Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

Research Article

Applying the Generalized Laplace Residual Power Series Method to the Time-Fractional Multi-Asset Black-Scholes European Option Pricing Model

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Received: 20 May 2025; **Revised:** 5 June 2025; **Accepted:** 6 June 2025

Abstract: It is a well-known fact that the Black-Scholes model is used in order to analyse the behavior of the financial market with regard to the pricing of options. An explicit analytical solution to the Black-Scholes equation is known as the Black-Scholes formula. The Black-Scholes equation is modified by mathematicians in the form of fractional Black-Scholes equations. Unfortunately, there are certain cases in which the fractional-order Black-Scholes equation does not have a closed-form formula. This article demonstrates the method for deriving analytical solutions to the fractional multi-asset Black-Scholes equation with the left-side Caputo-type Katugampola fractional derivative. The $\frac{t^\rho}{\rho}$-Laplace residual power series approach, which blends the residual power series method with the $\frac{t^\rho}{\rho}$-Laplace transform, is the methodology used to find analytical solutions to this equation. The suggested method is remarkably precise and efficient for the fractional multi-asset Black-Scholes equation, according to numerical analyses. This confirms that the $\frac{t^\rho}{\rho}$-Laplace residual power series method is among the most effective techniques for finding analytical solutions to fractional-order differential equations.

Keywords: residual power series method, generalized Laplace transform, fractional Black-Scholes equations, the left-side Caputo-type Katugampola fractional derivative

MSC: 91G20, 35R11

DOI: https://doi.org/10.37256/cm.6320257266

1. Introduction

Options are contracts that confer upon one party the right to purchase or sell an asset at a specified price and quantity at a future date, as stipulated in the contract. The option purchaser is required to compensate the seller for the rights conferred by the agreement. The option purchaser possesses the discretion to determine whether to exercise that right. Options are categorized into two types: call options and put options. The determinants influencing the price of an option encompass the prevailing spot price of the underlying asset (S), the exercise or strike price (K), the time to maturity (T), the volatility of the underlying asset (σ), the risk-free interest rate (r), and the dividend yield (q), as referenced in [1]. The Black-Scholes model is the most prevalent and impactful framework for option pricing in practice.

In the 1970s, three financial experts, Fischer Black, Myron Scholes, and Robert C. Merton, devised and refined a model for valuing net securities. This model connected the relationships between the stock market, the money market, and the derivatives market. The model calculates the value of options in the derivatives market by considering the price of underlying securities from the stock market and the interest rates in the money market. The model is predicated on the core assumption that the prospect of risk-free profit (arbitrage) is virtually unattainable or entirely nonexistent. The model has undergone continuous development, and to date, numerous intriguing new research studies have emerged. In 2025, D'Uggento et al. [2] published a study comparing the precision of option pricing between the conventional Black-Scholes model and machine learning models, including Artificial Neural Networks (ANN). The study sought to ascertain whether machine learning could supplant or augment the functionalities of the conventional model by utilizing authentic market data and statistical analytic methods. They employed a comparative analysis methodology, using actual option price data for experimentation, and juxtaposed the outcomes generated by the Black-Scholes model and machine learning algorithms (notably ANN), applying statistical performance metrics to assess the precision of option price predictions. The research indicated that employing machine learning methods, particularly artificial neural networks, enhances the precision of option pricing compared to the conventional Black-Scholes model. It also underscored the promise of machine learning as a feasible and efficient method for application in intricate financial markets.

Historically, numerous researchers have examined the Black-Scholes model using various analytical techniques [3–7], including the Homotopy Perturbation Method (HPM), the Adomian Decomposition Method (ADM) and the Variation Iteration Method (VIM), among others. Recently, a new approach called the Generalized Laplace Residual Power Series (GLRPS) has been presented as an effective numerical analysis method for tackling a range of issues [8, 9]. Another advantage of the GLRPS approach is its ability to solve nonlinear problems accurately and efficiently. The integration of analytical concepts with the Laplace transform in the GLRPS method improves solution accuracy and decreases problem-solving time. Obtaining precise solutions is crucial for understanding complex natural phenomena.

The history of fractional calculus, which applies the ideas of integrals and derivatives to non-integer orders, can be summed up as follows [10]: Earlier History (1695): The idea started when Gottfried Wilhelm Leibniz and Guillaume de l'Hôpital exchanged letters on what a half-order derivative meant. 18th-century mathematicians such as Pierre-Simon Laplace, Joseph-Louis Lagrange, and Leonhard Euler investigated and advanced the theoretical framework of fractional calculus. 19th century: Joseph Fourier applied fractional calculus to heat and diffusion problems. Liouville and Riemann further formalized the theory of fractional integrals. 20th-century: Mathematicians recognized its significance in various fields. Paul Lévy and Norbert Wiener used it to study Brownian motion. The 1960s and 1970s saw applications in control systems and signal processing.

Fractional differential equations are employed to represent several important scenarios in the real world, such as fluid dynamics, acoustics, electromagnetism, electrochemistry, and materials science. This leads to a relevant question in finance: 'Can fractional differential equations be applied in the financial market?' The answer is 'Yes.' Due to the self-similar nature of fractional derivatives, they are more effective in capturing long-range dependence than integer-order derivatives. As a result, these benefits are valuable for managing the fractal characteristics inherent in the financial market.

This article presents an approximate analytical solution for the fair price of a call option in the Black-Scholes equation, utilizing the fractional-order Caputo-type Katugampola fractional derivative, formulated through the GLRPS method. The Black-Scholes model for pricing European options with n underlying assets is expressed as follows [11, 12]:

$$\frac{\partial}{\partial \tau}u(S_1, S_2, ..., S_N, \tau)+\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{2}\frac{\partial^2 u}{\partial S_i \partial S_j}\sigma_i\sigma_j S_i S_j \varrho_{ij}+r\left(\sum_{j=1}^{N}S_j\frac{\partial u}{\partial S_j}-u\right)=0, \tag{1}$$

in $[0, \infty)^N \times [0, T)$. The terminal condition is given by:

$$u(S_1, S_2, ..., S_N, T)=\max\left(\sum_{i}^{N}\beta_i S_i - K, 0\right), \tag{2}$$

with the meanings of each variable explained in Table 1.

Table 1. Parameters identification of the Black-Scholes model for pricing European options with n underlying assets

Symbol	Identification
u	The value of the call option
S_i	The price of underlying asset i
σ_i	The volatility of underlying asset i
r	The risk free rate of interest
ϱ_{ij}	The correlation coefficients between σ_i and σ_j
T	The expiring date
β_i	The proportion of investment on asset i
K	$\max(K_i)$ where K_i is strike price of asset i

Subsequently, we will streamline the Black-Scholes European Option pricing model (1) and the terminal condition (2) by a variable transformation:

$$\zeta_i = ln(S_i)-\left(r-\frac{1}{2}\sigma_i^2\right)\tau, \quad t=T-\tau,$$

and

$$u=(S_1, S_2, ..., S_N, \tau)=e^{-r(T-\tau)}v(\zeta_1, \zeta_2, ..., \zeta_N, t).$$

The process of adjusting the Black-Scholes model to a simpler version can be found in [11], which we can rewrite in the form of:

$$\frac{\partial}{\partial t}v(\zeta_1, \zeta_2, ..., \zeta_N, t)=\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i\sigma_j\varrho_{ij}\frac{\partial^2}{\partial \zeta_i \partial \zeta_j}v(\zeta_1, \zeta_2, ..., \zeta_N, t), \tag{3}$$

in $\mathbb{R}^N \times (0, T]$ beginning with the state

$$v(\zeta_1, \zeta_2, ..., \zeta_N, 0) = \max\left(\sum_{i=1}^{N} c_i e^{\zeta_i} - K, 0\right) \tag{4}$$

where c_i are constants and $c_i = \beta_i e^{\left(r-\frac{1}{2}\sigma_i^2 T\right)}$ for $i = 1, 2, \ldots, N$.

This article will examine the multi-asset Black-Scholes equation by substituting the integer-order time derivative in Equation (3) with the left-side Caputo-type Katugampola fractional derivative with order $\alpha \in (0, 1]$, as follows:

$${}^{KC}\mathcal{D}_t^{\alpha,\rho} v(\zeta_1, \zeta_2, ..., \zeta_N) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \sigma_i \sigma_j \varrho_{ij} \frac{\partial^2}{\partial \zeta_i \partial \zeta_j} v(\zeta_1, \zeta_2, ..., \zeta_N, t), \tag{5}$$

in $D \times (0, T]$ beginning with the initial state

$$v(\zeta_1, \zeta_2, ..., \zeta_N, 0) = \max\left(\sum_{i=1}^{N} c_i e^{\zeta_i} - K, 0\right) \tag{6}$$

in $D = \prod_{i=1}^{N}[0, a_i]$ where a_i is the maximum print of underlying asset i. Additionally, ${}^{KC}D_t^{\alpha,\rho}$ represents the left-side Caputo-type Katugampola fractional derivative with order $0 < \alpha \leq 1$ with the parameter $\rho > 0$, while the constants c_i are specified by (4). This study employs the GLRPS method to derive analytical solutions for the multi-asset Black-Scholes European option model as presented in Equations (5) and (6).

The article is structured as follows: Section 2 delineates the concepts of fractional derivatives and integrals, generalized Laplace transforms, fractional power series, and specific properties utilized in this research. Section 3 investigates the existence and uniqueness of solutions to the time-fractional multi-asset Black-Scholes equation. Section 4 offers a comprehensive analysis of how the GLRPS method is applied to fractional differential equations. Section 5 delineates the analytical solution for the time-fractional multi-asset Black-Scholes equation. Section 6 shows the numerical results for the diverse parameter values. The outcome of this study is delineated in Section 7.

2. Fundamental definitions

In addition to introducing fractional calculus, generalized Laplace transforms, fractional power series, and some special functions, this section will also discuss certain properties that are essential for this article. Throughout this article, we suppose that α, ρ, and T are constants such that $0 < \alpha \leq 1$ and $\rho, T > 0$. Define the gamma function as $\Gamma(\cdot)$, and let Λ represent a bounded domain.

Definition 2.1 The left-side Caputo derivative operator with order α for $f : [0, \infty) \to \mathbb{R}$ is defined as follows, according to [13]:

$${}^{C}\mathcal{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\varepsilon)^{-\alpha} f'(\varepsilon) d\varepsilon.$$

Definition 2.2 According to [14], the left-side Katugampola integral operator with order α is defined as follows for $f : [0, \infty) \to \mathbb{R}$:

$$
{}^{KC}\mathcal{I}_t^{\alpha,\,\rho} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{t^\rho - \varepsilon^\rho}{\rho} \right)^{\alpha-1} f(\varepsilon) \frac{d\varepsilon}{\varepsilon^{1-\rho}}.
$$

Definition 2.3 The left-side Caputo-type Katugampola fractional derivative with order α for $f : [0,\, \infty) \to \mathbb{R}$ is defined as follows [15]:

$$
{}^{KC}\mathcal{D}_t^{\alpha,\,\rho} f(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t \left(\frac{t^\rho - \varepsilon^\rho}{\rho} \right)^{-\alpha} \frac{df(\varepsilon)}{d\varepsilon} \frac{d\varepsilon}{\varepsilon^{1-\rho}}.
$$

If you look at the situation where $\rho = 1$, you will see that the α-order left-side Caputo derivative operator simplifies to the left-side Caputo-type Katugampola fractional derivative.

This lemma outlines some properties of the left-side Caputo-type Katugampola fractional derivative, relevant to this investigation referred from [14, 15].

Lemma 2.1 Let β and ω be constants such that $0 < \beta \leq 1$ and $\omega \neq \alpha - 1$. Then,

1. ${}^{KC}\mathcal{D}_t^{\alpha,\,\rho} c = 0$ with c being a constant,
2. ${}^{KC}\mathcal{D}_t^{\alpha,\,\rho} \left(\frac{t^\rho}{\rho} \right)^{\omega} = \frac{\Gamma(\omega+1)}{\Gamma(\omega+1-\alpha)} \left(\frac{t^\rho}{\rho} \right)^{\omega-\alpha}$,
3. ${}^{KC}\mathcal{D}_t^{\alpha,\,\rho} \left({}^{KC}D_t^{\beta,\,\rho} f(t) \right) = {}^{KC}D_t^{\alpha+\beta,\,\rho} f(t)$, with f being continuous on $[0,\, \infty)$,
4. ${}^{KC}\mathcal{I}_t^{\alpha,\,\rho} \left({}^{KC}D_t^{\alpha,\,\rho} f(t) \right) = f(t) - f(0)$, with with f being continuous on $[0,\, \infty)$.

The following is the explanation of the $\frac{t^\rho}{\rho}$-Laplace transform.

Definition 2.4 The $\frac{t^\rho}{\rho}$-Laplace transform of the function $u : [0,\, \infty) \to \mathbb{R}$, represented as $\mathcal{L}_{\frac{t^\rho}{\rho}} \{u(t)\}(s)$ or $\mathcal{U}(s)$, is defined by [16].

$$
\mathcal{U}(s) = \mathcal{L}_{\frac{t^\rho}{\rho}} \{u(t)\}(s) := \int_0^\infty e^{-s\frac{t^\rho}{\rho}} u(t) \frac{dt}{t^{1-\rho}}, \quad s > 0, \tag{7}
$$

where the $\frac{t^\rho}{\rho}$-Laplace transform is denoted by s.

Here are some properties of the $\frac{t^\rho}{\rho}$-Laplace transform referred to in [17].

Lemma 2.2

1. $\mathcal{L}_{\frac{t^\rho}{\rho}} \{au(t) + bv(t)\}(s) = a\mathcal{L}_{\frac{t^\rho}{\rho}} \{u(t)\}(s) + b\mathcal{L}_{\frac{t^\rho}{\rho}} \{v(t)\}(s)$, where $a,\, b \in \mathbb{R}$,
2. $\mathcal{L}_{\frac{t^\rho}{\rho}} \left\{ \left(\frac{t^\rho}{\rho} \right)^{v} \right\}(s) = \frac{\Gamma(1+v)}{s^{1+v}}$, where $s > 0$, and $v \in \mathbb{R}$ with $v \in (-1,\, \infty)$,
3. $\mathcal{L}_{\frac{t^\rho}{\rho}} \{1\}(s) = \frac{1}{s}$, for $s > 0$,
4. $\mathcal{L}_{\frac{t^\rho}{\rho}} \left\{ {}^{KC}\mathcal{D}_t^{\alpha,\,\rho} u(t) \right\}(s) = s^\alpha \mathcal{U}(s) - s^{\alpha-1} u(0)$, where $\mathcal{U}(s) = \mathcal{L}_{\frac{t^\rho}{\rho}} \{u(t)\}(s)$,
5. $\mathcal{L}_{\frac{t^\rho}{\rho}} \left\{ {}^{KC}\mathcal{D}_t^{m\alpha,\,\rho} u(t) \right\}(s) = s^{m\alpha} \mathcal{U}(s) - \sum_{n=0}^{m-1} s^{(m-n)\alpha-1} \cdot {}^{KC}\mathcal{D}_t^{n\alpha,\,\rho} u(0)$, $0 < \alpha < 1$, where ${}^{KC}\mathcal{D}_t^{m\alpha,\,\rho} = \underbrace{{}^{KC}\mathcal{D}_t^{\alpha,\,\rho} \cdot {}^{KC}\mathcal{D}_t^{\alpha,\,\rho} \ldots {}^{KC}\mathcal{D}_t^{\alpha,\,\rho}}_{m-times}$.

Subsequently, we will provide the fractional power series, derived from the conceptual extension in reference [9].

Definition 2.5 The power series of fractional order α is defined in the following form:

$$v(\zeta_1,\, \zeta_2,\, ...,\, \zeta_N,\, t) = \sum_{n=0}^{\infty} h_n(\zeta_1,\, \zeta_2,\, ...,\, \zeta_N) \left(\frac{t^\rho}{\rho} - \frac{a^\rho}{\rho} \right)^{n\alpha},$$

where $h_n(\zeta_1, \zeta_2, ..., \zeta_N)$ is the coefficient of the fractional-order power series. Sometime, it is called the α fractional-order power series with variable ρ centered at $a \in [0,\, T)$ for $D \times [0,\, T]$.

We will now present the fractional-order Taylor series.

Theorem 2.1 Let $v(\zeta_1,\, \zeta_2,\, ...,\, \zeta_N,\, t)$ possess a fractional-order power series with variable ρ centered at $t = a$, expressed as:

$$v(\zeta_1,\, \zeta_2,\, ...,\, \zeta_N,\, t) = \sum_{n=0}^{\infty} h_n(\zeta_1,\, \zeta_2,\, ...,\, \zeta_N) \left(\frac{t^\rho}{\rho} - \frac{a^\rho}{\rho} \right)^{n\alpha}, \tag{8}$$

for any $(\zeta,\, t) \in D \times [0,\, T]$. Then, the coefficient h_n is as follows:

$$h_n(\zeta_1,\, \zeta_2,\, ...,\, \zeta_N) = \frac{{}^{KC}\mathcal{D}_t^{n\alpha,\,\rho} v(\zeta_1,\, \zeta_2,\, ...,\, \zeta_N,\, a)}{\Gamma(n\alpha+1)}, \quad \text{for any } \; n \in \mathbb{N} \cup \{0\},$$

if $v(\zeta_1,\, \zeta_2,\, ...,\, \zeta_N,\, t)$ and ${}^{KC}\mathcal{D}_t^{n\alpha,\,\rho} v(\zeta_1,\, \zeta_2,\, ...,\, \zeta_N,\, t)$ are continuous on $[0,\, T]$ and ${}^{KC}\mathcal{D}_t^{n\alpha,\,\rho} v(\zeta_1,\, \zeta_2,\, ...,\, \zeta_N,\, t)$ can be differentiated $(n-1)$ times on $(0,\, T)$.

When $a = 0$, the Taylor series of fractional order is referred to as the Maclaurin series of fractional order.

Theorem 2.2 Let v possess the fractional-order power series centered to $t = 0$, and let ${}^{KC}D_t^{n\alpha,\,\rho} v(\zeta_1,\, \zeta_2,\, ...,\, \zeta_N,\, t)$ exist on $D \times [0,\, T]$ for any $n \in \mathbb{N} \cup \{0\}$. Then,

$$\mathcal{V}(\zeta_1,\, \zeta_2,\, ...,\, \zeta_N,\, s) = \sum_{n=0}^{\infty} \frac{g_n(\zeta_1,\, \zeta_2,\, ...,\, \zeta_N)}{s^{1+n\alpha}}, \quad s > 0 \;\text{ and }\; \zeta_i \in [0,\, a_i],$$

where $\mathcal{V} = \mathcal{L}_{\frac{t^\rho}{\rho}} \{v\}(s)$, and $g_n(\zeta_1,\, \zeta_2,\, ...,\, \zeta_N) = {}^{KC}D_t^{n\alpha,\,\rho} v(\zeta_1,\, \zeta_2,\, ...,\, \zeta_N,\, 0)$ for any $n \in \mathbb{N} \cup \{0\}$.

Proof. The $\dfrac{t^\rho}{\rho}$-Laplace transform can be applied to both sides of (8) to produce the proof of Theorem 2.2.

We next will present the special functions utilized in this article.

Definition 2.6 The Mittag-Leffler (ML) function with one parameter $\alpha > 0$, denoted by E_α, is defined by:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)} \quad \text{for } z \text{ is any real number.}$$

Note that according to Definition 2.6, we have $E_1(z) = \sum_{n=0}^{\infty} \dfrac{z^n}{\Gamma(n+1)} = e^z$ when $\alpha = 1$.

3. Existence of solutions

Applying Banach's fixed-point theorem establishes the existence of the solution for the fractional Black-Scholes Equation (5) given the Initial Condition (IC) (6). The Banach space $\mathbf{C}(D \times [0, T])$ is first described in the following way:

$$\mathbf{C}(D \times [0, T]) := \{v \mid v \text{ is the continuous function on } D \times [0, T]\}.$$

and its norm

$$\|v\| = \sup_{(\zeta_1, \zeta_2, \ldots, \zeta_N, t) \in D \times [0, T]} |v(\zeta_1, \zeta_2, \ldots, \zeta_N, t)|.$$

Theorem 3.1 Let v be continuous and that the first and second partial derivatives of v be also continuous on $D \times [0, T]$. Furthermore, let

$$f(v(\zeta_1, \zeta_2, \ldots, \zeta_N, t)) = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_i \sigma_j \varrho_{ij} \frac{\partial^2}{\partial \zeta_i \partial \zeta_j} v(\zeta_1, \zeta_2, \ldots, \zeta_N, t), \tag{9}$$

is Lipschitz continuous with the Lipschitz constant L. Then, the fractional Black-Scholes differential equation given by (5) with the IC (6) has a solution $v(\zeta_1, \zeta_2, \ldots, \zeta_N, t)$ on $D \times [0, T]$, if $L < 1$.

Proof. The proof of this theorem is comparable to the one in reference [9].

4. The generalized Laplace residual power series method

This section aims to elucidate the GLRPS method and illustrate the convergence of solutions for fractional differential equations utilizing the left-side Caputo-type Katugampola fractional derivative.

4.1 *Routing general fractional differential equations with the GLRPS approach*

Let's start by considering the general fractional differential equation.

$$^{KC}\mathcal{D}_t^{\alpha, \rho} v(\zeta_1, \zeta_2, \ldots, \zeta_N, t) + R[v(\zeta_1, \zeta_2, \ldots, \zeta_N, t)] + N[v(\zeta_1, \zeta_2, \ldots, \zeta_N, t)]$$

$$= f(\zeta_1, \zeta_2, \ldots, \zeta_N, t), \tag{10}$$

for any $(\zeta_1, \zeta_2, \ldots, \zeta_N, t) \in D \times [0, T]$ as well as the IC

$$v(\zeta_1, \zeta_2, \ldots, \zeta_N, 0) = g(\zeta_1, \zeta_2, \ldots, \zeta_N) \text{ for } (\zeta_1, \zeta_2, \ldots, \zeta_N) \in D \tag{11}$$

in which f and g are given functions, $R(\zeta_1, \zeta_2, \ldots, \zeta_N, t)$ is a linear term, $N(\zeta_1, \zeta_2, \ldots, \zeta_N, t)$ is a nonlinear term.

The GLRPS procedure begins by presenting the solution v as follows:

$$v(\zeta_1, \zeta_2, ..., \zeta_N, t) = \sum_{n=0}^{\infty} \frac{g_n(\zeta_1, \zeta_2, ..., \zeta_N)}{\Gamma(1+n\alpha)} \left(\frac{t^\rho}{\rho}\right)^{n\alpha},$$

where the function $g_n(\zeta_1, \zeta_2, ..., \zeta_N)$ is determined by the procedure below, and it's obvious that $v(\zeta_1, \zeta_2, ..., \zeta_N, 0) = g(\zeta_1, \zeta_2, ..., \zeta_N) = g_0(\zeta_1, \zeta_2, ..., \zeta_N)$.

Step 1. By employing the $\frac{t^\rho}{\rho}$-Laplace transform concerning the time variable t on Equation (10), we derive:

$$\mathcal{L}_{\frac{t^\rho}{\rho}} \left\{ {}^{KC}D_t^{\alpha,\rho} v(\zeta_1, \zeta_2, ..., \zeta_N, t) \right\}(s) = \mathcal{L}_{\frac{t^\rho}{\rho}} \{ f(\zeta_1, \zeta_2, ..., \zeta_N, t) - R[v(\zeta_1, \zeta_2, ..., \zeta_N, t)]$$

$$-N[v(\zeta_1, \zeta_2, ..., \zeta_N, t)]\}(s).$$

Subsequently, using the property of the $\frac{t^\rho}{\rho}$-Laplace transform to the Katugampola fractional derivatives along with the IC (11), we obtain:

$$s^\alpha \mathcal{L}_{\frac{t^\rho}{\rho}} \{ v(\zeta_1, \zeta_2, ..., \zeta_N, t)\}(s) - s^{\alpha-1} v(\zeta_1, \zeta_2, ..., \zeta_N, 0)$$

$$= \mathcal{L}_{\frac{t^\rho}{\rho}} \{ f(\zeta_1, \zeta_2, ..., \zeta_N, t) - R[v(\zeta_1, \zeta_2, ..., \zeta_N, t)] - N[v(\zeta_1, \zeta_2, ..., \zeta_N, t)]\}(s),$$

or

$$\mathcal{V}(\zeta_1, \zeta_2, ..., \zeta_N, s) = \frac{g(\zeta_1, \zeta_2, ..., \zeta_N)}{s} + \frac{1}{s^\alpha} \mathcal{L}_{\frac{t^\rho}{\rho}} \{ f(\zeta_1, \zeta_2, ..., \zeta_N, t)$$

$$-R\left[\mathcal{L}^{-1}_{\frac{t^\rho}{\rho}}[\mathcal{V}(\zeta_1, \zeta_2, ..., \zeta_N, s)](t)\right] - N\left[\mathcal{L}^{-1}_{\frac{t^\rho}{\rho}}[\mathcal{V}(\zeta_1, \zeta_2, ..., \zeta_N, s)](t)\right]\Bigg\}(s), \tag{12}$$

where $\mathcal{V}(\zeta_1, \zeta_2, ..., \zeta_N, s) = \mathcal{L}_{\frac{t^\rho}{\rho}} \{ v(\zeta_1, \zeta_2, ..., \zeta_N, t)\}(s)$. It should be noted that the nonlinear PDE defined by (10) and (11) is corresponding to Equation (12).

Step 2. The function $\mathcal{V}(\zeta_1, \zeta_2, ..., \zeta_N, s)$ can be expressed as the following series using the GLRPS method:

$$\mathcal{V}(\zeta_1, \zeta_2, ..., \zeta_N, s) = \sum_{n=0}^{\infty} \frac{g_n(\zeta_1, \zeta_2, ..., \zeta_N)}{s^{1+n\alpha}} \quad \text{for } s > 0. \tag{13}$$

The infinite series defined by (13) is identified as the $\frac{t^\rho}{\rho}$-Laplace series solution. Furthermore, the limit $\lim_{s\to\infty} sV(\zeta_1, \zeta_2, ..., \zeta_N) = g_0(\zeta_1, \zeta_2, ..., \zeta_N)$ holds true. The k-th element of the $\frac{t^\rho}{\rho}$-Laplace series solution, represented as $\mathcal{V}_k(\zeta_1, \zeta_2, ..., \zeta_N, s)$, can be formulated as

$$\mathcal{V}_k(\zeta_1, \zeta_2, ..., \zeta_N, s) = \frac{g(\zeta_1, \zeta_2, ..., \zeta_N)}{s} + \sum_{n=1}^{k} \frac{g_n(\zeta_1, \zeta_2, ..., \zeta_N)}{s^{1+n\alpha}}, \quad s > 0 \tag{14}$$

To determine the coefficient $g_n(\zeta_1, \zeta_2, ..., \zeta_N)$ of the series solution (14), we introduce the $\frac{t^\rho}{\rho}$-Generalized Laplace Residual Function (GLRF). The GLRF for Equation (12) is delineated as follows:

$$\begin{aligned}
&\mathcal{L}_{\frac{t^\rho}{\rho}} Res(\zeta_1, \zeta_2, ..., \zeta_N, s) \\
&= \mathcal{V}(\zeta_1, \zeta_2, ..., \zeta_N, s) - \frac{g(\zeta_1, \zeta_2, ..., \zeta_N)}{s} \\
&- \frac{1}{s^\alpha} \mathcal{L}_{\frac{t^\rho}{\rho}} \left\{ f(\zeta_1, \zeta_2, ..., \zeta_N, t) - R\left[\mathcal{L}^{-1}_{\frac{t^\rho}{\rho}}[\mathcal{V}(\zeta_1, \zeta_2, ..., \zeta_N, s)]\right] - N\left[\mathcal{L}^{-1}_{\frac{t^\rho}{\rho}}[\mathcal{V}(\zeta_1, \zeta_2, ..., \zeta_N, s)]\right] \right\}(s)
\end{aligned} \tag{15}$$

and the k-th term of GLRF is given by:

$$\begin{aligned}
&\mathcal{L}_{\frac{t^\rho}{\rho}} Res_k(\zeta_1, \zeta_2, ..., \zeta_N, s) \\
&= \mathcal{V}_k(\zeta_1, \zeta_2, ..., \zeta_N, s) - \frac{g(\zeta_1, \zeta_2, ..., \zeta_N)}{s} \\
&- \frac{1}{s^\alpha} \mathcal{L}_{\frac{t^\rho}{\rho}} \left\{ f(\zeta_1, \zeta_2, ..., \zeta_N, t) - R\left[\mathcal{L}^{-1}_{\frac{t^\rho}{\rho}}[\mathcal{V}_k(\zeta_1, \zeta_2, ..., \zeta_N, s)]\right] - N\left[\mathcal{L}^{-1}_{\frac{t^\rho}{\rho}}[\mathcal{V}_k(\zeta_1, \zeta_2, ..., \zeta_N, s)]\right] \right\}(s)
\end{aligned} \tag{16}$$

Step 3. Properties of $\mathcal{L}_{\frac{t^\rho}{\rho}} Res(\zeta_1, \zeta_2, ..., \zeta_N, s)$ and $\mathcal{L}_{\frac{t^\rho}{\rho}} Res_k(\zeta_1, \zeta_2, ..., \zeta_N, s)$ that contribute to research are shown below:

1. $\mathcal{L}_{\frac{t^\rho}{\rho}} Res(\zeta_1, \zeta_2, ..., \zeta_N, s) = 0$ and $\lim_{k\to\infty} \mathcal{L}_{\frac{t^\rho}{\rho}} Res_k(\zeta_1, \zeta_2, ..., \zeta_N, s) = \mathcal{L}_{\frac{t^\rho}{\rho}} Res(\zeta_1, \zeta_2, ..., \zeta_N, s)$ for $s > 0$;
2. If $\lim_{s\to\infty} s\mathcal{L}_{\frac{t^\rho}{\rho}} Res(\zeta_1, \zeta_2, ..., \zeta_N, s) = 0$, then $\lim_{s\to\infty} s\mathcal{L}_{\frac{t^\rho}{\rho}} Res_k(\zeta_1, \zeta_2, ..., \zeta_N, s) = 0$;
3. If $\lim_{s\to\infty} s^{1+k\alpha}\mathcal{L}_{\frac{t^\rho}{\rho}} Res(\zeta_1, \zeta_2, ..., \zeta_N, s) = 0$, then $\lim_{s\to\infty} s^{1+k\alpha}\mathcal{L}_{\frac{t^\rho}{\rho}} Res_k(\zeta_1, \zeta_2, ..., \zeta_N, s) = 0$ for $k = 1, 2, 3, \ldots$.

Therefore, the coefficient function $g_n(\zeta_1, \zeta_2, ..., \zeta_N)$ can be found iteratively using this formula.

$$lim_{s\to\infty} \left(s^{1+k\alpha} \mathcal{L}_{\frac{t^\rho}{\rho}} Res_k(\zeta_1, \zeta_2, ..., \zeta_N, s) \right) = 0 \quad \text{for} \quad k = 1, 2, 3, \ldots. \tag{17}$$

The demonstration of these properties aligns closely with the methodologies presented in references [19, 20].

Step 4. The approximate analytical solution $v(\zeta_1, \zeta_2, ..., \zeta_N, t)$ of the fractional differential Equation (10) and the IC (11) is examined by applying the inverse of $\frac{t^\rho}{\rho}$-generalized Laplace transform to $\mathcal{V}(\zeta_1, \zeta_2, ..., \zeta_N, s)$.

4.2 *Examination of the convergence properties of the GLRPS method*

The solution of the nonlinear PDEs (10) and (11), which is represented as a series of $\sum_{n=0}^{\infty} \frac{g_n(\zeta_1, \zeta_2, \ldots, \zeta_N)}{\Gamma(1+n\alpha)} \left(\frac{t^\rho}{\rho}\right)^{n\alpha}$, converges when $g_n(\zeta_1, \zeta_2, \ldots, \zeta_N)$ is taken into account from the GLRPS process.

Theorem 4.1 $\sum_{n=0}^{\infty} \frac{g_n(\zeta_1, \zeta_2, \ldots, \zeta_N)}{\Gamma(1+n\alpha)} \left(\frac{t^\rho}{\rho}\right)^{n\alpha}$ is a convergent series. For any $0 < \zeta_i \leq a_i$, $i = 1, 2, \ldots, n$ and $t \in [0, T]$, is the solution of the nonlinear PDEs (10) and (11).

Proof. The demonstration of this theorem bears resemblance to the work cited in [9].

5. Solving the time-fractional multi-asset problem analytically the Black-Scholes model for pricing European options

In this part, we utilize the generalized Laplace residual power series method to address time-fractional multi-asset Black-Scholes differential equations employing the left-side Caputo-type Katugampola fractional derivative, as demonstrated in the subsequent equation:

$$ {}^{KC}\mathcal{D}_t^{\alpha,\rho} v(\zeta_1, \zeta_2, \ldots, \zeta_N) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \sigma_i\sigma_j\varrho_{ij} \frac{\partial^2}{\partial\zeta_i\partial\zeta_j} v(\zeta_1, \zeta_2, \ldots, \zeta_N, t), \tag{18} $$

with the IC:

$$ v(\zeta_1, \zeta_2, \ldots, \zeta_N, 0) = max\left(\sum_{i=1}^{N} c_i e^{\zeta_i} - K, 0\right), \tag{19} $$

where the constants c_i for $i = 1, 2, \ldots, N$ are defined by (4).

We will find that Equation (5) clearly contains terms from Equation (10), as follows:

$$ R\left[v(\zeta_1, \zeta_2, \ldots, \zeta_N, t\right] = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \sigma_i\sigma_j\varrho_{ij} \frac{\partial^2}{\partial\zeta_i\partial\zeta_j} v(\zeta_1, \zeta_2, \ldots, \zeta_N, t), $$

and $N\left[v(\zeta_1, \zeta_2, \ldots, \zeta_N, t\right]$, and $f(v(\zeta_1, \zeta_2, \ldots, \zeta_N, t)$ are zero.

Assume that the solution v of the time-fractional multi-asset Black-Scholes partial differential equation may be expressed as a Maclaurin series corresponding to the α fractional-order power series with variable ρ:

$$ v(\zeta_1, \zeta_2, \ldots, \zeta_N, t) = \sum_{n=0}^{\infty} \frac{g_n(\zeta_1, \zeta_2, \ldots, \zeta_N)}{\Gamma(1+n\alpha)} \left(\frac{t^\rho}{\rho}\right)^{n\alpha}. $$

From (19), it can be deduced that $v(\zeta_1, \zeta_2, \ldots, \zeta_N, 0) = g_0(\zeta_1, \zeta_2, \ldots, \zeta_N) = \max\left(\sum_{i=1}^{N} c_i e^{\zeta_i} - K, 0\right)$.

Step 1: Determine the analytical solution of Equation (5) given the initial condition (6) with the GLRPS method. Initially, we apply the $\frac{t^\rho}{\rho}$-Laplace transform to both sides of Equation (5) concerning the variable t.

$$\mathcal{L}_{\frac{t^\rho}{\rho}}\left\{{}^{KC}\mathcal{D}_t^{\alpha,\rho}v(\zeta_1,\zeta_2,\ldots,\zeta_N,t)\right\}(s)=\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i\sigma_j\varrho_{ij}\mathcal{L}_{\frac{t^\rho}{\rho}}\left\{\frac{\partial^2}{\partial\zeta_i\partial\zeta_j}v(\zeta_1,\zeta_2,\ldots,\zeta_N,t)\right\}(s), \tag{20}$$

for $t\geq 0$. From lemma 2.2, item 2, and the IC (6), we obtain Equation (20) in the form:

$$s^\alpha\mathcal{V}(\zeta_1,\zeta_2,\ldots,\zeta_N,s)-s^{\alpha-1}g_0(\zeta_1,\zeta_2,\ldots,\zeta_N)$$

$$=\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i\sigma_j\varrho_{ij}\mathcal{L}_{\frac{t^\rho}{\rho}}\left\{\frac{\partial^2}{\partial\zeta_i\partial\zeta_j}v(\zeta_1,\zeta_2,\ldots,\zeta_N,t)\right\},$$

$$\mathcal{V}(\zeta_1,\zeta_2,\ldots,\zeta_N,s)$$

$$=\frac{g_0(\zeta_1,\zeta_2,\ldots,\zeta_N)}{s}+\frac{1}{2s^\alpha}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i\sigma_j\varrho_{ij}\left\{\frac{\partial^2}{\partial\zeta_i\partial\zeta_j}\mathcal{V}(\zeta_1,\zeta_2,\ldots,\zeta_N,s)\right\},$$

or

$$\mathcal{V}(\zeta_1,\zeta_2,\ldots,\zeta_N,s)-\frac{g_0(\zeta_1,\zeta_2,\ldots,\zeta_N)}{s}-\frac{1}{2s^\alpha}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i\sigma_j\varrho_{ij}\left\{\frac{\partial^2}{\partial\zeta_i\partial\zeta_j}\mathcal{V}(\zeta_1,\zeta_2,\ldots,\zeta_N,s)\right\}=0, \tag{21}$$

where $\mathcal{V}(\zeta_1,\zeta_2,\ldots,\zeta_N,s)=\mathcal{L}_{\frac{t^\rho}{\rho}}\{v(\zeta_1,\zeta_2,\ldots,\zeta_N,t)\}(s)$, $s>0$.

Step 2: To obtain the solution of Equation (21), we will refer to Theorem 2.2. Therefore, $\mathcal{V}(\zeta_1,\zeta_2,\ldots,\zeta_N,s)$ will be represented in the following series form:

$$\mathcal{V}(\zeta_1,\zeta_2,\ldots,\zeta_N,s)=\sum_{n=0}^{\infty}\frac{g_n(\zeta_1,\zeta_2,\ldots,\zeta_N)}{s^{1+n\alpha}},\quad s>0. \tag{22}$$

Note that $g_0(\zeta_1,\zeta_2,\ldots,\zeta_N)=\lim_{s\to 0}s\mathcal{V}(\zeta_1,\zeta_2,\ldots,\zeta_N,s)=v(\zeta_1,\zeta_2,\ldots,\zeta_N,0)$. The solution $\mathcal{V}_k(\zeta_1,\zeta_2,\ldots,\zeta_N,s)$ can be written as:

$$\mathcal{V}_k(\zeta_1,\zeta_2,\ldots,\zeta_N,s)=\frac{g_0(\zeta_1,\zeta_2,\ldots,\zeta_N)}{s}+\sum_{n=1}^{k}\frac{g_n(\zeta_1,\zeta_2,\ldots,\zeta_N)}{s^{1+n\alpha}},\quad s>0. \tag{23}$$

In order to determine the coefficient $g_n(\zeta_1,\zeta_2,\ldots,\zeta_N)$ of the series in (23), we can construct the $\frac{t^\rho}{\rho}$-Laplace residual functions (GLRF) of the Equation (21) as stated below:

$$\mathcal{L}_{\frac{t^\rho}{\rho}} Res(\zeta_1, \zeta_2, ..., \zeta_N, s) = \mathcal{V}(\zeta_1, \zeta_2, ..., \zeta_N, s) - \frac{g_0(\zeta_1, \zeta_2, ..., \zeta_N)}{s}$$

$$- \frac{1}{2s^\alpha} \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_i \sigma_j \varrho_{ij} \left\{ \frac{\partial^2}{\partial \zeta_i \partial \zeta_j} \mathcal{V}(\zeta_1, \zeta_2, ..., \zeta_N, s) \right\}, \tag{24}$$

and the kth-GLRF as:

$$\mathcal{L}_{\frac{t^\rho}{\rho}} Res_k(\zeta_1, \zeta_2, ..., \zeta_N, s) = \mathcal{V}_k(\zeta_1, \zeta_2, ..., \zeta_N, s) - \frac{g_0(\zeta_1, \zeta_2, ..., \zeta_N)}{s}$$

$$- \frac{1}{2s^\alpha} \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_i \sigma_j \varrho_{ij} \left\{ \frac{\partial^2}{\partial \zeta_i \partial \zeta_j} \mathcal{V}_k(\zeta_1, \zeta_2, ..., \zeta_N, s) \right\}. \tag{25}$$

Step 3: To obtain the first coefficient $g_1(\zeta_1, \zeta_2, ..., \zeta_N)$ in Equation (23), we substitute $\mathcal{V}_1(\zeta_1, \zeta_2, ..., \zeta_N, s) = \frac{g_0(\zeta_1, \zeta_2, ..., \zeta_N)}{s} + \frac{g_1(\zeta_1, \zeta_2, ..., \zeta_N)}{s^{1+\alpha}}$ as the 1st-GLRF, $\mathcal{L}_{\frac{t^\rho}{\rho}} Res_1(\zeta_1, \zeta_2, ..., \zeta_N, s)$ to obtain

$$\mathcal{L}_{\frac{t^\rho}{\rho}} Res_1(\zeta_1, \zeta_2, ..., \zeta_N, s) = \mathcal{V}_1(\zeta_1, \zeta_2, ..., \zeta_N, s) - \frac{g_0(\zeta_1, \zeta_2, ..., \zeta_N)}{s}$$

$$- \frac{1}{2s^\alpha} \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_i \sigma_j \varrho_{ij} \left\{ \frac{\partial^2}{\partial \zeta_i \partial \zeta_j} \mathcal{V}_1(\zeta_1, \zeta_2, ..., \zeta_N, s) \right\}$$

$$= \frac{g_1(\zeta_1, \zeta_2, ..., \zeta_N)}{s^{1+\alpha}} - \frac{1}{2s^\alpha} \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_i \sigma_j \varrho_{ij} \left\{ \frac{1}{s} \frac{\partial^2}{\partial \zeta_i \partial \zeta_j} g(\zeta_1, \zeta_2, ..., \zeta_N) \right.$$

$$\left. + \frac{1}{s^{1+\alpha}} \frac{\partial^2}{\partial \zeta_i \partial \zeta_j} g_1(\zeta_1, \zeta_2, ..., \zeta_N) \right\}, \tag{26}$$

By multiplying both sides of Equation (26) by $s^{1+\alpha}$, we obtain

$$s^{1+\alpha} \mathcal{L}_{\frac{t^\rho}{\rho}} Res_1(\zeta_1, \zeta_2, ..., \zeta_N, s) = g_1(\zeta_1, \zeta_2, ..., \zeta_N)$$

$$- \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_i \sigma_j \varrho_{ij} \left\{ \frac{\partial^2}{\partial \zeta_i \partial \zeta_j} g_0(\zeta_1, \zeta_2, ..., \zeta_N) \right.$$

$$\left. + \frac{1}{s^\alpha} \frac{\partial^2}{\partial \zeta_i \partial \zeta_j} g_1(\zeta_1, \zeta_2, ..., \zeta_N) \right\}. \tag{27}$$

Evaluating the limit of both sides of Equation (27) as s tends to infinity and using Equation (17) at $k=1$, $lim_{s\to\infty}\left(s^{1+\alpha}\mathcal{L}_{\frac{t^\rho}{\rho}}Res_1(\zeta_1,\,\zeta_2,\,...,\,\zeta_N,\,s)\right)=0$, we then have:

$$g_1(\zeta_1,\,\zeta_2,\,...,\,\zeta_N)=\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i\sigma_j\varrho_{ij}\frac{\partial^2}{\partial x_i\partial x_j}g_0(\zeta_1,\,\zeta_2,\,...,\,\zeta_N)$$

$$=\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i\sigma_j\varrho_{ij}\left[\frac{\partial^2}{\partial\zeta_i\partial\zeta_j}\max\left(\sum_{i=1}^{N}c_ie^{\zeta_i}-K,\,0\right)\right]$$

$$=\frac{1}{2}\sum_{i=1}^{N}\sigma_i^2\varrho_{ii}\max\left(c_ie^{\zeta_i},\,0\right). \tag{28}$$

Similarly, the second coefficient $g_2(\zeta_1,\,\zeta_2,\,...,\,\zeta_N)$ is found by substituting $k=2$ into Equation (23), giving us $\mathcal{V}_2(\zeta_1,\,\zeta_2,\,...,\,\zeta_N,\,s)=\frac{g(\zeta_1,\,\zeta_2,\,...,\,\zeta_N)}{s}+\frac{g_1(\zeta_1,\,\zeta_2,\,...,\,\zeta_N)}{s^{1+\alpha}}+\frac{g_2(\zeta_1,\,\zeta_2,\,...,\,\zeta_N)}{s^{1+2\alpha}}$. We then substitute this into the 2nd-GLRF, $\mathcal{L}_{\frac{t^\rho}{\rho}}Res_2(\zeta_1,\,\zeta_2,\,...,\,\zeta_N,\,s)$, and use the result in Equation (28), $g_1(\zeta_1,\,\zeta_2,\,...,\,\zeta_N)=\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i\sigma_j\varrho_{ij}\frac{\partial^2}{\partial x_i\partial x_j}g_0$ $(\zeta_1,\,\zeta_2,\,...,\,\zeta_N)$, to obtain:

$$\mathcal{L}_{\frac{t^\rho}{\rho}}Res_2(\zeta_1,\,\zeta_2,\,...,\,\zeta_N,\,s)=\mathcal{V}_2(\zeta_1,\,\zeta_2,\,...,\,\zeta_N,\,s)-\frac{g_0(\zeta_1,\,\zeta_2,\,...,\,\zeta_N)}{s}$$

$$-\frac{1}{2s^{\alpha}}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i\sigma_j\varrho_{ij}\left\{\frac{\partial^2}{\partial\zeta_i\partial\zeta_j}\mathcal{V}_2(\zeta_1,\,\zeta_2,\,...,\,\zeta_N,\,s)\right\}$$

$$=\frac{g_2(\zeta_1,\,\zeta_2,\,...,\,\zeta_N)}{s^{1+2\alpha}}$$

$$-\frac{1}{2s^{1+2\alpha}}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i\sigma_j\varrho_{ij}\frac{\partial^2}{\partial\zeta_i\partial\zeta_j}g_1(\zeta_1,\,\zeta_2,\,...,\,\zeta_N)$$

$$-\frac{1}{2s^{1+3\alpha}}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i\sigma_j\varrho_{ij}\frac{\partial^2}{\partial\zeta_i\partial\zeta_j}g_2(\zeta_1,\,\zeta_2,\,...,\,\zeta_N). \tag{29}$$

Multiplying $s^{1+2\alpha}$ on both sides of Equation (29), we get

$$s^{1+2\alpha}\mathcal{L}_{t^\rho}^{\rho} Res_2(\zeta_1, \zeta_2, ..., \zeta_N, s) = g_2(\zeta_1, \zeta_2, ..., \zeta_N)$$

$$-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i\sigma_j\varrho_{ij}\frac{\partial^2}{\partial\zeta_i\partial\zeta_j}g_1(\zeta_1, \zeta_2, ..., \zeta_N)$$

$$-\frac{1}{s^\alpha}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i\sigma_j\varrho_{ij}\frac{\partial^2}{\partial\zeta_i\partial\zeta_j}g_2(\zeta_1, \zeta_2, ..., \zeta_N). \tag{30}$$

Evaluating the limit of both sides of equation of Eequation (30) as s tends to infinity.and using the condition from Equation (17) at $k = 2$, $lim_{s\to\infty}\left(s^{1+2\alpha}\mathcal{L}_{\frac{t^\rho}{\rho}} Res_2(\zeta_1, \zeta_2, ..., \zeta_N, s)\right) = 0$, and (28), we then obtain $g_2(\zeta_1, \zeta_2, ..., \zeta_N)$ as

$$g_2(\zeta_1, \zeta_2, ..., \zeta_N) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i\sigma_j\varrho_{ij}\frac{\partial^2}{\partial\zeta_i\partial\zeta_j}g_1(\zeta_1, \zeta_2, ..., \zeta_N)$$

$$= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i\sigma_j\varrho_{ij}\frac{\partial^2}{\partial\zeta_i\partial\zeta_j}\left[\frac{1}{2}\sum_{i=1}^{N}\sigma_i^2\varrho_{ii}\max\left(c_ie^{\zeta_i}, 0\right)\right]$$

$$= \left(\frac{1}{2}\right)^2\sum_{i=1}^{N}(\sigma_i^2\varrho_{ii})^2\max\left(c_ie^{\zeta_i}, 0\right). \tag{31}$$

Proceeding to obtain the third coefficient $g_3(\zeta_1, \zeta_2, ..., \zeta_N)$ from Equation (22), we substitute $\mathcal{V}_3(\zeta_1, \zeta_2, ..., \zeta_N, s) = \frac{g_0(\zeta_1, \zeta_2, ..., \zeta_N)}{s} + \frac{g_1(\zeta_1, \zeta_2, ..., \zeta_N)}{s^{1+\alpha}} + \frac{g_2(\zeta_1, \zeta_2, ..., \zeta_N)}{s^{1+2\alpha}} + \frac{g_3(\zeta_1, \zeta_2, ..., \zeta_N)}{s^{1+3\alpha}}$ into the 3rd-GLRF, $\mathcal{L}_{\frac{t^\rho}{\rho}} Res_3(\zeta_1, \zeta_2, ..., \zeta_N, s)$, and use the resulting expressions in Equations (28) and (31), we obtain

$$\mathcal{L}_{\frac{t^\rho}{\rho}} Res_3(\zeta_1, \zeta_2, ..., \zeta_N, s) = \mathcal{V}_3 - \frac{g_0(\zeta_1, \zeta_2, ..., \zeta_N)}{s} - \frac{1}{2s^\alpha}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i\sigma_j\varrho_{ij}\frac{\partial^2 V_3}{\partial x_i\partial x_j}$$

$$= \frac{g_3}{s^{1+3\alpha}} - \frac{1}{2s^{a+3\alpha}}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i\sigma_j\varrho_{ij}\frac{\partial^2}{\partial\zeta_i\partial\zeta_j}g_2(\zeta_1, \zeta_2, ..., \zeta_N)$$

$$-\frac{1}{2s^\alpha}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i\sigma_j\varrho_{ij}\frac{\partial^2}{\partial\zeta_i\partial\zeta_j}g_3(\zeta_1, \zeta_2, ..., \zeta_N). \tag{32}$$

By multiplying both sides of Equation (32) by $s^{1+3\alpha}$, we derive

$$s^{1+3\alpha}\mathcal{L}_{\frac{t^\rho}{\rho}} Res_3(\zeta_1, \zeta_2, ..., \zeta_N, s) = g_3(\zeta_1, \zeta_2, ..., \zeta_N)$$

$$- \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i\sigma_j\varrho_{ij}\frac{\partial^2}{\partial\zeta_i\partial\zeta_j}g_2(\zeta_1, \zeta_2, ..., \zeta_N)$$

$$- \frac{1}{2s^\alpha}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i\sigma_j\varrho_{ij}\frac{\partial^2}{\partial\zeta_i\partial\zeta_j}g_3(\zeta_1, \zeta_2, ..., \zeta_N). \quad (33)$$

Taking the limit as s approaches infinity for both sides of Equation (33) and using Equation (17) at $k = 3$, $lim_{s\to\infty}\left(s^{1+3\alpha}\mathcal{L}_{\frac{t^\rho}{\rho}} Res_3(\zeta_1, \zeta_2, ..., \zeta_N, s)\right) = 0$, and (31), we then obtain

$$g_3(\zeta_1, \zeta_2, ..., \zeta_N) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i\sigma_j\varrho_{ij}\frac{\partial^2}{\partial\zeta_i\partial\zeta_j}g_2(\zeta_1, \zeta_2, ..., \zeta_N)$$

$$= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i\sigma_j\varrho_{ij}\frac{\partial^2}{\partial\zeta_i\partial\zeta_j}\left[\left(\frac{1}{2}\right)^2\sum_{i=1}^{N}(\sigma_i^2\varrho_{ii})^2 \max\left(c_i e^{\zeta_i}, 0\right)\right]$$

$$= \left(\frac{1}{2}\right)^3\sum_{i=1}^{N}(\sigma_i^2\varrho_{ii})^3 \max\left(c_i e^{\zeta_i}, 0\right) \quad (34)$$

Similarly, the value of the coefficient $g_k(\zeta_1, \zeta_2, ..., \zeta_N)$ in Equation (22) can be determined by:

$$g_k(\zeta_1, \zeta_2, ..., \zeta_N) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i\sigma_j\varrho_{ij}\frac{\partial^2}{\partial\zeta_i\partial\zeta_j}g_{k-1}(\zeta_1, \zeta_2, ..., \zeta_N), \text{ for } k \geq 1,$$

where $g_0(\zeta_1, \zeta_2, ..., \zeta_N) = max\left(\sum_{i=1}^{N} c_i e^{\zeta_i} - K, 0\right)$ or,

$$g_k(\zeta_1, \zeta_2, ..., \zeta_N) = \left(\frac{1}{2}\right)^k\sum_{i=1}^{N}(\sigma_i^2\varrho_{ii})^k \max\left(c_i e^{\zeta_i}, 0\right), \text{ for } k \geq 1. \quad (35)$$

Thus, the series solution of Equation (21) can be summarized as

$$\mathcal{V}(\zeta_1, \zeta_2, ..., \zeta_N, s) = \frac{g_0(\zeta_1, \zeta_2, ..., \zeta_N)}{s}$$

$$+ \sum_{n=1}^{\infty}\frac{1}{s^{1+n\alpha}}\left(\frac{1}{2}\right)^n\sum_{i=1}^{N}(\sigma_i^2\varrho_{ii})^n \max\left(c_i e^{\zeta_i}, 0\right). \quad (36)$$

Step 4: Ultimately, execute the inverse Laplace transform on Equation (36) to derive the solution of the time-fractional multi-asset Black-Scholes Equation (5) with the beginning conditions (6), articulated as follows:

$$v(\zeta_1, \zeta_2, \ldots, \zeta_N, t) = \max\left(\sum_{i=1}^{N} c_i e^{\zeta_i} - K, 0\right)$$

$$+ \sum_{n=1}^{\infty} \frac{\left(\frac{t^\rho}{\rho}\right)^{n\alpha}}{\Gamma(1+n\alpha)} \left(\frac{1}{2}\right)^n \sum_{i=1}^{N} (\sigma_i^2 \varrho_{ii})^n \max\left(c_i e^{\zeta_i}, 0\right)$$

$$= \max\left(\sum_{i=1}^{N} c_i e^{\zeta_i} - K, 0\right)$$

$$+ \sum_{i=1}^{N} \max\left(c_i e^{\zeta_i}, 0\right) \sum_{n=1}^{\infty} \frac{1}{\Gamma(1+n\alpha)} \left(\frac{\sigma_i^2 \varrho_{ii}}{2} \left(\frac{t^\rho}{\rho}\right)^\alpha\right)^n$$

or

$$v(\zeta_1, \zeta_2, \ldots, \zeta_N, t) = \max\left(\sum_{i=1}^{N} c_i e^{\zeta_i} - K, 0\right)$$

$$+ \sum_{i=1}^{N} \max\left(c_i e^{\zeta_i}, 0\right) \left[E_\alpha\left(\frac{\sigma_i^2 \varrho_{ii}}{2} \left(\frac{t^\rho}{\rho}\right)^\alpha\right) - 1\right] \tag{37}$$

where $x_i \in [0, a_i]$, $t \in [0, T]$ and E_α is the ML function with order α.

6. Discussions and numerical outcomes

This section employs Matlab R2022b software to illustrate the analytical solution graph for a European call option for n-assets in the time-fractional Black-Scholes model, which is based on the left-side Caputo-type Katugampola fractional derivative. The graph of the solution will be illustrated using the example of two assets. The parameters have been established in accordance with [21], as illustrated in the Table 2.

Table 2. The parameter values utilized in the numerical calculations

Parameters	Value
The strike price (dollars): K	1.0
The risk free rate of interest: r	0.1
The expiration date (year): t	10.0
The volatility of the underlying stock 1: σ_1	0.2
The volatility of the underlying stock 2: σ_2	0.3
The proportion of investment on asset 1: β_1	0.6
The proportion of investment on asset 2: β_2	0.4
The correlation coefficients between σ_1 and σ_2: ρ_{12}	0.5

The approximate analytical solution graphs for (5) and (6) are plotted considering a series with 2 terms and setting the fractional derivative α to the values $\alpha = 0.25$, $\alpha = 0.5$, $\alpha = 1$, $\alpha = 2$, $\rho = 0.6$ when the assets are $0 < x, y < 1$ at time $t = 0$, as shown in Figures 1 to 4. The graph of option prices in Equation (37) corresponds to assets, x_1 and x_2 in Figures 5 and 6, it shows consistent results.

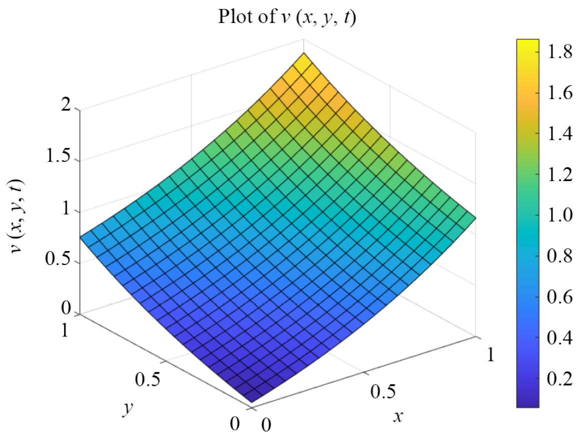

Figure 1. $\alpha = 0.25$

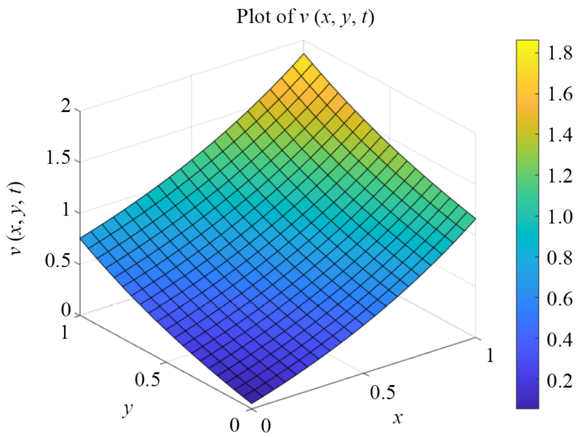

Figure 2. $\alpha = 0.5$

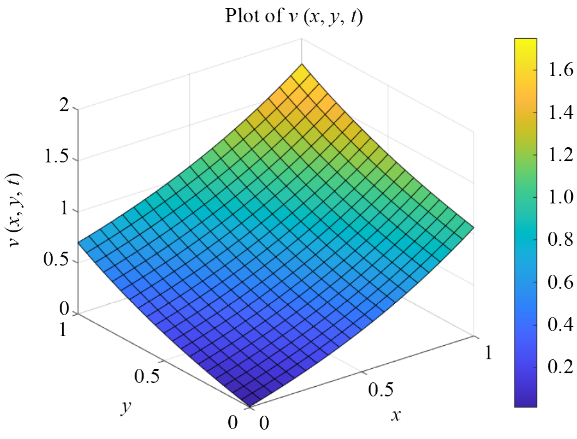

Figure 3. $\alpha = 0.75$

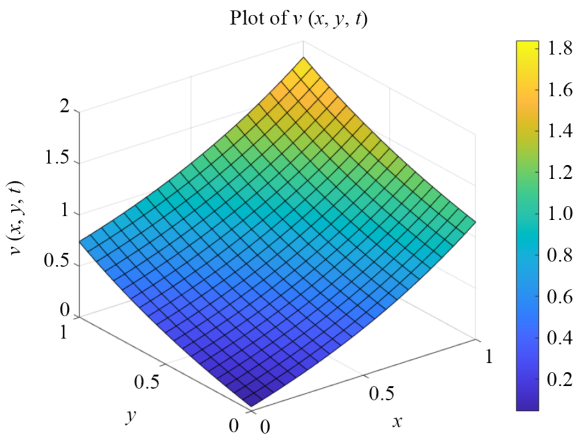

Figure 4. $\alpha = 1$

The value of European call options is lower when $\alpha < 1$ compared to when $\alpha = 1$, as shown in the Table 3 and Figures 5-6. This means that there is a direct variation relationship between α and v; that is, as the amount of α decreases, the value of European call options also decreases.

Table 3. The parameters utilized in the numerical calculations are assigned specific values

x	y	ρ	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$	x	y	ρ	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$
0	0	0.6	0.053981	0.053837	0.050023	0.044268	0.2	0	0.6	0.19655	0.19662	0.19246	0.18592
		0.8	0.066422	0.065993	0.060965	0.05362			0.8	0.21082	0.21066	0.20521	0.19694
		1	0.089045	0.087953	0.080517	0.070125			1	0.23654	0.23578	0.22781	0.21625
		2	0.90222	0.84476	0.70725	0.55313			2	1.1083	1.0526	0.91236	0.75239
	0.2	0.6	0.14477	0.14437	0.14007	0.13382		0.2	0.6	0.28734	0.28716	0.2825	0.27547
		0.8	0.15813	0.15734	0.15162	0.14358			0.8	0.30253	0.30201	0.29587	0.28689
		1	0.18267	0.181	0.17245	0.16093			1	0.33016	0.32883	0.31975	0.30705
		2	1.1173	1.0454	0.88013	0.69773			2	1.3234	1.2532	1.0852	0.89699
	0.4	0.6	0.25566	0.25495	0.25005	0.24319		0.4	0.6	0.39822	0.39774	0.39248	0.38485
		0.8	0.27014	0.26892	0.26236	0.25345			0.8	0.41454	0.41358	0.4066	0.39677
		1	0.29701	0.29464	0.28474	0.27183			1	0.44451	0.44247	0.43203	0.41796
		2	1.3799	1.2905	1.0913	0.87435			2	1.586	1.4983	1.2964	1.0736
	0.6	0.6	0.3911	0.39001	0.38438	0.37678		0.6	0.6	0.53366	0.5328	0.52682	0.51844
		0.8	0.40695	0.4052	0.3976	0.38765			0.8	0.55136	0.54986	0.54184	0.53096
		1	0.43668	0.43345	0.42189	0.4073			1	0.58418	0.58128	0.56918	0.55342
		2	1.7007	1.5898	1.3492	1.0901			2	1.9068	1.7976	1.5543	1.2893
	0.8	0.6	0.55653	0.55498	0.54846	0.53995		0.8	0.6	0.69909	0.69777	0.69089	0.6816
		0.8	0.57406	0.57165	0.56279	0.55156			0.8	0.71846	0.71631	0.70704	0.69487
		1	0.60727	0.60299	0.5894	0.57275			1	0.75476	0.75082	0.7367	0.71888
		2	2.0925	1.9554	1.6642	1.3536			2	2.2986	2.1632	1.8693	1.5528
	1	0.6	0.75858	0.75647	0.74886	0.73924		1	0.6	0.90115	0.89926	0.89129	0.8809
		0.8	0.77816	0.77495	0.76456	0.75175			0.8	0.92256	0.91961	0.9088	0.89507
		1	0.81562	0.81006	0.794	0.77484			1	0.96312	0.95789	0.9413	0.92096
		2	2.5711	2.402	2.0489	1.6754			2	2.7772	2.6097	2.254	1.8746
0.4	0	0.6	0.37068	0.37102	0.36642	0.35894	0.6	0	0.6	0.58336	0.58404	0.57891	0.57027
		0.8	0.38719	0.38735	0.38138	0.37199			0.8	0.60261	0.60316	0.59657	0.5858
		1	0.4167	0.41635	0.40772	0.39473			1	0.63674	0.63688	0.62746	0.61272
		2	1.3601	1.3063	1.1629	0.99577			2	1.6676	1.6163	1.4689	1.293
	0.2	0.6	0.46147	0.46156	0.45647	0.44849		0.2	0.6	0.67415	0.67457	0.66895	0.65982
		0.8	0.4789	0.4787	0.47204	0.46194			0.8	0.69432	0.69451	0.68723	0.67575
		1	0.51032	0.50939	0.49965	0.48553			1	0.73036	0.72993	0.71939	0.70353
		2	1.5751	1.507	1.3358	1.1404			2	1.8826	1.817	1.6418	1.4376
	0.4	0.6	0.57236	0.57214	0.56645	0.55787		0.4	0.6	0.78504	0.78515	0.77893	0.76919
		0.8	0.59091	0.59028	0.58277	0.57182			0.8	0.80634	0.80609	0.79796	0.78562
		1	0.62466	0.62303	0.61194	0.59644			1	0.84471	0.84357	0.83168	0.81443
		2	1.8378	1.7521	1.5469	1.317			2	2.1453	2.062	1.8529	1.6143

Table 3. (cont.)

x	y	ρ	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$	x	y	ρ	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$
	0.6	0.6	0.7078	0.7072	0.70078	0.69146		0.6	0.6	0.92048	0.92022	0.91327	0.90278
		0.8	0.72773	0.72655	0.71802	0.70601			0.8	0.94315	0.94237	0.9332	0.91982
		1	0.76433	0.76184	0.74909	0.7319			1	0.98437	0.98238	0.96883	0.94989
		2	2.1586	2.0514	1.8048	1.5327			2	2.4661	2.3614	2.1108	1.83
	0.8	0.6	0.87322	0.87217	0.86486	0.85426		0.8	0.6	1.0895	1.0852	1.0773	1.0659
		0.8	0.89483	0.893	0.88321	0.86992			0.8	1.1103	1.1088	1.0984	1.0837
		1	0.93492	0.93138	0.91661	0.89735			1	1.155	1.1519	1.1363	1.1153
		2	2.5504	2.417	2.1198	1.7962			2	2.8579	2.727	2.4258	2.0935
	1	0.6	1.0753	1.0737	1.0653	1.0539		1	0.6	1.288	1.2867	1.2777	1.2652
		0.8	1.0989	1.0963	1.085	1.0701			0.8	1.3144	1.3121	1.3002	1.2839
		1	1.1433	1.1385	1.1212	1.0994			1	1.3633	1.359	1.3409	1.3174
		2	3.029	2.8635	2.5046	2.118			2	3.3365	3.1735	2.8106	2.4153
0.8	0	0.6	0.84313	0.84421	0.83843	0.82838	1	0	0.6	1.1604	1.162	1.1554	1.1436
		0.8	0.86573	0.86676	0.85939	0.84694			0.8	1.1871	1.1887	1.1804	1.1659
		1	0.9055	0.90625	0.89585	0.87898			1	1.2338	1.2353	1.2237	1.2042
		2	2.0432	1.995	1.8426	1.6561			2	2.5019	2.4574	2.2991	2.0996
	0.2	0.6	0.93392	0.93475	0.92848	0.91793		0.2	0.6	1.2512	1.2525	1.2455	1.2332
		0.8	0.9574	0.95811	0.95005	0.93689			0.8	1.2788	1.2801	1.2711	1.2559
		1	0.99912	0.99929	0.98778	0.96978			1	1.3274	1.3283	1.3156	1.295
		2	2.2582	2.1956	2.0155	1.8007			2	2.7169	2.658	2.472	2.2442
	0.4	0.6	1.0448	1.0453	1.0385	1.0273		0.4	0.6	1.3621	1.3631	1.3555	1.3426
		0.8	1.0695	1.0697	1.0608	1.0468			0.8	1.3908	1.3916	1.3818	1.3657
		1	1.1135	1.1129	1.1001	1.0807			1	1.4417	1.4419	1.4279	1.4059
		2	2.5209	2.4407	2.2266	1.9773			2	2.9796	2.9031	2.6831	2.4208
	0.6	0.6	1.1803	1.1804	1.1728	1.1609		0.6	0.6	1.4975	1.4982	1.4898	1.4762
		0.8	1.2063	1.206	1.196	1.181			0.8	1.5276	1.5279	1.517	1.4999
		1	1.2531	1.2517	1.2372	1.2162			1	1.5814	1.5807	1.565	1.5414
		2	2.8416	2.74	2.4845	2.1931			2	3.3004	3.2024	2.941	2.6365
	0.8	0.6	1.3457	1.3454	1.3369	1.3241		0.8	0.6	1.663	1.6631	1.6539	1.6393
		0.8	1.3734	1.3724	1.3612	1.3449			0.8	1.6974	1.6944	1.6822	1.6638
		1	1.4237	1.4213	1.4047	1.3816			1	1.752	1.7503	1.7325	1.7068
		2	3.2335	3.1056	2.7995	2.4565			2	3.6922	3.568	3.256	2.4
	1	0.6	1.5477	1.5468	1.5373	1.5234		1	0.6	1.865	1.8646	1.8543	1.8386
		0.8	1.5775	1.5757	1.563	1.5451			0.8	1.8988	1.8977	1.884	1.864
		1	1.6321	1.6284	1.6093	1.5837			1	1.9603	1.9574	1.9371	1.9089
		2	3.712	3.5521	3.1843	2.7784			2	4.1708	4.0146	3.6408	3.2218

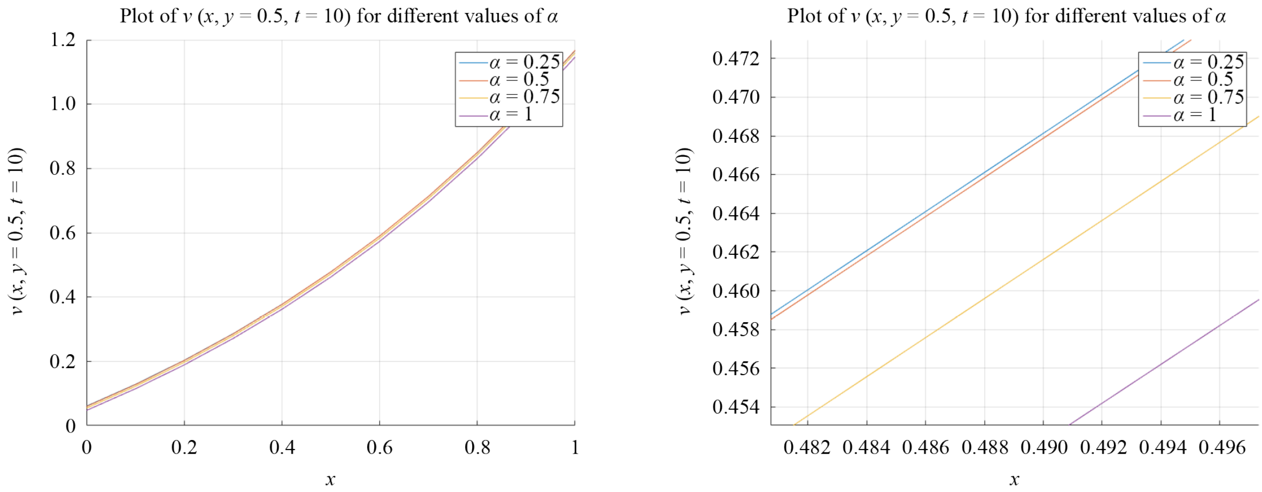

Figure 5. Options values for $\alpha = 0.25, 0.5, 0.75, 1$, years at $t = 10$, $y = 0.5$

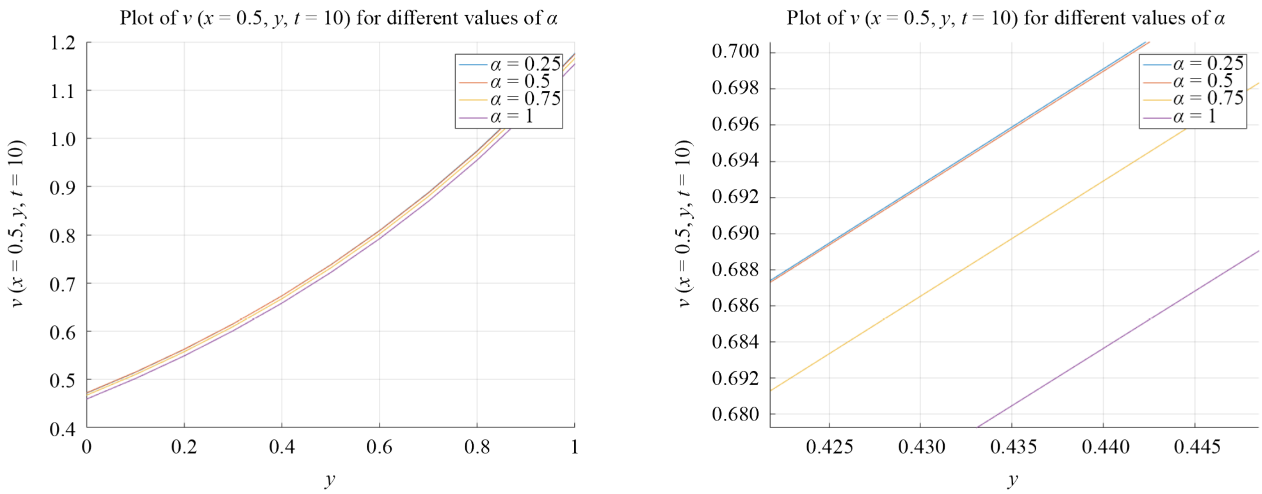

Figure 6. Options values for $\alpha = 0.25, 0.5, 0.75, 1$, years at $t = 10$, $x = 0.5$

7. Conclusion

In the field of option pricing theory, the Black-Scholes equation is widely considered one of the most important models. This article presents a modification of the classical Black-Scholes equation into the fractional multi-asset Black-Scholes equation, utilizing the left-side Caputo-type Katugampola fractional derivative. In this article, the modified multi-asset Black-Scholes equation is solved analytically using the $\frac{t^\rho}{\rho}$-Laplace residual power series. Furthermore, we demonstrate that the solution to the classical Black-Scholes equation can be regarded as a particular case of the proposed analytical solution. This indicates that the $\frac{t^\rho}{\rho}$-Laplace residual power series technique is a very successful strategy for identifying analytical solutions to fractional-order differential equations. The modified Black-Scholes equation offers the advantage of including two parameters in the formulation of the fractional derivative: ρ and α. By precisely calculating the values of these two parameters, the option prices derived from the modified form will closely align with the market values of the option prices. To determine the suitable values for two parameters, we can utilize the genetic algorithm or machine learning methods with the real value of the option. The determination of option pricing can be achieved through the solution of the modified Black-Scholes equation, contingent upon the establishment of suitable values for these two parameters.

Funding

This research was funded by King Mongkut's University of Technology North Bangkok, contract no KMUTNB-65-NEW-02.

Conflict of interest

The authors declare no competing financial interest.

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