UNIVERSAL WISER PUBLISHER

3-Quadratic Functions in Lipschitz Spaces

Ismail Nikoufar
Department of Mathematics, Payame Noor University, P.O. Box 19395-3697 Tehran, Iran
E-mail: nikoufar@pnu.ac.ir

Abstract: Stability of functional equations is a classical problem proposed by Ulam. In this paper, we prove the stability of the 3-quadratic functional equations in Lipschitz spaces.

Keywords: 3-quadratic functional equation, Lipschitz space, stability

1. Introduction

The first stability problem concerning group homomorphisms was raised by [20] in 1940 and affirmatively was answered for Banach spaces by [6].

The stability of the quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1}$$

was established by many authors in various spaces, see [5, 15, 8] and references therein. Any solution of (1) in the space of real numbers is of the form $g(x) = ax^2$ for all $x \in \mathbb{R}$, where $a \in \mathbb{R}$. The stability problem of the functional equation (1) has been verified in other spaces (see [17, 2, 4]).

The general solution and the stability of the following 2-variable quadratic functional equation f from a linear space X into a complete normed space Y was clarified by [7]:

$$f(x+z, y+w)+f(x-z, y-w)=2f(x, y)+2f(z, w). \tag{2}$$

Any solution of (2) is termed as a quadratic mapping. If $X = Y = \mathbb{R}$ the quadratic form $g(x, y) = ax^2 + bxy + cy^2$ is a solution of (2) for all $x, y \in \mathbb{R}$, where $a, b, c \in \mathbb{R}$.

When $b = c = 0$, we obtain the quadratic form $g(x, y) = ax^2$ satisfying in (1). So, any solution of (1) is a solution of (2), but not vice versa.

Ravi et al.[16] discussed the general solution and the stability of the 3-variable quadratic functional equation

$$f(x+y, z+w, u+v)+f(x-y, z-w, u-v)=2f(x, z, u)+2f(y, w, v). \tag{3}$$

We say that f is 3-quadratic if f satisfies in (3). Any solution of (3) in the space of real numbers is of the form

$$g(x, y, z) = a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5xz + a_6yz$$

for all $x, y, z \in \mathbb{R}$, where $a_1, ..., a_6 \in \mathbb{R}$. It is remarkable that any solution of either (1) or (2) is a solution of (3), but not vice versa.

Lipschitz spaces have a rich and beautiful algebra structure and these spaces possess various universal properties. These algebras present many opportunities for future research. Some of the open problems in this area are given in chapter 7 of [21].

In Lipschitz spaces, the stability of the quadratic functional equation (1) was verified by [4]. The stability type problems for some functional equations were also studied by [18, 19] in Lipschitz spaces. The author of the present paper proved the stability of the quadratic and cubic functional equations in Lipschitz spaces (see [9, 14]).

DOI: https://doi.org/10.37256/cm.13202072

We remark that this subject area has been considered less attention over the recent years and the current problem under consideration is important to verify the stability of the 3-quadratic functional equation in Lipschitz spaces. So, in this paper we focus on the stability of the quadratic functional equations of three variables which its solutions contain the solutions of the functional equations (1) and (2). We prove the stability of the 3-quadratic functional equation in Lipschitz spaces.

2. Main results

We start by introducing some quite standard notation. Let $G \times G \times G$ be the Cartesian product of an abelian group G with itself and denote by G^3. We denote By $B(G^3, S(V))$ the subset of all functions $f: G^3 \to V$ such that Imf $\subset A$ for some $A \in S(V)$, where V is a vector space, $S(V)$ is a family of subsets of V. The family $B(G^3, S(V))$ is a vector space and contains all constant functions. This family admits a left invariant mean (briefly LIM), if the subset $S(V)$ is linearly invariant, in the sense that $A + B \in S(V)$ for all A, B $\in S(V)$ and $x + \alpha A \in S(V)$ for all $x \in V$, $\alpha \in \mathbb{R}$, $A \in S(V)$ (see [1]), and there exists a linear operator $\Gamma : B(G^3, S(V)) \to V$ such that

(i) if Imf $\subset A$ for some $A \in S(V)$, then $\Gamma[f] \in A$,

(ii) if $f \in B(G^3, S(V))$ and $(a, b, c) \in G^3$, then $\Gamma[f^{a,b,c}] = \Gamma[f]$, where $f^{a,b,c}(x, y, z) = f(x + a, y + b, z + c)$.

Example 2.1. Let G be a finite group, V a vector space, and $S(V)$ any linearly invariant family of convex subsets of V. Let $f \in B(G^3, S(V))$. We define

$$\Gamma[f] := \frac{1}{|G^3|} \sum_{(x,\, y,\, z) \in G^3} f(x,\, y,\, z).$$

Then, Γ is a LIM on $B(G^3, S(V))$, where $|G|$ is the cardinality of G.

Example 2.2. Let G be a finite group, V a vector space, and $S(V)$ any linearly invariant family of subsets of V with one member. Let $f \in B(G^3, S(V))$. We define

$$\Gamma[f] := \sum_{(x,\, y,\, z) \in G^3} f(x,\, y,\, z).$$

Then, Γ is not a LIM on $B(G^3, S(V))$, where $|G|$ is the cardinality of G. Since $f \in B(G^3, S(V))$, there exists the subset $\{v\} \in S(V)$ such that Imf $= \{v\}$, but $\Gamma[f] = |G^3| v \notin \{v\}$.

Let $\mathbf{d} : G^3 \times G^3 \to S(V)$ be a set-valued function such that

$$\mathbf{d}((x + a, y + b, z + c), (u + a, v + b, w + c)) = \mathbf{d}((x, y, z), (u, v, w))$$

for all $(a, b, c), (x, y, z), (u, v, w) \in G^3$ (cf. [3, 14]). We say that a function $f: G^3 \to V$ is $\mathbf{d}$-Lipschitz if $f(x, y, z) - f(u, v, w) \in \mathbf{d}((x, y, z), (u, v, w))$ for all $(x, y, z), (u, v, w) \in G^3$. In particular, when (G^3, d) is a metric group and V a normed space, we define the function $m_f : \mathbb{R}^+ \to \mathbb{R}^+$ to be a module of continuity of the function $f: G^3 \to V$ if for all $\delta > 0$ and all $(x, y, z), (u, v, w) \in G^3$ the condition $d((x, y, z), (u, v, w)) \leq \delta$ implies $\|f(x, y, z) - f(u, v, w)\| \leq m_f(\delta)$. A function $f: G^3 \to V$ is called Lipschitz function of order α if there exists $L > 0$ such that

$$\| f(x,y,z) - f(u,v,w) \| \leq L d((x,y,z),(u,v,w))^{\alpha} \tag{4}$$

for all $(x, y, z), (u, v, w) \in G^3$. We consider $Lip(G^3, V)$ to be the Lipschitz space consisting of all bounded Lipschitz functions with the norm

$$\|f\|_{Lip} := \|f\|_{\infty} + L_{\alpha}(f)$$

where $\|.\|_{\infty}$ is the supremum norm and

$$L_\alpha(f) = \sup\left\{\frac{\|f(x,y,z)-f(u,v,w)\|}{d((x,y,z),(u,v,w))^\alpha} : (x,y,z),(u,v,w)\in G^3, (x,y,z)\neq(u,v,w)\right\}.$$

Let $(G^3, +)$ be an Abelian group. We say that a metric d on $(G^3, +)$ is invariant under translation if it satisfies the following condition

$$d((x+a, y+b, z+c), (u+a, v+b, w+c)) = d((x, y, z), (u, v, w))$$

for all $(a, b, c), (x, y, z), (u, v, z) \in G^3$. A metric $\tilde{d}$ on $G^3 \times G^3$ is called a product metric if it is an invariant metric and the following condition holds

$$\tilde{d}((a, b, c, x, y, z), (a, b, c, u, v, w)) = \tilde{d}((x, y, z, a, b, c), (u, v, w, a, b, c))$$

$$= d((x, y, z), (u, v, w))$$

for all $(a, b, c), (x, y, z), (u, v, w) \in G^3$.

In this section for the sake of a simplified writing and for a given function $f : G^3 \to V$ we define its 3-variable quadratic difference as follows:

$$T_f(x, y, z, u, v, w) := 2f(x, y, z) + 2f(u, v, w) - f(x+u, y+v, z+w) - f(x-u, y-v, z-w)$$

for all $(x, y, z), (u, v, w) \in G^3$. Note that f is 3-quadratic if and only if $T_f = 0$.

Theorem 2.3. Let G be an Abelian group and V a vector space. Assume that the family $B(G^3, S(V))$ admits LIM. If $f : G^3 \to V$ is a function and $T_f(r, t, s, \cdot, \cdot, \cdot) : G^3 \to V$ is **d**-Lipschitz for all $(r, t, s) \in G^3$, then there exists a function S such that $f - S$ is $\frac{1}{2}$ **d**-Lipschitz and $T_S = 0$.

Proof. By assumption the family $B(G^3, S(V))$ admits LIM and so there exists a linear operator $\Gamma : B(G^3, S(V)) \to V$ such that

(i) $\Gamma[Q] \in A$ for some $A \in S(V)$,

(ii) if for $(u, v, w) \in G^3$, $Q^{u,v,w} : G^3 \to V$ is defined by $Q^{u,v,w}(r, t, s) := Q(r+u, t+v, s+w)$ for every $(r, t, s) \in G^3$, then $Q^{u,v,w} \in B(G^3, S(V))$ and $\Gamma[Q] = \Gamma[Q^{u,v,w}]$.

Consider the function $Q_{a,b,c} : G^3 \to V$ by

$$Q_{a,b,c}(x,y,z) := \frac{1}{2}f(x+a,y+b,z+c) + \frac{1}{2}f(x-a,y-b,z-c) - f(x,y,z)$$

for all $(a, b, c) \in G^3$. We first prove that $Q_{a,b,c} \in B(G^3, S(V))$. We have

$$Q_{a,b,c}(x,y,z) := \frac{1}{2}f(x+a,y+b,z+c) + \frac{1}{2}f(x-a,y-b,z-c) - f(x,y,z) - f(a,b,c)$$

$$-\frac{1}{2}f(x,y,z) - \frac{1}{2}f(x,y,z) + f(x,y,z) + f(0,0,0) + f(a,b,c) - f(0,0,0)$$

$$= \frac{1}{2}T_f(x,y,z,0,0,0) - \frac{1}{2}T_f(x,y,z,a,b,c) + f(a,b,c) - f(0,0,0)$$

for all $(x, y, z), (a, b, c) \in G^3$. Then, $\operatorname{Im} Q_{a,b,c} \subset A$ and $A \in S(V)$, where $A := \frac{1}{2}\,\mathbf{d}((0, 0, 0), (a, b, c)) + f(a, b, c) - f(0, 0, 0)$.

We know that $B(G^3, S(V))$ contains constant functions. By using property (i) of Γ it is easy to verify that if $f : G^3 \to$

V is constant, i.e., $f(x, y, z) = k$ for $(x, y, z) \in G^3$, where $k \in V$, then $\Gamma[f] = k$. Define the function $S: G^3 \to V$ by $S(x, y, z) := \Gamma[Q_{x,y,z}]$ for $(x, y, z) \in G^3$.

We now show that $f - S$ is $\frac{1}{2}$ **d**-Lipschitz. For every $(x, y, z) \in G^3$ define the constant function $k_{x,y,z}: G^3 \to V$ by $k_{x,y,z}(u, v, w) := f(x, y, z)$ for all $(u, v, w) \in G^3$. Then,

$$(f(x,y,z) - S(x,y,z)) - (f(u,v,w) - S(u,v,w)) = (\Gamma[k_{x,y,z}] - \Gamma[Q_{x,y,z}]) - (\Gamma[k_{u,v,w}] - \Gamma[Q_{u,v,w}])$$

$$= \Gamma[k_{x,y,z} - Q_{x,y,z}] - \Gamma[k_{u,v,w} - Q_{u,v,w}]$$

$$= \Gamma[\frac{1}{2}T_f(\cdot,\cdot,\cdot,x,y,z)] - \frac{1}{2}T_f(\cdot,\cdot,\cdot,u,v,w)]$$

for all $(x, y, z), (u, v, w) \in G^3$.

In view of property (i) of Γ we conclude that

$$\Gamma[\frac{1}{2}T_f(\cdot,\cdot,\cdot,x,y,z) - \frac{1}{2}T_f(\cdot,\cdot,\cdot,u,v,w)] \in \frac{1}{2}\mathbf{d}((x,y,z)(u,v,w))$$

for all $(x, y, z), (u, v, w) \in G^3$. Therefore

$$(f(x,y,z) - S(x,y,z)) - (f(u,v,w) - S(u,v,w)) \in \frac{1}{2}\mathbf{d}((x,y,z)(u,v,w))$$

for all $(x, y, z), (u, v, w) \in G^3$. i.e., $f - S$ is a $\frac{1}{2}$ **d**-Lipschitz function. We have

$$2S(x,y,z) + 2S(u,v,w) = 2\Gamma[Q_{x,y,z}(r,t,s)] + 2\Gamma[Q_{u,v,w}(r,t,s)].$$

By using property (ii) of Γ, we obtain

$$\Gamma[Q_{x,y,z}] = \Gamma[Q_{x,y,z}^{u,v,w}], \Gamma[Q_{x,y,z}] = \Gamma[Q_{x,y,z}^{-u,-v,-w}]$$

for $(u, v, w) \in G^3$ and hence

$$2S(x,y,z) + 2S(u,v,w) = 2\Gamma[Q_{x,y,z}] + 2\Gamma[Q_{u,v,w}] = \Gamma[Q_{x,y,z}^{u,v,w}] + \Gamma[Q_{x,y,z}^{-u,-v,-w}] + 2\Gamma[Q_{u,v,w}].$$

On the other hand,

$$\Gamma[Q_{x,y,z}^{u,v,w}] + \Gamma[Q_{x,y,z}^{-u,-v,-w}] + 2\Gamma[Q_{u,v,w}]$$

$$= \Gamma[\frac{1}{2}f(r+x+u, t+y+v, s+z+w) + \frac{1}{2}f(r-x+u, t-y+v, s-z+w) - f(r+u, t+v, s+w)]$$

$$+\Gamma[\frac{1}{2}f(r+x-u, t+y-v, s+z-w) + \frac{1}{2}f(r-x-u, t-y-v, s-z-w) - f(r-u, t-v, s-w)]$$

$$+\Gamma[f(r+u, t+v, s+w) + f(r-u, t-v, s-w) - 2f(r,t,s)]$$

$$= S(x+u, y+v, z+w) + S(x-u, y-v, z-w).$$

Therefore S is 3-variable quadratic and so $T_S = 0$.

Remark 2.4. Under the hypotheses of Theorem 2.3, if Im $T_f \subset A$ for some $A \in S(V)$, then Im $(f - S) \subset \frac{1}{2}$ A. We know that

$$\operatorname{Im}(\frac{1}{2}T_f(x,y,z,\cdot,\cdot,\cdot)) \subset \operatorname{Im}(\frac{1}{2}T_f) \subset \frac{1}{2}A$$

and so $\frac{1}{2}$ $T_f(x, y, z, \cdot, \cdot, \cdot) \in B(G^3, S(V))$ for all $(x, y, z) \in G^3$. By property (i) of Γ, one gets

$$f(x,y,z) - S(x,y,z) = \Gamma[\frac{1}{2}T_f(\cdot,\cdot,\cdot,x,y,z)] \in \frac{1}{2}A$$

for all $(x, y, z) \in G^3$. Therefore, $\operatorname{Im}(f - S) \subset \frac{1}{2}$A.

Theorem 2.5. Let $(G^3, +, d, \tilde{d})$ be a product metric, and V a normed space such that $B(G^3, CB(V))$ admits LIM, where $CB(V)$ is the family of all closed balls with center at zero. If $f : G^3 \to V$ is a function, then there exists a 3-quadratic function $S : G^3 \to V$ such that $m_{f-S} = \frac{1}{2} m_{T_f}$. Moreover, if $T_f \in B(G^3 \times G^3, BC(V))$, then

$$\|f - S\|_\infty \le \frac{1}{2}\|T_f\|_\infty .$$

Proof. Let $\Theta : G^3 \times G^3 \to \mathbb{R}^+$ be a positive real-valued function defined by

$$\Theta((x,y,z),(u,v,w)) := \inf_{d((x,y,z),(u,v,w)) \le \delta} m_{T_f}(\delta)$$

for all $(x, y, z), (u, v, w) \in G^3$. Define the set-valued function $d : G^3 \times G^3 \to S(V)$ by

$$\mathbf{d}((x, y, z), (u, v, w)) := \Theta((x, y, z), (u, v, w)) B(0, 1)$$

for all $(x, y, z), (u, v, w) \in G^3$, where $B(0, 1)$ is the unit closed ball. Since m_{T_f} is the module of continuity of T_f,

$$\begin{aligned}\|T_f(r,s,t,x,y,z) - T_f(r,s,t,u,v,w)\| &\le \inf_{\tilde{d}((r,s,t,x,y,z),(r,s,t,u,v,w)) \le \delta} m_{T_f}(\delta) \\ &= \inf_{\tilde{d}((x,y,z),(u,v,w)) \le \delta} m_{T_f}(\delta) \\ &= \Theta((x,y,z),(u,v,w))\end{aligned}$$

for all $(x, y, z), (u, v, w) \in G^3$ and hence $T_f(r, s, t, \cdot, \cdot, \cdot)$ is a **d**-Lipschitz function.

On the other hand, Theorem 2.3 implies there exists a 3-quadratic function $S : G^3 \to V$ such that $f - S$ is a $\frac{1}{2}$ **d**-Lipschitz function. Thus,

$$(f(x,y,z) - S(x,y,z)) - (f(u,v,w) - S(u,v,w)) \in \frac{1}{2}\mathbf{d}((x,y,z),(u,v,w))$$

for all $(x, y, z), (u, v, w) \in G^3$. So,

$$\begin{aligned}\|(f(x,y,z) - S(x,y,z)) - (f(u,v,w) - S(u,v,w))\| &\le \frac{1}{2}\Theta((x,y,z),(u,v,w)) \\ &= \inf_{d((x,y,z),(u,v,w)) \le \delta} \frac{1}{2} m_{T_f}(\delta)\end{aligned}$$

for all $(x, y, z), (u, v, w) \in G^3$. Consequently,

$$m_{f-S} = \frac{1}{2} m_{T_f} .$$

Suppose that $T_f \in B(G^3 \times G^3, CB(V))$. Then, clearly $\operatorname{Im} T_f \subset \|T_f\|_\infty B(0, 1)$. By applying Remark 2.4, we get

$$\mathrm{Im}(f-S)\subset\frac{1}{2}\|T_f\|_\infty\, B(0,1)$$

and hence $\|f-S\|_\infty \le \frac{1}{2}\|T_f\|_\infty$.

Corollary 2.6. Let $(G^3, +, d, \tilde{d})$ be a product metric and V a normed space such that $B\,(G^3, CB(V))$ admits LIM, where $CB\,(V)$ is the family of all closed balls with center at zero. If $f: G^3 \to V$ is a function and $T_f \in Lip\,(G^3 \times G^3, V)$, then there exists a 3-quadratic function S such that

$$\|f-S\|_{Lip} \le \frac{1}{2}\|T_f\|_{Lip}\ .$$

Proof. Define the function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\Phi(t) := L_\alpha\,(T_f)t$$

for all $t \in \mathbb{R}^+$. Since $T_f \in Lip\,(G^3 \times G^3, V)$,

$$\left\|T_f(r,t,s,a,b,c)-T_f(x,y,z,u,v,w)\right\| \le L_\alpha\,(T_f)\tilde{d}((r,t,s,a,b,c),(x,y,z,u,v,w))$$

for all $(r, t, s), (a, b, c), (x, y, z), (u, v, z) \in G^3$. By using the definition of Φ, we see that

$$\left\|T_f(r,t,s,a,b,c)-T_f(x,y,z,u,v,w)\right\| \le \Phi(\tilde{d}((r,t,s,a,b,c),(x,y,z,u,v,w)))$$

for all $(r, t, s), (a, b, c), (x, y, z), (u, v, z) \in G^3$. It follows that Φ is the module of continuity of T_f and so Theorem 2.5 entails there exists a 3-quadratic function $S: G^3 \to V$ such that $m_{f-S} = \frac{1}{2}\Phi$. Hence,

$$\left\|(f(x,y,z)-S(x,y,z))-(f(u,v,w)-S(u,v,w))\right\| \le m_{f-S}(d((x,y,z),(u,v,w)))$$

$$=\frac{1}{2}\Phi(d((x,y,z),(u,v,w)))$$

$$=\frac{1}{2}L_\alpha\,(T_f)d((x,y,z),(u,v,w))$$

for all $(x, y, z), (u, v, z) \in G^3$. From the last inequality it follows that $f - S$ is a Lipschitz function and

$$L_\alpha\,(f-S) \le \frac{1}{2}L_\alpha\,(T_f)\ . \tag{5}$$

Since $T_f \in Lip\,(G^3 \times G^3, V)$, T_f is bounded and $\mathrm{Im}T_f \subset B(0, M)$ for some $M > 0$.
Therefore $T_f \in B\,(G^3 \times G^3, CB\,(V))$ and Theorem 2.5 implies

$$\|f-S\|_\infty \le \frac{1}{2}\|T_f\|_\infty\,. \tag{6}$$

From (6) it follows that the function $f - S$ is bounded and hence, $f - S \in Lip\,(G^3, V)$. By using (5) and (6) we obtain that

$$\|f-S\|_{Lip} = \|f-S\|_\infty + L_\alpha\,(f-S) \le \frac{1}{2}\|T_f\|_\infty + \frac{1}{2}L_\alpha\,(T_f) = \frac{1}{2}\|T_f\|_{Lip}\ .$$

References

[1] R. Badora. On some generalized invariant means and their application to the stability of the Hyers-Ulam type. *Ann. Polon. Math*. 1993; 58: 147-159.

[2] P. W. Cholewa. Remarks on the stability of functional equations. *Aequationes Math*. 1984; 27: 76-86.

[3] S. Czerwik. Functio*nal Equations and Inequalities in Several Variables*. 2002.

[4] S. Czerwik, K. Dlutek. Stability of the quadratic functional equation in Lipschitz spaces. *J. Math. Anal. Appl.* 2004; 293: 79-88.

[5] S. Czerwik. On the stability of the quadratic mapping in normed spaces. *Abh. Math. Sem. Univ. Hamburg.* 1992; 62: 59-64.

[6] D. H. Hyers. On the stability of the linear functional equation. *Proc. Natl. Acad. Sci*. 1941; 27: 222-224.

[7] J.-H. Bae, W.-G. Park, a functional equation originating from quadratic forms. *J. Math. Anal. Appl*. 2007; 326: 1142-1148.

[8] A. Najati. On the stability of a quartic functional equation. *J. Math. Anal. Appl*. 2008; 340: 569-574.

[9] I. Nikoufar. Quartic functional equations in Lipschitz spaces. *Rend. Circ. Mat. Palermo*. 2015; 64: 171-176.

[10] I. Nikoufar. Erratum to: Quartic functional equations in Lipschitz spaces. *Rend. Circ. Mat. Palermo*. 2016; 65: 345-350.

[11] I. Nikoufar. Lipschitz criteria for bi-quadratic functional equations. *Commun. Korean Math. Soc*. 2016; 31: 819-825.

[12] I. Nikoufar, Approximate tri-quadratic functional equations via Lipschitz conditions. *Math. Bohemica*. 2017; 142: 337-344.

[13] I. Nikoufar. Behavior of bi-cubic functions in Lipschitz spaces. *Lobachevskii J. Math*. 2018; 39: 803-808.

[14] I. Nikoufar. Stability of multi-quadratic functions in Lipschitz spaces. *Iran J. Sci. Technol. Trans. Sci*. 2019; 43(2): 621-625.

[15] J. M. Rassias. Solution of the Ulam stability problem for quartic mappings. *Glas. Mat. Ser*. 1999; 34: 243-252.

[16] K. Ravi, M. Arunkumar. Stability of a 3-variable quadratic functional equation. *Journal of Quality Measurement and Analysis*. 2008; 4(1): 97-107.

[17] F. Skof. Local properties and approximations of operators. *Rend. Sem. Mat. Fis. Milano*. 1983; 53: 113-129.

[18] J. Tabor. Lipschitz stability of the Cauchy and Jensen equations. *Results Math*. 1997; 32: 133-144.

[19] J. Tabor. Superstability of the Cauchy, Jensen and Isometry Equations. *Results Math*. 1999; 35: 355-379.

[20] S. M. Ulam. *A Collection of the Mathematical Problems*. New York: Interscience Publishing; 1940.

[21] N. Weaver. *Lipschitz Algebras*. Singapore. 1999.