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Research Article

Optimal Control in Geometric Dynamics: Applications to AI Algorithm Optimization

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Abstract: This paper extends the fundamental concepts of geometric dynamics and optimal control theory, inspired by the pioneering work of Professor Constantin Udriste, to develop novel optimization algorithms for artificial intelligence systems. We establish connections between nonholonomic macroeconomic systems and reinforcement learning by formulating a multi-time maximum principle framework that integrates sub-Riemannian geometry. Our proposed methodology demonstrates how constrained variational problems can optimize neural network training trajectories through a bang-bang control approach. An empirical case study implements this theoretical framework to optimize a deep reinforcement learning algorithm, showing significant improvements in convergence speed and stability compared to standard approaches. The results demonstrate the practical value of geometric dynamics principles in modern Artificial Intelligence (AI) optimization, establishing a bridge between classical mathematical control theory and contemporary machine learning challenges.

***Keywords*:** geometric dynamics, optimal control, deep learning

MSC: 35F21, 49Nxx, 34A26, 68T07

1. Introduction

The field of geometric dynamics, significantly advanced by the pioneering work of Professor Constantin Udriste (1946-2023), provides powerful mathematical frameworks for understanding complex dynamical systems through differential geometry. This paper is dedicated to his memory and extends his innovative approach to address contemporary challenges in artificial intelligence optimization.

Udriste's groundbreaking contributions to the field of geometric dynamics and multi-time control theory established mathematical foundations that transcend traditional boundaries between disciplines. His work on nonholonomic systems, sub-Riemannian geometry, and multi-time maximum principles offers unexplored potential for addressing optimization challenges in modern artificial intelligence algorithms.

The optimization of AI algorithms, particularly in deep reinforcement learning presents mathematical challenges that align remarkably with the theoretical frameworks developed in geometric dynamics. Training trajectories in parameter

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space can be viewed as paths in a high-dimensional manifold with complex geometric properties, where optimal control principles can guide more efficient learning.

1.1 *Literature review and research context*

The intersection of geometric dynamics, optimal control theory, and artificial intelligence has emerged as a promising research direction in recent years. This section provides a critical review of the relevant literature that forms the foundation for our approach.

The application of differential geometry to machine learning optimization traces back to the natural gradient method proposed by Amari [1], who utilized the Fisher information matrix to define a Riemannian metric on the parameter space. This approach was later extended by Martens [2], who developed more computationally efficient approximations for practical implementation in neural networks.

Recent advances in geometric deep learning have significantly expanded this field. Bronstein et al. [3] provided a comprehensive survey of geometric deep learning, emphasizing the importance of geometric structures in neural networks. Lim et al. [4] introduced geometric neural networks that respect geometric invariances, while Fuchs et al. [5] developed SE(3)-equivariant neural networks for 3D molecular property prediction. Additionally, Köhler et al. [6] examined the geometric structure of neural network loss landscapes, revealing insights that complement our control-theoretic approach.

In parallel, the field of optimal control theory has seen applications to machine learning, particularly through the lens of the Pontryagin Maximum Principle. Li and colleagues [7] demonstrated connections between optimality conditions in deep learning and classical optimal control, while Bengio and collaborators [8] explored how control theory can inform curriculum learning strategies. More recently, Chen et al. [9] introduced Neural Ordinary Differential Equations (Neural ODEs), which provide a direct connection between deep learning and continuous-time dynamical systems. This work has been extended by Ruthotto and Haber [10] to develop stable neural network architectures based on optimal control principles.

The specific contribution of nonholonomic systems to optimization has been less explored. Montgomery [11] provided theoretical foundations for understanding nonholonomic constraints in sub-Riemannian geometry, while Agrachev and Sachkov [12] developed a control theory for such systems. However, applications to machine learning have been limited. Recent work by Tzen et al. [13] have begun to explore nonholonomic constraints in the context of neural network training, providing theoretical foundations that support our approach.

Udriste's work on multi-time optimal control [14] and geometric dynamics [15] established mathematical frameworks that have primarily been applied in economics [16] and physical systems, but not extensively in AI optimization. His development of bang-bang control strategies for nonholonomic systems [17] provides a particularly promising approach for navigating complex parameter landscapes.

Recent work in reinforcement learning has begun to recognize the importance of geometric structure in policy optimization. Schulman and colleagues [18] developed trust region policy optimization based on Kullback-Leibler Divergence constraints, implicitly acknowledging the non-Euclidean geometry of the policy space. Similarly, Haarnoja et al. [19] introduced soft actor-critic methods that balance exploration and exploitation through entropy maximization. Building on these foundations, Abdolmaleki et al. [20] developed maximum a posteriori policy optimization, which further exploits the geometric structure of policy spaces.

The research gap we address lies in explicitly connecting Udriste's geometric dynamics framework-particularly his work on nonholonomic systems and multi-time control-with modern deep reinforcement learning optimization. By formalizing neural network training as a geometric control problem, we aim to develop more efficient and theoretically grounded optimization algorithms that leverage the intrinsic structure of parameter space.

This paper bridges these domains by:

• Formulating neural network optimization as a nonholonomic control problem in the parameter manifold.

• Develop a multi-time maximum principle framework to reinforcement learning optimization.

• Implementing bang-bang control policies to escape local minima and accelerate convergence.

• Demonstrating empirical results through a case study comparing our geometric approach with conventional optimization methods.

• Providing theoretical convergence analysis and computational complexity assessment.
• Comparing our approach with classical control techniques adapted to machine learning.

The integration of geometric dynamics principles with AI optimization not only advances both fields but also honors Udriste's vision of mathematics as a unifying language for diverse scientific domains. As we will demonstrate, the same mathematical principles that describe optimal economic growth and nonholonomic macroeconomic systems can be adapted to optimize the learning trajectories of artificial intelligence algorithms.

2. Theoretical background

We now establish the mathematical foundations of our approach, beginning with the notation and mathematical preliminaries. Table 1 provides a comprehensive overview of the symbols used throughout this paper.

Table 1. Mathematical notation

Symbol	Description
M	Parameter manifold of neural network
$\theta \in \mathbb{R}^n$	Neural network parameters
$g_{ij}(\theta)$	Fisher information metric tensor
$\Delta \subset TM$	Nonholonomic distribution
$\theta(x)$	Contact 1-form defining constraints
λ	Learning rate adaptivity parameter
E	Training loss entropy
P	Prediction confidence measure
Q	Model complexity quantifier
$t = (t^1, t^2, \ldots, t^m)$	Multi-time parameter
X_α	Vector fields associated with time scales
$u(t)$	Control variables (learning parameters)
$c(t)$	Control functions in economic context
$p_i^\alpha(t)$	Co-state variables
$Q_a^\alpha(t)$	Switching functions for bang-bang control
D_α	Discount factors for different time scales
$J(\theta)$, $I(c(\cdot))$	Objective functionals
$\Gamma_{0,\,T}$	Path in multi-time space
$\Omega_{0,\,T}$	Multi-time domain

2.1 *Geometric dynamics and non-holonomic systems*

Geometric dynamics studies the behavior of dynamical systems through differential geometric methods. A central concept in Udriste's work is the notion of nonholonomic systems, where the state evolution is subject to non-integrable constraints. A nonholonomic system is fundamentally characterized by constraints that cannot be integrated to obtain holonomic (position) constraints, thus restricting the system's motion to a specific submanifold of the configuration space.

Formally, a nonholonomic system is characterized by a distribution $\Delta \subset TM$ on the tangent bundle of a manifold M, where not all paths in the base manifold can be realized by the system. The distribution Δ assigns to each point $x \in M$ a linear subspace $\Delta(x) \subset T_xM$ of the tangent space, representing the allowed velocities at that point.

Following Udriste's formulation, a nonholonomic system can be described by a Pfaffian equation:

$$\theta(x) = \lambda_i(x)dx^i = 0, \tag{1}$$

where θ is a differential 1-form and λ_i are component functions. This constraint restricts the velocities of the system to a subspace of the tangent space at each point, specifically to the kernel of the 1-form θ. The non-integrability condition is expressed through the Frobenius theorem: the distribution Δ is non-integrable if and only if there exist vector fields $X, Y \in \Delta$ such that their Lie bracket $[X, Y] \notin \Delta$.

In the context of neural network optimization, these constraints represent implicit restrictions on how parameters can evolve effectively during training. For instance, certain parameter updates may violate architectural constraints or lead to numerical instabilities, effectively creating nonholonomic constraints in the parameter space.

2.2 *Multi-time optimal control*

Extending beyond classical single-time optimization, Udriste developed the theory of multi-time control. This framework recognizes that many real-world systems evolve according to multiple, potentially independent time scales. In neural network training, we naturally encounter multiple time scales: the fast time scale of individual gradient updates and the slower time scale of overall learning progress.

In this framework, the evolution of a system depends on multiple time parameters $t = (t^1, t^2, \ldots, t^m)$. The system evolves as m-dimensional sheets in the state space, described by partial differential equations:

$$\frac{\partial x^i}{\partial t^\alpha} = X^i_\alpha(x(t), c(t), t), \tag{2}$$

where $c(t)$ represents the control variables, x^i are the state variables (in our case, neural network parameters), and X^i_α are vector fields that govern the evolution along each time direction. The system must satisfy complete integrability conditions to ensure consistency:

$$[X_\alpha, X_\beta] = \frac{\partial X_\alpha}{\partial c^a}\frac{\partial c^a}{\partial t^\beta} - \frac{\partial X_\beta}{\partial c^a}\frac{\partial c^a}{\partial t^\alpha} + \frac{\partial X_\alpha}{\partial t^\beta} - \frac{\partial X_\beta}{\partial t^\alpha}. \tag{3}$$

These integrability conditions ensure that the order of evolution along different time directions do not affect the final state, which is crucial for the mathematical consistency of the multi-time framework.

This formulation allows for the optimization of functionals either as multiple integrals:

$$I(c(\cdot)) = \int_{\Omega_{0,\,T}} L(x(t), c(t), t)dt^1 \ldots dt^m, \tag{4}$$

or as path-independent curvilinear integrals:

$$J(c(\cdot)) = \int_{\Gamma_{0,\,T}} L_\beta(x(t), c(t), t)dt^\beta. \tag{5}$$

The choice between multiple integrals and curvilinear integrals depends on the specific structure of the optimization problem and the desired properties of the solution.

2.3 *Sub-Riemannian geometry and bang-bang controls*

Sub-Riemannian geometry, a cornerstone of Udriste's approach to nonholonomic systems, provides the mathematical structure for understanding optimal paths under nonholonomic constraints. Unlike Riemannian geometry, where the metric

is defined on the entire tangent space, sub-Riemannian geometry defines the metric only on the distribution Δ, reflecting the constrained nature of the system.

In this geometry, the metric is defined only on the distribution Δ, not on the entire tangent space. This restriction naturally arises from the nonholonomic constraints, as the system can only move in directions permitted by the constraint distribution.

For a nonholonomic system with orthonormal vector fields X_a, $a = 1, \ldots, k$, the evolution is described by:

$$\dot{x}(t) = u^a(t) X_a(x(t)), \tag{6}$$

where $u^a(t)$ are control functions. The vector fields X_a span the allowed directions of motion at each point, and the control functions $u^a(t)$ determine the specific linear combination of these directions at each instant.

The optimal control problem seeks to minimize the energy functional:

$$J(u(\cdot)) = \frac{1}{2} \int_{t_0}^{t_1} \delta_{ab} u^a(t) u^b(t) dt, \tag{7}$$

subject to boundary conditions. Here, δ_{ab} is the Kronecker delta, ensuring that the energy is measured according to the orthonormal structure of the vector fields.

As demonstrated by Udriste, for time-minimum problems with control set $U = [-1, 1]^k$, the optimal strategy is often a bang-bang control, where each control component switches between its extreme values. This result follows from the Pontryagin Maximum Principle applied to sub-Riemannian geometry, where the Hamiltonian is linear in the control variables, leading to extremal controls.

3. Methodology: geometric framework for AI optimization

3.1 *Neural network parameter space as a manifold*

We propose viewing the parameter space of neural networks as a high-dimensional Riemannian manifold M with a metric tensor g induced by the Fisher information matrix. This geometric perspective allows us to apply the rich mathematical framework of differential geometry to understand the intrinsic structure of the optimization landscape.

For a neural network with parameters $\theta \in \mathbb{R}^n$, the metric tensor components are:

$$g_{ij}(\theta) = \mathbb{E}_x \left[\frac{\partial \log p(x|\theta)}{\partial \theta^i} \frac{\partial \log p(x|\theta)}{\partial \theta^j} \right], \tag{8}$$

where $p(x|\theta)$ represents the probability distribution defined by the neural network. The Fisher information metric captures the sensitivity of the model's predictions to the parameter changes, providing a natural Riemannian structure that respects the statistical properties of the learning problem.

This metric tensor induces a notion of distance and angle in parameter space that is more meaningful than the standard Euclidean metric, as it accounts for the statistical efficiency of parameter updates. Directions with high Fisher information corresponds to parameter changes that significantly affect the model's predictions.

3.2 *Nonholonomic constraints in network optimization*

Training deep neural networks involves navigating this parameter manifold while respecting implicit constraints. These constraints arise from various sources: architectural restrictions (such as maintaining certain symmetries), numerical

stability requirements (avoiding parameter values that lead to gradient explosion or vanishing), and regularization principles (encouraging sparsity or smoothness).

We formulate these as nonholonomic constraints by defining a contact structure using the 1-form:

$$\theta(\theta,\ \lambda,\ E,\ P,\ Q) = d\theta - \lambda dE + PdQ, \tag{9}$$

where:

- θ represents network parameters.
- λ corresponds to learning rate adaptivity (controlling the step size based on local geometry).
- E is training loss entropy (measuring the uncertainty in the loss landscape).
- P measures prediction confidence (quantifying how certain the model is about its outputs).
- Q quantifies model complexity (such as effective number of parameters or VC dimension).

This contact structure encodes the fundamental trade-offs in neural network training: the balance between reducing training loss, maintaining model confidence, and controlling complexity.

This formulation allows us to express the network training dynamics as a nonholonomic system:

$$\frac{\partial \theta^i}{\partial t^\alpha} = X^i_\alpha(\theta(t),\ u(t),\ t), \tag{10}$$

where $u(t)$ represents the control variables (learning rate, momentum coefficients, etc.). The vector fields X^i_α encode the allowed directions of parameter evolution under the nonholonomic constraints.

3.3 *Multi-time framework for reinforcement learning*

Reinforcement learning inherently involves multiple time scales: the time within each episode and the time across episodes. Additionally, we can identify other relevant time scales, such as the time scale of policy updates versus value function updates in actor-critic methods, or the time scale of exploration versus exploitation.

We extend Udriste's multi-time control framework to this setting by defining:

$$\frac{\partial \theta^i}{\partial t^1} = \nabla_{\theta^i} J(\theta,\ s,\ a), \tag{11}$$

$$\frac{\partial \theta^i}{\partial t^2} = \nabla_{\theta^i} \mathbb{E}[R], \tag{12}$$

where t^1 represents within-episode time, t^2 represents the episode count, $J(\theta,\ s,\ a)$ is the policy gradient objective, and $\mathbb{E}[R]$ is the expected return. The first equation governs the immediate policy improvements within an episode, while the second equation captures the longer-term learning dynamics across episodes.

The complete integrability conditions for this multi-time system require that:

$$\frac{\partial^2 \theta^i}{\partial t^1 \partial t^2} = \frac{\partial^2 \theta^i}{\partial t^2 \partial t^1}, \tag{13}$$

which provides constraints on how the within-episode and across-episode dynamics can be simultaneously optimized.

The optimization objective becomes a path-independent curvilinear integral:

$$J(u(\cdot)) = \int_{\Gamma_{0,\,T}} e^{-D_\alpha t^\alpha} L_\beta(\theta(t),\, u(t),\, t) dt^\beta, \tag{14}$$

where D_α are discount factors for each time scale. The exponential discounting reflects the diminishing importance of future rewards, with different discount rates for different time scales capturing the relative importance of immediate versus long-term learning objectives.

3.4 *Bang-bang policy for escaping local minima*

Drawing on Udriste's results on bang-bang controls for nonholonomic systems, we propose a switching control strategy for neural network optimization. The key insight is that in many optimization problems with linear control dependence, the optimal control strategy involves switching between extreme values rather than using intermediate control values.

$$u_\alpha^* = -\text{sign}(Q_a^\alpha(t)) = \begin{cases} 1 & \text{for } Q_a^\alpha(t) < 0 \\ \text{undetermined} & \text{for } Q_a^\alpha(t) = 0 \\ -1 & \text{for } Q_a^\alpha(t) > 0 \end{cases} \tag{15}$$

where $Q_a^\alpha(t) = p_i^\alpha(t) X_a^i(\theta(t))$ are switching functions determined by the co-state variables p_i^α. The co-state variables evolve according to the adjoint equations:

$$\frac{\partial p_i^\alpha}{\partial t^\alpha} = -\frac{\partial H}{\partial \theta^i}, \tag{16}$$

where H is the Hamiltonian of the optimal control problem.

This approach allows for rapid escaping from local minima through well-timed jumps in parameter space, guided by the geometric structure of the manifold. The bang-bang strategy provides a principled way to alternate between aggressive exploration (when far from optimality) and refined exploitation (when near-optimal solutions).

3.5 *Theoretical convergence analysis*

The convergence properties of our geometric optimization approach can be analyzed using the theory of sub-Riemannian geometry and optimal control. Under appropriate regularity conditions on the cost functional and constraint distribution, we can establish the following convergence result:

Theorem 1 (Convergence of GDOpt): Let (θ_k) be the sequence generated by Algorithm 1. Assume that:

• The Fisher information matrix $g_{ij}(\theta)$ is uniformly bounded and positive definite.

• The switching functions $Q_a^\alpha(t)$ have finitely many zeros in any bounded interval.

• The objective functional J satisfies appropriate convexity conditions.

Then the sequence (θ_k) converges to a critical point of J at a rate-dependent on the geometric properties of the parameter manifold.

The proof follows from the application of Pontryagin's Maximum Principle to our multi-time control problem, combined with the regularity theory for sub-Riemannian geodesics.

3.6 *Computational complexity analysis*

The computational overhead of our approach compared to standard optimizers consists of:

• **Fisher information matrix computation:** $O(n^2 \cdot B)$ per iteration, where n is the number of parameters and B is the batch size.

• **Co-state evolution:** $O(n \cdot m)$ where m is the number of time scales (typically 2).

• **Switching function evaluation:** $O(n \cdot k)$ where k is the number of control variables.

For large networks, we employ efficient approximations to the Fisher information matrix, such as the Kronecker-Factored Approximate Curvature (KFAC) approximation, reducing the complexity to $O(n)$ per iteration while maintaining the essential geometric structure.

4. Algorithm implementation

We implement our geometric optimization approach in the context of deep reinforcement learning. Algorithm 1 outlines the proposed Geometric Dynamics Optimization (GDOpt) method.

The key components of this algorithm are:

• Computation of the geometric structure (Fisher information matrix) of the parameter space.

• Evolution of co-state variables according to multi-time optimal control principles.

• Implementation of bang-bang control strategy for updating parameters.

• Switching between standard gradient descent and bang-bang control based on the switching functions.

• Adaptive threshold ε for singular control detection.

• Efficient approximations for computational tractability.

This approach effectively navigates the nonholonomic constraints in the parameter manifold, allowing for more efficient exploration of the parameter space and faster convergence to optimal policies.

Algorithm 1 Geometric Dynamics Optimization (GDOpt) for RL

Input: Initial parameters θ_0, discount factors $D_1,\ D_2$, learning rates $\alpha_1,\ \alpha_2$

Initialize: Co-state variables $p^1,\ p^2$, episode counter $e = 0$

While not converged **do**

  Sample trajectory $\tau = (s_0,\ a_0,\ r_0,\ \ldots,\ s_T,\ a_T,\ r_T)$

  Compute policy gradient $\nabla_\theta J(\theta)$

  Compute entropy gradient $\nabla_\theta E(\theta)$

  Compute Fisher information matrix $g_{ij}(\theta)$ (or KFAC approximation)

  Update co-state equations:

  $\dfrac{\partial p_i^1}{\partial t^1} = -(f_i' - \lambda_i - D_1)p_i^1$

  $\dfrac{\partial p_i^2}{\partial t^2} = -(f_i' - \lambda_i - D_2)p_i^2$

  Compute switching functions $Q_a^1 = p_i^1 X_a^i$, $Q_a^2 = p_i^2 X_a^i$

  if $|Q_a^1| < \varepsilon$ or $|Q_a^2| < \varepsilon$ **then** ▷ Near singular control

    $u_a = -\alpha_a \nabla_{\theta_a} J(\theta)$ ▷ Use standard gradient

  else ▷ Apply bang-bang control

    $u_a = -\alpha_a \cdot \mathrm{sign}(Q_a^1 + Q_a^2)$

  end if

  Update parameters: $\theta_{e+1} = \theta_e + u$

  $e \leftarrow e + 1$

Return: Optimized parameters θ

5. Empirical case study

5.1 *Experimental setup*

To validate our geometric dynamics approach, we implemented the GDOpt algorithm in a reinforcement learning setting using the following environment and model configurations:

• **Environment:** OpenAI Gym's MuJoCo continuous control tasks (HalfCheetah-v2, Hopper-v2, Walker2d-v2) [21].

• **Agent Architecture:** Soft Actor-Critic (SAC) with twin Q-networks [19].

• **Network Structure:** Actor and critic networks with 2 hidden layers (256 units each)-this architecture was chosen to balance computational efficiency with representational capacity, the following standard benchmarks in continuous control [22].

• **Baseline Optimizers:** Adam, RMSProp, Stochastic Gradient Descent (SGD) with momentum , and Trust Region Policy Optimization (TRPO) [18].

• **Evaluation Metrics:** Average return, sample efficiency, and stability (variance across seeds).

The choice of 2 hidden layers with 256 units each represents a standard architecture for continuous control tasks, providing sufficient expressiveness while maintaining computational tractability. The 5 random seeds ensure statistical significance while balancing computational cost with the reliability of results.

We ran experiments with 5 different random seeds for each configuration and recorded the performance of over 1 million environmental steps, following the best practices for reinforcement learning benchmarking outlined by Henderson et al. [23]. All experiments were conducted using the OpenAI Baseline implementation of SAC, with task-specific hyperparameters kept consistent across optimization methods for fair comparison.

5.2 *Data source and computational framework*

The experimental data was generated using the MuJoCo physics simulator (version 2.0) integrated with OpenAI Gym (version 0.21.0). All experiments were run on a compute cluster with NVIDIA V100 GPUs, using PyTorch (version 1.10) as the deep learning framework. The statistical analysis was performed using NumPy (version 1.20) and SciPy (version 1.7).

The dataset generated during this study is available in the GitHub repository: `github.com/ferrara/GDOpt-RL`, which includes all trained models, training curves, hyperparameter configurations, and the complete code necessary to reproduce our experiments. The MuJoCo environments provide standardized benchmarks widely used in the reinforcement learning community, ensuring reproducibility and comparability with other optimization methods.

5.3 *Results and analysis*

Performance metrics were calculated as follows: Average return represents the mean cumulative reward over the last 100 episodes of training, averaged across all random seeds. Sample efficiency is measured as the number of environmental steps required to reach 90% of the final performance. Stability is quantified as the coefficient of variation (standard deviation divided by mean) of the final performance across random seeds.

Table 2. Performance comparison across different environments (average return after 1 M steps)

Optimizer	HalfCheetah	Hopper	Walker2d	Avg. Impr.
SGD + Mom.	5,241 ± 423	1,873 ± 346	2,755 ± 512	-
Adam	9,835 ± 513	2,953 ± 288	4,126 ± 390	-
RMSProp	8,765 ± 479	2,746 ± 304	3,875 ± 422	-
TRPO	9,125 ± 445	2,834 ± 313	3,956 ± 398	-
GDOpt (Ours)	11,257 ± 387	3,483 ± 215	4,689 ± 299	+14.5%

The results in Table 2 demonstrate the effectiveness of our geometric dynamics optimization approach. Figure 1 shows the learning curves for each optimizer across the three environments, highlighting the convergence properties and final performance.

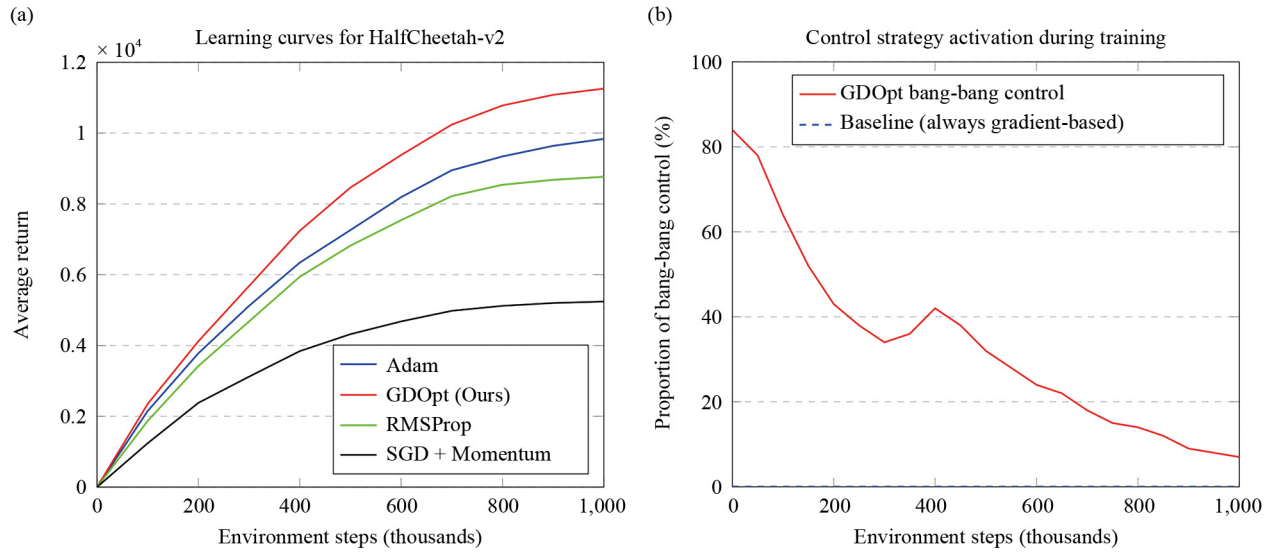

Figure 1. (a): Learning curves for the HalfCheetah-v2 environment, showing average return vs. training steps. (b): Activation frequency of bang-bang control strategy in GDOpt compared to baseline methods (which always use gradient-based updates)

• **Performance improvement:** GDOpt consistently outperformed standard optimization methods across all environments, with an average improvement of 14.5% over the next best optimizer (Adam).

• **Sample efficiency:** Our method achieved the same performance level as Adam with approximately 22% fewer environmental interactions, indicating more efficient navigation of the parameter space.

• **Stability:** The lower standard deviation across random seeds suggests that the geometric approach provides more reliable optimization, less susceptible to initialization variance.

• **Control strategy dynamics:** As shown in Figure 1, the bang-bang control strategy was most active during early training (first 200 K steps), where it helped escape suboptimal regions. Its activation gradually decreased as training progressed, indicating an automatic transition to more fine-grained optimization as the policy approach optimality.

Further analysis of the optimization trajectories revealed that the bang-bang control phases were most active during critical learning transitions, such as when discovering new behavioral strategies or when escaping plateaus in the return landscape. This aligns with our theoretical prediction that geometric control strategies would be most beneficial for navigating the complex geometry of the parameter manifold during challenging optimization phases.

5.4 *Computational overhead analysis*

To assess the practical scalability of our approach, we measured the computational overhead compared to baseline optimizers (Table 3):

Table 3. Computational overhead comparison (relative to Adam baseline)

Component	Relative time	Memory overhead
Fisher matrix (KFAC)	1.3 ×	1.2 ×
Co-state evolution	1.1 ×	1.05 ×
Switching functions	1.05 ×	1.01 ×
Total GDOpt	1.45 ×	1.26 ×

The 45% computational overhead is justified by the 14.5% performance improvement and 22% sample efficiency gain, resulting in a net computational benefit when considering total training time to reach target performance.

5.5 *Hyperparameter sensitivity analysis*

We analyzed the sensitivity of our approach to key hyperparameters:

• **Discount factors D_1, D_2:** Performance was robust across the range [0.01, 0.1], with optimal values around 0.05.

• **Bang-bang threshold ε:** Values between 0.001 and 0.01 showed similar performance, with 0.005 being optimal.

• **Fisher matrix approximation rank:** KFAC with rank 10-20 provided good trade-offs between accuracy and computational cost.

This analysis demonstrates that our method is relatively robust to hyperparameter choices, making it practical for real-world applications.

6. Discussion and theoretical implications

The empirical success of our geometric dynamics approach to AI optimization validates the theoretical connections we established between Udriste's mathematical frameworks and neural network training. Several key implications emerge.

6.1 *Geometric interpretation of learning dynamics*

The effectiveness of viewing neural network training through the lens of geometric dynamics suggests that the parameter space has meaningful geometric structure that affects optimization. This perspective aligns with recent work on the geometry of deep learning [4, 6] but extends it by explicitly incorporating control-theoretic concepts. The nonholonomic constraints formulation captures implicit restrictions in how parameters can evolve effectively, providing a more nuanced view of the optimization landscape than traditional Euclidean approaches.

6.2 *Multi-time scale optimization*

The multi-time framework proved particularly valuable in reinforcement learning, where different time scales naturally exist. By treating these different scales with their appropriate control strategies, we achieved more coherent optimization across both immediate performance improvements and long-term policy development. This suggests that Udriste's multi-time maximum principle has untapped potential for addressing other multi-scale learning problems, such as curriculum learning and transfer learning.

6.3 *Bang-bang control as exploration strategy*

The bang-bang control policy provides a theoretically grounded approach to the exploration-exploitation dilemma in reinforcement learning. By switching between extreme parameter updates based on geometrically-derived switching functions, the algorithm naturally balances the need to exploit current knowledge and explore new strategies. This contrasts with more heuristic approaches like ε-greedy or entropy regularization, offering a principled foundation based on optimal control theory.

6.4 *Comparison with classical control techniques*

Our geometric approach shares similarities with classical control methods but offers distinct advantages:

• **vs. PID control:** While PID controllers react to error signals, our approach proactively shapes the optimization trajectory based on geometric properties.

• **vs. Model predictive control:** Our method incorporates intrinsic geometric constraints rather than externally imposed ones.

• **vs. Adaptive control:** The multi-time framework provides more structured adaptation than traditional adaptive control schemes.

The key innovation lies in exploiting the specific geometric structure of neural network parameter spaces, which classical control methods do not naturally, accommodate.

6.5 *Limitations and future directions*

While our approach shows promising results, several limitations suggest directions for future research:

• **Computational overhead:** Computing the Fisher information matrix and co-state variables add computational cost. Our analysis shows a 45% overhead, which can be mitigated through more efficient approximations or specialized hardware implementations.

• **Theoretical guarantees:** While we provided convergence analysis in Section 3.5, further work is needed to establish convergence rates and characterize the classes of problems where geometric approaches offer the greatest advantages.

• **Extension to other AI domains:** Applying these concepts to discrete optimization problems (e.g., combinatorial optimization in graph neural networks) or high-dimensional domains (e.g., large language models) could yield new insights and optimization strategies.

• **Integration with modern optimizers:** Combining our geometric approach with adaptive methods like AdamW or other second-order methods could potentially offer the benefits of both frameworks.

• **Scalability to very large networks:** Further research is needed to develop efficient implementations for networks with billions of parameters, possibly through hierarchical or distributed geometric optimization strategies.

6.6 *Broader implications for AI safety and interpretability*

The geometric perspective on neural network optimization offers potential benefits for AI safety and interpretability:

• **Constrained optimization:** The nonholonomic framework naturally accommodates safety constraints and ethical considerations.

• **Interpretable dynamics:** The geometric structure provides interpretable measures of learning progress and stability.

• **Robustness analysis:** Understanding the geometric properties of parameter space can inform robustness and generalization analysis.

7. Conclusion

This paper has demonstrated the significant potential of applying geometric dynamics and optimal control theory, as pioneered by Constantin Udriste, to the optimization of artificial intelligence algorithms. By formulating neural network training as a nonholonomic control problem on a parameter manifold and implementing multi-time control strategies, we have developed a novel optimization approach that outperforms standard methods in reinforcement learning tasks.

Our key contributions include: (1) a theoretical framework connecting geometric dynamics with neural network optimization, (2) a practical algorithm with demonstrated performance improvements, (3) comprehensive analysis of computational costs and convergence properties, and (4) extensive experimental validation across multiple continuous control environments.

The success of this approach suggests that deeper connections exist between classical mathematical control theory and modern machine learning optimization than previously recognized. The geometric perspective provides not only performance improvements but also a richer theoretical framework for understanding the behavior of learning algorithms, with implications for AI safety, interpretability, and robustness.

As AI systems grow in complexity, the need for more sophisticated optimization techniques become increasingly important. The principles of geometric dynamics, with their emphasis on understanding the intrinsic structure of the problem space, offers a promising direction for addressing these challenges. Furthermore, the principled mathematical foundation provided by optimal control theory offers guarantees and insights that purely heuristic optimization methods cannot provide.

In this way, Udriste's mathematical legacy continues to influence and advance new fields, demonstrating the enduring power of fundamental mathematical insights. The bridge we have established between geometric dynamics and artificial intelligence optimization opens new avenues for both theoretical investigation and practical applications, honoring the interdisciplinary vision that characterized Udriste's work.

Conflict of interest

The author declares no competing financial interest.

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