}}g,\quad \xi_1=\left[-\frac{a}{6\left(b_1-b_2v+b_3v^2\right)}\right]^{\frac{1}{4}}\xi, \tag{23}$$

Eq. (22) comes out as

$$\left(h_{\xi_1}\right)^2=h^4+d_3h^2+d_2h+d_1, \tag{24}$$

where

$$d_3 = 6v\left[-6a\left(b_1 - b_2 v + b_3 v^2\right)\right]^{-\frac{1}{2}},$$

$$d_2 = 12c_1(-a)^{-\frac{1}{4}}\left[6\left(b_1 - b_2 v + b_3 v^2\right)\right]^{-\frac{3}{4}}, \tag{25}$$

$$d_1 = c_2\left(b_1 - b_2 v + b_3 v^2\right)^{-1}.$$

Rewrite Eq. (24) as

$$\pm(\xi_1 - \xi_0) = \int \frac{dh}{\sqrt{F(h)}}, \tag{26}$$

where

$$F(h) = h^4 + d_3 h^2 + d_2 h + d_1. \tag{27}$$

Next, we give the discriminant system

$$\begin{aligned}
&D_1 = 1,\\
&D_2 = -d_1,\\
&D_3 = -2d_1^3 + 8d_1 d_3 - 9d_2^2,\\
&D_4 = -d_1^3 d_2^2 + 4d_1^4 d_3 + 36 d_1 d_2^2 d_3 - 32 d_1^2 d_3^2 - \frac{27}{4} d_2^4 + 64 d_3^3,\\
&E_2 = 9d_2^2 - 32 d_1 d_3.
\end{aligned} \tag{28}$$

By classifying the roots of $F(h)$, we arrive
(1) $D_4 > 0$ & $((D_2 > 0$ & $D_3 \leq 0) \| D_2 \leq 0)$, then $F(h) = \left[(h-\varepsilon_1)^2 + \varepsilon_2^2\right]\left[(h-\varepsilon_3)^2 + \varepsilon_4^2\right]$,
(2) $D_4 < 0$ & $((D_2 < 0$ & $D_3 < 0) \| (D_2 = 0$ & $D_3 \leq 0) \| D_2 > 0)$, then $F(h) = (h-\varepsilon_1)(h-\varepsilon_2)\left[(h-\varepsilon_3)^2 + \varepsilon_4^2\right]$,
(3) $D_4 > 0$, $D_3 > 0$, $D_2 > 0$, then $F(h) = (h-\varepsilon_1)(h-\varepsilon_2)(h-\varepsilon_3)(h-\varepsilon_4)$,
(4) $D_4 = 0$, $D_3 < 0$, then $F(h) = (h-\varepsilon_1)^2\left[(h-\varepsilon_2)^2 + \varepsilon_3^2\right]$,
(5) $E_2 = D_4 = D_3 = 0$, $D_2 > 0$, then $F(h) = (h-\varepsilon_1)^3(h-\varepsilon_2)$,
(6) $E_2 < 0$, $D_4 = D_3 = 0$, $D_2 < 0$, then $F(h) = \left[(h-\varepsilon_1)^2 + \varepsilon_2^2\right]^2$,
(7) $D_4 = 0$, $D_3 > 0$, $D_2 > 0$, then $F(h) = (h-\varepsilon_1)^2(h-\varepsilon_2)(h-\varepsilon_3)$,
(8) $E_2 > 0$, $D_4 = D_3 = 0$, $D_2 > 0$, then $F(h) = (h-\varepsilon_1)^2(h-\varepsilon_2)^2$,

(9) $D_4 = 0$, $D_3 = 0$, $D_2 = 0$, then $F(h) = h^4$, where ε_i $(i \leq i \leq 4)$ are constants.

Case 1: When $D_2 < 0$, $D_3 = D_4 = 0$, $E_2 < 0$, a singular periodic solution turns into

$$q_1(x, t) = \left[-\frac{a}{6(b_1 - b_2 v + b_3 v^2)}\right]^{-\frac{1}{4}} \left[\varepsilon_2 \tan\left\{\varepsilon_2 \left\{-\frac{a}{6(b_1 - b_2 v + b_3 v^2)}\right\}^{\frac{1}{4}} \xi - \xi_0\right\} + \varepsilon_1\right]. \tag{29}$$

Case 2: When $D_4 = 0$, $D_3 = 0$, $D_2 = 0$, singular rational solution turns into

$$q_2(x, t) = \left[-\frac{a}{6(b_1 - b_2 v + b_3 v^2)}\right]^{-\frac{1}{4}} \left[\left\{-\frac{a}{6(b_1 - b_2 v + b_3 v^2)}\right\}^{\frac{1}{4}} \xi - \xi_0\right]^{-1}. \tag{30}$$

Case 3: When $E_2 > 0$, $D_4 = D_3 = 0$, $D_2 > 0$, singular solitary wave and shock wave are yielded as:

$$q_3(x, t) = \left[-\frac{a}{6(b_1 - b_2 v + b_3 v^2)}\right]^{-\frac{1}{4}}$$

$$\left[\frac{\varepsilon_2 - \varepsilon_1}{2}\left\{\coth \frac{(\varepsilon_1 - \varepsilon_2)\left[\left\{-\frac{a}{6(b_1 - b_2 v + b_3 v^2)}\right\}^{\frac{1}{4}} \xi - \xi_0\right]}{2} - 1\right\} + \varepsilon_2\right], \tag{31}$$

and

$$q_4(x, t) = \left[-\frac{a}{6(b_1 - b_2 v + b_3 v^2)}\right]^{-\frac{1}{4}}$$

$$\left[\frac{\varepsilon_2 - \varepsilon_1}{2}\left\{\tanh \frac{(\varepsilon_1 - \varepsilon_2)\left[\left\{-\frac{a}{6(b_1 - b_2 v + b_3 v^2)}\right\}^{\frac{1}{4}} \xi - \xi_0\right]}{2} - 1\right\} + \varepsilon_2\right], \tag{32}$$

respectively.

Case 4: When $D_4 = 0$, $D_3 > 0$, $D_2 > 0$, solitary wave and singular solitary wave emerge as:

$$q_5(x, t) = \frac{2\left\{-\frac{a}{6(b_1 - b_2 v + b_3 v^2)}\right\}^{-\frac{1}{4}} (\varepsilon_1 - \varepsilon_2)(\varepsilon_1 - \varepsilon_3)}{(\varepsilon_2 - \varepsilon_3)\cosh\left[\sqrt{(\varepsilon_1 - \varepsilon_2)(\varepsilon_1 - \varepsilon_3)}\left\{\left\{-\frac{a}{6(b_1 - b_2 v + b_3 v^2)}\right\}^{\frac{1}{4}} \xi - \xi_0\right\}\right] - (2\varepsilon_1 - \varepsilon_2 - \varepsilon_3)}, \tag{33}$$

and

$$q_6(x,t) = \frac{2\left\{-\dfrac{a}{6\left(b_1 - b_2 v + b_3 v^2\right)}\right\}^{-\frac{1}{4}} (\varepsilon_1 - \varepsilon_2)(\varepsilon_1 - \varepsilon_3)}{(\varepsilon_2 - \varepsilon_3)\sinh\left[\sqrt{(\varepsilon_1 - \varepsilon_2)(\varepsilon_1 - \varepsilon_3)}\left\{\left\{-\dfrac{a}{6\left(b_1 - b_2 v + b_3 v^2\right)}\right\}^{\frac{1}{4}} \xi - \xi_0\right\}\right] - (2\varepsilon_1 - \varepsilon_2 - \varepsilon_3)}, \tag{34}$$

respectively.

Case 5: When $D_2 > 0$, $D_3 = D_4 = 0$, $E_2 = 0$, the plane wave solution is given as

$$q_7(x,t) = \left[-\frac{a}{6\left(b_1 - b_2 v + b_3 v^2\right)}\right]^{-\frac{1}{4}} \left[\varepsilon_1 + \frac{4(\varepsilon_1 - \varepsilon_2)}{(\varepsilon_2 - \varepsilon_1)^2 \left\{\left\{-\dfrac{a}{6\left(b_1 - b_2 v + b_3 v^2\right)}\right\}^{\frac{1}{4}} \xi - \xi_0\right\}^2 - 4}\right]. \tag{35}$$

Case 6: When $D_3 < 0$, $D_4 = 0$, an exponential solution turns into

$$q_8(x,t) = \left[-\frac{a}{6\left(b_1 - b_2 v + b_3 v^2\right)}\right]^{-\frac{1}{4}}$$

$$\left[\frac{e^{\pm\sqrt{(\varepsilon_1\varepsilon_1 - \varepsilon_2)^2 + \varepsilon_3^2}\left\{\left\{-\frac{a}{6\left(b_1 - b_2 v + b_3 v^2\right)}\right\}^{\frac{1}{4}} \xi - \xi_0\right\}} - \dfrac{\varepsilon_1 - 2\varepsilon_2}{\sqrt{(\varepsilon_1 - \varepsilon_2)^2 + \varepsilon_3^2}} + 2\sqrt{(\varepsilon_1 - \varepsilon_2)^2 + \varepsilon_3^2} - (\varepsilon_1 - 2\varepsilon_2)}{\left\{e^{\pm\sqrt{(\varepsilon_1 - \varepsilon_2)^2 + \varepsilon_3^2}\left\{\left\{-\frac{a}{6\left(b_1 - b_2 v + b_3 v^2\right)}\right\}^{\frac{1}{4}} \xi - \xi_0\right\}} - \dfrac{\varepsilon_1 - \varepsilon_2}{\sqrt{(\varepsilon_1 - \varepsilon_2)^2 + \varepsilon_3^2}}\right\}^2 - 1}\right]. \tag{36}$$

Case 7: When $D_4 > 0$, $D_3 > 0$, $D_2 > 0$, two doubly periodic solutions turn into

$$q_9(x,t) = \left[-\frac{a}{6\left(b_1 - b_2 v + b_3 v^2\right)}\right]^{-\frac{1}{4}}$$

$$\left[\frac{\varepsilon_2(\varepsilon_1 - \varepsilon_4)\,\mathrm{sn}^2\left[\dfrac{\sqrt{(\varepsilon_1 - \varepsilon_3)(\varepsilon_2 - \varepsilon_4)}}{2}\left\{\left\{-\dfrac{a}{6\left(b_1 - b_2 v + b_3 v^2\right)}\right\}^{\frac{1}{4}} \xi - \xi_0\right\}, m\right] - \varepsilon_1(\varepsilon_2 - \varepsilon_4)}{(\varepsilon_1 - \varepsilon_4)\,\mathrm{sn}^2\left[\dfrac{\sqrt{(\varepsilon_1 - \varepsilon_3)(\varepsilon_2 - \varepsilon_4)}}{2}\left\{\left\{-\dfrac{a}{6\left(b_1 - b_2 v + b_3 v^2\right)}\right\}^{\frac{1}{4}} \xi - \xi_0\right\}, m\right] - (\varepsilon_2 - \varepsilon_4)}\right], \tag{37}$$

and

$$q_{10}(x,t)=\left[-\frac{a}{6\left(b_1-b_2v+b_3v^2\right)}\right]^{-\frac{1}{4}}$$

$$\left[\frac{\varepsilon_4(\varepsilon_2-\varepsilon_3)\,\mathrm{sn}^2\left[\frac{\sqrt{(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_4)}}{2}\left\{\left\{-\frac{a}{6\left(b_1-b_2v+b_3v^2\right)}\right\}^{\frac{1}{4}}\xi-\xi_0\right\},m\right]-\varepsilon_3(\varepsilon_2-\varepsilon_4)}{(\varepsilon_2-\varepsilon_3)\,\mathrm{sn}^2\left[\frac{\sqrt{(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_4)}}{2}\left\{\left\{-\frac{a}{6\left(b_1-b_2v+b_3v^2\right)}\right\}^{\frac{1}{4}}\xi-\xi_0\right\},m\right]-(\varepsilon_2-\varepsilon_4)}\right], \tag{38}$$

where

$$m^2=\frac{(\varepsilon_1-\varepsilon_4)(\varepsilon_2-\varepsilon_3)}{(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_4)}.$$

Case 8: When $D_4<0$ & $((D_2<0$ & $D_3<0)\|(D_2=0$ & $D_3\leq 0)\|D_2>0)$, another doubly periodic solution turns into

$$q_{11}(x,t)=\left[-\frac{a}{6\left(b_1-b_2v+b_3v^2\right)}\right]^{-\frac{1}{4}}$$

$$\left[\frac{\varepsilon_1\mathrm{cn}^2\left[\frac{\sqrt{-2\varepsilon_4 o_1(\varepsilon_1-\varepsilon_2)}}{2o_1o}\left\{\left\{-\frac{a}{6\left(b_1-b_2v+b_3v^2\right)}\right\}^{\frac{1}{4}}\xi-\xi_0\right\},o\right]+\varepsilon_2}{\varepsilon_3\mathrm{cn}^2\left[\frac{\sqrt{-2\varepsilon_4 o_1(\varepsilon_1-\varepsilon_2)}}{2o_1o}\left\{\left\{-\frac{a}{6\left(b_1-b_2v+b_3v^2\right)}\right\}^{\frac{1}{4}}\xi-\xi_0\right\},o\right]+\varepsilon_4}\right], \tag{39}$$

where

$$\varepsilon_1=\frac{1}{2}(\varepsilon_1+\varepsilon_2)\varepsilon_3-\frac{1}{2}(\varepsilon_1-\varepsilon_2)\varepsilon_4,$$

$$\varepsilon_2=\frac{1}{2}(\varepsilon_1+\varepsilon_2)\varepsilon_4-\frac{1}{2}(\varepsilon_1-\varepsilon_2)\varepsilon_3,$$

$$\varepsilon_3=\varepsilon_1-\varepsilon_3-\frac{\varepsilon_4}{o_1},$$

$$\varepsilon_4=\varepsilon_1-\varepsilon_3-\varepsilon_4 o_1,$$

$$E=\frac{\varepsilon_4^2+(\varepsilon_1-\varepsilon_3\varepsilon)(\varepsilon_2-\varepsilon_3)}{\varepsilon_4(\varepsilon_1-\varepsilon_2)},$$

$$o_1 = E \pm \sqrt{E^2+1},$$

$$o^2 = \frac{1}{1+o_1^2}. \tag{40}$$

Case 9: When $D_4 > 0$ & $((D_2 > 0$ & $D_3 \leq 0)\|D_2 \leq 0)$, another doubly periodic solution is:

$$q_{12}(x, t)$$

$$= \left[-\frac{a}{6\left(b_1-b_2v+b_3v^2\right)}\right]^{-\frac{1}{4}}$$

$$\left[\frac{\varepsilon_1 \operatorname{cn}\left[L\left\{\left\{-\frac{a}{6\left(b_1-b_2v+b_3v^2\right)}\right\}^{\frac{1}{4}}\xi-\xi_0\right\}, o\right]+\varepsilon_2 \operatorname{cn}\left[L\left\{\left\{-\frac{a}{6\left(b_1-b_2v+b_3v^2\right)}\right\}^{\frac{1}{4}}\xi-\xi_0\right\}, o\right]}{\varepsilon_3 \operatorname{sn}\left[L\left\{\left\{-\frac{a}{6\left(b_1-b_2v+b_3v^2\right)}\right\}^{\frac{1}{4}}\xi-\xi_0\right\}, o\right]+\varepsilon_4 \operatorname{sn}\left[L\left\{\left\{-\frac{a}{6\left(b_1-b_2v+b_3v^2\right)}\right\}^{\frac{1}{4}}\xi-\xi_0\right\}, o\right]}\right], \tag{41}$$

where

$$\varepsilon_1 = \varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_4,$$

$$\varepsilon_2 = \varepsilon_1\varepsilon_4-\varepsilon_2\varepsilon_3,$$

$$\varepsilon_3 = -\varepsilon_2-\frac{\varepsilon_4}{o_1},$$

$$\varepsilon_4 = \varepsilon_1-\varepsilon_3,$$

$$E = \frac{(\varepsilon_1-\varepsilon_3)^2+\varepsilon_2^2+\varepsilon_4^2}{2\varepsilon_2\varepsilon_4}, \tag{42}$$

$$o_1 = E+\sqrt{E^2-1},$$

$$o = \sqrt{\frac{o_1^2-1}{o_1^2}},$$

$$L = \frac{\varepsilon_2\sqrt{(\varepsilon_3^2+\varepsilon_4^2)(\rho_1^2\varepsilon_3^2+\varepsilon_4^2)}}{\varepsilon_3^2+e_4^2}.$$

3.3 *Gardner's equation*

Here, for any real-valued constants a_j where $j = 1, 2$, the function $F(q)$ is given by $F(q) = a_1 q + a_2 q^2$. Based on (1), this results in [2]

$$q_t + \left(a_1 q + a_2 q^2\right) q_x + b_1 q_{xxx} + b_2 q_{xxt} + b_3 q_{xtt} = 0, \tag{43}$$

which means that Eq. (5) transforms into

$$6c_2 + 12c_1 g + 6vg^2 - (2a_1 + a_2 g) g^3 - 6\left(b_1 - b_2 v + b_3 v^2\right)\left(g'\right)^2 = 0. \tag{44}$$

Rewrite the above equation as

$$\left(g'\right)^2 = s_4 g^4 + s_3 g^3 + s_2 g^2 + s_1 g + s_0, \tag{45}$$

where

$$\begin{aligned} s_0 &= \frac{c_2}{b_1 - b_2 v + b_3 v^2}, \\ s_1 &= \frac{2c_1}{b_1 - b_2 v + b_3 v^2}, \\ s_2 &= \frac{v}{b_1 - b_2 v + b_3 v^2}, \\ s_3 &= -\frac{a_1}{3(b_1 - b_2 v + b_3 v^2)}, \\ s_4 &= -\frac{a_2}{6(b_1 - b_2 v + b_3 v^2)}. \end{aligned} \tag{46}$$

Transformations are given by

$$f = (s_4)^{\frac{1}{4}}\left(g + \frac{s_3}{4s_4}\right), \quad \xi_1 = (s_4)^{\frac{1}{4}}\xi. \tag{47}$$

Thus, Eq. (45) turns into

$$(f_{\xi_1})^2 = f^4 + z_3 f^2 + z_2 f + z_1, \tag{48}$$

where

$$z_3 = -\frac{3}{8}(s_3)^2(s_4)^{-\frac{3}{2}} + s_2(s_4)^{-\frac{1}{2}},$$

$$z_2 = \left(\frac{(s_3)^3}{8(s_4)^2} - \frac{s_2 s_3}{2s_4} + s_1\right)(s_4)^{-\frac{1}{4}}, \tag{49}$$

$$z_1 = -\frac{3(s_3)^4}{256(s_4)^3} + \frac{s_2(s_3)^2}{16(s_4)^2} - \frac{s_1 s_3}{4s_4} + s_0.$$

Based on the fourth-order polynomial discrimination system (28), we present the following results.
Case 1: When $D_2 < 0$, $D_3 = D_4 = 0$, $E_2 < 0$, a singular periodic solution reads as

$$q_1(x, t) = \left[-\frac{a_2}{6(b_1 - b_2 v + b_3 v^2)}\right]^{-\frac{1}{4}} \left[\varepsilon_2 \tan\left\{\varepsilon_2\left\{-\frac{a_2}{6(b_1 - b_2 v + b_3 v^2)}\right\}^{\frac{1}{4}} \xi - \xi_0\right\} + \varepsilon_1\right] - \frac{a_1}{2a_2}. \tag{50}$$

Case 2: When $D_4 = 0$, $D_3 = 0$, $D_2 = 0$, a singular rational solution reads as

$$q_2(x, t) = \left\{-\frac{a_2}{6(b_1 - b_2 v + b_3 v^2)}\right\}^{-\frac{1}{4}} \left[\left\{-\frac{a_2}{6(b_1 - b_2 v + b_3 v^2)}\right\}^{\frac{1}{4}} \xi - \xi_0\right]^{-1} - \frac{a_1}{2a_2}. \tag{51}$$

Case 3: When $E_2 > 0$, $D_4 = D_3 = 0$, $D_2 > 0$, the singular solitary wave and shock wave solutions read as:

$$q_3(x, t) = \left[-\frac{a_2}{6(b_1 - b_2 v + b_3 v^2)}\right]^{-\frac{1}{4}} \left[\frac{\varepsilon_2 - \varepsilon_1}{2}\left\{\coth\frac{(\varepsilon_1 - \varepsilon_2)\left[\left\{-\frac{a_2}{6(b_1 - b_2 v + b_3 v^2)}\right\}^{\frac{1}{4}} \xi - \xi_0\right]}{2} - 1\right\} + \varepsilon_2\right] - \frac{a_1}{2a_2}, \tag{52}$$

and

$$q_4(x,t) = \left[-\frac{a_2}{6(b_1 - b_2 v + b_3 v^2)}\right]^{-\frac{1}{4}}$$

$$\left[\frac{\varepsilon_2 - \varepsilon_1}{2}\left\{\tanh\frac{(\varepsilon_1 - \varepsilon_2)\left[\left\{-\frac{a_2}{6(b_1 - b_2 v + b_3 v^2)}\right\}^{\frac{1}{4}}\xi - \xi_0\right]}{2} - 1\right\} + \varepsilon_2\right] - \frac{a_1}{2a_2}, \tag{53}$$

respectively.

Case 4: When $D_4 = 0$, $D_3 > 0$, $D_2 > 0$, solitary wave and singular solitary wave solutions read as

$$q_5(x,t) = \frac{2\left\{-\frac{a_2}{6(b_1 - b_2 v + b_3 v^2)}\right\}^{-\frac{1}{4}}(\varepsilon_1 - \varepsilon_2)(\varepsilon_1 - \varepsilon_3)}{(\varepsilon_2 - \varepsilon_3)\cosh\left[\sqrt{(\varepsilon_1 - \varepsilon_2)(\varepsilon_1 - \varepsilon_3)}\left\{\left\{-\frac{a_2}{6(b_1 - b_2 v + b_3 v^2)}\right\}^{\frac{1}{4}}\xi - \xi_0\right\}\right] - (2\varepsilon_1 - \varepsilon_2 - \varepsilon_3)} - \frac{a_1}{2a_2}, \tag{54}$$

and

$$q_6(x,t) = \frac{2\left\{-\frac{a_2}{6(b_1 - b_2 v + b_3 v^2)}\right\}^{-\frac{1}{4}}(\varepsilon_1 - \varepsilon_2)(\varepsilon_1 - \varepsilon_3)}{(\varepsilon_2 - \varepsilon_3)\sinh\left[\sqrt{(\varepsilon_1 - \varepsilon_2)(\varepsilon_1 - \varepsilon_3)}\left\{\left\{-\frac{a_2}{6(b_1 - b_2 v + b_3 v^2)}\right\}^{\frac{1}{4}}\xi - \xi_0\right\}\right] - (2\varepsilon_1 - \varepsilon_2 - \varepsilon_3)} - \frac{a_1}{2a_2}, \tag{55}$$

respectively.

Case 5: When $D_2 > 0$, $D_3 = D_4 = 0$, $E_2 = 0$, a plane wave solution reads as

$$q_7(x,t) = \left[-\frac{a_2}{6(b_1 - b_2 v + b_3 v^2)}\right]^{-\frac{1}{4}}$$

$$\left[\varepsilon_1 + \frac{4(\varepsilon_1 - \varepsilon_2)}{(\varepsilon_2 - \varepsilon_1)^2\left[\left\{-\frac{a_2}{6(b_1 - b_2 v + b_3 v^2)}\right\}^{\frac{1}{4}}\xi - \xi_0\right]^2 - 4}\right] - \frac{a_1}{2a_2}. \tag{56}$$

Case 6: When $D_3 < 0$, $D_4 = 0$, an exponential function solution reads:

$$q_8(x,\,t)$$

$$= \left[-\frac{a_2}{6\,(b_1-b_2v+b_3v^2)}\right]^{-\frac{1}{4}}$$

$$\left[\frac{e^{\pm\sqrt{(\varepsilon_1\varepsilon_1-\varepsilon_2)^2+\varepsilon_3^2}\left\{\left\{-\frac{a_2}{6\left(b_1-b_2v+b_3v^2\right)}\right\}^{\frac{1}{4}}\xi-\xi_0\right\}}-\frac{\varepsilon_1-2\varepsilon_2}{\sqrt{(\varepsilon_1-\varepsilon_2)^2+\varepsilon_3^2}}+2\sqrt{(\varepsilon_1-\varepsilon_2)^2+\varepsilon_3^2}-(\varepsilon_1-2\varepsilon_2)}{\left\{e^{\pm\sqrt{(\varepsilon_1-\varepsilon_2)^2+\varepsilon_3^2}\left\{\left\{-\frac{a_2}{6\left(b_1-b_2v+b_3v^2\right)}\right\}^{\frac{1}{4}}\xi-\xi_0\right\}}-\frac{\varepsilon_1-\varepsilon_2}{\sqrt{(\varepsilon_1-\varepsilon_2)^2+\varepsilon_3^2}}\right\}^2-1}\right]-\frac{a_1}{2a_2}. \tag{57}$$

Case 7: When $D_4>0$, $D_3>0$, $D_2>0$, two doubly periodic solutions are:

$$q_9(x,\,t)$$

$$= \left[-\frac{a_2}{6\,(b_1-b_2v+b_3v^2)}\right]^{-\frac{1}{4}}$$

$$\left[\frac{\varepsilon_2(\varepsilon_1-\varepsilon_4)\,\mathrm{sn}^2\left[\frac{\sqrt{(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_4)}}{2}\left\{\left\{-\frac{a_2}{6\,(b_1-b_2v+b_3v^2)}\right\}^{\frac{1}{4}}\xi-\xi_0\right\},\,m\right]-\varepsilon_1(\varepsilon_2-\varepsilon_4)}{(\varepsilon_1-\varepsilon_4)\,\mathrm{sn}^2\left[\frac{\sqrt{(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_4)}}{2}\left\{\left\{-\frac{a_2}{6\,(b_1-b_2v+b_3v^2)}\right\}^{\frac{1}{4}}\xi-\xi_0\right\},\,m\right]-(\varepsilon_2-\varepsilon_4)}\right]-\frac{a_1}{2a_2}, \tag{58}$$

and

$$q_{10}(x,\,t)$$

$$= \left[-\frac{a_2}{6\,(b_1-b_2v+b_3v^2)}\right]^{-\frac{1}{4}}$$

$$\left[\frac{\varepsilon_4(\varepsilon_2-\varepsilon_3)\,\mathrm{sn}^2\left[\frac{\sqrt{(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_4)}}{2}\left\{\left\{-\frac{a_2}{6\,(b_1-b_2v+b_3v^2)}\right\}^{\frac{1}{4}}\xi-\xi_0\right\},\,m\right]-\varepsilon_3(\varepsilon_2-\varepsilon_4)}{(\varepsilon_2-\varepsilon_3)\,\mathrm{sn}^2\left[\frac{\sqrt{(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_4)}}{2}\left\{\left\{-\frac{a_2}{6\,(b_1-b_2v+b_3v^2)}\right\}^{\frac{1}{4}}\xi-\xi_0\right\},\,m\right]-(\varepsilon_2-\varepsilon_4)}\right]-\frac{a_1}{2a_2}, \tag{59}$$

where

$$m^2 = \frac{(\varepsilon_1-\varepsilon_4)(\varepsilon_2-\varepsilon_3)}{(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_4)}.$$

Case 8: When $D_4 < 0$ & $((D_2 < 0$ & $D_3 < 0)\|(D_2 = 0$ & $D_3 \leq 0)\|D_2 > 0)$, a doubly periodic solution reads as

$$\begin{aligned} &q_{11}(x, t) \\ &= \left[-\frac{a_2}{6(b_1-b_2v+b_3v^2)}\right]^{-\frac{1}{4}} \\ &\left[\frac{\varepsilon_1 \mathrm{cn}^2\left[\frac{\sqrt{-2\varepsilon_4 o_1(\varepsilon_1-\varepsilon_2)}}{2o_1 o}\left\{\left\{-\frac{a_2}{6(b_1-b_2v+b_3v^2)}\right\}^{\frac{1}{4}}\xi-\xi_0\right\}, o\right]+\varepsilon_2}{\varepsilon_3 \mathrm{cn}^2\left[\frac{\sqrt{-2\varepsilon_4 o_1(\varepsilon_1-\varepsilon_2)}}{2o_1 o}\left\{\left\{-\frac{a_2}{6(b_1-b_2v+b_3v^2)}\right\}^{\frac{1}{4}}\xi-\xi_0\right\}, o\right]+\varepsilon_4}\right]-\frac{a_1}{2a_2}, \end{aligned} \tag{60}$$

where

$$\begin{aligned} \varepsilon_1 &= \frac{1}{2}(\varepsilon_1+\varepsilon_2)\varepsilon_3-\frac{1}{2}(\varepsilon_1-\varepsilon_2)\varepsilon_4, \\ \varepsilon_2 &= \frac{1}{2}(\varepsilon_1+\varepsilon_2)\varepsilon_4-\frac{1}{2}(\varepsilon_1-\varepsilon_2)\varepsilon_3, \\ \varepsilon_3 &= \varepsilon_1-\varepsilon_3-\frac{\varepsilon_4}{o_1}, \\ \varepsilon_4 &= \varepsilon_1-\varepsilon_3-\varepsilon_4 o_1, \\ E &= \frac{\varepsilon_4^2+(\varepsilon_1-\varepsilon_3\varepsilon)(\varepsilon_2-\varepsilon_3)}{\varepsilon_4(\varepsilon_1-\varepsilon_2)}, \\ o_1 &= E \pm \sqrt{E^2+1}, \\ o^2 &= \frac{1}{1+o_1^2}. \end{aligned} \tag{61}$$

Case 9: When $D_4 > 0$ & $((D_2 > 0$ & $D_3 \leq 0)\|D_2 \leq 0)$, another doubly periodic solution reads:

$$
q_{12}(x, t)
$$

$$
= \left[-\frac{a_2}{6\left(b_1 - b_2 v + b_3 v^2\right)} \right]^{-\frac{1}{4}}
$$

$$
\left[\frac{\varepsilon_1 \operatorname{cn}\left[L \left\{ \left\{ -\frac{a_2}{6\left(b_1 - b_2 v + b_3 v^2\right)} \right\}^{\frac{1}{4}} \xi - \xi_0 \right\}, o \right] + \varepsilon_2 \operatorname{cn}\left[L \left\{ \left\{ -\frac{a_2}{6\left(b_1 - b_2 v + b_3 v^2\right)} \right\}^{\frac{1}{4}} \xi - \xi_0 \right\}, o \right]}{\varepsilon_3 \operatorname{sn}\left[L \left\{ \left\{ -\frac{a_2}{6\left(b_1 - b_2 v + b_3 v^2\right)} \right\}^{\frac{1}{4}} \xi - \xi_0 \right\}, o \right] + \varepsilon_4 \operatorname{sn}\left[L \left\{ \left\{ -\frac{a_2}{6\left(b_1 - b_2 v + b_3 v^2\right)} \right\}^{\frac{1}{4}} \xi - \xi_0 \right\}, o \right]} \right] \tag{62}
$$

$$
- \frac{a_1}{2a_2},
$$

where

$$
\varepsilon_1 = \varepsilon_1 \varepsilon_3 + \varepsilon_2 \varepsilon_4,
$$

$$
\varepsilon_2 = \varepsilon_1 \varepsilon_4 - \varepsilon_2 \varepsilon_3,
$$

$$
\varepsilon_3 = -\varepsilon_2 - \frac{\varepsilon_4}{o_1},
$$

$$
\varepsilon_4 = \varepsilon_1 - \varepsilon_3,
$$

$$
E = \frac{(\varepsilon_1 - \varepsilon_3)^2 + \varepsilon_2^2 + \varepsilon_4^2}{2\varepsilon_2 \varepsilon_4}, \tag{63}
$$

$$
o_1 = E + \sqrt{E^2 - 1},
$$

$$
o = \sqrt{\frac{o_1^2 - 1}{o_1^2}},
$$

$$
L = \frac{\varepsilon_2 \sqrt{(\varepsilon_3^2 + \varepsilon_4^2)(\rho_1^2 \varepsilon_3^2 + \varepsilon_4^2)}}{\varepsilon_3^2 + e_4^2}.
$$

In addition to the enlisted parameter constraints, one must note that for the KdV and mKdV equations, the constraint condition given by

$$a\left(b_1 - b_2 v + b_3 v^2\right) > 0 \tag{64}$$

must hold. Then, for the Gardners equation, one must have

$$a_2\left(b_1 - b_2 v + b_3 v^2\right) > 0 \tag{65}$$

to remain valid for the spectrum of waves to exist. Eqs. (64) and (65) are specified to guarantee the existence of singular solitary wave, shock wave, and solitary wave.

4. Results and discussion

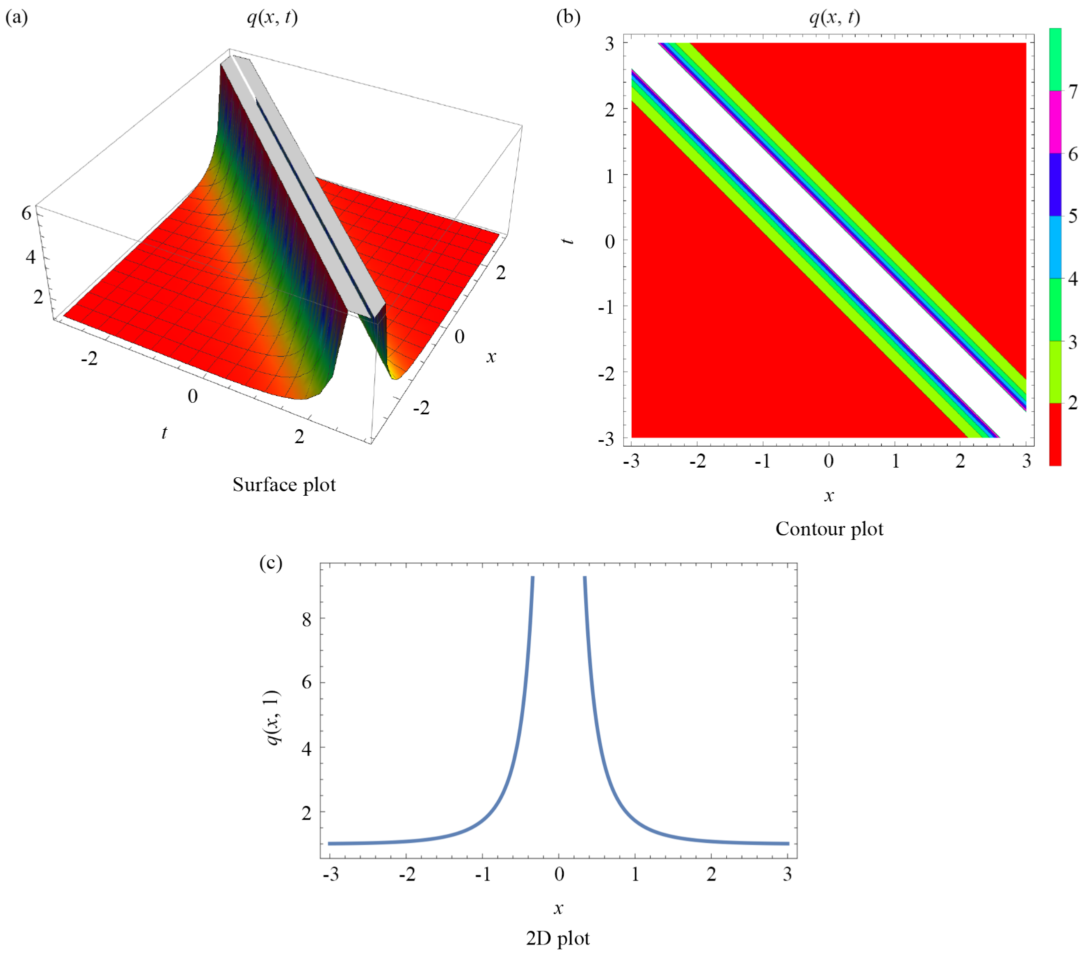

Figure 1. A singular solitary wave

Figures 1 through 3 present the graphical analysis of three distinct wave structures that are singular solitary wave, shock wave, and solitary wave each characterized by unique functional forms and described by the analytical solutions

(31), (32), and (33), respectively. All figures are evaluated at a fixed temporal snapshot, $t = 1$, under the consistent parameter set: $\varepsilon_2 = 2$, $\varepsilon_1 = 1$, $\xi_0 = 1$, $a = -1$, $b_1 = 1$, $b_2 = 1$, $b_3 = 1$, and $v = 1$. These parameter choices ensure coherent comparative visualization across all three wave types.

Figure 1 illustrates the behavior of the singular solitary wave derived from solution (31). Figure 1a presents a surface plot, clearly showcasing the singular nature of the solution. The wave exhibits a vertical asymptote near the origin due to the inherent singularity in the hyperbolic cotangent function. This is further emphasized in the contour plot in Figure1b, where the contours become densely packed near the singular line, indicating a steep gradient. The 2D plot in Figure 1c offers a direct view of the wave profile along a fixed direction. It clearly reveals the sharp discontinuity, characteristic of singular solitary waves, which distinguishes them from regular smooth solitons. The presence of this non-removable singularity marks an essential qualitative difference from other wave types analyzed.

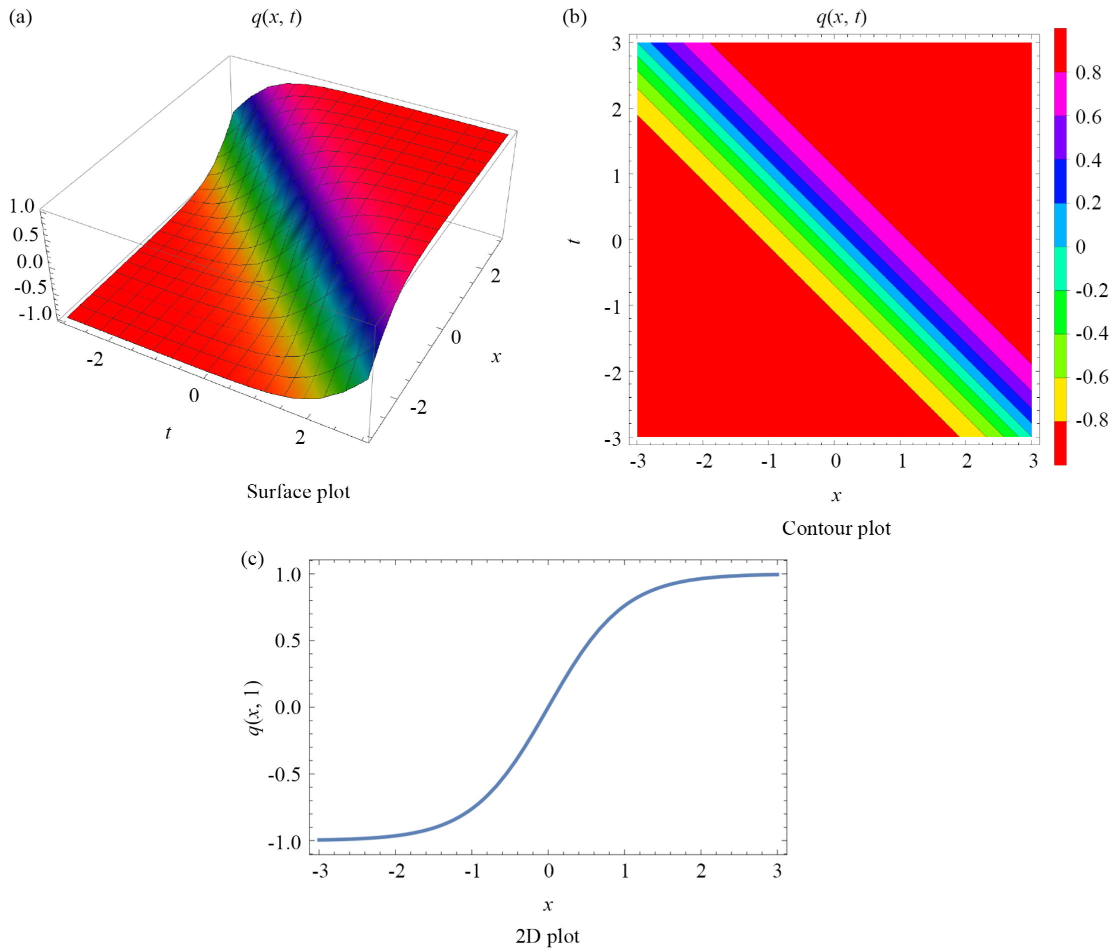

Figure 2. A shock wave

Figure 2 corresponds to the shock wave structure based on solution (32). In the surface plot of Figure 2a, the wave exhibits a sharp transition between two plateau regions, indicating the presence of a steep wavefront that is a hallmark of shock wave behavior. The transition region is relatively narrow, reflecting high nonlinearity. The contour plot in Figure 2b supports this observation by showing closely spaced contour lines at the center, which then broaden out, highlighting the abrupt change in amplitude. In the 2D profile presented in Figure 2c, the typical S-shaped curve of the hyperbolic tangent function is evident, transitioning smoothly but rapidly from one asymptotic value to another. This wave form captures the dissipative-like, non-oscillatory nature of shock waves, where the wavefront represents a sharp interface between two distinct states.

Figure 3 visualizes the classic solitary wave derived from solution (33). The surface plot in Figure 3a demonstrates the bell-shaped, localized nature of the solitary wave. The amplitude reaches its maximum at the wave center and decays symmetrically and rapidly toward zero, which is a defining characteristic of solitary waves. Figure 3b's contour plot depicts concentric contours around the central peak, highlighting the smooth and localized energy distribution. The 2D plot in Figure 3c further confirms this, exhibiting a symmetric and smooth peak. Unlike the singular solitary or shock wave profiles, this solution does not exhibit any singularity or sharp discontinuity, reaffirming its identity as a regular, localized, and non-singular soliton-like structure.

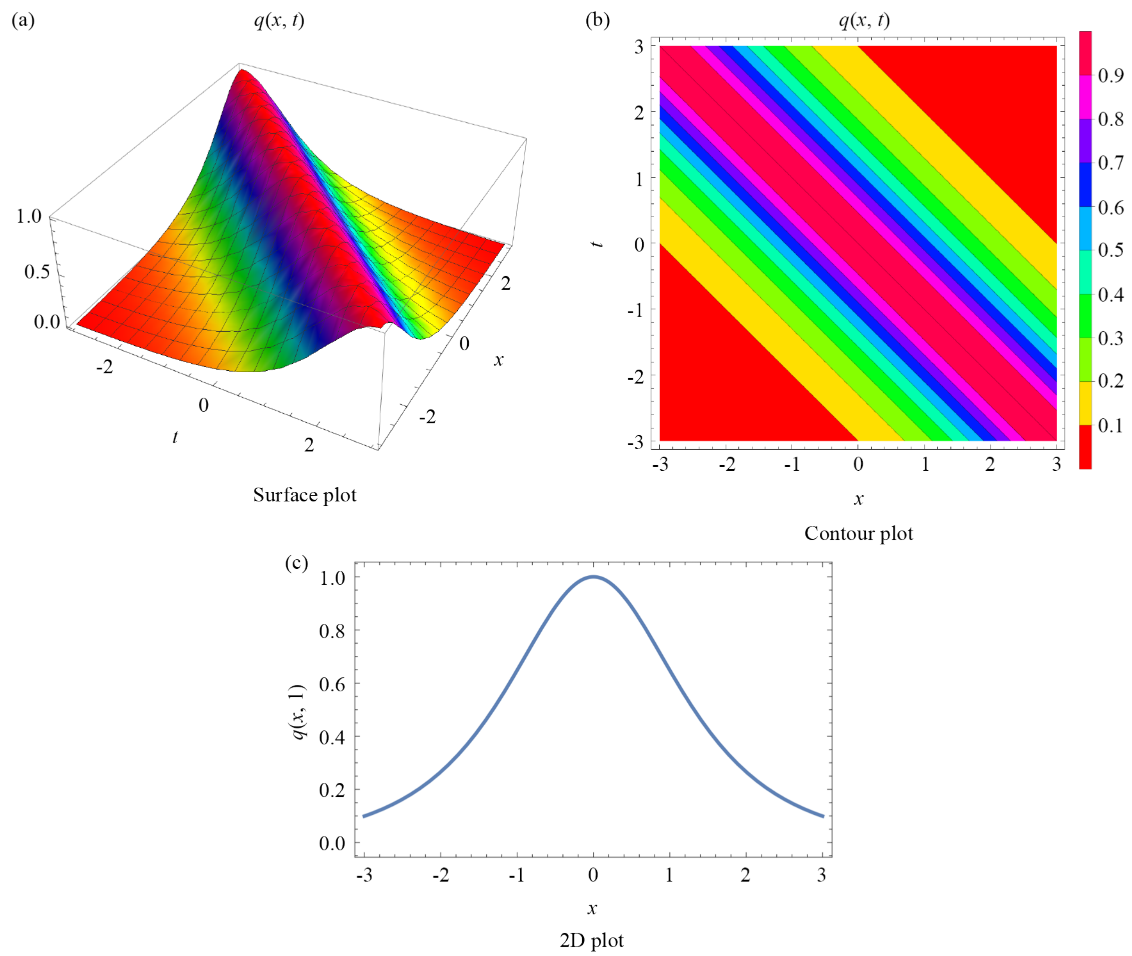

Figure 3. A solitary wave

In summary, the three figures collectively highlight the distinctive nature of the analyzed wave structures. The singular solitary wave in Figure 1 features a non-removable singularity, making it inherently discontinuous and non-physical in certain contexts. The shock wave of Figure 2, while continuous, shows a rapid transition suggestive of energy dissipation or steep nonlinear behavior. Finally, the solitary wave in Figure 3 stands out due to its smooth, localized, and symmetric profile, making it ideal for modeling undistorted wave propagation in nonlinear media. The comparison underscores the diverse range of nonlinear wave solutions captured within the employed analytical framework.

5. Conclusions

This paper has successfully recovered a variety of wave structures including solitary waves, shock waves, plane waves, and cnoidal waves for several prominent nonlinear evolution equations such as the KdV equation, the mKdV equation, and the Gardner equation. The utilization of the complete discriminant approach was instrumental in systematically classifying and deriving these exact solutions, thereby reinforcing its effectiveness as a robust analytical technique in nonlinear wave theory.

One of the key novelties of this study lies in the unified treatment of multiple wave structures across different integrable models using a single algebraic framework. This unified perspective not only deepens the understanding of wave propagation phenomena but also opens avenues for efficiently generating a broader class of exact solutions. Moreover, the approach highlights the discriminant structure as a decisive factor in determining the nature of the solutions, which could be further exploited in the analysis of other nonlinear systems.

However, the study is not without limitations. Notably, the adopted integration algorithm failed to produce meaningful solutions for the power-law KdV and power-law Gardner equations. This limitation underscores the potential inflexibility of the current scheme when applied to equations with non-polynomial or generalized nonlinearities. It suggests that while the discriminant method is powerful, it may require significant adaptation or supplementation when extended to equations of non-standard form or higher complexity.

To address this gap, future work will focus on developing or incorporating alternative integration schemes possibly involving Lie symmetry analysis, the extended tanh method, or numerical continuation techniques that are better suited for handling power-law nonlinearities. Additionally, extending the methodology to study higher-dimensional or perturbed versions of the KdV and Gardner equations could provide deeper insight into the stability and modulation of the derived waveforms. Another promising direction would involve exploring the interplay between nonlinearity and dispersion in the presence of stochastic forcing or external potentials, which is of growing interest in realistic physical systems.

As a result, the findings of this study contribute to the growing body of exact solution techniques in mathematical physics, and the limitations encountered offer fertile ground for further analytical and computational advancements.

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Conflict of interest

The authors claim that there is no conflict of interest.

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