Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

Research Article

On the Shrinkage Estimators for a Multivariate Normal Mean Vector with Unknown Diagonal Covariance Matrix

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Received: 12 May 2025; **Revised:** 8 July 2025; **Accepted:** 8 July 2025

Abstract: The results presented in this work focus on the construction of two classes of shrinkage estimators for a Multivariate Normal Mean (MNM) and studying their performance according to the Balanced Loss Function (BLF). First, we introduce a class of estimators derived from the Maximum Likelihood Estimator (MLE) and provide a sufficient condition on the shrinkage function to improve upon the MLE. Then, from the MLE and the James-Stein Estimator (JSE) we build a new class of estimators, and under a simple practical condition, we show that their risks are no greater than those of the JSE, which explains why they perform better. We conclude the paper with numerical results that confirm the performance of the proposed estimators.

***Keywords*:** balanced loss function, estimator of James-Stein, minimax estimator, multivariate normal distribution, risk function, shrinkage estimators

MSC: 62C05, 62H10, 62J07

Abbreviation

MNM	Multivariate Normal Mean
BLF	Balanced Loss Function
MLE	Maximum Likelihood Estimator
JSE	James-Stein Estimator
MSE	Mean Squared Error
QLF	Quadratic Loss Function

DOI: https://doi.org/10.37256/cm.6420257216

1. Introduction

In classical and Bayesian statistics, the multivariate normal distribution has been used extensively in many applications and statistical tests. In fact, estimating the MNM plays a major role in nearly every field. Among different methods, the MLE is of interest. The latter has used in many application. Early references concerning the use of the MLE can be found in Gillariose et al. [1, 2]. Elbatal et al. [3] derived the unknown parameters of the Slash inverse distribution model utilizing the maximum likelihood approach.

Stein [4] played a pivotal role in studying of the MNM estimation by providing sufficient criteria for minimaxity. He used the so-called shrinkage estimator method. James and Stein [5] applied this method to evaluate the performance of the MLE for a MNM in $\mathbb{R}^q$, under the Mean Squared Error (MSE) criterion. Their results revealed that the MLE remains admissible only in low-dimensional settings (specifically, for $q \leq 2$), while it turns out to be inadmissible when the dimension q is greater than or equal to 3. Therefore, many researchers have adopted the shrinkage estimators techniques as a means to enhance the performance of the MLE. We cite for example [6–13]. Tassopoulou et al. [14] introduce deep kernel regression with Adaptive Shrinkage Estimation for predicting personalized biomarker trajectories via posterior correction. Chen et al. [15] critically examine the widespread practice of using the empirical Bayes shrinkage estimator to correct for measurement errors in regression models that incorporate individual latent effects. Alahmadi et al. [16] conducted a thorough investigation into the estimation of the mean of multivariate normal distribution from a Bayesian perspective using the BLF. Hamdaoui and Benmansour [17], studied the estimation of the MNM using shrinkage strategies under appropriate condition of the covariance matrix. They considered the model $\mathrm{Z} \sim \mathscr{N}_q\left(\nu,\ \tau^2 I_q\right)$, where τ^2 is unknown, and treated the minimaxity property of the estimators $\Lambda^{\xi}(\mathrm{Z},\ \mathrm{S}^2) = (1 - l\xi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\mathrm{S}^2/\|\mathrm{Z}\|^2)\mathrm{Z}$ where the parameter l is a real positive constant and S^2 is the natural estimator of τ^2. They demonstrated that if the shrinkage function ξ meets certain conditions, the estimators $\Lambda^{\xi}(\mathrm{Z},\ \mathrm{S}^2)$ dominate the MLE, so they are minimax. All works cited above are based on risk functions relative to the Quadratic Loss Function (QLF).

Many authors have extended published results on shrinkage estimators using the BLF instead of the QLF. Examples include [18–20]. Benkhaled et al. [21], used the same model given by Hamdaoui and Benmansour [17], and introduced a class of estimators which deduced from the MLE and the JSE that expressed as: $\Lambda^{(2)}_{\beta,\ js}(\mathrm{Z},\ \mathrm{S}^2) = \Lambda_{js}(\mathrm{Z},\ \mathrm{S}^2) + \beta\left(\mathrm{S}^2/\|\mathrm{Z}\|^2\right)^2\mathrm{Z}$, with $\Lambda_{js}(\mathrm{Z},\ \mathrm{S}^2)$ is the JSE and β is a real constant that can depend on n and q. To measure the quality of the proposed estimators the authors adopted the risk function relatively to the BLF given by: for any estimator Λ of ν,

$$\ell_{\omega}(\Lambda,\ \nu) = \omega\|\Lambda - \Lambda_0\|^2 + (1-\omega)\|\Lambda - \nu\|^2, \tag{1}$$

where Λ_0 is the target estimator (in this case Λ_0 to be the MLE).

The balanced loss function, introduced by Zellner [18], provides a framework for evaluating estimators that considers both goodness of fit and precision of estimation. The parameter ω $(0 \leq \omega < 1)$ controls the relative weight assigned to goodness of fit, leaving $1-\omega$ as the weight given to the precision of estimation. It is also worth noting that the QLF emerges as a special case of the BLF when $\omega = 0$.

The risk function associate to this loss function is

$$\mathfrak{R}_{\omega}(\Lambda,\ \nu) = \mathbb{E}(\ell_{\omega}(\Lambda,\ \nu)). \tag{2}$$

This article focuses on the estimation of the mean vector of a multivariate normal distribution in $\mathbb{R}^q$, under the assumption that $q \geq 3$ and that the covariance matrix is diagonal and unknown. Specifically, we consider the model $\mathrm{Z} \sim \mathscr{N}_q(\nu,\ \tau^2 I_q)$, where τ^2 is an unknown parameter estimated by the statistic S^2, such that $\mathrm{S}^2 \sim \tau^2\chi^2_n$. Our goal is to construct the new shrinkage estimators of the mean vector $\nu = (\nu_1,\ \nu_2,\ \ldots,\ \nu_q)^T$, and under the BLF ℓ_{ω} defined in (1), we study their improvement over both MLE and JSE.

We divide this work as follows. In section 2, using the Stein's identity mentioned in Lemma 1 of Stein [22] and relatively to the BLF ℓ_ω defined in formula (1), we give a sufficient condition on the shrinkage function $\zeta(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)$ so that the estimator $\Lambda_\zeta(\mathrm{Z},\ \mathrm{S}^2)=\left(1-\zeta(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\dfrac{\mathrm{S}^2}{\|\mathrm{Z}\|^2}\right)\mathrm{Z}$ is minimax, but this is realized with the constraint that this condition is expressed as an expectation of a complicated measurable function related with the shrinkage function. We finally this section by dealing with the special case where $\zeta(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)=\gamma$ and γ is a real parameter, which allow us to confirm the minimaxity of the JSE. In section 3, we introduce a new class of shrinkage estimators that generalize the class of estimators $\Lambda^{(2)}_{\beta,\ js}(\mathrm{Z},\ \mathrm{S}^2)$ considered by Benkhaled et al. [21]. Then, we give a simple practical condition on the shrinkage function for that the current estimators do not only minimax but they also improve the JSE. Furthermore, this sufficient condition related directly to the shrinkage function $\zeta(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)$. We conclude this article with section 4, which contains numerical results that show the out-performance of some proposed approaches over the JSE.

2. Minimax estimators through the Stein's identity

In this part, we consider a general class of shrinkage estimators of the mean vector $\nu=(\nu_1,\ \nu_2,\ \ldots,\ \nu_q)^T$ and study their improvement over the MLE. Based on Stein's identity (Lemma 1 of Stein [22]) and under BLF, we compute their risks's values which allow us to provide sufficient conditions on the shrinkage function to improve the MLE. And consequently, we give sufficient conditions that confirm the minimaxity of these estimators.

First, we recall from Hamdaoui et al. [23] that the MLE of the unknown vector ν is $\Lambda_0=\mathrm{Z}$, and its risk relatively to the BLF is equal to $(1-\omega)q\tau^2$. In the literature, it is clear that if $q\geq 3$ the estimator Λ_0 is minimax and inadmissible, therefore each estimator Λ which improves Λ_0 (i.e. $\mathfrak{R}_\omega(\Lambda,\ \nu)\leq\mathfrak{R}_\omega(\mathrm{Z},\ \nu)$) is minimax.

Now, consider the estimator

$$\Lambda_\xi(\mathrm{Z},\ \mathrm{S}^2)=(1-\xi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2))\mathrm{Z},\tag{3}$$

where the measurable function $\xi(.,\ \|\mathrm{Z}\|^2)$ is differentiable.

Lemma 1 Under the BLF ℓ_ω, the risk function of $\Lambda_\xi(\mathrm{Z},\ \mathrm{S}^2)$ given in (3) is

$$\mathfrak{R}_\omega(\Lambda_\xi(\mathrm{Z},\ \mathrm{S}^2),\ \nu)=(1-\omega)q\tau^2+\mathbb{E}\left\{\xi^2(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\|\mathrm{Z}\|^2-2q\tau^2(1-\omega)\xi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\right\}$$

$$-4\tau^2(1-\omega)\mathbb{E}\left(\|\mathrm{Z}\|^2\xi'_{\|\mathrm{Z}\|^2}(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\right),$$

where $\xi'_{\|\mathrm{Z}\|^2}$ denotes the derivative of the function $\xi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)$ with respect to the variable $\|\mathrm{Z}\|^2$.

Proof. From the definition of the risk function associated to the BLF given in Equation (1), and that of the usual inner product in $\mathbb{R}^q$, we have

$$\mathfrak{R}_\omega(\Lambda_\xi(\mathrm{Z},\ \mathrm{S}^2),\ \nu)=\omega\mathbb{E}(\|\Lambda_\xi(\mathrm{Z},\ \mathrm{S}^2)-\mathrm{Z}\|^2)+(1-\omega)\mathbb{E}(\|\Lambda_\xi(\mathrm{Z},\ \mathrm{S}^2)-\nu\|^2)$$

$$=\omega\mathbb{E}\left(\langle(1-\xi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2))\mathrm{Z}-\mathrm{Z},\ (1-\xi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2))\mathrm{Z}-\mathrm{Z}\rangle\right)$$

$$+(1-\omega)\mathbb{E}\left(\langle(1-\xi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2))\mathrm{Z}-\nu,\ (1-\xi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2))\mathrm{Z}-\nu\rangle\right)$$

$$= \omega\mathbb{E}(\xi^2(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\|\mathrm{Z}\|^2) + (1-\omega)\mathbb{E}\left\{\|\mathrm{Z}-\nu\|^2 + \xi^2(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\|\mathrm{Z}\|^2\right\}$$

$$-2(1-\omega)\mathbb{E}\left(\left\langle \mathrm{Z}-\nu,\ \xi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\mathrm{Z}\right\rangle\right)$$

$$= (1-\omega)\mathbb{E}(\|\mathrm{Z}-\nu\|^2) + \mathbb{E}(\xi^2(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\|\mathrm{Z}\|^2)$$

$$-2(1-\omega)\mathbb{E}\left(\sum_{i=1}^{q}(\mathrm{Z}_i-\nu_i)\xi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\mathrm{Z}_i\right).$$

And according to the linearity of the expectation, we obtain

$$\mathfrak{R}_\omega(\Lambda_\xi(\mathrm{Z},\ \mathrm{S}^2),\ \nu) = (1-\omega)\mathbb{E}(\|\mathrm{Z}-\nu\|^2) + \mathbb{E}(\xi^2(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\|\mathrm{Z}\|^2)$$

$$-2(1-\omega)\sum_{i=1}^{q}\mathbb{E}\left\{(\mathrm{Z}_i-\nu_i)\xi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\mathrm{Z}_i\right\}. \tag{4}$$

Using the change of variable $y = (y_1,\ \ldots,\ y_q)^t = \dfrac{\mathrm{Z}}{\tau}$, we deduce that $y_i = \dfrac{\mathrm{Z}_i}{\tau} \sim \mathcal{N}\left(\dfrac{\nu_i}{\tau},\ 1\right)$ for each $i = 1,\ \ldots,\ q$, and then

$$\sum_{i=1}^{q}\mathbb{E}\left\{(\mathrm{Z}_i-\nu_i)\xi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\mathrm{Z}_i\right\} = \tau^2\sum_{i=1}^{q}\mathbb{E}\left\{\left(y_i-\frac{\nu_i}{\tau}\right)\xi(\mathrm{S}^2,\ \tau^2\|y\|^2)y_i\right\}, \tag{5}$$

The properties of the conditional expectation and Lemma 1 of Stein [22], leads to

$$\mathbb{E}\left\{\left(y_i-\frac{\nu_i}{\tau}\right)\xi(\mathrm{S}^2,\tau^2\|y\|^2)y_i\right\} = \mathbb{E}\left\{\mathbb{E}\left(\frac{\partial}{\partial y_i}\xi(\mathrm{S}^2,\tau^2\|y\|^2)y_i|\mathrm{S}^2\right)\right\}$$

$$= \mathbb{E}\left\{\mathbb{E}\left(\xi(\mathrm{S}^2,\tau^2\|y\|^2) + 2\tau^2y_i^2\xi'_{\|\mathrm{Z}\|^2}(\mathrm{S}^2,\tau^2\|y\|^2)|\mathrm{S}^2\right)\right\}$$

$$= \mathbb{E}\left\{\xi(\mathrm{S}^2,\|\mathrm{Z}\|^2) + 2\mathrm{Z}_i^2\xi'_{\|\mathrm{Z}\|^2}(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\right\}. \tag{6}$$

Then, we put (4), (5) and (6) together to get the result. □

The following Proposition is deduced immediately from the previous Lemma.

Proposition 1 Relatively to the BLF ℓ_ω, a sufficient condition for that $\Lambda_\xi(\mathrm{Z},\ \mathrm{S}^2)$ given in (3), dominating the MLE (thus it is minimax), is

$$\mathbb{E}\left\{\xi^2(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\|\mathrm{Z}\|^2 - 2\tau^2(1-\omega)\left(q\xi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2) + 2\|\mathrm{Z}\|^2\xi'_{\|\mathrm{Z}\|^2}(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\right)\right\} \le 0.$$

Rewrite the estimator defined in (3) by letting $\xi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)=\zeta(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\dfrac{\mathrm{S}^2}{\|\mathrm{Z}\|^2}$, leads to

$$\Lambda_\zeta(\mathrm{Z},\ \mathrm{S}^2)=\left(1-\zeta(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\frac{\mathrm{S}^2}{\|\mathrm{Z}\|^2}\right)\mathrm{Z}, \tag{7}$$

where $\zeta(.,\ \|\mathrm{Z}\|^2)$ is differentiable and measurable function.

Using the last Proposition, one can show easily that a sufficient condition for that $\Lambda_\zeta(\mathrm{Z},\ \mathrm{S}^2)$ given in (7) dominating the MLE (so it is minimax), is

$$\mathbb{E}\left\{\zeta^2(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\frac{(\mathrm{S}^2)^2}{\|\mathrm{Z}\|^2}-2\tau^2(1-\omega)\left((p-2)\zeta(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\frac{\mathrm{S}^2}{\|\mathrm{Z}\|^2}+2\mathrm{S}^2\zeta'_{\|\mathrm{Z}\|^2}(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\right)\right\}\leq 0. \tag{8}$$

In the special case when $\zeta(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)=\gamma$, (i.e. $\Lambda_\gamma(\mathrm{Z},\ \mathrm{S}^2)=\left(1-\gamma\dfrac{\mathrm{S}^2}{\|\mathrm{Z}\|^2}\right)$ Z) with the real constant γ can depend on n, q and ω. We deduce immediately from the inequality (8) that a sufficient condition for that $\Lambda_\gamma(\mathrm{Z},\ \mathrm{S}^2)$ dominating the MLE, is

$$0\leq\gamma\leq\frac{2(1-\omega)(q-2)}{n+2}.$$

The convexity of the function $\mathfrak{R}_\omega(\Lambda_\gamma(\mathrm{Z},\ \mathrm{S}^2),\ \nu)$ on γ allows us to conclude that the optimal value of γ which makes the risk function $\mathfrak{R}_\omega(\Lambda_\gamma(\mathrm{Z},\ \mathrm{S}^2),\ \nu)$ takes its minimum is

$$\widehat{\gamma}=\frac{(1-\omega)(q-2)}{n+2}. \tag{9}$$

If we replace γ by $\widehat{\gamma}$, we get the JSE

$$\Lambda_{js}(\mathrm{Z},\ \mathrm{S}^2)=\left(1-\widehat{\gamma}\frac{\mathrm{S}^2}{\|\mathrm{Z}\|^2}\right)\mathrm{Z}=\left(1-\frac{(1-\omega)(q-2)}{n+2}\frac{\mathrm{S}^2}{\|\mathrm{Z}\|^2}\right)\mathrm{Z}. \tag{10}$$

As, $\dfrac{\|\mathrm{Z}\|^2}{\tau^2}\sim\chi^2_q\left(\dfrac{\|\nu\|^2}{\tau^2}\right)$ where $\chi^2_q\left(\dfrac{\|\nu\|^2}{\tau^2}\right)$ indicate the chi-square non-central distribution with q degrees of freedom and non-centrality parameter $\dfrac{\|\nu\|^2}{\tau^2}$, and from Lemma 1, the independence of random variables S^2 and $\|\mathrm{Z}\|^2$ and the Definition 1 of Hamdaoui et al. [24], we deduce that the risk of the JSE is equal to

$$\begin{aligned}\mathfrak{R}_\omega(\Lambda_{js}(\mathrm{Z},\ \mathrm{S}^2),\ \nu)&=(1-\omega)q\tau^2-(1-\omega)^2(q-2)^2\frac{n}{n+2}\tau^2\mathbb{E}\left(\frac{1}{\|\mathrm{Z}\|^2}\right)\\&=(1-\omega)q\tau^2-(1-\omega)^2(q-2)^2\frac{n}{n+2}\tau^2\mathbb{E}\left(\frac{1}{q-2+2\mathrm{K}}\right),\end{aligned} \tag{11}$$

where $\mathrm{K} \sim P\left(\frac{\|\nu\|^2}{2\tau^2}\right)$ is the Poisson distribution of parameter $\frac{\|\nu\|^2}{2\tau^2}$.

From the formula (11), we see that $\mathfrak{R}_\omega(\Lambda_{js}(\mathrm{Z},\ \mathrm{S}^2),\ \nu) \leq \mathfrak{R}_\omega(\mathrm{Z},\ \nu)$, then $\Lambda_{js}(\mathrm{Z},\ \mathrm{S}^2)$ improves the MLE Z, and therefore it is minimax.

3. New approach to improving the JSE

In the previous section, we provided a sufficient condition which appears in the inequality (8), for that the estimator defined in (7) to be minimax. However, this condition is expressed as an expectation of a complicated measurable function related with the shrinkage function. In this section, we establish a simple sufficient condition that related directly to the shrinkage function and it is not presented as a function of the expectation of a complicate measurable function. Furthermore, under this simple condition we obtain a class of estimators which do not only minimax, but they also dominates the JSE.

Let's take the estimator defined in (7) again, but in this time we substitute $\zeta(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)$ by $\widehat{\gamma} - \psi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)$, where $\widehat{\gamma}$ is provided in (9) and $\psi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)$ is a measurable function. This leads to,

$$
\begin{aligned}
\Lambda_{\psi,\ \widehat{\gamma}}(\mathrm{Z},\ \mathrm{S}^2) &= \left(1 - (\widehat{\gamma} - \psi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\,\frac{\mathrm{S}^2}{\|\mathrm{Z}\|^2}\right)\mathrm{Z} \\
&= \left(1 - \widehat{\gamma}\frac{\mathrm{S}^2}{\|\mathrm{Z}\|^2}\right)\mathrm{Z} + \psi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\frac{\mathrm{S}^2}{\|\mathrm{Z}\|^2}\mathrm{Z} \\
&= \Lambda_{js}(\mathrm{Z},\ \mathrm{S}^2) + \psi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\frac{\mathrm{S}^2}{\|\mathrm{Z}\|^2}\mathrm{Z}. \qquad (12)
\end{aligned}
$$

The following lemma establishes an explicit formula of the risk function of the estimator $\Lambda_{\psi,\ \widehat{\gamma}}(\mathrm{Z},\ \mathrm{S}^2)$ relatively to the BLF ℓ_ω defined in (1), and consequently conducts us to extract sufficient conditions on the shrinkage function to ensure the domination of the estimator $\Lambda_{\psi,\ \widehat{\gamma}}(\mathrm{Z},\ \mathrm{S}^2)$ over the JSE.

Lemma 2 Relatively to the BLF ℓ_ω, the risk function of the estimator $\Lambda_{\psi,\ \widehat{\gamma}}(\mathrm{Z},\ \mathrm{S}^2)$ given in (12) is

$$
\begin{aligned}
\mathfrak{R}_\omega(\Lambda_{\psi,\ \widehat{\gamma}}(\mathrm{Z},\ \mathrm{S}^2),\ \nu) = \mathfrak{R}_\omega(\Lambda_{js}(\mathrm{Z},\ \mathrm{S}^2),\ \nu) \\
&+ \omega\mathbb{E}\left\{\psi^2(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\frac{(\mathrm{S}^2)^2}{\|\mathrm{Z}\|^2} - 2\widehat{\gamma}\psi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\frac{(\mathrm{S}^2)^2}{\|\mathrm{Z}\|^2}\right\} \\
&+ (1-\omega)\mathbb{E}\left\{\psi^2(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\frac{(\mathrm{S}^2)^2}{\|\mathrm{Z}\|^2} - 2\widehat{\gamma}\psi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\frac{(\mathrm{S}^2)^2}{\|\mathrm{Z}\|^2}\right\} \\
&- 2(1-\omega)\mathbb{E}\left\{\langle \nu,\ \mathrm{Z}\rangle\phi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\frac{\mathrm{S}^2}{\|\mathrm{Z}\|^2} - \mathrm{S}^2\psi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2\right\}.
\end{aligned}
$$

Proof. Relatively to the BLF ℓ_ω given in Equation (1), we have

$$\mathfrak{R}_\omega(\Lambda_{\psi,\,\widehat{\gamma}}(\mathrm{Z},\ \mathrm{S}^2),\ \nu) = \omega\mathbb{E}\left(\|\Lambda_{js}(\mathrm{Z},\ \mathrm{S}^2) + \psi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\frac{\mathrm{S}^2}{\|\mathrm{Z}\|^2}\mathrm{Z} - \mathrm{Z}\|^2\right)$$

$$+ (1-\omega)\mathbb{E}\left(\|\Lambda_{js}(\mathrm{Z},\ \mathrm{S}^2) + \psi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\frac{\mathrm{S}^2}{\|\mathrm{Z}\|^2}\mathrm{Z} - \nu\|^2\right).$$

From the definition of the usual norm and that of the usual inner product in $\mathbb{R}^q$, we get

$$\mathfrak{R}_\omega(\Lambda_{\psi,\,\widehat{\gamma}}(\mathrm{Z},\ \mathrm{S}^2),\ \nu) = \omega\mathbb{E}\left\{\|\Lambda_{js}(\mathrm{Z},\ \mathrm{S}^2) - \mathrm{Z}\|^2 + \psi^2(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\frac{(\mathrm{S}^2)^2}{\|\mathrm{Z}\|^2}\right\}$$

$$+ 2\omega\mathbb{E}\left\{\left\langle \Lambda_{js}(\mathrm{Z},\ \mathrm{S}^2) - \mathrm{Z},\ \psi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\frac{\mathrm{S}^2}{\|\mathrm{Z}\|^2}\mathrm{Z}\right\rangle\right\}$$

$$+ (1-\omega)\mathbb{E}\left\{\|\Lambda_{js}(\mathrm{Z},\ \mathrm{S}^2) - \nu\|^2 + \psi^2(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\frac{(\mathrm{S}^2)^2}{\|\mathrm{Z}\|^2}\right\}$$

$$+ 2(1-\omega)\mathbb{E}\left\{\left\langle \Lambda_{js}(\mathrm{Z},\ \mathrm{S}^2) - \nu,\ \psi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\frac{\mathrm{S}^2}{\|\mathrm{Z}\|^2}\mathrm{Z}\right\rangle\right\}$$

$$= \omega\mathbb{E}\left\{\|\Lambda_{js}(\mathrm{Z},\ \mathrm{S}^2) - \mathrm{Z}\|^2\right\} + (1-\omega)\mathbb{E}\left\{\|\Lambda_{js}(\mathrm{Z},\ \mathrm{S}^2) - \nu\|^2\right\}$$

$$+ \omega\mathbb{E}\left\{\psi^2(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\frac{(\mathrm{S}^2)^2}{\|\mathrm{Z}\|^2} - 2\widehat{\gamma}\psi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\frac{(\mathrm{S}^2)^2}{\|\mathrm{Z}\|^2}\right\}$$

$$+ (1-\omega)\mathbb{E}\left\{\psi^2(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\frac{(\mathrm{S}^2)^2}{\|\mathrm{Z}\|^2} - 2\widehat{\gamma}\psi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\frac{(\mathrm{S}^2)^2}{\|\mathrm{Z}\|^2})\right\}$$

$$- 2(1-\omega)\mathbb{E}\left\{\langle \nu,\ \mathrm{Z}\rangle\phi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\frac{\mathrm{S}^2}{\|\mathrm{Z}\|^2} - \mathrm{S}^2\psi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2\right\}.$$

As,

$$\mathfrak{R}_\omega(\Lambda_{js}(\mathrm{Z},\ \mathrm{S}^2) = \omega\mathbb{E}\left\{\|\Lambda_{js}(\mathrm{Z},\ \mathrm{S}^2) - \mathrm{Z}\|^2\right\} + (1-\omega)\mathbb{E}\left\{\|\Lambda_{js}(\mathrm{Z},\ \mathrm{S}^2) - \nu\|^2\right\}.$$

Then, we obtain the desirable result. □

Next, we present a theorem that provides a simple practical sufficient condition on the function ψ, ensures that the estimator $\Lambda_{\psi,\,\widehat{\gamma}}(\mathrm{Z},\ \mathrm{S}^2)$ outperforms $\Lambda_{js}(\mathrm{Z},\ \mathrm{S}^2)$ under the BLF by adaptively adjusting the shrinkage function ψ. Unlike the JSE, which is optimized for squared error loss, this condition tailors the shrinkage function to balance the goodness-of-fit and the estimation precision, as prioritized by the weight ω of the BLF. By keeping ψ non-negative and capping it based on the signal strength S^2 and the parameter $\|\mathrm{Z}\|^2$, the condition prevents excessive or insufficient shrinkage. This reduces expected loss when the signal is moderate. Therefore, due to its adaptability, the estimator is more effective in applications where fitting data and precise parameter estimation are critical. It outperforms the JSE's fixed shrinkage rule.

Theorem 1 Relatively to the BLF ℓ_ω, a sufficient condition for that $\Lambda_{\psi,\ \widehat{\gamma}}(\mathrm{Z},\ \mathrm{S}^2)$ given in (12), dominating $\Lambda_{js}(\mathrm{Z},\ \mathrm{S}^2)$, is

$$0 \leq \psi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2) \leq 2\left(\widehat{\gamma} - \frac{\|\mathrm{Z}\|^2}{\mathrm{S}^2}\right)\mathbb{I}_{\frac{\widehat{\gamma}\mathrm{S}^2}{\|\mathrm{Z}\|^2} \geq 1},$$

where $\mathbb{I}_{\widehat{\gamma}\frac{\mathrm{S}^2}{\|\mathrm{Z}\|^2} \geq 1}$ is the indicating function of $\left(\widehat{\gamma}\dfrac{\mathrm{S}^2}{\|\mathrm{Z}\|^2} \geq 1\right)$.

Proof. The conditional expectation and the Equation (2.7) defined by Benmansour and Mourid in [25], lead to

$$\begin{aligned}\mathbb{E}\left\{\langle \nu,\ \mathrm{Z}\rangle \psi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\frac{\mathrm{S}^2}{\|\mathrm{Z}\|^2}\right\} &= \mathbb{E}\left\{\mathbb{E}\left(\sum_{i=1}^{p}\left(\psi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\frac{\mathrm{S}^2}{\|\mathrm{Z}\|^2}\mathrm{Z}_i\nu_i\right)|\mathrm{S}^2\right)\right\}\\ &= \lambda\mathbb{E}\left\{\psi(\tau^2\chi_n^2,\ \tau^2\chi_{q+2}^2(\lambda))\frac{\chi_n^2}{\chi_{q+2}^2(\lambda)}\right\}\\ &= \lambda\mathbb{E}\left\{\psi(\tau^2\chi_n^2,\ \tau^2\chi_{q+2K}^2)\frac{\chi_n^2}{\chi_{q+2K}^2}\right\},\end{aligned} \tag{13}$$

where $\mathrm{K} \sim P\left(\dfrac{\lambda}{2} = \dfrac{\|\nu\|^2}{2\tau^2}\right)$.

Then from Lemma 2 and Equation (13), we can deduce that the sufficient conditions for which $\Lambda_{\psi,\ \widehat{\gamma}}(\mathrm{Z},\ \mathrm{S}^2)$ dominates $\Lambda_{js}(\mathrm{Z},\ \mathrm{S}^2)$, are

$$\mathbb{E}\left\{\psi^2(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\frac{(\mathrm{S}^2)^2}{\|\mathrm{Z}\|^2} - 2\widehat{\gamma}\psi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\frac{(\mathrm{S}^2)^2}{\|\mathrm{Z}\|^2}\right\} \leq 0 \tag{14}$$

and

$$\begin{aligned}&\mathbb{E}\left\{\psi^2(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\frac{(\mathrm{S}^2)^2}{\|\mathrm{Z}\|^2} - 2\widehat{\gamma}\psi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\frac{(\mathrm{S}^2)^2}{\|\mathrm{Z}\|^2} + 2\mathrm{S}^2\psi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\right\}\\ &-2\lambda\mathbb{E}\left\{\psi(\tau^2\chi_n^2,\ \tau^2\chi_{q+2K}^2)\frac{\chi_n^2}{\chi_{q+2K}^2}\right\} \leq 0.\end{aligned} \tag{15}$$

The inequality (14) can be written as follows

$$\mathbb{E}\left\{\psi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2)\frac{(\mathrm{S}^2)^2}{\|\mathrm{Z}\|^2}\left(\psi(\mathrm{S}^2,\ \|\mathrm{Z}\|^2) - 2\widehat{\gamma}\right)\right\} \leq 0. \tag{16}$$

For inequality (15) to be satisfied, it suffices that the function ψ be positive and

$$\mathbb{E}\left\{\psi^2(S^2,\ \|Z\|^2)\frac{(S^2)^2}{\|Z\|^2}-2\widehat{\gamma}\psi(S^2,\ \|Z\|^2)\frac{(S^2)^2}{\|Z\|^2}+2S^2\psi(S^2,\ \|Z\|^2)\right\}$$

$$=\mathbb{E}\left\{\psi(S^2,\ \|Z\|^2)\frac{S^2}{\|Z\|^2}\left(S^2\psi(S^2,\ \|Z\|^2)-2\widehat{\gamma}S^2+2\|Z\|^2\right)\right\}\leq 0. \tag{17}$$

From (16) and (17), we can prove easily that $\Lambda_{\psi,\ \widehat{\gamma}}(Z,\ S^2)$ dominates $\Lambda_{js}(Z,\ S^2)$, if $0\leq\psi(S^2,\ \|Z\|^2)\leq 2\widehat{\gamma}$ and $0\leq\psi(S^2,\ \|Z\|^2)\leq 2\left(\widehat{\gamma}-\frac{\|Z\|^2}{S^2}\right)\mathbb{I}_{\frac{\widehat{\gamma}S^2}{\|Z\|^2}\geq 1}$. Thus, a sufficient condition for that $\Lambda_{\psi,\ \widehat{\gamma}}(Z,\ S^2)$ dominates $\Lambda_{js}(Z,\ S^2)$, is

$$0\leq\psi(S^2,\ \|Z\|^2)\leq 2\left(\widehat{\gamma}-\frac{\|Z\|^2}{S^2}\right)\mathbb{I}_{\frac{\widehat{\gamma}S^2}{\|Z\|^2}\geq 1}.$$

□

Example 1 Let $\psi_1(S^2,\ \|Z\|^2)=\left(\widehat{\gamma}-\frac{\|Z\|^2}{S^2}\right)\mathbb{I}_{\frac{\widehat{\gamma}S^2}{\|Z\|^2}\geq 1}$, and therefore

$$\Lambda_{\psi_1,\ \widehat{\gamma}}(Z,\ S^2)=\Lambda_{js}(Z,\ S^2)+\psi_1(S^2,\ \|Z\|^2)\frac{S^2}{\|Z\|^2}Z$$

$$=\Lambda_{js}(Z,\ S^2)+\left(\widehat{\gamma}\frac{S^2}{\|Z\|^2}-1\right)\mathbb{I}_{\frac{\widehat{\gamma}S^2}{\|Z\|^2}\geq 1}Z, \tag{18}$$

It is clear that the function $\psi_1(S^2,\ \|Z\|^2)$ satisfies the condition of Theorem 1, then the estimator $\Lambda_{\psi_1,\ \widehat{\gamma}}(Z,\ S^2)$ dominates $\Lambda_{js}(Z,\ S^2)$.

4. Numerical results

4.1 *On simulated data*

We remember the explicit formula of the risk function associated to the BLF of the JSE given by the formula (11),

$$\mathfrak{R}_\omega(\Lambda_{js}(Z,\ S^2),\ \nu)=(1-\omega)q\tau^2-(1-\omega)^2(q-2)^2\frac{n}{n+2}\tau^2\mathbb{E}\left(\frac{1}{\|Z\|^2}\right), \tag{19}$$

which can be also written as,

$$\mathfrak{R}_\omega(\Lambda_{js}(Z,\ S^2),\ \nu)=(1-\omega)q\tau^2-(1-\omega)^2(q-2)^2\frac{n}{n+2}\tau^2\mathbb{E}\left(\frac{1}{q-2+2K}\right), \tag{20}$$

with $K \sim P\left(\frac{\|\nu\|^2}{2\tau^2}\right)$. We can show that the risk function of the estimator $\Lambda_{\psi_1,\,\widehat{\gamma}}(Z,\ S^2)$ defined in (18) associated to ℓ_ω is

$$\mathfrak{R}_\omega(\Lambda_{\psi_1,\,\widehat{\gamma}}(Z,\ S^2),\ \nu) = \mathfrak{R}_\omega(\Lambda_{js}(Z,\ S^2),\ \nu)$$

$$+\mathbb{E}\left[\left(\|Z\|^2 - \frac{\widehat{\gamma}^2 S^4}{\|Z\|^2} + \frac{2\widehat{\gamma}(1-\omega)(q-2)\tau^2 S^2}{\|Z\|^2} - 2(1-\omega)q\tau^2\right)\mathbb{I}_{\widehat{\gamma}\frac{S^2}{\|Z\|^2}\geq 1}\right]. \tag{21}$$

We also recall that the risk function of the estimator $\Lambda^{(2)}_{\widehat{\beta},\,js}(Z,\ S^2)$ introduced by Benkhaled et al. [21] is

$$\mathfrak{R}_\omega(\Lambda^{(2)}_{\widehat{\beta},\,js}(Z,\ S^2),\ \nu) = \mathfrak{R}_\omega(\Lambda_{js}(Z,\ S^2),\ \nu) + 2\widehat{\beta}n(n+2)(1-\omega)\tau^2\left[(q-4) - \frac{(q-2)(n+4)}{n+2)}\right]\mathbb{E}\left(\frac{1}{\|y\|^4}\right)$$

$$+(\widehat{\beta})^2 n(n+2)(n+4)(n+6)\tau^2\mathbb{E}\left(\frac{1}{\|y\|^6}\right), \tag{22}$$

where $y = \frac{Z}{\tau} = (y_1,\ \ldots,\ y_q)^t$ and $\widehat{\beta} = \frac{2(1-\omega)(q-6)}{(n+4)(n+6)}$.

In this subsection, we will graphically examine and compare the risk ratios of JSE:$= \Lambda_{js}(Z,\ S^2)$, Est1:$= \Lambda^{(2)}_{\widehat{\beta},\,js}(Z,\ S^2)$ and Newest:$= \Lambda_{\psi_1,\,\widehat{\gamma}}(Z,\ S^2)$ relatively to the MLE, denoted by $\frac{\mathfrak{R}_\omega(\Lambda_{js}(Z,\ S^2),\ \nu)}{\mathfrak{R}_\omega(Z,\ \nu)}$, $\frac{\mathfrak{R}_\omega(\Lambda^{(2)}_{\widehat{\beta},\,js}(Z,\ S^2),\ \nu)}{\mathfrak{R}_\omega(Z,\ \nu)}$ and $\frac{\mathfrak{R}_\omega(\Lambda_{\psi_1,\,\widehat{\gamma}}(Z,\ S^2),\ \nu)}{\mathfrak{R}_\omega(Z,\ \nu)}$, as functions of $d = \frac{\|\nu\|^2}{2\tau^2}$ for diverse values of n, q and ω. One can check to see the dominance of one over the other to confirm our study. To graph these curves, we determinate the true values of these risks ratios as functions of $d = \frac{\|\nu\|^2}{2\tau^2}$ by evaluating the expectations $\mathbb{E}\left(\frac{1}{(\|y\|^2)^m}\right) = \mathbb{E}\left(\frac{1}{\left[\chi^2_q\left(\frac{\|\nu\|^2}{\tau^2}\right)\right]^m}\right)$ $(m = 1,\ 2,\ 3)$, using the definition of the density of the non-central χ^2 with the non-centrality parameter $\frac{\|\nu\|^2}{\tau^2}$ provided in formula (1.2) page 9 by Arnold [26].

Figures 1-3, show that: first, for a fixed value of $\omega = 0.2$, if we increase n and fixes q, we obtain a significant dominance of $\Lambda_{js}(Z,\ S^2)$, $\Lambda^{(2)}_{\widehat{\beta},\,js}(Z,\ S^2)$ and $\Lambda_{\psi_1,\,\widehat{\gamma}}(Z,\ S^2)$ over the MLE Z, and less domination of $\Lambda_{\psi_1,\,\widehat{\gamma}}(Z,\ S^2)$ over $\Lambda_{js}(Z,\ S^2)$, this improvement tends to disappear when d large. A similar interpretation can be shared for fixing n and growing q. Secondly, when we vary ω and fixes n and q, we notice that: if ω increases the improvement of $\Lambda_{js}(Z,\ S^2)$ and $\Lambda_{\psi_1,\,\widehat{\gamma}}(Z,\ S^2)$ over the MLE decrease. The refinement becomes negligible and tends towards 0 if d grows. We conclude that all parameters (n, q and ω) influence the performance of $\Lambda_{js}(Z,\ S^2)$ and $\Lambda_{\psi_1,\,js}(Z,\ S^2)$.

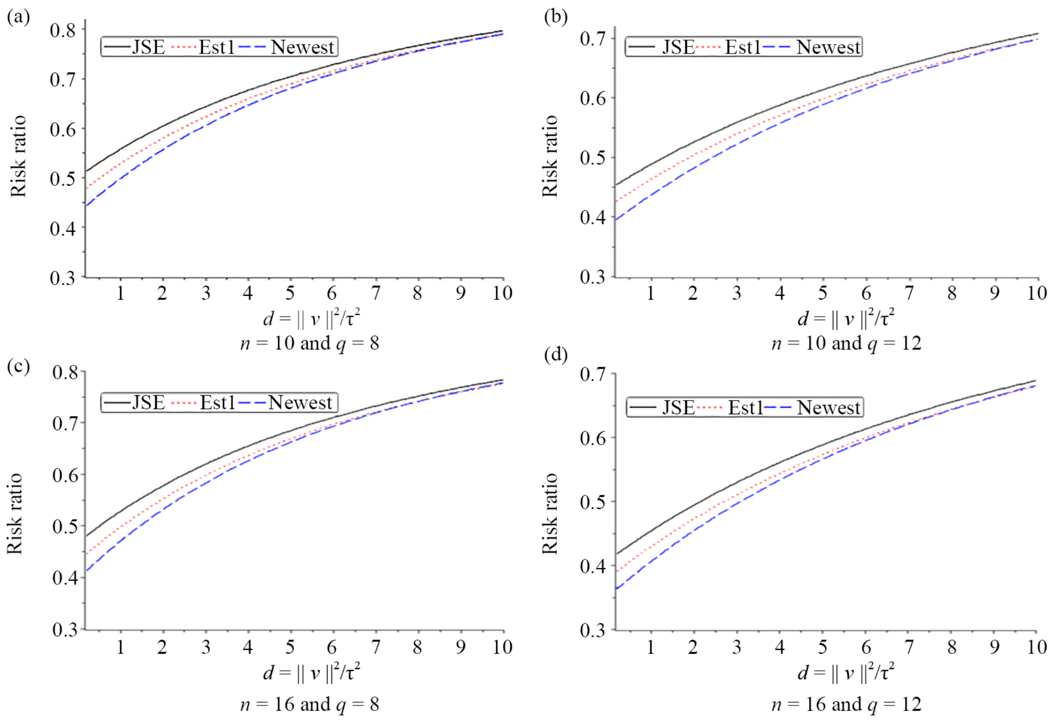

Figure 1. $\frac{\mathfrak{R}_\omega(\Lambda_{js}(Z,\ S^2),\ v)}{\mathfrak{R}_\omega(Z,\ v)}$, $\frac{\mathfrak{R}_\omega(\Lambda^{(2)}_{\widehat{\beta},\ js}(Z,\ S^2),v)}{\mathfrak{R}_\omega(Z,\ v)}$ and $\frac{\mathfrak{R}_\omega(\Lambda_{\psi_1,\ \widehat{\gamma}}(Z,\ S^2),\ v)}{\mathfrak{R}_\omega(Z,\ v)}$ as function of $d = \|v\|^2/\tau^2$ for $n = 10,\ 16,\ q = 8,\ 12$ and $\omega = 0.2$

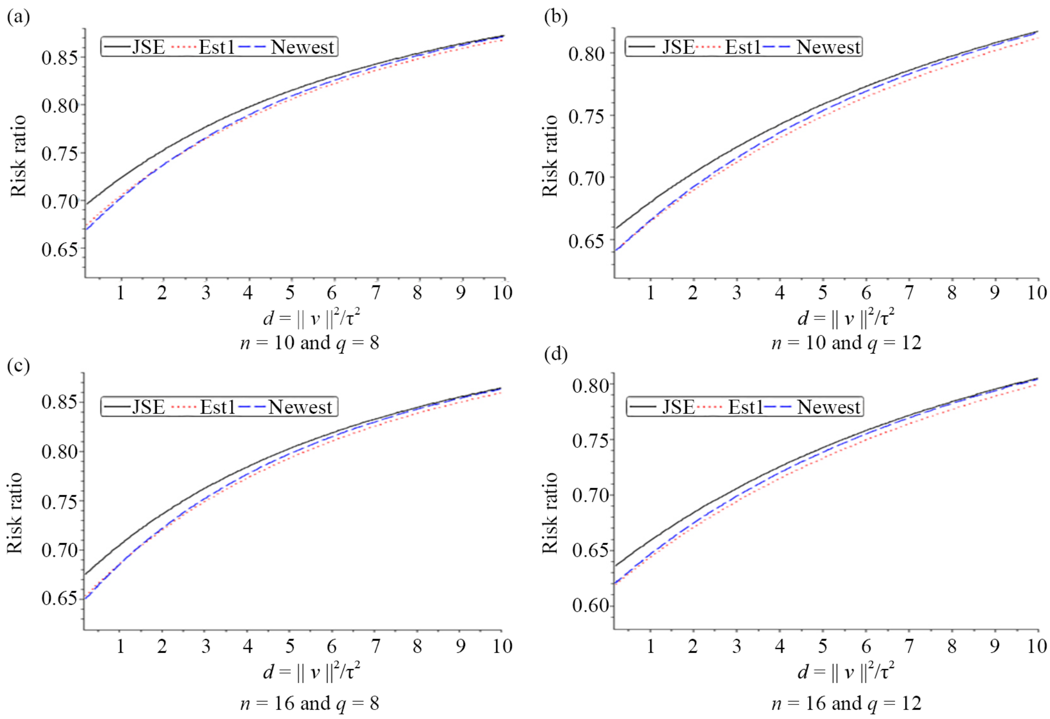

Figure 2. $\frac{\mathfrak{R}_\omega(\Lambda_{js}(Z,\ S^2),\ v)}{\mathfrak{R}_\omega(Z,\ v)}$, $\frac{\mathfrak{R}_\omega(\Lambda^{(2)}_{\widehat{\beta},\ js}(Z,\ S^2),\ v)}{\mathfrak{R}_\omega(Z,\ v)}$ and $\frac{\mathfrak{R}_\omega(\Lambda_{\psi_1,\ \widehat{\gamma}}(Z,\ S^2),\ v)}{\mathfrak{R}_\omega(Z,\ v)}$ as function of $d = \|v\|^2/\tau^2$ for $n = 10,\ 16,\ q = 8,\ 12$ and $\omega = 0.5$

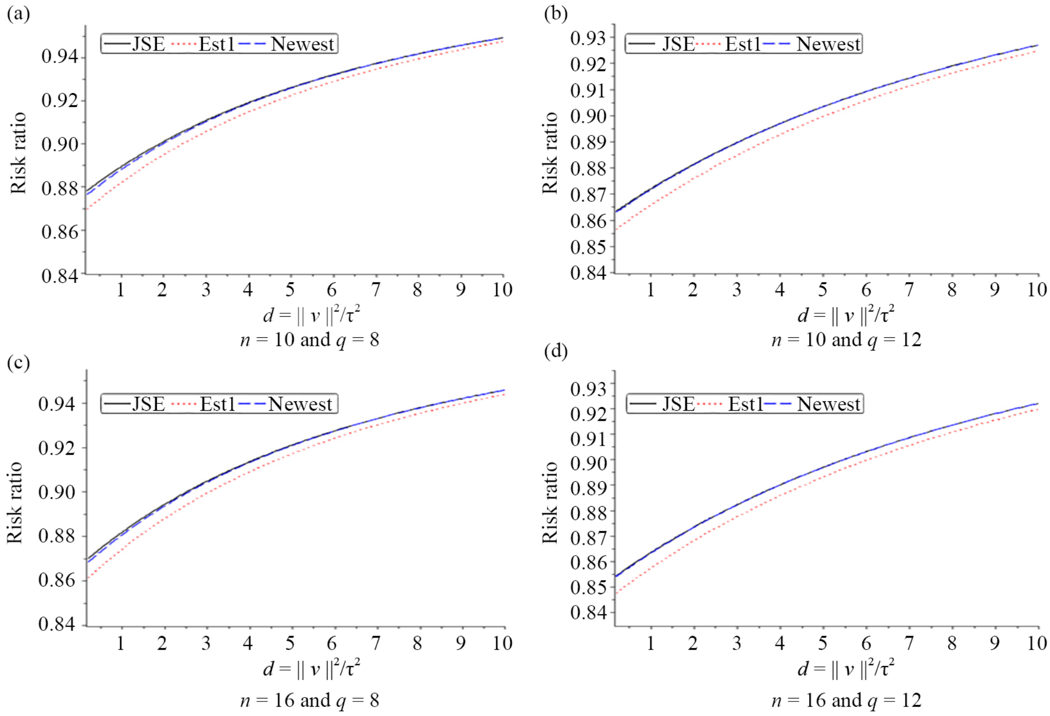

Figure 3. $\frac{\mathfrak{R}_\omega(\Lambda_{js}(Z, S^2), v)}{\mathfrak{R}_\omega(Z, v)}$, $\frac{\mathfrak{R}_\omega(\Lambda^{(2)}_{\widehat{\beta}, js}(Z, S^2), v)}{\mathfrak{R}_\omega(Z, v)}$ and $\frac{\mathfrak{R}_\omega(\Lambda_{\psi_1, \widehat{\gamma}}(Z, S^2), v)}{\mathfrak{R}_\omega(Z, v)}$ as function of $d = \|v\|^2/\tau^2$ for $n = 10, 16$, $q = 8, 12$ and $\omega = 0.8$

Concerning the comparison between the performance of estimators $\Lambda^{(2)}_{\widehat{\beta}, js}(Z, S^2)$ and $\Lambda_{\psi_1, \widehat{\gamma}}(Z, S^2)$, we note that there are no specific cases for the values of parameters n, q and ω in which one estimator performs better than the other. We may be able to suggest a new problem about what is the form of the optimal function ψ that minimizes the risk function $\mathfrak{R}_\omega(\Lambda_{\psi, \widehat{\gamma}}(Z, S^2), v)$ of the class of estimators $\Lambda_{\psi, \widehat{\gamma}}(Z, S^2)$ given in (12).

Now, we present tables that show the values of risks ratios $\frac{\mathfrak{R}_\omega(\Lambda_{js}(Z, S^2), v)}{\mathfrak{R}_\omega(Z, v)}$ and $\frac{\mathfrak{R}_\omega(\Lambda_{\psi_1, \widehat{\gamma}}(Z, S^2), v)}{\mathfrak{R}_\omega(Z, v)}$, for chosen values of n and q ($n = 10$, $q = 4$, $n = 20$, $q = 4$, $n = 10$, $q = 12$ and $n = 20$, $q = 12$) and diverse values of ω and d. The first entry is $\frac{\mathfrak{R}_\omega(\Lambda_{js}(Z, S^2), v)}{\mathfrak{R}_\omega(Z, v)}$ and the second entry is $\frac{\mathfrak{R}_\omega(\Lambda_{\psi_1, \widehat{\gamma}}(Z, S^2), v)}{\mathfrak{R}_\omega(Z, v)}$.

Table 1. This table shows the evaluation of the risk ratios as following: first entry is $\frac{\mathfrak{R}_\omega(\Lambda_{js}(Z, S^2), v)}{\mathfrak{R}_\omega(Z, v)}$, next is $\frac{\mathfrak{R}_\omega(\Lambda_{\psi_1, \widehat{\gamma}}(Z, S^2), v)}{\mathfrak{R}_\omega(Z, v)}$, where $n = 10$, $q = 4$ and d, ω take different values

d	ω						
	0.1	0.2	0.3	0.5	0.6	0.8	0.9
1.2418	0.7206 0.6450	0.7517 0.6876	0.7827 0.7301	0.8448 0.8140	0.8758 0.8547	0.9379 0.9318	0.9690 0.9673
2.4948	0.7857 0.7340	0.8095 0.7668	0.8333 0.7992	0.8810 0.8620	0.9048 0.8921	0.9523 0.9489	0.9762 0.9753
5.0019	0.8623 0.8393	0.8776 0.8594	0.8929 0.8789	0.9235 0.9164	0.9388 0.9343	0.9694 0.9683	0.9847 0.9844
9.9901	0.9254 0.9213	0.9337 0.9307	0.9420 0.9398	0.9586 0.9576	0.9668 0.9663	0.9834 0.9833	0.9917 0.9917
20.0000	0.9625 0.9624	0.9667 0.9666	0.9708 0.9708	0.9792 0.9791	0.9833 0.9833	0.9917 0.9917	0.9958 0.9958

Table 2. This table shows the evaluation of the risk ratios as following: first entry is $\frac{\mathfrak{R}_\omega(\Lambda_{js}(Z,\ S^2),\ \nu)}{\mathfrak{R}_\omega(Z,\ \nu)}$, next is $\frac{\mathfrak{R}_\omega(\Lambda_{\psi_1,\ \hat{\gamma}}(Z,\ S^2),\ \nu)}{\mathfrak{R}_\omega(Z,\ \nu)}$, where $n = 20$, $q = 4$ and d, ω take different values

d	ω						
	0.1	0.2	0.3	0.5	0.6	0.8	0.9
1.2418	0.6952 0.6162	0.7291 0.6622	0.7630 0.7082	0.83069 0.7987	0.8645 0.8427	0.9323 0.9260	0.9661 0.9645
2.4948	0.7662 0.7125	0.7922 0.7479	0.8182 0.7828	0.8701 0.8505	0.8961 0.8830	0.9480 0.9445	0.9740 0.9731
5.0019	0.8498 0.8262	0.8665 0.8478	0.8832 0.8689	0.9548 0.9093	0.9333 0.9286	0.9666 0.9655	0.9833 0.9830
9.9901	0.9186 0.9146	0.9277 0.9246	0.9367 0.9345	0.9548 0.9538	0.9638 0.9633	0.9819 0.9818	0.9910 0.9909
20.0000	0.9591 0.9590	0.9636 0.9636	0.9682 0.9681	0.9773 0.9773	0.9818 0.9818	0.9909 0.9909	0.9954 0.9954

Table 3. This table shows the evaluation of the risk ratios as following: first entry is $\frac{\mathfrak{R}_\omega(\Lambda_{js}(Z,\ S^2),\ \nu)}{\mathfrak{R}_\omega(Z,\ \nu)}$, next is $\frac{\mathfrak{R}_\omega(\Lambda_{\psi_1,\ \hat{\gamma}}(Z,\ S^2),\ \nu)}{\mathfrak{R}_\omega(Z,\ \nu)}$, where $n = 10$, $q = 12$ and d, ω take different values

d	ω						
	0.1	0.2	0.3	0.5	0.6	0.8	0.9
1.2418	0.4344 0.3721	0.4972 0.4478	0.5601 0.5238	0.6857 0.6721	0.7486 0.7424	0.8743 0.8740	0.9371 0.9371
2.4948	0.4849 0.4328	0.5422 0.5022	0.5994 0.5711	0.7139 0.7040	0.7711 0.7668	0.8855 0.8853	0.9428 0.9428
5.0019	0.5651 0.5301	0.6134 0.5882	0.6617 0.6450	0.7584 0.7534	0.8067 0.8047	0.9033 0.9033	0.9517 0.9517
9.9901	0.6710 0.6567	0.7075 0.6983	0.7441 0.7386	0.8172 0.8160	0.8538 0.8533	0.9269 0.9269	0.9634 0.9634
20.0000	0.7817 0.7799	0.8060 0.8050	0.8302 0.8298	0.8787 0.8787	0.9030 0.9030	0.9515 0.9515	0.9757 0.9757

Table 4. This table shows the evaluation of the risk ratios as following: first entry is $\frac{\mathfrak{R}_\omega(\Lambda_{js}(Z,\ S^2),\ \nu)}{\mathfrak{R}_\omega(Z,\ \nu)}$, next is $\frac{\mathfrak{R}_\omega(\Lambda_{\psi_1,\ \hat{\gamma}}(Z,\ S^2),\ \nu)}{\mathfrak{R}_\omega(Z,\ \nu)}$, where $n = 20$, $q = 12$ and d, ω take different values

d	ω						
	0.1	0.2	0.3	0.5	0.6	0.8	0.9
	0.1	0.2	0.3	0.5	0.6	0.8	0.9
1.2418	0.3829 0.3263	0.4515 0.4074	0.5205 0.4886	0.6572 0.6463	0.7257 0.7211	0.8629 0.8627	0.9314 0.9314
2.4948	0.4381 0.3915	0.5005 0.4657	0.5630 0.5391	0.6878 0.6802	0.7503 0.7471	0.8751 0.8750	0.9376 0.9376
5.0019	0.5255 0.4954	0.5782 0.5572	0.6310 0.6176	0.7364 0.7328	0.7891 0.7877	0.8946 0.8945	0.9473 0.9473
9.9901	0.6410 0.6300	0.6810 0.6741	0.7208 0.7169	0.8006 0.7998	0.8405 0.8402	0.9202 0.9202	0.9601 0.9601
20.0000	0.7619 0.7608	0.7884 0.7878	0.8148 0.8145	0.8677 0.8677	0.8942 0.8942	0.9471 0.9471	0.9735 0.9735

In Table 1 to Table 4, for small values of ω and d, we get a tremendous amelioration of $\Lambda_{js}(Z,\ S^2)$ and $\Lambda_{\psi_1,\ \hat{\gamma}}(Z,\ S^2)$ over the MLE. This amelioration decays and tends to 0 for large values of ω and d. Furthermore, we also notice that the improvement of $\Lambda_{\psi_1,\ \hat{\gamma}}(Z,\ S^2)$ for $\Lambda_{js}(Z,\ S^2)$ turns out to be insignificant for high values of ω and d. We note that $\Lambda_{\psi_1,\ \hat{\gamma}}(Z,\ S^2)$ beats $\Lambda_{js}(Z,\ S^2)$ for each selected value of ω and d.

4.2 *Real data application*

The basic target of this subsection is to contrast the effectiveness of the suggested estimators to that of the MLE, using a dataset from the real world. For this purpose application, we consider the Batting dataset from the Lahman package in the R program, which contains pitching, batting, and fielding statistics for Major League Baseball from 1871 through 2023. Our foremost objective is to assess how well the new estimators perform. Specifically, we want to know how well they can estimate batting averages. To do so, we will establish a setting that involves sampling a limited number of observations from a group of players. Afterwards, we will calculate the risk ratio of each estimator relative to the MLE using the BLF to evaluate the predictive accuracy of the proposed estimators and determine how closely they approximate the true values. As shown in Figure 4, for different values of n, q and ω, all of the proposed estimators consistently produce risk ratios below one. This suggests that all estimators outperform the MLE in terms of efficiency by offering reduced variance through shrinkage. In applied settings, selecting an appropriate value for ω is essential for effective implementation. One data-driven approach is to select ω via cross-validation. By partitioning the data into training and validation sets, one can evaluate the risk function related to the BLF across a grid of ω values. The value of ω that yields the best average performance on the validation sets is chosen. This method is computationally intensive, yet robust, especially when sufficient data is available. In the absence of context-specific information, we recommend using a default value of 0.5 for ω. This value strikes a reasonable balance between goodness of fit and precision of estimation.

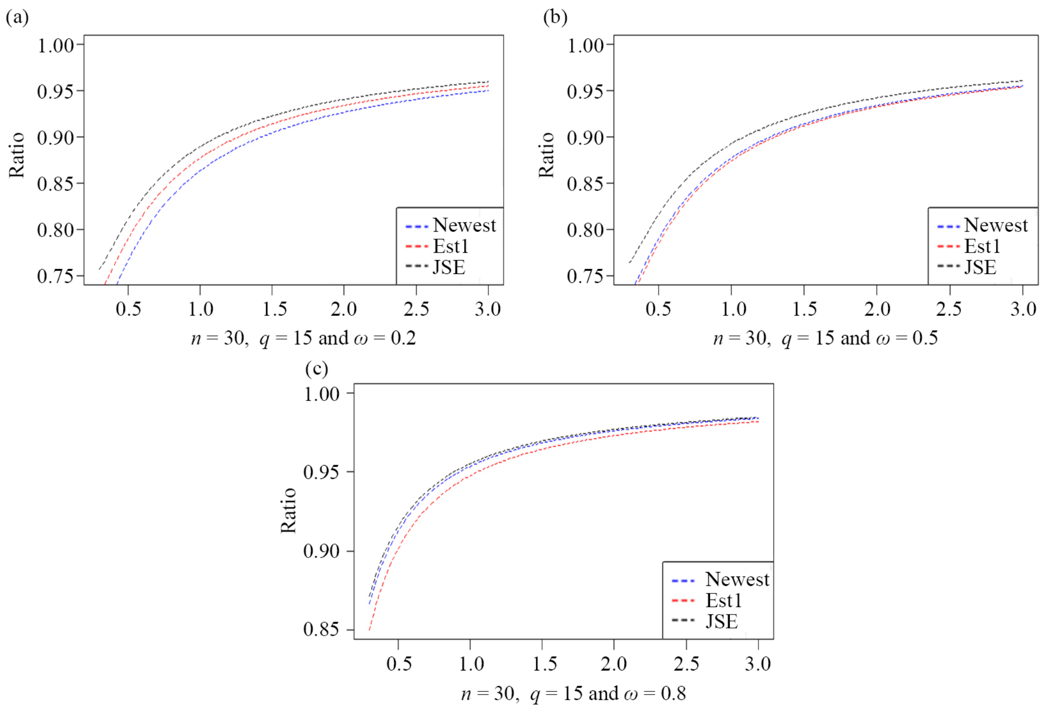

Figure 4. $\frac{\mathfrak{R}_\omega(\Lambda_{js}(Z,\ S^2),\ \nu)}{\mathfrak{R}_\omega(Z,\ \nu)}$, $\frac{\mathfrak{R}_\omega(\Lambda^{(2)}_{\hat{\beta},\ js}(Z,\ S^2),\ \nu)}{\mathfrak{R}_\omega(Z,\ \nu)}$ and $\frac{\mathfrak{R}_\omega(\Lambda_{\psi_1,\ \hat{\gamma}}(Z,\ S^2),\ \nu)}{\mathfrak{R}_\omega(Z,\ \nu)}$ for $n = 30$, $q = 15$ and $\omega = 0.2,\ 0.5,\ 0.8$

5. Conclusion

In this article, we studied the estimation of the MNM distribution $Z \sim \mathcal{N}_p\left(\nu,\ \tau^2 I_q\right)$ based on the BLF. We established sufficient conditions for that the estimator $\Lambda_\zeta = \left(1 - \zeta(S^2,\ \|Z\|^2)S^2/\|Z\|^2\right)Z$ dominates the MLE (so it is minimax) and we deduce the minimaxity of the JSE. Then we deal with the class of shrinkage estimators $\Lambda_{\psi,\ \hat{\gamma}}(Z,\ S^2) = \Lambda_{js}(Z,\ S^2) + \psi(S^2,\ \|Z\|^2)(S^2/\|Z\|^2)Z$ which do not only dominate the MLE but also dominate the JSE ($\Lambda_{js}(Z,\ S^2)$). We gave a simple condition on the function ψ for which the estimators $\Lambda_{\psi,\ \hat{\gamma}}(Z,\ S^2)$ improve $\Lambda_{js}(Z,\ S^2)$. Furthermore this condition does not depend on the differentiability of the function ψ. Consequently, we established a condition of the minimaxity of estimators $\Lambda_{\psi,\ \hat{\gamma}}(Z,\ S^2)$ without using the condition of the differentiability of the function ψ. As an extension of this work, we can look for analogous results and examine the performance of the proposed estimators in the general case of the covariance matrix. This work can also be investigated under the Bayesian framework.

Acknowledgement

The Researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University for financial support (QU-APC-2025).

Conflict of interest

The authors declare that there are not conflict of interest.

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