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Research Article

Ulam Stability for Nonlinear Fractional Differential Equations with Multi-Term and Nonlocal Multi-Point Boundary Value Problem

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Abstract: We focus on Ulam type stability for fractional differential equations with nonlocal multi-point and multi-term boundary conditions. The application of Ulam stability to any type of boundary condition causes some misunderstandings which are mainly connected with the solutions of the applied inequality and the corresponding boundary condition. The main idea of Ulam type stability is the closeness between any solution of the corresponding differential inequality and the solution of the studied problem. Also, both solutions have to be deeply connected. To avoid misunderstandings in the literature we suggest three different approaches. In all of them we study the appropriately defined Ulam type stability.

Keywords: Caputo fractional derivative, multi-term, nonlocal multi-point integral boundary conditions, Ulam stability

MSC: 34A34, 34A08, 34D20

1. Introduction

Boundary Value Problems (BVP) for fractional differential equations were studied by many authors and there was a special focus on boundary value problems involving multi-term fractional differential equations and some existence results were obtained (see, for example [1–7]). The interest is connected with their applications in problems of physics, mechanics and other applied sciences (see, for example [8–10]).

One of the main qualitative questions about the behavior of the solutions of differential equations is connected with stability property. Often it is connected with the essential question whether for a given function almost satisfying the given equation there exists a solution of the equation which is close to the given function. Initially, this problem was defined to functional equations by Ulam and Hyers (see [11, 12]) and later it is used for differential equations. Despite this problem is a comparatively old one, recently many authors studied it for various types of differential equations and applied it to several dynamical models (see, for example some recent works [13–16]).

The main purpose of this paper is to consider Ulam Stability (US) for nonlocal multi-point and multi-term boundary conditions for a nonlinear fractional differential equation. Note that when US is studied for initial value problems, the basis of the study is the appropriately defined differential inequality where any solution is close enough to the corresponding solution of the studied problem. The main point in this case is that the initial value of corresponding solution of the

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given problem is taken to be equal to the initial value of the chosen solution of the inequality. In this way there is a deep connection between both the chosen solution of the inequality and the solution of the studied problem. When this type of stability is applied to a BVP there are some misunderstandings in the literature. We suggest three different approaches to study correctly Ulam type stability for the given BVP. In the first approach, to keep the idea of the classical definitions of US we introduce parameters in the boundary conditions. Then for any particular solution of the fractional differential inequality we choose these parameters to depend on its values. In this way the solution of the given BVP and the particular solution of the differential inequality are deeply connected. Note this idea was applied to linear BVP for Caputo fractional differential equations in [17, 18]. In the second approach we suggest the differential inequality be replaced by an integral based on the integral presentation of the solution of the studied problem. In the third approach we consider a differential inequality with inequalities in the boundary conditions. In the last two approaches we consider the given fractional differential equation with boundary conditions without any parameters. The solution of this BVP is unique and it has to be close enough to any solution of the corresponding inequality. In these two approaches we modify the definition for US. In all three approaches some sufficient conditions are obtained.

2. Statement of the problem and some preliminaries

We use the norm $||x|| = \sup_{\tau\in[0,\,1]} |x(\tau)|$ for any function $x \in C([0,\,1],\,\mathbb{R})$.

Definition 1 [19, 20] The Riemann-Liouville fractional integral of order $\omega > 0$ for the function $\Upsilon : [0,\,1] \to \mathbb{R}$ is defined by

$$I_0^{\omega}\Upsilon(\tau) = \frac{1}{\Gamma(\omega)}\int_0^{\tau}\frac{\Upsilon(\vartheta)}{(\tau-\vartheta)^{1-\omega}}d\vartheta, \quad \tau \in (0,\,1],$$

where Γ is the Euler gamma function.

Definition 2 [19, 20] Let $\omega > 0 : N-1 < \omega < N$, $N = [\omega]+1$ with $[\omega]$ equal to the integer part of ω. The Caputo fractional derivative of fractional order $\omega > 0$ for the function $\Upsilon : [0,\,1] \to \mathbb{R}$, is defined by

$${}^{c}D^{\omega}\Upsilon(\tau) = \frac{1}{\Gamma(N-\omega)}\int_0^{\tau}\frac{\Upsilon^{(N)}(\vartheta)}{(\tau-\vartheta)^{1+\omega-N}}d\vartheta, \ \ \tau \in (0,\,1].$$

We consider the nonlinear fractional differential equations with multi-term and nonlocal multi-point boundary conditions (BVP):

$$q_2\,{}^{C}D^{\rho+2}\chi(\tau) + q_1\,{}^{C}D^{\rho+1}\chi(\tau) + q_0\,{}^{C}D^{\rho}\chi(\tau) = F(\tau,\,\chi(\tau)), \quad \tau \in (0,\,1], \tag{1}$$

$$\chi(0) = h(\chi(.)), \quad \chi(\zeta) = \sum_{m=1}^{n}\Psi_m\chi(\xi_m), \quad \chi(1) = \lambda\int_0^{\varpi}\chi(s)ds, \tag{2}$$

where $\rho \in (0,\,1)$, $F : [0,\,1]\times\mathbb{R} \to \mathbb{R}$ and $h : C([0,\,1],\,\mathbb{R}) \to \mathbb{R}$ are given continuous functions, $0 < \varpi < \zeta < \xi_1 < \xi_2 < \dots < \xi_n < 1$, $\lambda \in \mathbb{R}$, $q_0,\,q_1,\,q_2 \in \mathbb{R}$, $q_2 \neq 0$, $\Psi_i \in \mathbb{R}$, $i = 1,\,2,\,\dots,\,n$.

Some existence results concerning (1), (2) are proved in [1].

In our further investigations we use the following conditions:

(C1) $|F(\tau,\,u) - F(\tau,\,v)| \le \mathbb{K}|u-v|$, $u,\,v \in \mathbb{R}$, $\tau \in [0,\,1]$ where $\mathbb{K}$ is a positive constant;

(C2) $|h(w_1(.) - h(w_2(.))| \le \mathbb{H}||w_1 - w_2||$ for $w_1,\,w_2 \in C([0,\,1],\,\mathbb{R})$ where $\mathbb{H}$ is a positive constant;

(C3) $q_1^2 - 4q_0q_2 > 0$, $q_0 \neq 0$, $q_2 \neq 0$.

Consider the boundary conditions (2) with parameters:

$$\chi(0) = h(\chi) + p, \quad \chi(\zeta) = \sum_{i=m}^{n} \Psi_m \chi(\xi_m) + Q, \quad \chi(1) = \lambda \int_0^{\varpi} \chi(s)ds + P, \tag{3}$$

where p, P, Q are parameters.

Define the operator $\mathcal{W} : C([0, 1], \mathbb{R}) \to C([0, 1], \mathbb{R})$ whose fixed point is a solution of (1), (3) (similar to the one defined in [1] but with some changes because of the presence of parameters in (3)):

$$\begin{aligned}(\mathcal{W}\chi)(\tau) = {} & \frac{D_1(\chi)W_4 - D_2(\chi)W_2}{W_1W_4 - W_2W_3} \left[\frac{\mathcal{X}(M_2, M_1, \tau) - \mathcal{X}(M_1, M_2, \tau)}{q_0\sqrt{q_1^2 - 4q_0q_2}} \right] \\ & + \frac{D_2(\chi)W_1 - D_1(\chi)W_3}{W_1W_4 - W_2W_3}(e^{M_1\tau} - e^{M_2\tau}) + h(\chi)e^{M_2\tau} + pe^{M_2\tau} \\ & + \mathcal{Z}(\tau, \chi) \end{aligned} \tag{4}$$

where

$$\mathcal{Z}(\tau, \chi) = \frac{1}{\sqrt{q_1^2 - 4q_0q_2}} \int_0^{\tau} \left(e^{M_2(\tau-\xi)} - e^{M_1(\tau-\xi)}\right) \left(\int_0^{\xi} \frac{(\xi-u)^{\rho-1}}{\Gamma(\rho)} F(\varsigma, x(\varsigma))d\varsigma\right) d\xi, \tag{5}$$

$$\mathcal{X}(a, b, \tau) = a(1 - e^{b\tau}), \tag{6}$$

$$M_1 = \frac{-q_1 - \sqrt{q_1^2 - 4q_0q_2}}{2q_2}, \quad M_2 = \frac{-q_1 + \sqrt{q_1^2 - 4q_0q_2}}{2q_2},$$

$$W_1 = \frac{1}{q_0\sqrt{q_1^2 - 4q_0q_2}} \Big[\mathcal{X}(M_2, M_1, \zeta) - \mathcal{X}(M_1, M_2, \zeta) - \sum_{i=m}^{n} \Psi_m \Big(\mathcal{X}(M_2, M_1, \xi_m) - \mathcal{X}(M_1, M_2, \xi_m) \Big) \Big],$$

$$W_2 = (e^{M_1\zeta} - e^{M_2\zeta}) - \sum_{i=m}^{n} \Psi_m (e^{M_1\xi_m} - e^{M_2\xi_m}),$$

$$W_3 = \frac{1}{q_0\sqrt{q_1^2 - 4q_0q_2}} \Big[\mathcal{X}(M_2, M_1, 1) - \mathcal{X}(M_1, M_2, 1) - \lambda \int_0^{\varpi} \Big(\mathcal{X}(M_2, M_1, s) - \mathcal{X}(M_1, M_2, s) \Big) ds \Big],$$

$$W_4 = e^{M_1} - e^{M_2} - \lambda \int_0^{\varpi} c_2(e^{M_1s} - e^{M_2s})ds, \tag{7}$$

$$D_1(\chi) = Q - (h(\chi)+p)\left(e^{M_2\zeta} - \sum_{i=m}^{n}\Psi_m e^{M_2\xi_m}\right) + \sum_{i=m}^{n}\Psi_m \mathcal{Z}(\xi_m,\,\chi) - \mathcal{Z}(\zeta,\,\chi),$$

$$D_2(\chi) = P + (h(\chi)+p)\left(\lambda\int_0^{\varpi} e^{M_2 s}ds - e^{M_2}\right) + \lambda\int_0^{\varpi}\mathcal{Z}(s,\,\chi)ds - \mathcal{Z}(1,\,\chi). \tag{8}$$

We will prove the existence and uniqueness of the solution of BVP (1), (3) by Banach contraction principle. Note this principle is applied to a BVP without any parameter (1), (2) in [1]. Since the boundary condition (3) is a generalization of (2) we will provide the main points in the proof.

Theorem 1 Let the conditions (C1), (C2), (C3) be satisfied and $\sup_{\tau\in[0,\,1]}\Lambda(\tau) < 1$ where

$$\Lambda(\tau) = \frac{\mathbb{L}W_4 + \mathcal{L}W_2}{|W_1W_4 - W_2W_3|}\left[\frac{\mathcal{X}(M_2,\,M_1,\,\tau) - \mathcal{X}(M_1,\,M_2,\,\tau)}{q_0\sqrt{q_1^2 - 4q_0q_2}}\right]$$

$$+ \frac{\mathcal{L}W_1 + \mathbb{L}W_3}{|W_1W_4 - W_2W_3|}|e^{M_1\tau} - e^{M_2\tau}| + \mathbb{H}e^{M_2\tau} + \frac{\mathbb{K}}{\Gamma(1+\rho)\sqrt{q_1^2 - 4q_0q_2}}\mathcal{P}(\tau), \tag{9}$$

$\mathcal{X}_1(a,\,b,\,\tau)$ is defined in (6), and the positive constants $\mathcal{L}$, $\mathbb{L}$ are defined by

$$\mathcal{L} = \mathbb{H}\left|\lambda\frac{e^{M_2\varpi} - 1}{M_2} - e^{M_2}\right|$$

$$+ \lambda\frac{\mathbb{K}}{\Gamma(1+\rho)\sqrt{q_1^2 - 4q_0q_2}}\int_0^{\varpi}\mathcal{P}(s)ds + \frac{\mathbb{K}}{\Gamma(1+\rho)\sqrt{q_1^2 - 4q_0q_2}}|\mathcal{P}(1)|, \tag{10}$$

and

$$\mathbb{L} = \mathbb{H}\left|e^{M_2\zeta} - \sum_{i=m}^{n}\Psi_m e^{M_2\xi_m}\right| + \frac{\mathbb{K}}{\Gamma(1+\rho)\sqrt{q_1^2 - 4q_0q_2}}\left(\sum_{i=m}^{n}\Psi_m|\mathcal{P}(\xi_m)| + |\mathcal{P}(\zeta)|\right), \tag{11}$$

$$\mathcal{P}(\tau) = \int_0^{\tau}\left(e^{M_2(\tau-\xi)} - e^{M_1(\tau-\xi)}\right)\xi^{\rho}d\xi. \tag{12}$$

Then the BVP (1), (3) has a unique solution for any real parameters p, P, Q.

Proof. For any α, $\beta \in C([0,\,1],\,\mathbb{R})$ we have

$$
\begin{aligned}
&|\mathcal{Z}(\tau,\alpha)-\mathcal{Z}(\tau,\beta)| \\
&\le \frac{\mathbb{K}||\alpha-\beta||}{\sqrt{q_1^2-4q_0q_2}}\int_0^\tau \left(e^{M_2(\tau-\xi)}-e^{M_1(\tau-\xi)}\right)\left(\int_0^\xi \frac{(\xi-\varsigma)^{\rho-1}}{\Gamma(\rho)}d\varsigma\right)d\xi \\
&= \frac{\mathbb{K}||\alpha-\beta||}{\Gamma(1+\rho)\sqrt{q_1^2-4q_0q_2}}\int_0^\tau \left(e^{M_2(\tau-\xi)}-e^{M_1(\tau-\xi)}\right)\xi^\rho d\xi \\
&= \frac{\mathbb{K}||\alpha-\beta||}{\Gamma(1+\rho)\sqrt{q_1^2-4q_0q_2}}\mathcal{P}(\tau), \qquad (13)
\end{aligned}
$$

and

$$
\begin{aligned}
|D_1(\alpha)-D_1(\beta)| &\le |h(\alpha)-h(\beta)|\left|e^{M_2\zeta}-\sum_{i=m}^n \Psi_m e^{M_2\xi_m}\right| \\
&\quad +\sum_{i=m}^n \Psi_m|\mathcal{Z}(\xi_m,\alpha)-\mathcal{Z}(\xi_m,\beta)| \\
&\quad +|\mathcal{Z}(\zeta,\alpha)-\mathcal{Z}(\zeta,\beta)| \\
&\le \mathbb{H}\left|e^{M_2\zeta}-\sum_{i=m}^n \Psi_m e^{M_2\xi_m}\right| ||\alpha-\beta|| \\
&\quad +\frac{\mathbb{K}||\alpha-\beta||}{\Gamma(1+\rho)\sqrt{q_1^2-4q_0q_2}}\left(\sum_{i=m}^n \Psi_m|\mathcal{P}(\xi_m)|+|\mathcal{P}(\zeta)|\right) \\
&= \mathbb{L}||\alpha-\beta||, \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
|D_2(\alpha)-D_2(\beta)| \le |h(\alpha)-h(\beta)|\left|\lambda \frac{e^{M_2\varpi}-1}{M_2}-e^{M_2}\right| \\
+\lambda \int_0^{\varpi} |\mathcal{Z}(s,\,\alpha)-\mathcal{Z}(s,\,\beta)|\,ds+|\mathcal{Z}(1,\,\alpha)-\mathcal{Z}(1,\,\beta)| \\
\le \mathbb{H}||\alpha-\beta||\left|\lambda \frac{e^{M_2\varpi}-1}{M_2}-e^{M_2}\right| \\
+\lambda \int_0^{\varpi} \frac{\mathbb{K}||\alpha-\beta||}{\Gamma(1+\rho)\sqrt{q_1^2-4q_0q_2}}\mathcal{P}(s)ds+\frac{\mathbb{K}||\alpha-\beta||}{\Gamma(1+\rho)\sqrt{q_1^2-4q_0q_2}}|\mathcal{P}(1)| \\
=\mathcal{L}||\alpha-\beta||, \qquad (15)
\end{aligned}
$$

where $\mathcal{L}$ and $\mathbb{L}$ are defined by (10) and (11) respectively.

According to inequalities (14) and (15) we get for $\tau \in [0,\,1]$ that

$$
\begin{aligned}
&|(\mathcal{W}\alpha)(\tau)-(\mathcal{W}\beta)(\tau)| \\
&\le \frac{|D_1(\alpha)-D_1(\beta)|W_4+|D_2(\alpha)-D_2(\beta)|W_2}{|W_1W_4-W_2W_3|}\left[\frac{\mathcal{X}(M_2,\,M_1,\,\tau)-\mathcal{X}(M_1,\,M_2,\,\tau)}{q_0\sqrt{q_1^2-4q_0q_2}}\right] \\
&\quad+\frac{|D_2(\alpha)-D_2(\beta)|W_1+|D_1(\alpha)-D_1(\beta)|W_3}{|W_1W_4-W_2W_3|}|e^{M_1\tau}-e^{M_2\tau}|+|h(\alpha)-h(\beta)|e^{M_2\tau} \\
&\quad+|\mathcal{Z}(\tau,\,\alpha)-\mathcal{Z}(\tau,\,\beta)| \\
&\le ||\alpha-\beta||\left(\frac{\mathbb{L}W_4+\mathcal{L}W_2}{|W_1W_4-W_2W_3|}\left[\frac{\mathcal{X}(M_2,\,M_1,\,\tau)-\mathcal{X}(M_1,\,M_2,\,\tau)}{q_0\sqrt{q_1^2-4q_0q_2}}\right]\right. \\
&\quad\left.+\frac{\mathcal{L}W_1+\mathbb{L}W_3}{|W_1W_4-W_2W_3|}|e^{M_1\tau}-e^{M_2\tau}|+\mathbb{H}e^{M_2\tau}+\frac{\mathbb{K}}{\Gamma(1+\rho)\sqrt{q_1^2-4q_0q_2}}\mathcal{P}(\tau)\right) \\
&\le \sup_{\tau\in[0,\,1]} \Lambda(\tau)||\alpha-\beta||,
\end{aligned}
$$

where $\Lambda(t)$ is defined in (9).

The proof of Theorem 1 is completed via the Banach contraction principle. □

Corollary 1 [1] Let condition of Theorem 1 be satisfied. Then the BVP (1), (2) has an unique solution and the function $x \in C([0, 1], \mathbb{R})$ satisfies $(\mathcal{Q}x)t = x(t)$, $\tau \in [0, 1]$ iff $x(.)$ is a solution of BVP (1), (2), where $\mathcal{Q} = \mathcal{W}|_{p=P=Q=0}$.

3. Ulam type stability

We will consider three possible approaches to study Ulam type stability.

3.1 *Fractional differential inequality and Ulam stability*

Let $\gamma >$ be an arbitrary number. Consider the fractional differential inequality

$$|q_2\, {}^C D^{\rho+2} v(\tau) + q_1\, {}^C D^{\rho+1} v(\tau) + q_0\, {}^C D^{\rho} v(\tau) - F(\tau, v(\tau))| \leq \gamma, \quad \tau \in (0, 1]. \tag{16}$$

In this case we will use the classical definition for Ulam-Hyers stability (see, for example [11, 12]) adopted to the studied problem:

Definition 3 The BVP (1), (3) is called Ulam-Hyers (UH) stable, if there exists a constant $\mathcal{C} > 0$ such that, for any $\gamma > 0$ and any solution $v \in C([0, 1], \mathbb{R})$ of the fractional differential inequality (16) there exists a solution $\chi^*(.)$ of BVP (1), (3) such that the inequality $||v - \chi^*|| \leqslant \mathcal{C}\gamma$ holds.

We will obtain sufficient conditions for UH.

Theorem 2 Let conditions (C1), (C2), (C3) be satisfied and $\sup_{t\in[0, 1]} \Lambda(t) < 1$, where $\Lambda(t)$ is defined in (9). Then the BVP (1), (3) is UH.

Proof. Let $v \in C([0, 1], \mathbb{R})$ be a solution of the differential inequality (16). Then there exists a function $\varsigma : [0, 1] \to \mathbb{R} : |\varsigma(\tau)| \leq \gamma$ such that

$$q_2\, {}^C D^{\rho+2} v(\tau) + q_1\, {}^C D^{\rho+1} v(\tau) + q_0\, {}^C D^{\rho} v(\tau) = F(\tau, v(\tau)) + \varsigma(\tau), \quad \tau \in [0, 1]. \tag{17}$$

We define the parameters p, P, Q in (3) depending on the values of the function $v(t)$.

- Define $p = v(0) - h(v(.))$;
- Define $P = v(1) - \lambda \int_0^{\varpi} v(s)ds$;
- Define $Q = v(\zeta) - \sum_{m=1}^{n} \Psi_m v(\xi_m)$.

From the definitions of the parameters p, P, Q it follows that the function $v(.)$ satisfies the boundary conditions (3).

Then, from (17) and the choice of the parameters p, P, Q it follows that the function $v(.)$ is a fixed point of the operator $\mathbb{S} : C([0, 1], \mathbb{R}) \to C([0, 1], \mathbb{R})$ defined by (compare with (4)):

$$(\mathbb{S}\chi)(\tau) = \frac{D_1(\chi)W_4 - D_2(\chi)W_2}{W_1W_4 - W_2W_3} \left[\frac{\mathcal{X}(M_2, M_1, \tau) - \mathcal{X}(M_1, M_2, \tau)}{q_0\sqrt{q_1^2 - 4q_0q_2}} \right]$$

$$+ \frac{D_2(\chi)W_1 - D_1(\chi)W_3}{W_1W_4 - W_2W_3}(e^{M_1\tau} - e^{M_2\tau}) + h(\chi)e^{M_2\tau} + pe^{M_2\tau} + \mathcal{V}(\tau, \chi), \tag{18}$$

where

$$\mathcal{V}(\tau, \chi) = \frac{1}{\sqrt{q_1^2 - 4q_0q_2}} \int_0^{\tau} \left(e^{M_2(\tau-\xi)} - e^{M_1(\tau-\xi)}\right) \left(\int_0^{\xi} \frac{(\xi-u)^{\rho-1}}{\Gamma(\rho)} F(u, x(u)) du\right) d\xi$$

$$+ \frac{1}{\sqrt{q_1^2 - 4q_0q_2}} \int_0^{\tau} \left(e^{M_2(\tau-\xi)} - e^{M_1(\tau-\xi)}\right) \left(\int_0^{\xi} \frac{(\xi-u)^{\rho-1}}{\Gamma(\rho)} \varsigma(u) du\right) d\xi, \tag{19}$$

and W_1, W_2, W_3, W_4, $D_1(\chi)$, $D_2(\chi)$ are defined by equalities (7), (8).

According to Theorem 1 there exists an unique solution $\chi^*(.)$ of BVP (1), (3) with the chosen parameters p, P, Q in the boundary condition (3). This solution is a fixed point of the operator $\mathcal{W}$ defined in (4).

Note we have

$$|\mathcal{V}(\tau, v) - \mathcal{Z}(\tau, \chi^*)|$$

$$\leq \frac{\mathbb{K}||v - \chi^*||}{\sqrt{q_1^2 - 4q_0q_2}} \int_0^{\tau} \left(e^{M_2(\tau-\xi)} - e^{M_1(\tau-\xi)}\right) \left(\int_0^{\xi} \frac{(\xi-\varsigma)^{\rho-1}}{\Gamma(\rho)} d\varsigma\right) d\xi$$

$$+ \gamma \frac{1}{\sqrt{q_1^2 - 4q_0q_2}} \int_0^{\tau} \left(e^{M_2(\tau-\xi)} - e^{M_1(\tau-\xi)}\right) \left(\int_0^{\xi} \frac{(\xi-u)^{\rho-1}}{\Gamma(\rho)} du\right) d\xi$$

$$= \frac{\mathbb{K}||v - \chi^*||}{\Gamma(1+\rho)\sqrt{q_1^2 - 4q_0q_2}} \mathcal{P}(\tau) + \gamma \mathcal{B}(\tau), \tag{20}$$

where

$$\mathcal{B}(\tau) = \frac{1}{\Gamma(1+\rho)\sqrt{q_1^2 - 4q_0q_2}} \int_0^{\tau} \left(e^{M_2(\tau-\xi)} - e^{M_1(\tau-\xi)}\right) \xi^{\rho} d\chi = \frac{1}{\Gamma(1+\rho)\sqrt{q_1^2 - 4q_0q_2}} \mathcal{P}(\tau).$$

Then for $\tau \in [0, 1]$ we get

$$|v(\tau) - \chi^*(\tau)| = |(\mathbb{S}v)(\tau) - (\mathcal{W}\chi^*)(\tau)|$$

$$\leq \frac{|D_1(v) - D_1(\chi^*)|W_4 - |D_2(v) - D_2(\chi^*)|W_2}{|W_1W_4 - W_2W_3|} \left[\frac{\mathcal{X}(M_2, M_1, \tau) - \mathcal{X}(M_1, M_2, \tau)}{q_0\sqrt{q_1^2 - 4q_0q_2}}\right]$$

$$+\frac{|D_2(v)-D_2(\chi^*)|W_1-|D_1(v)-D_1(\chi^*)|W_3}{|W_1W_4-W_2W_3|}(e^{M_1\tau}-e^{M_2\tau})$$

$$+|h(v)-h(\chi^*)|e^{M_2\tau}+pe^{M_2\tau}$$

$$+|\mathcal{V}(\tau,\, v)-\mathcal{Z}(\tau,\, \chi^*)|$$

$$\leq \sup_{\tau\in[0,\, 1]} \Lambda(\tau)||v-\chi^*||$$

$$+\gamma\left(\mathcal{B}(\tau)+e^{M_2\tau}\right),$$

where $\Lambda(t)$ is defined in (9).

From the above inequality it follows that

$$||v-\chi^*||\leq \gamma\frac{\sup_{\tau\in[0,\, 1]}\left(\mathcal{B}(\tau)+e^{M_2\tau}\right)}{1-\sup_{\tau\in[0,\, 1]}\Lambda(\tau)} \tag{21}$$

Therefore, BVP (1), (3) is Ulam-Hyers stable with constant $\mathcal{C}=\dfrac{\sup_{\tau\in[0,\, 1]}\left(\mathcal{B}(\tau)+e^{M_2\tau}\right)}{1-\sup_{\tau\in[0,\, 1]}\Lambda(\tau)}$. Note the constant $\mathcal{C}$ does not depend on any solution of (16). □

Remark 1 The parameters in the boundary conditions (2) are very important in the application of US. These parameters allow us to consider a solution of the studied BVP which depends on the chosen solution of the inequality and to keep the main idea of US.

3.2 *Fractional integral inequality and Ulam stability*

In this case we consider BVP (1), (2) without any parameters. Then this BVP has a unique solution under some conditions (see Theorem 1 with $p=P=Q=0$).

Denote by $\mathcal{Q}=\mathcal{W}|_{p=P=Q=0}$, i.e.,

$$(\mathcal{Q}\chi)(\tau)=\frac{\hat{D}_1(\chi)W_4-\hat{D}_2(\chi)W_2}{W_1W_4-W_2W_3}\left[\frac{\mathcal{X}(M_2,\, M_1,\, \tau)-\mathcal{X}(M_1,\, M_2,\, \tau)}{q_0\sqrt{q_1^2-4q_0q_2}}\right]$$

$$+\frac{\hat{D}_2(\chi)W_1-\hat{D}_1(\chi)W_3}{W_1W_4-W_2W_3}(e^{M_1\tau}-e^{M_2\tau})+h(\chi)e^{M_2\tau}+\mathcal{Z}(\tau,\, \chi) \tag{22}$$

where $\mathcal{Z}(\tau,\, \chi)$, W_k, $k=1,\, 2,\, 3,\, 4$ are defined by (5), (7) respectively and

$$\hat{D}_1(\chi) = -h(\chi)\left(e^{M_2\zeta} - \sum_{i=m}^{n} \Psi_m e^{M_2\xi_m}\right) + \sum_{i=m}^{n} \Psi_m \mathcal{Z}(\xi_m, \chi) - \mathcal{Z}(\zeta, \chi),$$

$$\hat{D}_2(\chi) = h(\chi)\left(\lambda \int_0^{\varpi} e^{M_2 s} ds - e^{M_2}\right) + \lambda \int_0^{\varpi} \mathcal{Z}(s, \chi) ds - \mathcal{Z}(1, \chi). \tag{23}$$

Let $\gamma > 0$ be a given number. Based on the defined operator $\mathcal{Q}$ we will consider the integral inequality:

$$|v(\tau) - (\mathcal{Q}v)\tau| \leq \gamma, \quad v \in C([0, 1], \mathbb{R}), \quad \tau \in [0, 1]. \tag{24}$$

Remark 2 Let $v(.)$ be a solution of (24). Then there exists a function $\varsigma \in C([0, 1], \mathbb{R}) : |\varsigma(\tau)| \leq \gamma$, $\tau \in [0, 1]$ (this function depends on $v(.)$) such that

$$v(\tau) = (\mathcal{Q}v)\tau + \varsigma(\tau), \quad \tau \in [0, 1]. \tag{25}$$

Note (24) has many solution for any given $\gamma > 0$. At the same time, under some conditions BVP (1), (2) has a unique solution (see Corollary 1). If we consider US, then this solution has to be close enough to any solution of the inequality (24). This is different than the classical definition of US.

Definition 4 The BVP (1), (2) is called Integral Modified Ulam-Hyers Stable (IMUS), if there exists a constant $\mathcal{C} > 0$ such that, for any $\gamma > 0$ and any solution $v \in C([0, 1], \mathbb{R})$ of the fractional integral inequality (24) the inequality $||v - \chi^*|| \leqslant \mathcal{C}\gamma$ holds where $\chi^*(.)$ is the solution of BVP (1), (2).

Theorem 3 Let conditions (C1), (C2), (C3) be satisfied and $\sup_{t\in[0, 1]} \Lambda(\tau) < 1$, where $\Lambda(\tau)$ is defined in (9). Then the BVP (1), (2) is IMUS.

Proof. Let $\gamma > 0$ be an arbitrary number and $v \in C([0, 1], \mathbb{R})$ be a solution of the fractional integral inequality (24). According to Remark 2 the equality (25) holds.

According Corollary 1 BVP (1), (2) has a unique solution $\chi^*(.)$ which is a fixed point of the operator $\mathcal{Q}$ defined by (22).

Then for any $\tau \in [0, 1]$ we have

$$|v(\tau) - \chi^*(\tau)| = |(\mathcal{Q}v)\tau + \varsigma(\tau) - (\mathcal{Q}\chi^*)\tau| \leq |(\mathcal{Q}v)\tau - (\mathcal{Q}\chi^*)\tau| + \gamma. \tag{26}$$

As in the proof of Theorem 2 we obtain the inequality

$$|v(\tau) - \chi^*(\tau)| \leq \gamma + \Lambda(\tau)||v - \chi^*||, \quad \tau \in [0, 1],$$

where $\Lambda(\tau)$ is defined by (9).

Then

$$||v - \chi^*|| \leq \mathcal{C}\gamma,$$

where $\mathcal{C} = \dfrac{1}{1 - \max_{\tau\in[0,\,1]} \Lambda(\tau)}$. □

Remark 3 Note in both Theorem 2 and Theorem 3 we prove $||v - \chi^*|| \le \mathcal{C}\gamma$ but in Theorem 2 the solution χ^* is a solution of fractional differential equation (1) with appropriately chosen parameters in the boundary conditions (3), whereas in Theorem 3 the solution χ^* is a solution of fractional differential equation (1) with boundary conditions (2) without any parameters. Also note in Theorem 2 the function $v(.)$ is a solution of differential inequality (16) whereas in Theorem 3 the function $v(.)$ is a solution of integral inequality (24).

3.3 *Fractional differential inequality with boundary condition and Ulam stability*

Now we consider BVP (1), (2).

As in the previous section, denote by $\mathcal{Q} = \mathcal{W}|_{p=P=Q=0}$, i.e., by (22).

Let $\gamma > 0$ be a given number. Consider the system of fractional differential inequality with inequalities for boundary conditions:

$$|q_2\, {}^C D^{\rho+2} v(\tau) + q_1\, {}^C D^{\rho+1} v(\tau) + q_0\, {}^C D^{\rho} v(\tau) - F(\tau,\, v(\tau))| \le \gamma, \quad \tau \in (0,\, 1]$$

$$|v(0) - h(x)| \le \gamma, \quad \left| v(\zeta) - \sum_{m=1}^{n} \Psi_m v(\xi_m) \right| \le \gamma, \quad \left| v(1) - \lambda \int_0^{\varpi} v(s)ds \right| \le \gamma. \tag{27}$$

Remark 4 Let $v(.)$ be a solution of the inequality (27). Then there exist a function $\varsigma \in C([0,\, 1],\, \mathbb{R}) : |\varsigma(.)| \le \gamma$ and constants $A,\, B,\, C : |A| \le \gamma,\, |B| \le \gamma,\, |C| \le \gamma$ ($\varsigma(.),\, A,\, B,\, C$ depend on the solution $v(.)$) such that

$$q_2\, {}^C D^{\rho+2} v(\tau) + q_1\, {}^C D^{\rho+1} v(\tau) + q_0\, {}^C D^{\rho} v(\tau) = F(\tau,\, v(\tau)) + \varsigma(\tau), \quad \tau \in (0,\, 1],$$

$$v(0) = h(x) + A, \quad v(\zeta) = \sum_{m=1}^{n} \Psi_m v(\xi) + B, \quad v(1) = \lambda \int_0^{\varpi} v(s)ds + C. \tag{28}$$

As in the previous section, (27) has many solution for any given $\gamma > 0$. At the same time, under some conditions, BVP (1), (2) has a unique solution (see Theorem 1). If we consider US, then this solution has to be close enough to any solution of the inequality (27). This is different than the classical definition of US.

Definition 5 The BVP (1), (2) is called differential modified Ulam-Hyers stable, if there exists a constant $\mathcal{C} > 0$ such that, for any $\gamma > 0$ and any solution $v \in C([0,\, 1],\, \mathbb{R})$ of the fractional differential inequality (27) the inequality $||v - \chi^*|| \leqslant \mathcal{C}\gamma$ holds where $\chi^*(t)$ is a solution of BVP (1), (2).

Theorem 4 Let conditions (C1), (C2), (C3) be satisfied and $\sup_{t\in[0,\,1]} \Lambda(t) < 1$, where $\Lambda(t)$ is defined in (9). Then the BVP (1), (2) is differential modified Ulam-Hyers stable.

Proof. According to Corollary 1 BVP (1), (2) has an unique solution χ^* which is a fixed point of the operator $\mathcal{Q}$ defined by (22).

Let $\gamma > 0$ be an arbitrary number and $v \in C([0,\, 1],\, \mathbb{R})$ be a solution of the differential inequality (27). According to Remark 4 the function $v(.)$ is a solution of BVP (28).

Therefore, the function $v(.)$ is a fixed point of the operator $\mathbb{Q} : C([0,\, 1],\, \mathbb{R}) \to C([0,\, 1],\, \mathbb{R})$ defined by

$$(\mathbb{Q}x)(\tau)=\frac{\tilde{D}_1(\chi)W_4-\tilde{D}_2(\chi)W_2}{W_1W_4-W_2W_3}\left[\frac{\mathcal{X}(M_2,\,M_1,\,\tau)-\mathcal{X}(M_1,\,M_2,\,\tau)}{q_0\sqrt{q_1^2-4q_0q_2}}\right]$$

$$+\frac{\tilde{D}_2(\chi)W_1-\tilde{D}_1(\chi)W_3}{W_1W_4-W_2W_3}(e^{M_1\tau}-e^{M_2\tau})+h(\chi)e^{M_2\tau}+Ae^{M_2\tau}+\mathcal{V}(\tau,\,\chi), \tag{29}$$

where $\mathcal{X}(a,\,b,\,\tau)$, M_1, M_2, W_i, $i=1,\,2,\,3,\,4$ and $\mathcal{V}(\tau,\,\chi)$ are defined by (6), (7), (19), respectively, and

$$\tilde{D}_1(\chi)=B-(h(\chi)+A)\left(e^{M_2\zeta}-\sum_{i=m}^{n}\Psi_m e^{M_2\xi_m}\right)+\sum_{i=m}^{n}\Psi_m\mathcal{V}(\xi_m,\,\chi)-\mathcal{V}(\zeta,\,\chi),$$

$$\tilde{D}_2(\chi)=C+(h(\chi)+A)\left(\lambda\int_0^{\varpi}e^{M_2s}ds-e^{M_2}\right)+\lambda\int_0^{\varpi}\mathcal{V}(s,\,\chi)ds-\mathcal{V}(1,\,\chi). \tag{30}$$

From definitions (23), (30) and inequalities (20) we have

$$|\tilde{D}_1(v)-\hat{D}_1(\chi^*)|\le\gamma+\mathbb{H}||\chi^*-v||\left|e^{M_2\zeta}-\sum_{i=m}^{n}\Psi_m e^{M_2\xi_m}\right|+\gamma\left|e^{M_2\zeta}-\sum_{i=m}^{n}\Psi_m e^{M_2\xi_m}\right|$$

$$+\sum_{i=m}^{n}\Psi_m\left(\frac{\mathbb{K}\mathcal{P}(\xi_m)}{\Gamma(1+\rho)\sqrt{q_1^2-4q_0q_2}}||\chi^*-v||+\gamma\frac{\mathcal{P}(\xi_m)}{\Gamma(1+\rho)\sqrt{q_1^2-4q_0q_2}}\right)$$

$$+\frac{\mathbb{K}\mathcal{P}(\zeta)}{\Gamma(1+\rho)\sqrt{q_1^2-4q_0q_2}}||\chi^*-v||+\gamma\frac{\mathcal{P}(\zeta)}{\Gamma(1+\rho)\sqrt{q_1^2-4q_0q_2}}$$

$$\le\mathbb{L}||\chi^*-v||+\gamma\mathbb{C}_1, \tag{31}$$

and

$$|\tilde{D}_2(v)-\hat{D}_2(\chi^*)|\le\gamma+\mathbb{H}||\chi^*-v||\left|\lambda\frac{e^{M_2\varpi}-1}{M_2}-e^{M_2}\right|+\gamma\left|\lambda\frac{e^{M_2\varpi}-1}{M_2}-e^{M_2}\right|$$

$$+\lambda\left(\frac{\mathbb{K}}{\Gamma(1+\rho)\sqrt{q_1^2-4q_0q_2}}||\chi^*-v||+\gamma\frac{1}{\Gamma(1+\rho)\sqrt{q_1^2-4q_0q_2}}\right)\int_0^{\varpi}\mathcal{P}(s)ds$$

$$+\left(\frac{\mathbb{K}}{\Gamma(1+\rho)\sqrt{q_1^2-4q_0q_2}}||\chi^*-v||+\gamma\frac{1}{\Gamma(1+\rho)\sqrt{q_1^2-4q_0q_2}}\right)\mathcal{P}(1)$$

$$\leq \mathcal{L}||\chi^*-v||+\gamma\mathbb{C}_2, \tag{32}$$

where $\mathcal{L}$ and $\mathbb{L}$ are defined by (10), (11), respectively, and

$$\mathbb{C}_1=1=\left|e^{M_2\zeta}-\sum_{i=m}^{n}\Psi_m e^{M_2\xi_m}\right|+\sum_{i=m}^{n}\Psi_m\frac{\mathcal{P}(\xi_m)}{\Gamma(1+\rho)\sqrt{q_1^2-4q_0q_2}}+\frac{\mathcal{P}(\zeta)}{\Gamma(1+\rho)\sqrt{q_1^2-4q_0q_2}},$$

and

$$\mathbb{C}_2=1+\left|\lambda\frac{e^{M_2\varpi}-1}{M_2}-e^{M_2}\right|+\frac{\lambda}{\Gamma(1+\rho)\sqrt{q_1^2-4q_0q_2}}\int_0^{\varpi}\mathcal{P}(s)ds+\frac{1}{\Gamma(1+\rho)\sqrt{q_1^2-4q_0q_2}}\mathcal{P}(1).$$

Then for any $\tau\in[0, 1]$ from inequalities (31) and (32) we have

$$|v(\tau)-\chi^*(\tau)|=|(\mathbb{Q}v)(\tau)-(\mathcal{Q}\chi^*)(\tau)|$$

$$\leq \mathbb{H}||v-\chi^*||e^{M_2\tau}+\gamma e^{M_2\tau}+\frac{\mathbb{K}\mathcal{P}(\tau)}{\Gamma(1+\rho)\sqrt{q_1^2-4q_0q_2}}||\chi^*-v||+\gamma\frac{\mathcal{P}(\tau)}{\Gamma(1+\rho)\sqrt{q_1^2-4q_0q_2}}$$

$$+\frac{(\mathbb{L}||\chi^*-v||+\gamma\mathbb{C}_1)|W_4|+(\mathcal{L}||\chi^*-v||+\gamma\mathbb{C}_2)|W_2|}{|W_1W_4-W_2W_3|}\left[\frac{|\mathcal{X}(M_2, M_1, \tau)-\mathcal{X}(M_1, M_2, \tau)|}{|q_0|\sqrt{q_1^2-4q_0q_2}}\right]$$

$$+\frac{(\mathcal{L}||\chi^*-v||+\gamma\mathbb{C}_2)|W_1|+(\mathbb{L}||\chi^*-v||+\gamma\mathbb{C}_1)|W_3|}{|W_1W_4-W_2W_3|}|e^{M_1\tau}-e^{M_2\tau}|$$

$$\leq \sup_{\tau\in[0, 1]}\Lambda(\tau)+\gamma\mathcal{B}, \tag{33}$$

where $\Lambda(\tau)$ is defined by (9) and

$$\mathcal{B} = e^{M_2\tau} + \frac{\mathcal{P}(\tau)}{\Gamma(1+\rho)\sqrt{q_1^2 - 4q_0q_2}} + \frac{\mathbb{C}_1|W_4| + \mathbb{C}_2|W_2|}{|W_1W_4 - W_2W_3|}\left[\frac{|\mathcal{X}(M_2, M_1, \tau) - \mathcal{X}(M_1, M_2, \tau)|}{|q_0|\sqrt{q_1^2 - 4q_0q_2}}\right]$$

$$+ \frac{\mathbb{C}_2|W_1| + \mathbb{C}_1|W_3|}{|W_1W_4 - W_2W_3|}|e^{M_1\tau} - e^{M_2\tau}|.$$

Therefore BVP (1), (2) is differential modified Ulam-Hyers stable with a constant $\mathcal{C} = \dfrac{\mathcal{B}}{1 - \sup_{\tau\in[0,\,1]}\Lambda(\tau)}$. □

4. Applications

Consider the nonlinear fractional differential equations with multi-term and nonlocal multi-point boundary conditions:

$$^{C}D^{2.5}\chi(\tau) - {}^{C}D^{0.5}\chi(\tau) = \tau\sin(\chi(\tau)), \quad \tau \in (0,\, 1], \tag{34}$$

$$\chi(0) = 0.001\chi(0.7), \quad \chi(0.2) = 0.5\chi(0.5), \quad \chi(1) = 0. \tag{35}$$

The problem (34), (35) is a partial case of BVP (1), (2) with $\rho = 0.5 \in (0,\, 1)$, $q_2 = 1$, $q_1 = 0$, $q_0 = -1$, $n = 1$, $0 < 0.1 = \varpi < 0.2 = \zeta < 0.5 = \xi_1 < 1$, $h(x.) = 0.01x(0.7)$ for any $x(.) \in C([0,\, 1],\, \mathbb{R})$, $\lambda = 0$, $\Psi_1 = 0.5$.

Then $q_1^2 - 4q_0q_2 = 4 > 0$, $|F(\tau,\, x) - F(\tau,\, y)| = \tau|\sin(x),\, -\sin(y)| \leq |x - y|$, $x,\, y \in \mathbb{R}$, $|h(w_1(.)) - h(w_2(.))| = 0.001|w_1(0.7) - w_2(0.7)| \leq ||w_1 - w_2||$, $w_1,\, w_2 \in C([0,\, 1],\, \mathbb{R})$, i.e., conditions (C1), (C2) and (C3) are satisfied with $\mathbb{H} = 0.001$, $\mathbb{K} = 1$.

In this case the equalities (5) are reduced to

$$M_1 = -1, \quad M_2 = 1,$$

$$W_1 = \frac{1}{-2}\left[2 - e^{-0.2} - e^{0.2} - 0.1\left(2 - e^{-0.5} - e^{0.5}\right)\right] \approx -0.0073041,$$

$$W_2 = (e^{-0.2} - e^{0.2}) - 0.1(e^{-0.5} - e^{0.5}) \approx -0.298453,$$

$$W_3 = \frac{1}{-2}(2 - e^{-1} - e) \approx 0.543081, \quad W_4 = e^{-1} - e^{1} \approx -2.3504,$$

$$\mathcal{P}(1) = \int_0^1 \left(e^{(1-\xi)} - e^{-(1-\xi)}\right)\xi^{0.5}d\xi \approx 0.568158,$$

$$\mathcal{P}(0.5) = \int_0^{0.5} \left(e^{(0.5-\xi)} - e^{-(0.5-\xi)}\right)\xi^{0.5}d\xi \approx 0.0957879,$$

$$\mathcal{P}(0.2) = \int_0^{0.2} \left(e^{(0.2-\xi)} - e^{-(0.2-\xi)}\right) \xi^{0.5} d\xi \approx 0.00956481$$

$$\mathbb{L} = 0.001|e^{0.2} - 0.5e^{0.5}| + \frac{1}{2\Gamma(1.5)} (0.5|\mathcal{P}(0.5)| + |\mathcal{P}(0.2)|) \approx 0.0321532,$$

$$\mathcal{L} = 0.001e + \frac{1}{2\Gamma(1.5)}|\mathcal{P}(1)| \approx 0.323267,$$

$$\begin{aligned}\Lambda(\tau) &= \frac{\mathbb{L}W_4 + \mathcal{L}W_2}{|W_1W_4 - W_2W_3|} \left[\frac{|2 - e^{-\tau} - e^{\tau}|}{2}\right] \\ &\quad + \frac{\mathcal{L}W_1 + \mathbb{L}W_3}{|W_1W_4 - W_2W_3|}|e^{-\tau} - e^{\tau}| + 0.001e^{\tau} + \frac{1}{2\Gamma(1.5)}\mathcal{P}(\tau) \\ &\leq 0.988786 < 1. \end{aligned} \tag{36}$$

According to Theorem 1 BVP (34), (35) has unique solution $\chi^*(.)$. According to Theorem 3 BVP (34), (35) is integral modified Ulam-Hyers stable, i.e., for any $\gamma > 0$ the inequality $||v - \chi^*|| \leq \gamma\frac{1}{1 - 0.988786} = 89.1742\gamma$ holds, where v is any solution of the inequality

$$|v(\tau) - (\mathcal{Q}v)\tau| \leq \gamma, \quad v \in C([0, 1], \mathbb{R}), \quad \tau \in [0, 1]$$

with

$$\begin{aligned}(\mathcal{Q}\chi)(\tau) &= \frac{D_1(\chi)W_4 - D_2(\chi)W_2}{W_1W_4 - W_2W_3}\left[\frac{2 - e^{-\tau} - e^{\tau}}{-2}\right] \\ &\quad + \frac{D_2(\chi)W_1 - D_1(\chi)W_3}{W_1W_4 - W_2W_3}(e^{-\tau} - e^{\tau}) + \chi(0.7)e^{\tau} + \mathcal{Z}(\tau, \chi),\end{aligned}$$

$$D_1(\chi) = -\chi(0.7)(e^{0.2} - 0.5e^{0.5}) + 0.5\mathcal{Z}(0.5, \chi) - \mathcal{Z}(-0.2, \chi),$$

$$D_2(\chi) = -\mathcal{Z}(1, \chi),$$

$$\mathcal{Z}(\tau, \chi) = \frac{1}{2}\int_0^{\tau} \left(e^{\tau-\xi} - e^{\xi-\tau}\right)\left(\int_0^{\xi} \frac{(\xi - u)^{-0.5}}{\Gamma(0.5)}\varsigma \sin(\chi(\varsigma))d\varsigma\right) d\xi.$$

5. Conclusion

In this paper we study the application of US on a special type of BVP for a Caputo fractional differential equation with several terms. The boundary conditions consist of multi-term and nonlocal multi-points. We suggest three different approaches to avoid some misunderstandings in the literature. We define in appropriate approaches Ulam-Hyers stability in all considered cases and we discuss them and we obtain sufficient conditions for the defined types of stability. Also, to emphasize on US we consider only the case of $\mathcal{W} = q_1^2 - 4q_0q_2 > 0$. The suggested three approaches easily could be applied to the other cases of $\mathcal{W}$.

We note that other types of US, such Ulam-Hyers-Rassias stability, generalized Ulam-Hyers-Rassias stability, could be defined in appropriate approaches and studied following the suggested three approaches in this paper.

We note that the suggested ideas could be succesfylly applied to study boundary value problems of differential equations with other types of fractional derivatives with slight appropriate changes.

In addition the suggested approaches could be also applied to other types of BVP for different types of differential equations.

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Conflict of interest

The authors declare no competing financial interest.

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