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Research Article

New High-Dimensional Generalizations of Nesbitt's Inequality and Relative Applications

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Abstract: Two kinds of novel generalizations of Nesbitt's inequality are explored in various cases regarding dimensions and parameters in this article. Some other cases are also discussed elaborately by using the semiconcave-semiconvex theorem. The general inequalities are then employed to deduce some cyclic inequalities and mathematical competition questions. At last, a relation about Hurwitz-Lerch zeta functions is obtained.

Keywords: Nesbitt's inequality, Jensen's inequality, Chebyshev's inequality, semiconcave-semiconvex theorems, Hurwitz-Lerch zeta functions

MSC: 26D15, 26D10, 11M35

1. Introduction

Inequalities play a significant and fundamental role in the development of modern science, technology and education [1]. As an ancient Chinese proverb goes, "A very tiny difference within a millimeter can lead to an error of more than thousands of miles", which is just like a fatal tornado caused by a butterfly's flapping wings. Since it is impossible to measure and constrain the real things in the absolute sense, the most important issue we have to face is how to estimate and ascertain the terrible unknown outcomes. In this process, inequalities have showcased extraordinary application value [2–17].

In area of education, inequalities are of particular effectiveness to practice and test the intelligence of students in high school [18–26]. Typical ones of such inequalities include cyclic inequalities, mean value inequalities, and Radon's inequality. Amongst these inequalities, Nesbitt's inequality (see [27])

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\geqslant\frac{3}{2},\qquad x,\ y,\ z>0,$$

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has been known as a famous one and generalized to different forms since 1903. In recent two decades, increasing attention has been paid to generalizations and relative applications of Nesbitt's inequality. Bencze et al. in [28, 29] gave one kind of generalization with weights and refinements of Nesbitt's inequality, which is written as

$$\frac{\left(\sum_{k=1}^{n} p_k x_k\right)^{\alpha}}{\sum_{k=1}^{n} p_k (s - x_k)} \leqslant \sum_{k=1}^{n} \frac{p_k x_k^{\alpha}}{s - x_k},$$

where $s = x_1 + \cdots + x_n$. Batinetu-Giurgiu and Stanciu presented various concrete examples of generalizations with weights and analogous form of Nesbitt's inequality in [30–32] and proved the following inequality: If $n \in \mathbb{N}^+ - \{1\}$, $a, v \geqslant 0$, $b, c, d, m, p, x_k > 0$, $k = 1, \cdots, n$, $t \in [1, \infty)$, $X_{n,\, m} = \sum_{k=1}^{n} x_k^m$, $X_{n,\, p} = \sum_{k=1}^{n} x_k^p$, such that $cX_{n,\, p} > d \max_{1 \leqslant k \leqslant n} x_k^p$, then

$$\sum_{k=1}^{n} \frac{(aX_{n,\, m} + bx_k^m)^t}{(cX_{n,\, p} - dx_k^p)^v} \geqslant \frac{n^{v-t}(an+b)^t X_{n,\, m}^t}{(cn-d)X_{n,\, p}^v}. \tag{1}$$

An iconic generalization of Nesbitt's inequality was a high-dimensional version given by Wang in [33] and read as follows:

$$\sum_{i=1}^{n} \frac{a_i^m}{ts - a_i^p} \geqslant \frac{\sum_{i=1}^{n} a_i^{m-p}}{nt - 1}, \tag{2}$$

where $a_1, \cdots, a_n > 0$, $m, n, p, t \in \mathbb{N}^+$, $m \geqslant p$, $n \geqslant 2$ and $s = \sum_{i=1}^{n} a_i^p$. Chu also generalized Nesbitt's inequality independently on dimensions and powers in [34] to the following results: Let $a_1, \cdots, a_n > 0$, $s = a_1 + \cdots + a_n$. Then

$$\sum_{i=1}^{n} \left[\frac{(n-1)a_i}{s - a_i}\right]^k \geqslant n, \text{ if } k \geqslant \frac{n-1}{n}, \text{ and}$$

$$\sum_{i=1}^{n} \frac{a_i^k}{(s - a_i)^{\beta}} \geqslant \frac{\sum_{i=1}^{n} a_i^{k-\beta}}{(n-1)^{\beta}}, \text{ if } k \geqslant \beta \geqslant 1.$$

With the same notations, Jiang in [21, 22] gave the following generalizations:

$$\sum_{i=1}^{n}\left(\frac{a_i}{s-a_i}\right)^k \geqslant \frac{n}{(n-1)^k},$$

if k is an integer, and

$$\left(\frac{a_1}{a_2+a_3}\right)^k+\left(\frac{a_2}{a_3+a_1}\right)^k+\left(\frac{a_3}{a_1+a_2}\right)^k \geqslant \frac{1}{2}\left[\left(\frac{a_1}{a_2+a_3}\right)^{k-1}+\left(\frac{a_2}{a_3+a_1}\right)^{k-1}+\left(\frac{a_3}{a_1+a_2}\right)^{k-1}\right],$$

if k is a positive integer. The Nesbitt's inequality was also concerned with in the study of other inequalities [20, 32, 35, 36]. What is even more interesting is that Nesbitt's inequality can be also applied to other fields such as theories of matrices and numbers [13, 37]. However, up to our knowledge, until now there have been no more generalizations of Nesbitt's inequality in the literature much more general than the results mentioned above.

Among the generalizations of Nesbitt's inequality, we can see some are approached from perspective of coefficients, and some are from the perspective of exponents, however, each subject to specific constraints on the parameter ranges. Based on these observations, it becomes both interesting and necessary to summarize, synthesize and develop more general extensions of the aforementioned generalizations.

In this article, we further develop and generalize Nesbitt's inequality with more parameters and high dimensions in different forms (as a further generalization of (2)). The newly generalized versions of Nesbitt's inequality cover most generalized versions given before and even include the situations that derive inverse inequalities. Specifically, we consider the algebraic expression

$$\sum_{i=1}^{n}\frac{a_i^m}{(ts-ra_i^p)^\beta}, \text{ where } s=\sum_{i=1}^{n}a_i^p, \tag{3}$$

and compare (3) with

$$\frac{n^{\beta+1-\frac{m}{p}}}{(nt-r)^\beta}\left(\sum_{i=1}^{n}a_i^p\right)^{\frac{m}{p}-\beta} \text{ and } \frac{1}{(nt-r)^\beta}\sum_{i=1}^{n}a_i^{m-\beta p}, \tag{4}$$

where $n\in\mathbb{N}^+$, $a_1,\ \cdots,\ a_n>0$, $m,\ p,\ \beta,\ t,\ r\in\mathbb{R}$ with $t\geqslant 0$ and $ts>ra_i^p$ for all $i=1,\ \cdots,\ n$. The inequality (2) is a simple relation of (3) and (the second expression of) (4) for the case when $\beta=1$.

Our main goal in this article is to study the relation between (3) and (4), which differs greatly in different cases. To compare (3) and the first algebraic expression of (4) in Theorem 3.1, the Jensen's inequality is a powerful tool. To determine the relation of (3) and the second one of (4), we employ Theorem 3.1, rearrangement inequality, Chebyshev's inequality and Jensen's inequality and give definitive results in various different cases in Theorem 3.2 and 3.3. Most significantly, we not only generalize the result of [13], but also obtain a great deal of cases such that the reverse inequalities of the previously-mentioned inequalities are true, which are completely new consequences. Due to the high complexity of the parameters in (3), it is impossible to discuss all the cases or to conclude a definitive inequality consequence for each case merely by the classical inequalities mentioned above. Fortunately, for some other special cases that can guarantee definitive inequalities, a useful theorem-Semiconcave-semiconvex theorem from [38] is turned out to help a lot in the new situations.

The newly generalized Nesbitt's inequalities can be applied to prove many cyclic inequalities in different forms regarding dimensions, parameters and exponents. In particular, some competitive contest questions, including International Mathematical Olympiad (IMO for short) questions, can be easily obtained only by picking certain specific parameters in the generalized inequalities.

At last, we also consider the applications of the obtained inequalities in the study of Hurwitz-Lerch zeta functions. In [13], Wang obtained the minimum value related to Riemann's and Hurwitz's zeta function by using his main inequality

$$\frac{\left(\sum_{k=1}^{n} p_k x_k\right)^{\alpha}}{\left(M - \sum_{k=1}^{n} p_k x_k\right)^{\beta}} \leqslant \sum_{k=1}^{n} \frac{p_k x_k^{\alpha}}{(M - x_k)^{\beta}},$$

where $\alpha \geqslant 1$, $\beta \geqslant 0$, $0 < x_k < M < +\infty$, $p_k \in [0,\ 1]$, $k = 1, \cdots, n$ with $p_1 + \cdots + p_n = 1$. In our work, we further study the relation of different Hurwitz-Lerch zeta functions by using our generalized inequalities.

The remainder of this article is organized as follows. In Section 2, some necessary inequalities are presented for the following argument. In Section 3, the main theorems are proved and some examples of other cases are given for clarity. In Section 4, we apply the main theorems to some inequality problems and competition questions. In Section 5, we apply the main theorems to obtaining some relations about different Hurwitz-Lerch zeta functions.

2. Preliminaries

In this section, we present some necessary basic inequalities.

First we recall the Rearrangement Inequality. Let $a_i, b_i \in \mathbb{R}$ $(1 \leqslant i \leqslant n)$ with

$$a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n \text{ and } b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n, \tag{5}$$

and $\{c_i\}_{1 \leqslant i \leqslant n}$ be a rearrangement of $\{b_i\}_{1 \leqslant i \leqslant n}$. Then it holds that

$$\sum_{i=1}^{n} a_i b_{n+1-i} \leqslant \sum_{i=1}^{n} a_i c_i \leqslant \sum_{i=1}^{n} a_i b_i.$$

Applying the rearrangement inequality stated above, one can easily obtain the Chebyshev's inequality: for $\{a_i\}_{1\leqslant i\leqslant n}$, $\{b_i\}_{1\leqslant i\leqslant n}$ given in (5), it holds that

$$\sum_{i=1}^{n} a_i b_i \geqslant \frac{1}{n}\sum_{i=1}^{n} a_i \cdot \sum_{i=1}^{n} b_i \geqslant \sum_{i=1}^{n} a_i b_{n+1-i}.$$

We then recall the famous Jensen's inequality (see [39, Section 1.11] and [1, Section 7.1]). Let $I \subset \mathbb{R}$ be an interval, $\varphi : I \to \mathbb{R}$ a convex function, $\psi : I \to \mathbb{R}$ a concave one, then for each $n \in \mathbb{N}$, $x_1, \cdots, x_n \in I$ and positive weights $\lambda_1, \cdots, \lambda_n$ with $\lambda_1 + \cdots + \lambda_n = 1$, the following inequalities hold:

$$\varphi\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leqslant \sum_{i=1}^{n} \lambda_i \varphi(x_i) \text{ and } \psi\left(\sum_{i=1}^{n} \lambda_i x_i\right) \geqslant \sum_{i=1}^{n} \lambda_i \psi(x_i).$$

In this article, we often take $p_1 = \cdots = p_n = 1/n$. As special cases, if we consider the convex function x^p with $p \in (-\infty,\ 0) \cup [1,\ +\infty)$, for $x_1,\ \cdots,\ x_n > 0$,

$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^p \leqslant \frac{1}{n}\sum_{i=1}^{n} x_i^p, \text{ i.e., } \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} \geqslant n^{\frac{1}{p}-1}\sum_{i=1}^{n} x_i \text{ for } p \geqslant 1,\ \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} \leqslant n^{\frac{1}{p}-1}\sum_{i=1}^{n} x_i \text{ for } p < 0; \tag{6}$$

for the concave function x^p with $p \in (0,\ 1]$ and $x_1,\ \cdots,\ x_n > 0$, we also have

$$\left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} \leqslant n^{\frac{1}{p}-1}\sum_{i=1}^{n} x_i; \tag{7}$$

for $\ln x$, which is a concave function, we have

$$\ln\sum_{i=1}^{n} p_i x_i \geqslant \sum_{i=1}^{n} p_i \ln x_i, \text{ i.e., } \sum_{i=1}^{n} p_i x_i \geqslant \prod_{i=1}^{n} x_i^{p_i}, \tag{8}$$

where $x_1,\ \cdots,\ x_n$ are positive. Actually (8) can be regarded as a generalized version of mean value inequality.

The following Semiconcave-semiconvex Theorem can be found in [38, Theorem 7.4]. And this theorem is very effective to deduce more general inequalities.

Theorem 2.1 Let $a < b$ and $x_1, \cdots, x_n \in [a, b]$ such that

(1) $x_1 \leqslant \cdots \leqslant x_n$;

(2) $x_1 + \cdots + x_n = C$, where C is a constant.

Let $f : [a, b] \to \mathbb{R}$ be a function with $c \in (a, b)$ such that f is concave (resp. convex) on $[a, c]$ and convex (resp. concave) on $[c, b]$, and

$$F(x_1, \cdots, x_n) = f(x_1) + \cdots + f(x_n).$$

Then if F achieves its minimum (resp. maximum) at some point $x = (x_1, \cdots, x_n)$, then x satisfies $x_1 = \cdots = x_{k-1} = a$, $x_{k+1} = \cdots = x_n$, $k = 1, \cdots, n$; if F achieves its maximum (resp. minimum) at some point $x = (x_1, \cdots, x_n)$, then x satisfies $x_1 = \cdots = x_{k-1}$, $x_{k+1} = \cdots = x_n = b$, $k = 1, \cdots, n$.

In the sequel, when it comes to the derivatives of a function $f(x)$ on the bottom a of an interval, we still use $f'(a)$ to denote the unilateral derivatives for convenience if defined.

3. Main inequalities

3.1 *Main theorems*

In this subsection we are to present the main inequalities in various cases and prove them. First we present some parameters T, Θ_1 and Θ_2 dependent on the parameters in (3) and these parameters will be used frequently in this article.

In the sequel, we always assume that:

$a_1, a_2, \cdots, a_n > 0$,

$m, p, t, r, \beta \in \mathbb{R}$ with $t \geqslant 0$,

$s := \sum_{i=1}^{n} a_i^p$ with $ts > ra_i^p$, for each $i = 1, \cdots, n$.

$$T := \begin{cases} s, & \text{if } r \leqslant t, \\ ts/r, & \text{if } r > t, \end{cases}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a parabolic function such that

$$f(\theta) = \theta^2 + \frac{2m}{(\beta+1)rp}\theta + \frac{m(m-p)}{\beta(\beta+1)r^2p^2},$$

with β, r, p chosen appropriately. When there exist two different real solutions to the equation $f(\theta) = 0$, we set Θ_1 and Θ_2 to be the two solutions with $\Theta_1 < \Theta_2$, i.e.,

$$\Theta_1 = -\frac{m}{(\beta+1)rp} - \sqrt{\frac{m(\beta p + p - m)}{\beta(\beta+1)^2 r^2 p^2}} \text{ and } \Theta_2 = -\frac{m}{(\beta+1)rp} + \sqrt{\frac{m(\beta p + p - m)}{\beta(\beta+1)^2 r^2 p^2}}. \tag{9}$$

We first compare

$$\sum_{i=1}^{n} \frac{a_i^m}{(ts - ra_i^p)^\beta} \text{ and } \frac{n^{\beta+1-\frac{m}{p}}}{(nt-r)^\beta} \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{m}{p}-\beta}$$

in Theorem 3.1, where clearly it is required that $p \neq 0$.

Theorem 3.1 (1) Suppose that β, r, m, p and $t \in \mathbb{R}$ satisfy each one of the following four cases:

(i) $\beta r = 0$ and $m(m-p) \geqslant 0$;

(ii) $\beta = -1$, $r \neq 0$ and either

(ii.1) $m(m-p)t \geqslant 0$ and $m(m+p)r \leqslant 0$, or

(ii.2) $m(m-p)t > 0$, $m(m+p)r > 0$ and $\frac{(m-p)ts}{(m+p)r} \geqslant T$;

(iii) $\beta \in (-\infty,\ -1) \cup (0,\ +\infty)$, $r \neq 0$ and one of the following cases holds,

(iii.1) $\beta m[m-(\beta+1)p] \geqslant 0$,

(iii.2) $\beta m[m-(\beta+1)p] < 0$, $mrp\beta \geqslant 0$ and $m(m-p) \geqslant 0$,

(iii.3) $t > r$, $\beta m[m-(\beta+1)p] < 0$, $mrp\beta < 0$, $m(m-p) > 0$ and $(t-r)\Theta_1 \geqslant 1$;

(iv) $\beta \in (-1,\ 0)$, $m[m-(\beta+1)p] > 0$, $m(m-p) \geqslant 0$, $t > r \neq 0$ and $(t-r)\Theta_2 \geqslant 1$.

Then

$$\sum_{i=1}^{n} \frac{a_i^m}{(ts - ra_i^p)^\beta} \geqslant \frac{n^{\beta+1-\frac{m}{p}}}{(nt-r)^\beta} \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{m}{p}-\beta}. \tag{10}$$

(2) Suppose that β, r, m, p and $t \in \mathbb{R}$ satisfy one of the following four cases:

(v) $\beta r = 0$ and $m(m-p) \leqslant 0$;

(vi) $\beta = -1$, $r \neq 0$ and either

(vi.1) $m(m-p)t \leqslant 0$ and $m(m+p)r \geqslant 0$, or

(vi.2) $m(m-p)t < 0$, $m(m+p)r < 0$ and $\frac{(m-p)ts}{(m+p)r} \geqslant T$;

(vii) $\beta \in (-1,\ 0)$, $r \neq 0$ and one of the following cases holds,

(vii.1) $0 \leqslant pm \leqslant (\beta+1)p^2$,

(vii.2) $r \geqslant 0$ and $(\beta+1)p^2 < pm \leqslant p^2$,

(vii.3) $r < 0$, $(\beta+1)p^2 < pm \leqslant p^2$ and $(t-r)\Theta_1 \geqslant 1$;

(viii) $\beta \in (-\infty,\ -1) \cup (0,\ +\infty)$, $t > r \neq 0$, $\beta m[m-(\beta+1)p] < 0$, $m(m-p) \leqslant 0$ and $(t-r)\Theta_2 \geqslant 1$.

Then

$$\sum_{i=1}^{n} \frac{a_i^m}{(ts - ra_i^p)^\beta} \leqslant \frac{n^{\beta+1-\frac{m}{p}}}{(nt-r)^\beta} \left(\sum_{i=1}^{n} a_i^p \right)^{\frac{m}{p}-\beta}. \tag{11}$$

Proof. Noting that

$$ts > r \max_{1 \leqslant i \leqslant n} \{a_i^p\},$$

we know that

$$nt \geqslant \frac{ts}{\max\limits_{1 \leqslant i \leqslant n} \{a_i^p\}} > r,$$

which implies that the right sides of (10) and (11) make sense.

We consider the function $g : (0,\ T) \to \mathbb{R}$ with such that

$$g(x) = \frac{x^{\frac{m}{p}}}{(ts - rx)^\beta}. \tag{12}$$

Then we know that for each $x \in (0,\ T)$,

$$g''(x) = \frac{x^{\frac{m}{p}-2}}{(ts-rx)^\beta} \left[\frac{m}{p} \left(\frac{m}{p} - 1 \right) + \frac{2m\beta r}{p} \frac{x}{ts-rx} + \beta(\beta+1) r^2 \frac{x^2}{(ts-rx)^2} \right]. \tag{13}$$

If $\beta = -1$,

$$g''(x) = \frac{m}{p^2} x^{\frac{m}{p}-2} \left[(m-p)ts - (m+p)rx \right]; \tag{14}$$

if $\beta r = 0$,

$$g''(x) = \frac{m}{p} \left(\frac{m}{p} - 1 \right) \frac{x^{\frac{m}{p}-2}}{(ts-rx)^\beta}; \tag{15}$$

if $(\beta+1)\beta r \neq 0$,

$$g''(x) = (\beta+1)\beta r^2 \frac{x^{\frac{m}{p}-2}}{(ts-rx)^{\beta}} f\left(\frac{x}{ts-rx}\right)$$

$$= (\beta+1)\beta r^2 \frac{x^{\frac{m}{p}-2}}{(ts-rx)^{\beta}} \left[\left(\frac{x}{ts-rx} + \frac{m}{(\beta+1)rp}\right)^2 + \frac{m(m-\beta p-p)}{\beta(\beta+1)^2 r^2 p^2}\right]. \tag{16}$$

First note that

$$n\frac{(s/n)^{\frac{m}{p}}}{(ts-rs/n)^{\beta}} = \frac{n^{\beta+1-\frac{m}{p}}}{(nt-r)^{\beta}} \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{m}{p}-\beta}.$$

Hence, by Jensen's inequality, if $g(x)$ is convex, we can obtain (10) and if $g(x)$ is concave, we can obtain (11). We thus only need to guarantee the convexity and concavity of $g(x)$ to give the conclusions. In this mind, we first split it into two parts to discuss the convexity and concavity of $g(x)$.

Part 1 The situations where $(\beta+1)\beta r = 0$ to show the conclusions for the cases (i), (ii), (v) and (vi).

Actually, by (14) and (15), one easily sees that (i) and (ii) can be obtained by $g''(x) \geqslant 0$ on $(0,\ T)$ and (v) and (vi) can be obtained by $g''(x) \leqslant 0$ on $(0,\ T)$, respectively. The convexity and concavity of $g(x)$ follow immediately for (i), (ii) and (v), (vi), respectively.

Part 2 The situations where $(\beta+1)\beta r \neq 0$ to show the conclusions for the cases (iii), (iv), (vii) and (viii).

In this situation, the convexity and concavity of $g(x)$ are ascribed to sign of the parabolic function $f(\theta)$ with $\theta \geqslant 0$. We can further consider four situations.

(a) The parabolic curve $f(\theta)$ has at most one intersection point with the real axis. In this situation, $m\beta[m-(\beta+1)p] \geqslant 0$, and if $(\beta+1)\beta > 0$, $g(x)$ is convex; if $(\beta+1)\beta < 0$, $g(x)$ is concave. Then (iii.1) and (vii.1) can be obtained.

(b) The parabolic curve $f(\theta)$ has exactly two different intersection points with the real axis, but neither one lies in the positive semi-axis ($\theta > 0$). Then $m\beta[m-(\beta+1)p] < 0$, the symmetry axis $-\dfrac{m}{(\beta+1)rp} \leqslant 0$, and if $(\beta+1)\beta > 0$, and $f(0) \geqslant 0$, $g(x)$ is convex on $(0,\ T)$; if $(\beta+1)\beta < 0$, and $f(0) \geqslant 0$, $g(x)$ is concave on $(0,\ T)$. Then (iii.2) and (vii.2) are obtained.

(c) The parabolic curve $f(\theta)$ has exactly two different intersection points with the real axis, but only one intersection point lies the positive semi-axis ($\theta > 0$). In this situation, we can set $t > r$,

$$m\beta[m-(\beta+1)p] < 0,\ \Theta_2 \geqslant \sup_{x\in[0,\ T]} \frac{x}{ts-rx} = \frac{1}{t-r},$$

and if $\beta(\beta+1)<0$ and $f(0)\leqslant 0$, $g(x)$ is convex on $(0,\ T)$; if $\beta(\beta+1)>0$ and $f(0)\leqslant 0$, $g(x)$ is concave on $(0,\ T)$. Thus (iv) and (vii) are obtained.

(d) The parabolic curve $f(\theta)$ has exactly two different intersection points with the real axis and both lie in the positive semi-axis ($\theta>0$). In this situation, we can set $t>r$,

$$m\beta[m-(\beta+1)p]<0,\ \Theta_1\geqslant\sup_{x\in[0,\ T]}\frac{x}{ts-rx}=\frac{1}{t-r},$$

and if $\beta(\beta+1)>0$ and $f(0)>0$, $g(x)$ is convex; if $\beta(\beta+1)<0$ and $f(0)>0$, $g(x)$ is concave. Thus (iii.3) and (vii.3) are obtained. The proof is hence accomplished now. □

Next we compare

$$\sum_{i=1}^{n}\frac{a_i^m}{(ts-ra_i^p)^\beta}\text{ and }\frac{1}{(nt-r)^\beta}\sum_{i=1}^{n}a_i^{m-\beta p}.$$

First, we observe that when $\beta=0$, $p=0$ or $t=0$, it always holds that

$$\sum_{i=1}^{n}\frac{a_i^m}{(ts-ra_i^p)^\beta}=\frac{1}{(nt-r)^\beta}\sum_{i=1}^{n}a_i^{m-\beta p}.$$

Hence we only consider the cases when $\beta pt\neq 0$ in the following.

Theorem 3.2 The inequality

$$\sum_{i=1}^{n}\frac{a_i^m}{(ts-ra_i^p)^\beta}\geqslant\frac{1}{(nt-r)^\beta}\sum_{i=1}^{n}a_i^{m-\beta p}\tag{17}$$

holds in the following cases:

(i) $\beta>0$ and one of the following cases holds,

(i.1) $\beta\geqslant 1$, $r\geqslant 0$ and $\beta p^2\leqslant pm$ or $m=(\beta+1)p$,

(i.2) $\beta\geqslant 1$, $r<0$ and $(\beta+1)p^2\leqslant pm$,

(i.3) $\beta\in(0,\ 1)$, $r\geqslant 0$ and $p^2\leqslant pm\leqslant(\beta+1)p^2$,

(i.4) $r<0$, $(\beta\vee 1)p^2\leqslant pm<(\beta+1)p^2$ and $(t-r)\Theta_1\geqslant 1$;

(ii) $\beta\in(-1,\ 0)$ and one of the following cases holds,

(ii.1) $r=0$ and $\beta p^2<pm\leqslant 0$,

(ii.2) $t\geqslant r$ and $pm\leqslant\beta p^2$,

(ii.3) $t>r\neq 0$, $\beta p^2<pm\leqslant 0$ and $(t-r)\Theta_2\geqslant 1$;

(iii) $\beta=-1$ and one of the following cases holds,

(iii.1) $pm \leqslant -p^2$, or $m=0$,

(iii.2) $r \geqslant 0$ and $-p^2 < pm < 0$,

(iii.3) $r<0$, $-p^2 < pm < 0$ and $\dfrac{(m-p)t}{(m+p)r} \geqslant 1$;

(iv) $\beta < -1$ and one of the following cases holds,

(iv.1) $pm \leqslant \beta p^2$, or $m=(\beta+1)p$,

(iv.2) $r \geqslant 0$ and $\beta p^2 < pm < (\beta+1)p^2$,

(iv.3) $r<0$, $\beta p^2 < pm < (\beta+1)p^2$ and $(t-r)\Theta_1 \geqslant 1$.

Here $a \vee b$ means the bigger one of $a,\ b \in \mathbb{R}$.

Proof. We first show (17) in the cases (i.1) (with $\beta p^2 \leqslant pm$), (i.2), (ii.2), (iii.1) and (iv.1) (with $pm \leqslant \beta p^2$) in the first three parts and for other cases in the fourth part.

Part 1 (1) We first consider (i.1) (with $\beta p^2 \leqslant pm$) and (i.2) in this part and prove (17) in the following cases in advance:

(i.1a) $\beta \geqslant 1$, $r \geqslant 0$, $p>0$ and $\beta p \leqslant m$;

(i.2a) $\beta \geqslant 1$, $r<0$, $p>0$ and $(\beta+1)p \leqslant m$.

We can assume that $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$ for writing convenience. Set

$$A_i = \frac{a_i^{m-p}}{(ts-ra_i^p)^\beta}. \tag{18}$$

Then we know that $a_1^p \leqslant a_2^p \leqslant \cdots \leqslant a_n^p$ and $A_1 \leqslant A_2 \leqslant \cdots \leqslant A_n$, since the function $x^{m-p}/(ts-rx^p)^\beta$ is non-decreasing in $x>0$ in each case. By the rearrangement inequality, we have

$$t\sum_{i=1}^n a_i^p A_i \geqslant t\sum_{i=1}^n a_{i+k}^p A_i, \tag{19}$$

for all $k=1,\ \cdots,\ n-1$, and

$$(t-r)\sum_{i=1}^n a_i^p A_i = (t-r)\sum_{i=1}^n a_i^p A_i, \tag{20}$$

where when $i+k>n$, a_{i+k} is taken to be a_{i+k-n}. Adding all inequalities in (19) for each $k=1,\ \cdots,\ n-1$ and (20) up, we obtain

$$(nt-r)\sum_{i=1}^n a_i^p A_i \geqslant \sum_{i=1}^n (ts-ra_i^p)A_i \text{ and } \sum_{i=1}^n \frac{a_i^m}{(ts-ra_i^p)^\beta} \geqslant \frac{1}{nt-r}\sum_{i=1}^n \frac{a_i^{m-p}}{(ts-ra_i^p)^{\beta-1}}. \tag{21}$$

Now when $\beta \geqslant 1$, noticing that for each $r \in \mathbb{R}$,

$$a_1^{m-\beta p} \leqslant a_2^{m-\beta p} \leqslant \cdots \leqslant a_n^{m-\beta p} \tag{22}$$

and

$$\left(\frac{a_1^p}{ts - ra_1^p}\right)^\beta \leqslant \left(\frac{a_2^p}{ts - ra_2^p}\right)^\beta \leqslant \cdots \leqslant \left(\frac{a_n^p}{ts - ra_n^p}\right)^\beta, \tag{23}$$

we can adopt the Chebyshev's inequality and have

$$\sum_{i=1}^n \frac{a_i^m}{(ts - ra_i^p)^\beta} = \sum_{i=1}^n a_i^{m-\beta p}\left(\frac{a_i^p}{ts - ra_i^p}\right)^\beta \geqslant \sum_{i=1}^n a_i^{m-\beta p} \cdot \frac{1}{n}\sum_{i=1}^n \left(\frac{a_i^p}{ts - ra_i^p}\right)^\beta. \tag{24}$$

Since x^β is a convex increasing function, we can use the Jensen's inequality and obtain that

$$\frac{1}{n}\sum_{i=1}^n \left(\frac{a_i^p}{ts - ra_i^p}\right)^\beta \geqslant \left(\frac{1}{n}\sum_{i=1}^n \frac{a_i^p}{ts - ra_i^p}\right)^\beta \geqslant \frac{1}{(nt-r)^\beta}, \tag{25}$$

where we have used (21) by setting $\beta = 1$ and $m = p$. We have actually obtained (17) for these two cases by combining (24) and (25).

(2) We then prove (17) for the cases:

(i.1b) $\beta \geqslant 1$, $r \geqslant 0$, $p < 0$ and $m \leqslant \beta p$; (i.2b) $\beta \geqslant 1$, $r < 0$, $p < 0$ and $m \leqslant (\beta+1)p$.

We also assume that $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$ and set A_i as (18). Then we know

$$a_1^p \geqslant a_2^p \geqslant \cdots \geqslant a_n^p \text{and } A_1 \geqslant A_2 \geqslant \cdots \geqslant A_n, \tag{26}$$

since the functions x^p and $x^{m-p}/(ts - rx^p)^\beta$ are both non-increasing in $x > 0$ in each case. By the rearrangement inequality, we can similarly obtain (19), (20) and (21).

Then if $\beta \in \mathbb{N}^+$, we can similarly deduce (21). If $\beta \geqslant 1$, noticing also that

$$a_1^{m-\beta p} \geqslant a_2^{m-\beta p} \geqslant \cdots \geqslant a_n^{m-\beta p} \tag{27}$$

and

$$\left(\frac{a_1^p}{ts-ra_1^p}\right)^{\beta} \geqslant \left(\frac{a_2^p}{ts-ra_2^p}\right)^{\beta} \geqslant \cdots \geqslant \left(\frac{a_n^p}{ts-ra_n^p}\right)^{\beta}, \tag{28}$$

we can also adopt the Chebyshev's inequality and have (24), (25) and then (17), finally.

Part 2 Now we consider the case (ii.2). We first prove (17) for the case when $\beta \in (-1,\ 0)$, $t \geqslant r$, $p > 0$ and $m \leqslant \beta p$. Similarly, we assume that $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$. Noticing that (27) and (28) still hold for this case, and then we again obtain (24). By the mean value inequality, we can see that

$$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{a_i^p}{ts-ra_i^p}\right)^{\beta} \geqslant \left(\prod_{i=1}^{n}\frac{a_i^p}{ts-ra_i^p}\right)^{\frac{\beta}{n}} = \left(\frac{\prod\limits_{i=1}^{n}a_i^p}{\prod\limits_{i=1}^{n}(ts-ra_i^p)}\right)^{\frac{\beta}{n}} \tag{29}$$

and by (8),

$$\begin{aligned} ts-ra_i^p =& ta_1^p+\cdots+ta_{i-1}^p+(t-r)a_i^p+ta_{i+1}^p+\cdots+ta_n^p \\ \geqslant& (nt-r)a_i^{\frac{p(t-r)}{nt-r}}\prod_{k=1,\ k\neq i}^{n}a_k^{\frac{pt}{nt-r}}. \end{aligned}$$

Hence by (29)

$$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{a_i^p}{ts-ra_i^p}\right)^{\beta} \geqslant \left(\frac{\prod\limits_{i=1}^{n}a_i^p}{(nt-r)^n\prod\limits_{k=1}^{n}a_k^{\frac{p(nt-r)}{nt-r}}}\right)^{\frac{\beta}{n}} = (nt-r)^{-\beta}. \tag{30}$$

and then (17) follows from (30) and (24).

For the case when $\beta \in (-1,\ 0)$, $t \geqslant r$, $p < 0$ and $m \geqslant \beta p$, we can similarly deduce (22), (23) and (24). Then since $\beta < 0$, (29) and (30) are also valid and (17) holds true for this case.

Part 3 We then consider the cases (iii.1) and (iv.1) with $pm \leqslant \beta p^2$. We first consider the case when $\beta \leqslant -1$, $r \in \mathbb{R}$, $p > 0$ and $m \leqslant \beta p$. We similarly assume that $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$ and then for all $r \in \mathbb{R}$, (24), (27) and (28) are valid. Since $x^{-\beta}$ is a convex increasing function, we can use the Jensen's inequality and obtain that

$$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{a_i^p}{ts-ra_i^p}\right)^{\beta}=\frac{1}{n}\sum_{i=1}^{n}\left(t\sum_{k=1}^{n}\frac{a_k^p}{a_i^p}-r\right)^{-\beta}\geqslant\left(\frac{t}{n}\sum_{k=1}^{n}\sum_{i=1}^{n}\frac{a_k^p}{a_i^p}-r\right)^{-\beta}$$

$$\geqslant\left(\frac{t}{n}n^2-r\right)^{-\beta}=\frac{1}{(nt-r)^{\beta}}. \tag{31}$$

Then (17) is proved for all $\beta\leqslant -1$ by combining (24) and (31).

For the case when $\beta\leqslant -1$, $r\in\mathbb{R}$, $p<0$ and $m\geqslant\beta p$, we can similarly obtain (22) and (23). Then with the same argument as above, we can show (17) in this case.

Part 4 We now consider other cases, in which cases we adopt Theorem 3.1 to show (17). Actually, in other cases, it holds that $(m-\beta p)/p\in[0,\ 1]$. This implies that the function $x^{\frac{m}{p}-\beta}$ is concave. Then by Jensen's inequality, we deduce that

$$\left(\sum_{i=1}^{n}a_i^p\right)^{\frac{m}{p}-\beta}\geqslant n^{\frac{m}{p}-1-\beta}\sum_{i=1}^{n}a_i^{m-\beta p}. \tag{32}$$

Next we observe that (i.1) with $m=(\beta+1)p$ satisfies (iii.1) of Theorem 3.1, (i.3) and (iv.2) satisfies (i) or (iii.2) of Theorem 3.1, (i.4) and (iv.3) satisfy the case (iii.3) of Theorem 3.1, (ii.1) satisfies (i) of Theorem 3.1, (ii.3) satisfies (iv) of Theorem 3.1, (iii.1) (with $m=0$) and (iii.2) satisfy (ii.1) of Theorem 3.1, (iii.3) satisfies (ii.2) of Theorem 3.1, and (iv.1) (with $m=(\beta+1)p$) satisfies (iii.1) of Theorem 3.1. Then (17) follows from (32) and the result (10) in these cases. The proof is complete. □

Theorem 3.3 The inequality

$$\sum_{i=1}^{n}\frac{a_i^m}{(ts-ra_i^p)^{\beta}}\leqslant\frac{1}{(nt-r)^{\beta}}\sum_{i=1}^{n}a_i^{m-\beta p} \tag{33}$$

holds in the following cases:

(i) $\beta>0$ and either

(i.1) $r\leqslant 0$ and $pm\leqslant(\beta\wedge 1)p^2$, or

(i.2) $t>r>0$, $0\leqslant pm\leqslant(\beta\wedge 1)p^2$ and $(t-r)\Theta_2\geqslant 1$;

(ii) $\beta\in(-1,0)$ and one of the following cases holds,

(ii.1) $m=(\beta+1)p$,

(ii.2) $r>0$ and $(\beta+1)p^2<pm\leqslant p^2$,

(ii.3) $r<0$, $(\beta+1)p^2<pm\leqslant p^2$ and $(t-r)\Theta_1\geqslant 1$;

(iii) $\beta\leqslant -1$ and one of the following cases holds, (iii.1) $\beta = -1$, $r > 0$ and $0 \leqslant pm \leqslant p^2$,

(iii.2) $r = 0$ and $0 \leqslant pm \leqslant p^2$,

(iii.3) $\beta \in \mathbb{Z} \backslash \mathbb{N}$, $r < 0$ and $pm \geqslant 0$,

(iii.4) $\beta < -1$, $t > r > 0$, $0 \leqslant pm \leqslant p^2$ and $(t-r)\Theta_2 \geqslant 1$.

Here $a \wedge b$ means the smaller one of $a,\ b \in \mathbb{R}$.

Proof. We split the proof into three parts.

Part 1 We first show (33) for (i.1) and the case when $\beta \geqslant 1$, $r \leqslant 0$, $p > 0$ and $p \geqslant m$. In this case, the function x^p is non-decreasing and $x^{m-p}/(ts - rx^p)^\beta$ is non-increasing. Hence by setting $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$ and (18), we have

$$a_1^p \leqslant a_2^p \leqslant \cdots \leqslant a_n^p \text{ and } A_1 \geqslant A_2 \geqslant \cdots \geqslant A_n. \tag{34}$$

By the rearrangement inequality, we obtain

$$t\sum_{i=1}^{n} a_i^p A_i \leqslant t\sum_{i=1}^{n} a_{i+k}^p A_i, \tag{35}$$

for all $k = 1,\ \cdots,\ n-1$, and

$$(t-r)\sum_{i=1}^{n} a_i^p A_i = (t-r)\sum_{i=1}^{n} a_i^p A_i, \tag{36}$$

where it is also taken that $a_{i+k} = a_{i+k-n}$, when $i+k > n$. Adding all inequalities in (35) for each $k = 1,\ \cdots,\ n-1$ and (36) up, we can further obtain

$$\sum_{i=1}^{n} \frac{a_i^m}{(ts - ra_i^p)^\beta} \leqslant \frac{1}{nt-r}\sum_{i=1}^{n} \frac{a_i^{m-p}}{(ts - ra_i^p)^{\beta-1}}. \tag{37}$$

Next we consider $\beta \in [0,\ 1)$, $r \leqslant 0$, $p > 0$ and $\beta p \geqslant m$. Since $x^{m-\beta p}$ is non-increasing and $x/(ts - rx)$ is non-decreasing, we have

$$a_1^{m-\beta p} \geqslant a_2^{m-\beta p} \geqslant \cdots \geqslant a_n^{m-\beta p} \tag{38}$$

and

$$\left(\frac{a_1^p}{ts-ra_1^p}\right)^\beta \leqslant \left(\frac{a_2^p}{ts-ra_2^p}\right)^\beta \leqslant \cdots \leqslant \left(\frac{a_n^p}{ts-ra_n^p}\right)^\beta. \tag{39}$$

By the Chebyshev's inequality, we obtain

$$\sum_{i=1}^n \frac{a_i^m}{(ts-ra_i^p)^\beta} \leqslant \sum_{i=1}^n a_i^{m-\beta p} \cdot \frac{1}{n} \sum_{i=1}^n \left(\frac{a_i^p}{ts-ra_i^p}\right)^\beta. \tag{40}$$

By Jensen's inequality, we have

$$\frac{1}{n} \sum_{i=1}^n \left(\frac{a_i^p}{ts-ra_i^p}\right)^\beta \leqslant \left(\frac{1}{n} \sum_{i=1}^n \frac{a_i^p}{ts-ra_i^p}\right)^\beta \leqslant \frac{1}{(nt-r)^\beta}, \tag{41}$$

where we used (37) with $\beta = 1$ and $m = p$. Then (33) follows from (40) and (41).

Now we consider $\beta \geqslant 1$, $r \leqslant 0$, $p > 0$ and $p \geqslant m$. One can see by (37) that

$$\sum_{i=1}^n \frac{a_i^m}{(ts-ra_i^p)^\beta} \leqslant \frac{1}{nt-r} \sum_{i=1}^n \frac{a_i^{m-p}}{(ts-ra_i^p)^{\beta-1}} \leqslant \cdots \leqslant \frac{1}{(nt-r)^{\lfloor\beta\rfloor}} \sum_{i=1}^n \frac{a_i^{m-\lfloor\beta\rfloor p}}{(ts-ra_i^p)^{\beta-\lfloor\beta\rfloor}}, \tag{42}$$

where $\lfloor\beta\rfloor$ is the largest integer no more than β. Noting that $0 \leqslant \beta - \lfloor\beta\rfloor < 1$, we infer from (33) for the case (i.1) with $\beta \in [0, 1)$ that

$$\frac{1}{(nt-r)^{\lfloor\beta\rfloor}} \sum_{i=1}^n \frac{a_i^{m-\lfloor\beta\rfloor p}}{(ts-ra_i^p)^{\beta-\lfloor\beta\rfloor}} \leqslant \frac{1}{(nt-r)^\beta} \sum_{i=1}^n a_i^{m-\beta p}, \tag{43}$$

which is exactly (33) in this case.

Hereafter we consider the case when $\beta > 0$, $r \leqslant 0$, $p < 0$ and $\beta p \leqslant m$ and prove (33) for $\beta \geqslant 1$ first. In this case the function x^p is non-increasing and $x^{m-p}/(ts-rx^p)^\beta$ is non-decreasing. Similarly by setting $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$ and (18),

$$a_1^p \geqslant a_2^p \geqslant \cdots \geqslant a_n^p \quad \text{and} \quad A_1 \leqslant A_2 \leqslant \cdots \leqslant A_n, \tag{44}$$

which implies (37) in this case. Then for $\beta \in [0, 1)$, $r \leqslant 0$, $p < 0$ and $\beta p \leqslant m$, we have

$$a_1^{m-\beta p} \leqslant a_2^{m-\beta p} \leqslant \cdots \leqslant a_n^{m-\beta p} \tag{45}$$

and

$$\left(\frac{a_1^p}{ts-ra_1^p}\right)^\beta \geqslant \left(\frac{a_2^p}{ts-ra_2^p}\right)^\beta \geqslant \cdots \geqslant \left(\frac{a_n^p}{ts-ra_n^p}\right)^\beta, \tag{46}$$

and (41) and (42) are obtained. Hence (33) is proved. For $\beta \geqslant 1$, $r \leqslant 0$, $p < 0$ and $p \leqslant m$, (33) can be deduced by similar argument.

Part 2 Next we prove (33) for other cases except (iii.3), each of which satisfies

$$\frac{m-\beta p}{p} \in (-\infty,\ 0] \cup [1,\ +\infty). \tag{47}$$

For these cases, we need to employ the results from Theorem 3.1. Note that as long as these cases satisfy the cases in (2) of Theorem 3.1 and (47), which implies

$$\left(\sum_{i=1}^n a_i^p\right)^{\frac{m}{p}-\beta} \leqslant n^{\frac{m}{p}-1-\beta} \sum_{i=1}^n a_i^{m-\beta p}, \tag{48}$$

the inequality (33) follows directly from (48).

Actually, it is not hard to check that (i.2) and (iii.4) satisfy (viii) of Theorem 3.1, (ii.1) satisfies (vii.1) of Theorem 3.1, (ii.2) satisfies (vii.2) of Theorem 3.1, (ii.3) satisfies (vii.3) of Theorem 3.1, (iii.1) satisfies (vi.1) of Theorem 3.1 and (iii.2) satisfies (v) of Theorem 3.1. The proof of Part 2 ends here thereby.

Part 3 At last we show (33) in the case (iii.3). We consider the case when $\beta \leqslant -1$, $r < 0$ and $m,\ p \geqslant 0$ in advance. Indeed, in this case, we can also see that the functions x^p and $x^m/(ts-rx^p)^{\beta+1}$ are both non-decreasing in $x > 0$. Then similar to the argument from (18) to (21), we have

$$a_1^p \leqslant a_2^p \leqslant \cdots \leqslant a_n^p \tag{49}$$

and

$$\frac{a_1^m}{(ts-ra_1^p)^{\beta+1}} \leqslant \frac{a_2^m}{(ts-ra_2^p)^{\beta+1}} \leqslant \cdots \leqslant \frac{a_n^m}{(ts-ra_n^p)^{\beta+1}}. \tag{50}$$

Analogously, we obtain

$$(nt-r)\sum_{i=1}^{n}\frac{a_i^{m+p}}{(ts-ra_i^p)^{\beta+1}} \geqslant \sum_{i=1}^{n}\frac{a_i^m}{(ts-ra_i^p)^{\beta}},$$

i.e.,

$$\sum_{i=1}^{n}\frac{a_i^m}{(ts-ra_i^p)^{\beta}} \leqslant \frac{1}{(nt-r)^{-1}}\sum_{i=1}^{n}\frac{a_i^{m+p}}{(ts-ra_i^p)^{\beta+1}}. \tag{51}$$

Then if $\beta \leqslant -1$,

$$\sum_{i=1}^{n}\frac{a_i^m}{(ts-ra_i^p)^{\beta}} \leqslant \cdots \leqslant \frac{1}{(nt-r)^{\lceil\beta\rceil}}\sum_{i=1}^{n}\frac{a_i^{m-\lceil\beta\rceil p}}{(ts-ra_i^p)^{\beta-\lceil\beta\rceil}}, \tag{52}$$

where $\lceil\beta\rceil$ means the smallest integer no less than β. Therefore, when $\beta \in \mathbb{Z}\backslash\mathbb{N}$, (52) is exactly (33).

For the case when $\beta \in \mathbb{Z}\backslash\mathbb{N}$, $r<0$ and $m,\ p \leqslant 0$, the function x^p and $x^m/(ts-rx^p)^{\beta+1}$ are both non-increasing in $x>0$. Hence (49) and (50) hold with all $\leqslant$'s replaced by $\geqslant$'s. Then (51) and (52) are also valid. Finally, we can similarly deduce (33) in this case. The proof is complete now. □

Remark 3.4 In the proofs of Theorems 3.2 and 3.3, we also obtain some interesting inequalities (21) and (51) in corresponding cases presented therein. These inequalities can also be viewed as generalizations of those obtained in [22].

3.2 *Other cases*

Up to now we have proved the main theorems, but there are still other cases which can guarantee the inequalities (10), (11), (17) and (33). For example, when $t>r\neq 0$ and one of the followings holds:

(A) $\beta \in (-\infty,\ -1)\cup(0,\ +\infty)$, $mrp\beta<0$, $m(m-p)>0$, $\beta m[m-(\beta+1)p]<0$ and $(t-r)\Theta_1<1\leqslant(t-r)\Theta_2$,

(B) $\beta \in (-1,\ 0)$, $pm \geqslant p^2$ or $\beta p^2 < pm \leqslant 0$ and $0<(t-r)\Theta_2<1$,

(C) $\beta \in (-1,\ 0)$, $r<0$, $(\beta+1)p^2 < pm \leqslant p^2$ and $0<(t-r)\Theta_1<1\leqslant(t-r)\Theta_2$,

(D) $\beta \in (-\infty,\ -1)\cup(0,\ +\infty)$, $m(m-p)\leqslant 0$, $\beta m[m-(\beta+1)p]<0$ and $0<(t-r)\Theta_2<1$,

$f(x)$ can not stay non-positive or non-negative on the whole interval $[0,\ T]$. As a result, we can not directly use Jensen's inequality, but the Semiconcave-semiconvex Theorem brings us some hope.

However, as n increases or the parameters $\beta,\ t,\ r,\ m,\ p$ take more general values, the difficulty also increases greatly. Therefore, we only present some concrete examples for these cases as follows.

Example 3.1 Under the conditions $r<0$ and (B), we let $n=4$, $m=\beta=-1/2$, $p=2$, $t=1$ and $r=-3$. Then (10) and (17) hold, i.e.,

$$\sum_{i=1}^{4}\left(\frac{s+3a_i^2}{a_i}\right)^{\frac{1}{2}} \geqslant 2\sqrt{14}s^{\frac{1}{4}} \geqslant \sqrt{7}\sum_{i=1}^{4}a_i^{\frac{1}{2}}, \tag{53}$$

where $s=a_1^2+a_2^2+a_3^2+a_4^2$.

Proof. Since $\dfrac{m-\beta p}{p}=0.5\in(0,\ 1)$, we can use Jensen's inequality and obtain the second inequality of (53). In the following, we only consider the first inequality of (53).

According to Theorem 3.1, we know that

$$(t-r)\Theta_2=\frac{2(\sqrt{6}-1)}{3}\in(0,\ 1).$$

And hence $g(x)$ is convex on $\left(0,\ \dfrac{\Theta_2 s}{1-3\Theta_2}\right]$ and concave on $\left[\dfrac{\Theta_2 s}{1-3\Theta_2},\ s\right)$. We take arbitrarily a positive $\varepsilon<\min\left\{\dfrac{\Theta_2}{1-3\Theta_2},\ 1-\dfrac{\Theta_2}{1-3\Theta_2}\right\}$ and set $a_1\leqslant a_2\leqslant a_3\leqslant a_4$. Denote the left hand side of (53) by $F(a_1,\ a_2,\ a_3,\ a_4)$ with $a_1,\ a_2,\ a_3,\ a_4\in[\varepsilon\sqrt{s},\ (1-\varepsilon)\sqrt{s}]$. By Theorem 2.2, we know that $F(a_1,\ a_2,\ a_3,\ a_4)$ achieves its possible minimum in four cases in the following.

The first case is that $a_1^2=a_2^2=a_3^2=x^2$ and $a_4^2=s-3x^2\geqslant x^2$ with $x\in[\varepsilon\sqrt{s},\ \sqrt{s}/2]$. Let $\xi=x^2/s\in[\varepsilon^2,\ 1/4]$ and

$$h(\xi):=\frac{1}{\sqrt[4]{s}}F(a_1,\ a_2,\ a_3,\ a_4)=\frac{3\sqrt{s+3x^2}}{\sqrt[4]{sx^2}}+\frac{\sqrt{4s-9x^2}}{\sqrt[4]{s(s-3x^2)}}=\frac{3(1+3\xi)^{\frac{1}{2}}}{\xi^{\frac{1}{4}}}+\frac{(4-9\xi)^{\frac{1}{2}}}{(1-3\xi)^{\frac{1}{4}}}.$$

Then

$$h'(\xi)=-\frac{3}{4}(1+3\xi)^{-\frac{1}{2}}(1-3\xi)^{-\frac{1}{4}}\left[\left(\frac{1-3\xi}{\xi}\right)^{\frac{5}{4}}+\frac{2-9\xi}{1-3\xi}\left(\frac{1+3\xi}{4-9\xi}\right)^{\frac{1}{2}}\right].$$

Let $\eta=(1-3\xi)/\xi\in[1,\ +\infty)$ and

$$\tilde{h}(\eta)=\left(\frac{1-3\xi}{\xi}\right)^{\frac{5}{4}}+\frac{2-9\xi}{1-3\xi}\left(\frac{1+3\xi}{4-9\xi}\right)^{\frac{1}{2}}=\eta^{\frac{5}{4}}+\frac{2\eta-3}{\eta}\left(\frac{\eta+6}{4\eta+3}\right)^{\frac{1}{2}}.$$

It is easy to see that $\tilde{h}(\eta) > 0$ when $\eta \in [3/2,\ +\infty)$. When $\eta \in [1,\ 3/2]$, we see that

$$\eta^{\frac{5}{2}} - \left(\frac{2\eta-3}{\eta}\right)^2 \frac{\eta+6}{4\eta+3} = \frac{1}{\eta^2(4\eta+3)}\left[(\eta-1)\left(7\eta^4+7\eta^3+3\eta^2+9(6-\eta)\right)+\eta^{\frac{9}{2}}(\eta^{\frac{1}{2}}-1)(4\eta^{\frac{1}{2}}-3)\right] \geqslant 0,$$

which implies that $\tilde{h}(\eta) \geqslant 0$ for all $\eta \in [1,\ +\infty)$ and $h'(\xi) \leqslant 0$ for all $\xi \in [\varepsilon^2,\ 1/4]$. Hence

$$h(\xi) \geqslant h\left(\frac{1}{4}\right) = 2\sqrt{14}.$$

The second case is $a_1^2 = a_2^2$, $a_4^2 = (1-\varepsilon)^2 s$. Since

$$a_1^2 < \frac{1}{2}(s - a_4^2) = \varepsilon s - \frac{1}{2}\varepsilon s^2 < \varepsilon s,$$

we know

$$\frac{1}{\sqrt[4]{s}} F(a_1,\ a_2,\ a_3,\ a_4) > \frac{1}{\sqrt[4]{s}}\left(\frac{s+3a_1^2}{a_1}\right)^{\frac{1}{2}} > \frac{1}{\sqrt[4]{s}}\left(\frac{s}{a_1}\right)^{\frac{1}{2}} > \varepsilon^{-\frac{1}{4}} > 2\sqrt{14},$$

if ε is sufficiently small. Thus, we fix a sufficiently small $\varepsilon_0 \in (0,\ 1/5)$ such that

$$F(a_1,\ a_2,\ a_3,\ (1-\varepsilon_0)\sqrt{s}) > 2\sqrt{14}s^{\frac{1}{4}}.$$

The third case is $a_3 = a_4 = (1-\varepsilon_0)\sqrt{s}$. In this case $a_3^2 + a_4^2 = 2(1-\varepsilon_0)^2 s > s$, which is impossible. The fourth case is $a_2 = a_3 = a_4 = (1-\varepsilon_0)\sqrt{s}$, which can be excluded either. Eventually, we have proved (53) now. □

Example 3.2 Under the conditions $r < 0$ and (C), we let $n = 4$, $\beta = -2/3$, $m = 2/3$, $p = t = 1$ and $r = -1$. Then (11) and (33) hold, i.e.,

$$\sum_{i=1}^{4}\left[(s+a_i)a_i\right]^{\frac{2}{3}} \leqslant \sqrt[3]{\frac{25s^4}{4}} \leqslant \sqrt[3]{25}\sum_{i=1}^{4} a_i^{\frac{4}{3}}, \tag{54}$$

where $s = a_1 + a_2 + a_3 + a_4$.

Proof. The second inequality of (54) is obviously correct. In this case, we have

$$(t-r)\Theta_1 = 2(2-\sqrt{3}) \in (0,\ 1).$$

Hence $g(x)$ is concave on $\left(0,\ \frac{\Theta_1 s}{1-\Theta_1}\right]$ and convex on $\left[\frac{\Theta_1 s}{1-\Theta_1},\ s\right)$. Set $a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$. Denote the left hand side of (54) by $F(a_1,\ a_2,\ a_3,\ a_4)$ with $a_1,\ a_2,\ a_3,\ a_4 \in [0,\ s]$. By Theorem 2.2, we know that $F(a_1,\ a_2,\ a_3,\ a_4)$ achieves its possible maximum in four cases in the following.

The first case is $a_1 = a_2 = a_3 = x$ and $a_4 = s-3x \geqslant x$ with $x \in [0,\ s/4]$. Let $\xi = x/s \in [0,\ 1/4]$ and

$$h(\xi) = s^{-\frac{4}{3}} F(a_1,\ a_2,\ a_3,\ a_4) = 3[(1+\xi)\xi]^{\frac{2}{3}} + [(2-3\xi)(1-3\xi)]^{\frac{2}{3}}.$$

Then

$$h'(\xi) = 2(1+\xi)^{-\frac{1}{3}}(1-3\xi)^{-\frac{1}{3}}(2\xi+1)\left[\left(\frac{1-3\xi}{\xi}\right)^{\frac{1}{3}} + \frac{3(2\xi-1)}{2\xi+1}\left(\frac{1+\xi}{2-3\xi}\right)^{\frac{1}{3}}\right].$$

Still we let $\eta = (1-3\xi)/\xi \in [1,\ +\infty)$ and

$$\tilde{h}(\eta) = \left(\frac{1-3\xi}{\xi}\right)^{\frac{1}{3}} + \frac{3(2\xi-1)}{2\xi+1}\left(\frac{1+\xi}{2-3\xi}\right)^{\frac{1}{3}} = \eta^{\frac{1}{3}} - \frac{3(\eta+1)}{\eta+5}\left(\frac{\eta+4}{2\eta+3}\right)^{\frac{1}{3}}.$$

Since

$$\eta - \frac{27(\eta+4)}{2\eta+3}\left(\frac{\eta+1}{\eta+5}\right)^3 = \frac{2(\eta-1)(\eta^4+4\eta^3+7\eta^2+42\eta+54)}{(2\eta+3)(\eta+5)^3} \geqslant 0,$$

we can conclude that $h'(\xi) > 0$ on $(0,\ 1/4]$ and $h(\xi)$ is strictly increasing on $[0,\ 1/4]$, which implies that

$$h(\xi) \leqslant h\left(\frac{1}{4}\right) = \sqrt[3]{\frac{25}{4}}.$$

The second case is $a_1 = a_2$ and $a_4 = s$, in which case $a_1 = a_2 = a_3 = 0$ and

$$F(a_1,\ a_2,\ a_3,\ a_4) = F(0,\ 0\ ,0,\ s) = 2^{\frac{2}{3}} s^{\frac{4}{3}} < \sqrt[3]{\frac{25s^4}{4}}.$$

The third case is $a_3 = a_4 = s$, and the fourth case is $a_2 = a_3 = a_4 = s$. Both of the cases are impossible. As a result, we conclude (54). □

Example 3.3 Under the conditions $t > r > 0$ and (D), we let $n = 3$, $m = \beta \in (0,\ 1)$, $p = r = 1$ and $t = 2$. Then there is $\beta_0 \in (0.5,\ 1)$ such that when $\beta \in (0,\ \beta_0)$, (11) holds, i.e.,

$$\left(\frac{a_1}{a_1+2a_2+2a_3}\right)^\beta + \left(\frac{a_2}{2a_1+a_2+2a_3}\right)^\beta + \left(\frac{a_3}{2a_1+2a_2+a_3}\right)^\beta \leqslant \frac{3}{5^\beta}. \tag{55}$$

Proof. Following the proof of Theorem 3.1, we know that $\Theta_2 = (1-\beta)/(1+\beta)$ and hence $g(x)$ is concave on $(0,\ (1-\beta)s]$ and convex on $[(1-\beta)s,\ s)$. We can as well set $a_1 \leqslant a_2 \leqslant a_3$. Then by Theorem 2.2, we know the left hand side of (60), denoted by $F(a_1,\ a_2,\ a_3)$ for writing convenience, achieves its possible maximum in three cases as follows.

The first case is $a_1 = a_2 = x$ and $a_3 = s - 2x \geqslant x$. Then $x \in [0,\ s/3]$ and

$$F(a_1,\ a_2,\ a_3) = 2\left(\frac{x}{2s-x}\right)^\beta + \left(\frac{s-2x}{s+2x}\right)^\beta.$$

We let $\xi = x/(2s-x) \in [0,\ 1/5]$ and

$$h(\xi) = F(a_1,\ a_2,\ a_3) = 2\xi^\beta + \left(\frac{1-3\xi}{1+5\xi}\right)^\beta,$$

$$h'(\xi) = \frac{2\beta}{\xi^{1-\beta}(1+5\xi)^{1+\beta}}\left[(1+5\xi)^{1+\beta} - 4\left(\frac{\xi}{1-3\xi}\right)^{1-\beta}\right].$$

Now let $\eta = 1+5\xi \in [1,\ 2]$ and we can see that

$$(1+5\xi)^{\frac{1+\beta}{1-\beta}} - \frac{4^{\frac{1}{1-\beta}}\xi}{1-3\xi} = \frac{3\eta^{\frac{2}{1-\beta}} - 8\eta^{\frac{1+\beta}{1-\beta}} + 4^{\frac{1}{1-\beta}}\eta - 4^{\frac{1}{1-\beta}}}{3\eta - 8}.$$

Setting

$$\tilde{h}(\eta) = 3\eta^{\frac{2}{1-\beta}} - 8\eta^{\frac{1+\beta}{1-\beta}} + 4^{\frac{1}{1-\beta}}\eta - 4^{\frac{1}{1-\beta}},$$

we have $\tilde{h}(1) = -5$, $\tilde{h}(2) = 0$,

$$\tilde{h}'(\eta) = \frac{6}{1-\beta}\eta^{\frac{1+\beta}{1-\beta}} - \frac{8(1+\beta)}{1-\beta}\eta^{\frac{2\beta}{1-\beta}} + 4^{\frac{1}{1-\beta}},$$

$$\tilde{h}''(\eta) = \frac{6(1+\beta)}{(1-\beta)^2}\eta^{\frac{2\beta}{1-\beta}} - \frac{16\beta(1+\beta)}{(1-\beta)^2}\eta^{\frac{3\beta-1}{1-\beta}} = \frac{2(1+\beta)}{(1-\beta)^2}\eta^{\frac{3\beta-1}{1-\beta}}(3\eta - 8\beta), \tag{56}$$

$$\tilde{h}'(1) = 4^{\frac{1}{1-\beta}} - \frac{2(1+4\beta)}{1-\beta},\ \ \tilde{h}'(2) = \frac{2-3\beta}{1-\beta}\cdot 2^{\frac{2}{1-\beta}}.$$

We can see from (56) that only when $\beta \in (3/8,\ 3/4)$, $h'(\eta)$ can reach its least value in $(1,\ 2)$, i.e.,

$$\min_{\eta\in[1,\ 2]}\tilde{h}'(\eta) = \tilde{h}'\left(\frac{8\beta}{3}\right) = 2^{\frac{2}{1-\beta}} - \frac{9}{8\beta^2}\left(\frac{8\beta}{3}\right)^{\frac{2}{1-\beta}}. \tag{57}$$

Actually, in this process, we have to require $\tilde{h}'(\eta) \geqslant 0$ on $[1,\ 2]$, which implies $\tilde{h}$ is non-decreasing on $[1,\ 2]$ and so is $h'(\xi)$. With these results, it yields that

$$\max_{\xi\in[1,\ 1/5]} h(\xi) = h\left(\frac{1}{5}\right) = \frac{3}{5^\beta}.$$

To this end, we only need to require

$$\tilde{h}'(1) \geqslant 0 \text{ when } \beta \in (0,\ 3/8],\ \tilde{h}'(2) \geqslant 0 \text{ when } \beta \in [3/4,\ 1) \tag{58}$$

$$\text{and } \tilde{h}'\left(\frac{8\beta}{3}\right) \geqslant 0,\ \ \text{when } \beta \in (3/8,\ 3/4). \tag{59}$$

It is not hard to deduce from (58) that $\beta \in (0,\ 3/8]$. From (57) and (59), we get

$$\frac{8\beta^2}{9} \geqslant \left(\frac{4\beta}{3}\right)^{\frac{2}{1-\beta}} \Leftrightarrow \frac{1}{2} \geqslant \left(\frac{4\beta}{3}\right)^{\frac{2\beta}{1-\beta}} \Leftrightarrow 2\log_2 \frac{3}{\beta} \geqslant \frac{1}{\beta} + 3.$$

Let $\alpha = 1/\beta \in (4/3,\ 8/3)$ and

$$j(\alpha) = 2\log_2(3\alpha) - \alpha - 3 \text{ and so } j'(\alpha) = \frac{2}{\alpha \ln 2} - 1 > 0.$$

Since $j(4/3) = -1/3$ and $j(8/3) = 1/3$, we can see that there is a unique $\alpha_0 \in (4/3,\ 8/3)$ such that $j(\alpha_0) = 0$ and $j(\alpha) \geqslant 0$ when $\alpha \in (\alpha_0,\ 8/3)$. By calculation using computers, we find

$$\beta_0 = 1/\alpha_0 = 0.5887287\cdots \in (0.5,\ 1).$$

And hence it follows from (59) that $\beta \in (3/8,\ \beta_0)$. As a result, we deduce that in this case when $\beta \in (0,\ \beta_0]$, (55) holds true.

The second case is $a_3 = s$. Then $a_1 = a_2 = 0$ and $F(0,\ 0,\ s) = 1 < 3/5^{\beta_0}$. The third case is $a_2 = a_3 = s$, which is impossible. Consequently, we have obtained (55). □

In the example above, it has been proved that although the parameters satisfy the condition (D), the conclusions of Theorems 3.1, 3.2 and 3.3 need not always hold. Actually, under the cases (A), (B), (C) and (D), it is still possible that none of the inequalities (10), (11), (17) and (33) is valid. The following example gives us a counterexample.

Example 3.4 Under the conditions $r < 0$ and (A), we let $n = 3$, $m = \beta = 3/2$, $p = t = 1$, $r = -1$ and

$$F(a_1,\ a_2,\ a_3) = \left(\frac{a_1}{2a_1 + a_2 + a_3}\right)^{\frac{3}{2}} + \left(\frac{a_2}{a_1 + 2a_2 + a_3}\right)^{\frac{3}{2}} + \left(\frac{a_3}{a_1 + a_2 + 2a_3}\right)^{\frac{3}{2}},$$

in which case $(t - r)\Theta_1 = 2/5 < 1 < 2 = (t - r)\Theta_2$. Then the right hand sides of (10), (11), (17) and (33) are the same, i.e., $3/8$. However, we observe that

$$\lim_{x \to 0^+} F(x,\ x,\ 1) = F(0,\ 0,\ 1) = \frac{1}{2\sqrt{2}} < \frac{3}{8} \text{ and } \lim_{x \to 0^+} F(x,\ 1,\ 1) = F(0,\ 1,\ 1) = \frac{2}{3\sqrt{3}} > \frac{3}{8}.$$

This situation thereby conflicts with all the inequalities (10), (11), (17) and (33).

4. Applications on inequality questions

We employ the theorems in Section 3 to prove some interesting examples and some mathematical competition questions in this section.

4.1 *Extensions on some inequalities*

The first example is a dimensional generalization of Example 7.19 of [38]. This consequence also includes the result of Corollary 2.2 of [13].

Example 4.1 Suppose that $a_1, \cdots a_n > 0$, $n \in \mathbb{N}^+$ and $n \geqslant 2$. Let

$$S_\beta = S_\beta(a_1, \cdots, a_n) := \sum_{i=1}^{n} \left(\frac{a_i}{s - a_i} \right)^\beta$$

and

$$\beta_n = \frac{\ln n - \ln(n-1)}{\ln(n-1) - \ln(n-2)} \text{ for } n > 2. \tag{60}$$

where $s = a_1 + \cdots + a_n$. Then β_n is increasing in n and

$$\inf S_\beta(a_1, \cdots, a_n) = \begin{cases} 2, & \beta \in (0, \beta_3); \\ \dfrac{k}{(k-1)^\beta}, & \beta \in [\beta_k, \beta_{k+1}); \; k = 3, \cdots, n-1; \\ \dfrac{n}{(n-1)^\beta}, & \beta \in (-\infty, 0] \cup [\beta_n, +\infty). \end{cases} \tag{61}$$

Proof. First by definition (61), using Cauchy mean value theorem, we know that for each $n > 2$, there is $\theta_n \in (0, 1)$ such that

$$\beta_n = \frac{1/(n-1+\theta_n)}{1/(n-2+\theta_n)} = \frac{n-2+\theta_n}{n-1+\theta_n}.$$

Hence β_n is increasing as n increases. Moreover,

$$\lim_{n \to 2^+} \beta_n = 0 \text{ and } \lim_{n \to +\infty} \beta_n = 1,$$

and

$$\frac{n-1}{(n-2)^{\beta_n}} = \frac{n}{(n-1)^{\beta_n}} \text{ and } \beta_n > \frac{n-2}{n}.$$

And hence

$$\text{when } \beta \in [\beta_n, 1), \ \frac{n-1}{(n-2)^{\beta}} \geqslant \frac{n}{(n-1)^{\beta}}; \tag{62}$$

$$\text{when } \beta \in (0, \beta_n), \ \frac{n-1}{(n-2)^{\beta}} < \frac{n}{(n-1)^{\beta}}. \tag{63}$$

Next we show (61). In accord with Theorem 3.1, the conditions presented in this example is $p = t = r = 1$ and $m = \beta$. Then by the cases (ii.1) and (iii.2) of Theorem 3.2 and (ii.2) of Theorem 3.2, we know that if $\beta \geqslant 1$ or $\beta \leqslant 0$, (61) can be deduced (it is obvious when $\beta = 0$). Next we only consider the case when $\beta \in (0, 1)$.

Following the proof of Theorem 3.1, we know that

$$g''(x) = \frac{\beta s x^{\beta-2}}{(s-x)^{\beta+2}} [2x - (1-\beta)s], \ x \in [0, s].$$

This means that $g(x) = (x/(s-x))^{\beta}$ is concave on $[0, (1-\beta)s/2]$ and convex on $[(1-\beta)s/2, s]$. Let a_k be non-decreasing when k increases. By Theorem 2.2, we pick one arbitrary possible minimum point of $S_{\beta}(a_1, \cdots, a_n)$ (Here we allow $a_k = 0$ for $k \leqslant n-2$) such that

$$a_1 = \cdots = a_{n-k} = 0 < a_{n-k+1} = s - (k-1)x \leqslant a_{n-k+2} = \cdots = a_n = x,$$

with $k \geqslant 2$. It is easy to see that $x \in [s/k, s/(k-1))$. We let $\xi = x/s \in [1/k, 1/(k-1))$ and

$$h(\xi) = S_{\beta}(a_1, \cdots, a_n) = \left(\frac{1-(k-1)\xi}{(k-1)\xi}\right)^{\beta} + (k-1)\left(\frac{\xi}{1-\xi}\right)^{\beta}.$$

We know that

$$\text{if } k = 2, \ h(\xi) \geqslant 2 \text{ and } h(\xi) = 2 \text{ if and only if } \xi = \frac{1}{2}. \tag{64}$$

In the following, we consider the case when $k \geqslant 3$. Let $\eta = \dfrac{1-(k-1)\xi}{(k-1)\xi} \in \left(0,\ \dfrac{1}{k-1}\right]$ and

$$i(\eta) := h(\xi) = \eta^{\beta} + \frac{k-1}{[(k-1)\eta + k-2]^{\beta}}.$$

Then

$$i'(\eta) = \frac{\beta}{[(k-1)\eta + k-2]^{\beta+1}} \left[[(k-1)\eta + k-2]^{\beta+1}\eta^{\beta-1} - (k-1)^2\right].$$

Let

$$j(\eta) := [(k-1)\eta + k-2]^{\beta+1}\eta^{\beta-1} - (k-1)^2$$

and then

$$j'(\eta) = [(k-1)\eta + k-2]^{\beta}\eta^{\beta-2}[2\beta(k-1)\eta + (\beta-1)(k-2)].$$

Setting $\eta_0 = \dfrac{(1-\beta)(k-2)}{2\beta(k-1)}$, we split it into two situations for discussing.
If $\eta_0 \geqslant \dfrac{1}{k-1}$, i.e., $\beta \in \left(0,\ \dfrac{k-2}{k}\right]$, then

$$j'(\eta) \leqslant 0,\ j(\eta) \geqslant j\left(\frac{1}{k-1}\right) = 0,\ i'(\eta) \geqslant 0$$

and

$$h(\xi) = i(\eta) > \lim_{\eta \to 0^+} i(\eta) = \frac{k-1}{(k-2)^{\beta}}. \tag{65}$$

If $\eta_0 \in \left(0,\ \dfrac{1}{k-1}\right)$, i.e., $\beta \in \left(\dfrac{k-2}{k},\ 1\right)$, then

$$j'(\eta) < 0 \ \text{ as } \ \eta \in (0,\ \eta_0),\ j'(\eta_0) = 0,\ j'(\eta) > 0 \text{ as } \eta \in \left(\eta_0,\ \frac{1}{k-1}\right).$$

Then j reaches its minimum at $\eta = \eta_0$. Noting that

$$\lim_{\eta \to 0^+} j(\eta) = +\infty \text{ and } j\left(\frac{1}{k-1}\right) = 0,$$

we know there exists $\eta_1 \in (0,\ \eta_0)$ such that

$$j(\eta) > 0 \ \text{ as } \ \eta \in (0,\ \eta_1),\ j(\eta_1) = 0,\ j(\eta) < 0 \ \text{ as } \ \eta \in \left(\eta_1,\ \frac{1}{k-1}\right).$$

This also means that

$$i'(\eta) > 0 \ \text{ as } \ \eta \in (0,\ \eta_1),\ i'(\eta_1) = i'\left(\frac{1}{k-1}\right) = 0,\ i'(\eta) < 0 \ \text{ as } \ \eta \in \left(\eta_1,\ \frac{1}{k-1}\right).$$

Hence i is increasing on $(0,\ \eta_1]$ and decreasing on $\left[\eta_1,\ \frac{1}{k-1}\right]$. Therefore, recalling (65), we obtain for all $\beta \in (0,\ 1)$,

$$S_\beta(a_1,\ \cdots,\ a_n) \geqslant \min\left\{\lim_{\eta \to 0^+} i(\eta),\ i\left(\frac{1}{k-1}\right)\right\} = \min\left\{\frac{k-1}{(k-2)^\beta},\ \frac{k}{(k-1)^\beta}\right\}. \tag{66}$$

Now we consider $n \geqslant 2$ and combine (64), (66), (62) and (63) to obtain the following consequences. When $\beta \in [\beta_n,\ 1)$, by (64), (66) and (62), we know

$$S_\beta(a_1,\ \cdots,\ a_n) = \min\left\{2,\ \frac{3}{2^\beta},\ \frac{4}{3^\beta},\ \cdots,\ \frac{n}{(n-1)^\beta}\right\} = \frac{n}{(n-1)^\beta}. \tag{67}$$

When $\beta \in [\beta_k,\ \beta_{k+1})$, $k = 3,\ \cdots,\ n-1$, by (64), (66), (62) and (63), we know

$$S_\beta(a_1,\ \cdots,\ a_n) = \min\left\{2,\ \frac{3}{2^\beta},\ \frac{4}{3^\beta},\ \cdots,\ \frac{n}{(n-1)^\beta}\right\} = \frac{k}{(k-1)^\beta}. \tag{68}$$

When $\beta \in (0,\ \beta_3)$, by (64), (66), (62) and (63), we know

$$S_\beta(a_1,\ \cdots,\ a_n) = \min\left\{2,\ \frac{3}{2^\beta},\ \frac{4}{3^\beta},\ \cdots,\ \frac{n}{(n-1)^\beta}\right\} = 2. \tag{69}$$

At last, (61) follows from (67), (68) and (69). The proof is complete. □

Based on Theorem 3.1 in Section 3, we can also get a more general result as follows. The following example is a new generalization of Mitrinović inequality (see [2]).

Example 4.2 Under the conditions of Theorem 3.1, we further pick $k \in \mathbb{N}^+ \cap [1,\ n-1]$. Then in the cases (i), (ii), (iii) and (iv) with $t > r$ replaced by $t > kr$ where $t > r$ appears,

$$\sum_{i=1}^{n} \frac{(a_i^p + \cdots + a_{i+k-1}^p)^m}{[ts - r(a_i^p + \cdots + a_{i+k-1}^p)]^\beta} \geqslant \frac{k^m n^{\beta+1-m}}{(nt-kr)^\beta} \left(\sum_{i=1}^{n} a_i^p \right)^{m-\beta}; \tag{70}$$

in the cases (v), (vi), (vii) and (viii) with $t > r$ replaced by $t > kr$ where $t > r$ appears,

$$\sum_{i=1}^{n} \frac{(a_i^p + \cdots + a_{i+k-1}^p)^m}{[ts - r(a_i^p + \cdots + a_{i+k-1}^p)]^\beta} \leqslant \frac{k^m n^{\beta+1-m}}{(nt-kr)^\beta} \left(\sum_{i=1}^{n} a_i^p \right)^{m-\beta}. \tag{71}$$

Here a_k is supposed to be a_{k-n} if $k > n$.

Proof. We only show (70) since (71) can be proved similarly. Let $A_i := a_i^p + \cdots + a_{i+k-1}^p$ for $i = 1,\ \cdots,\ n$ and $S := A_1 + \cdots + A_n = ks$. Since $t > kr$ implies $t/k > r$, then under the cases (i), (ii), (iii) and (iv), we can use Theorem 3.1 and obtain

$$\sum_{i=1}^{n} \frac{(a_i^p + \cdots + a_{i+k-1}^p)^m}{[ts - r(a_i^p + \cdots + a_{i+k-1}^p)]^\beta} = \sum_{i=1}^{n} \frac{A_i^m}{\left(\frac{t}{k}S - rA_i\right)^\beta} \geqslant \frac{n^{\beta+1-m}}{\left(\frac{nt}{k} - r\right)^\beta} S^{m-\beta} = \frac{k^m n^{\beta+1-m}}{(nt-kr)^\beta} \left(\sum_{i=1}^{n} a_i^p \right)^{m-\beta},$$

which ends the proof. □

The theorems in Section 3 can be used to prove inequalities concerning with the sides of triangles.

Example 4.3 Let a, b and c be three sides of a triangle. Then

$$\frac{a^2}{b+c-a} + \frac{b^2}{c+a-b} + \frac{c^2}{a+b-c} \geqslant a+b+c. \tag{72}$$

Proof. Since a, b and c are the sides of a triangle, the sum of each two of a, b and c is bigger than the other. Therefore, we can take $n = 3$, $m = 2$, $p = t = \beta = 1$ and $r = 2$ in (10) in the case (iii.1) of Theorem 3.1 and directly obtain (72). □

4.2 *Applications on competition questions*

In the following, we present some mathematical competition questions that can be obtained by the main theorems in Section 3. For writing convenience, we denote the left hand side of some inequality by LHS.

Example 4.4 [28th IMO Pre-selection Question] Let a, b and c be the sides of a triangle and $2S = a+b+c$. Prove that

$$\frac{a^m}{b+c}+\frac{b^m}{c+a}+\frac{c^m}{a+b} \geqslant \left(\frac{2}{3}\right)^m S^{m-1},$$

where $m \geqslant 1$. Particularly, when $m = 2$, this is a question of 19th Nordic Mathematical Olympiad Contest in 2005.

Proof. It is a simple example of (10) by picking $n = 3$, $m \geqslant 2$ and $p = \beta = t = r = 1$ in the case (iii.1) of Theorem 3.1, and an example of (18) by picking $n = 3$ and $m = p = \beta = t = r = 1$ in the case (i.1) of Theorem 3.2 (This is also the famous Nesbitt's inequality). □

Example 4.5 [31st IMO Pre-selection Question] Let a, b, c and d be positive real numbers such that $ab+bc+cd+da = 1$. Prove

$$\frac{a^3}{b+c+d}+\frac{b^3}{c+d+a}+\frac{c^3}{d+a+b}+\frac{d^3}{a+b+c} \geqslant \frac{1}{3}.$$

This example was also selected in Chinese Mathematical Olympiad in Senior (Xinjiang Division) Preliminary Contest in 2020.

Proof. Pick $n = 4$, $m = 3$ and $p = \beta = t = r = 1$ in (18) in the case (i.1) of Theorem 3.2. Then we have

$$\begin{aligned}
\text{LHS} &\geqslant \frac{1}{3}(a^2+b^2+c^2+d^2) \\
&= \frac{1}{3}\frac{(a^2+b^2)+(b^2+c^2)+(c^2+d^2)+(d^2+a^2)}{2} \\
&\geqslant \frac{1}{3}(ab+bc+cd+da) = \frac{1}{3}.
\end{aligned}$$

The proof is hence over. □

Example 4.6 Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)} \geqslant \frac{3}{2}.$$

Proof. Picking $n = 3$, $m = -2$, $p = -1$ and $\beta = t = r = 1$ in (10) in the case (iii.1) of Theorem 3.1, we obtain

$$\text{LHS} = \frac{bc}{a^2(b+c)} + \frac{ca}{b^2(c+a)} + \frac{ab}{c^2(a+b)}$$

$$= \frac{1}{a^2(b^{-1}+c^{-1})} + \frac{1}{b^2(c^{-1}+a^{-1})} + \frac{1}{c^2(a^{-1}+b^{-1})} \geqslant \frac{1}{2}(a^{-1}+b^{-1}+c^{-1})$$

$$\geqslant \frac{3}{2}\sqrt[3]{(abc)^{-1}} = \frac{3}{2},$$

where we have also used the mean value inequality. □

Example 4.7 [Serbian Math Olympiad in 2005] Let x, y and z be positive numbers. Prove

$$\frac{x}{\sqrt{y+z}} + \frac{y}{\sqrt{z+x}} + \frac{z}{\sqrt{x+y}} \geqslant \sqrt{\frac{3}{2}(x+y+z)}.$$

Proof. Pick $n = 3$, $m = p = t = r = 1$ and $\beta = 1/2$ in (10) in the case (iii.2) of Theorem 3.1, we easily obtain the conclusion. □

5. Applications on Hurwitz-Lerch zeta functions

The Hurwitz-Lerch zeta function $\zeta(z,\ \beta,\ a)$ is defined by

$$\zeta(z,\ \beta,\ a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^{\beta}},$$

where a $\in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\beta \in \mathbb{C}$ when $|z| < 1$ and $\Re(\beta) > 1$ when $|z| = 1$. Here $\mathbb{C}$ is the set of complex numbers, $\mathbb{Z}_0^-$ is the set of nonpositive integers and $\Re(\beta)$ means the real part of $\beta \in \mathbb{C}$.

In the following theorem, we only discuss the relation about Hurwitz-Lerch zeta functions with real variables.

Theorem 5.1 Let $x = (x_1,\ \cdots,\ x_n) \in \mathbb{R}^n$, $\{a_n\}_{n\in\mathbb{N}^+}$ be a positive sequence,

$$X = \{x \in \mathbb{R}^n : x_i > 0 \ \text{ and } \ x_1 + \cdots + x_n = 1\},\ \alpha = \inf_{n\in\mathbb{N}^+} a_n,$$

$z > 0$ and $r \in \mathbb{R}$ such that $\alpha > r$. Then in the case when $\beta r > 0$,

$$\min_{x\in X}\sum_{i=1}^{n} x_i\zeta\left(z,\ \beta,\ a_n-rx_i\right)=\zeta\left(z,\ \beta,\ a_n-\frac{r}{n}\right) \tag{73}$$

in the following cases

(1) $\beta>0$, $r<0$ and $2(\alpha-r)\geqslant-(\beta+1)r$,

(2) $\beta<-1$, $r>0$ and $2(\alpha-r)\geqslant-(\beta+1)r$,

(3) $\beta\in[-1,\ 0)$ and $r>0$,

$$\max_{x\in X}\sum_{i=1}^{n} x_i\zeta\left(z,\ \beta,\ a_n-rx_i\right)=\zeta\left(z,\ \beta,\ a_n-\frac{r}{n}\right). \tag{74}$$

Here each Hurwitz-Lerch zeta function in (73) and (74) is supposed to be convergent.

Remark 5.2 In Theorem 5.1, we only consider the case when $\beta r\neq 0$, since when $\beta r=0$, $\zeta(z,\ \beta,\ a_n-rx_i)$ does not depend on x_i, and it is obvious that

$$\sum_{i=1}^{n} x_i\zeta\left(z,\ \beta,\ a_n-rx_i\right)=\zeta\left(z,\ \beta,\ a_n-\frac{r}{n}\right).$$

Proof. In this proof, we always assume that $m=p=1$. For the case when $\beta r>0$, we first consider the case when $\beta>0$, $r>0$ or $\beta<-1$, $r<0$. In this case, (iii.2) of Theorem 3.1 is satisfied. Thus we obtain by (10) that

$$\sum_{i=1}^{n}\frac{x_i}{(j+a_n-rx_i)^{\beta}}\geqslant\frac{n^{\beta}}{[n(j+a_n)-r]^{\beta}}=\frac{1}{\left(j+a_n-\frac{r}{n}\right)^{\beta}}. \tag{75}$$

Then multiplying (75) by z^j and adding the results for $j=0,\ 1,\ \cdots,\ k$ together with $k\in\mathbb{N}^{+}$, we have

$$\sum_{i=1}^{n}x_i\sum_{j=0}^{k}\frac{z^j}{(j+a_n-rx_i)^{\beta}}=\sum_{j=0}^{k}z^j\sum_{i=1}^{n}\frac{x_i}{(j+a_n-rx_i)^{\beta}}\geqslant\sum_{j=0}^{k}\frac{z^j}{(j+a_n-\frac{r}{n})^{\beta}}. \tag{76}$$

Let k tend to the infinity, we conclude that

$$\sum_{i=1}^{n}x_i\zeta(z,\ \beta,\ a_n-rx_i)\geqslant\zeta(z,\ \beta,\ a_n-\frac{r}{n}), \tag{77}$$

which implies (73). For the case when $\beta = -1$ and $r < 0$, (ii.1) of Theorem 3.1 is satisfied; for the case when $\beta \in (-1,\ 0)$ and $r < 0$, we see that

$$(j + a_n - r)\Theta_2 = \frac{2(j + a_n - r)}{(\beta+1)(-r)} > \frac{2}{\beta+1} > 1$$

and (iv) is satisfied. In these two cases, we also have (75) and hence (73).

Next we show (74) for the case when $\beta r < 0$. When $\beta > 0$, $r < 0$ and $2(\alpha - r) \geqslant -(\beta+1)r$ (the proof for the case (2) is the same), we have

$$(j + a_n - r)\Theta_2 = -\frac{2(j + a_n - r)}{(\beta+1)r} \geqslant -\frac{2(\alpha - r)}{(\beta+1)r} \geqslant 1.$$

Thus (viii) of Theorem 3.1 is satisfied. Similar to the discussion of (75), (76) and (77), we can use (11) to obtain (74). When $\beta = -1$ and $r > 0$, (vi.1) of Theorem 3.1 is satisfied. When $\beta \in (-1,\ 0)$ and $r > 0$, (vii.2) of Theorem 3.1 is satisfied. As a result, (74) can be similarly obtained. The proof is hence finished now.

Remark 5.3 Theorem 5.1 is a generalization of Theorem 3.2 of [13]. Specifically, when $a_n = 1 + \frac{1}{n}$, $r = 1$ and $z = 1$, (73) is the result obtained in [13].

Remark 5.4 In Theorem 3.3, we have obtained the result (33) for the case (iii.3), i.e., $\beta \in \mathbb{Z}\backslash\mathbb{N}$, $r < 0$ and $pm \geqslant 0$, which is much stronger than other cases, since it is required that $\beta \in \mathbb{Z}\backslash\mathbb{N}$, but not $\beta \in (-\infty,\ -1)$. In fact, many examples imply that the inequality (33) probably holds true for all $\beta \in (-\infty,\ -1)$, $r < 0$ and $pm \geqslant 0$. However, this is just our conjecture that we are unable to prove yet now. This is an interesting and open problem.

Remark 5.5 This article mainly generalizes Nesbitt's inequality in respect of dimensions and parameters and gives different results in various cases. The argument also provides a series of methods to estimate algebraic expressions analogous to (3). This article is not concerning with the inequalities with weights like [13, 28–32]. Actually, it is still interesting to study the generalizations of inequalities (10), (11), (17) and (33) with weights and other types of Nesbitt's inequality such as (1) [31] and

$$\left(\frac{x_1}{x_1+x_2}\right)^{\alpha} + \left(\frac{x_2}{x_2+x_3}\right)^{\alpha} + \cdots + \left(\frac{x_n}{x_n+x_1}\right)^{\alpha} \leqslant \frac{n}{2^{\alpha}},$$

with $x_1,\ \cdots,\ x_n > 0$, $n \geqslant 2$ and $\alpha \leqslant 1/2$ [40].

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Conflict of interest

The authors declare no conflict of interest.

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