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Research Article

The Weak Irreducibility Markov Chains

Ren Zhang[1], Shiling Zhao[1], Xia Wang[1], Kaijun Leng[2*]

[1]Primary Education College, Wuhan City Polytechnic, WuHan, China
[2]Research Center of Hubei Logistics Development, Hubei University of Economics, WuHan, China
E-mail: lengkaijun@hbue.edu.cn

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Abstract: Irreducibility serves as the foundational concept in the theoretical study of Markov chains. This paper introduces a generalized notion termed weak irreducibility and investigates its stochastic stability properties for Markov chains defined on countable state spaces. We establish the existence of invariant measures and stationary distributions for weak irreducibility Markov chains and provide an illustrative example demonstrating its practical relevance. Moreover, we extend the framework of weak irreducibility to general state spaces without requiring the prior assumptions of separability and the existence of a reference measure.

***Keywords*:** weak irreducibility markov chains, times series analysis method, communicate, accessible, small set

MSC: 60J20, 60J10

1. Introduction

The stochastic stability theory of Markov chains has achieved significant progress over the past century, with applications spanning time series analysis, Markov chain Monte Carlo methods, statistical physics, and economics [1–5]. These developments rely on fundamental stability properties-including irreducibility, recurrence, existence of invariant measures and stationary distributions, and ergodicity-which underpin the theoretical framework of Markov chains.

Irreducibility serves as the foundational concept in the theoretical study of Markov chain stability, with our analysis focusing specifically on its structural properties. We consider a Markov chain $Z = \{Z_n\}_{n\geq 0}$ defined on a state space Y with generated Borel σ-field $\Delta(Y)$. The motion of Z is governed by an overall probability law $\mathbb{P}$. The one-step probability kernel are denoted by

$$P(y, D) := \mathbb{P}(Z_{n+1} \in D \mid Z_n = y) \text{ for all } y \in Y \text{ and } D \in \Delta(Y).$$

The n-step transition probabilities are recursively determined by the Chapman-Kolmogorov equations and satisfy

$$P^n(y, D) = \mathbb{P}(Z_n \in D \mid Z_0 = y).$$

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The notion of irreducibility was formalized by Kolmogorov [6] through the concept of communicating classes on countable spaces, later extended by Chung [7] and Feller [8] to analyze state properties within equivalence classes. To characterize irreducibility on countable spaces, we need two key relations:

• Accessibility: State $i \in Y$ is accessible from $k \in Y$ (denoted $k \to i$; see Figure 1a) if there exists $t > 0$ such that $P^t(k, \{i\}) > 0$. This implies that Z can transition from k to i in finitely many steps.

• Communication: States k and i communicate (denoted $k \leftrightarrow i$) when both $k \to i$ and $i \to k$ hold (Figure 1b).

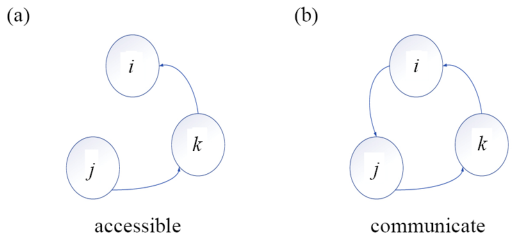

Figure 1. Accessible and communicate

A Markov chain Z (or equivalently, its state space Y) is called irreducible on countable spaces if the communication relation generates a single equivalence class $C(k) = \{i \in Y : k \leftrightarrow i\} = Y$ for some $k \in Y$. This implies that all states $j, k \in Y$ communicate (a form of "double accessibility"), establishing the fundamental structural property of irreducibility.

However, the notion of communicating classes cannot be directly extended to general state spaces due to the absence of a well-defined communication framework. To address this limitation, ψ-irreducibility-a foundational concept in Markov chain theory-was fundamentally established by Doeblin [9, 10] for general state spaces and further developed by subsequent authors [11–14]. Building on this framework, Jain and Jamison [15] demonstrated the existence of small sets for ψ-irreducibility Markov chains. Leveraging these small sets, Nummelin [16] introduced the split chain $\check{Z}$ associated with a Markov chain Z, which retains nearly identical stochastic stability properties as Z. A key feature of $\check{Z}$ is the presence of reachable atoms, enabling the analysis of ergodic properties on general spaces through methodologies traditionally applied to countable spaces (see monograph [17]). Our primary objective is to unify the theoretical framework of stochastic stability for irreducible Markov chains across both countable and general state spaces.

According to monograph [17], the defining property of ψ-irreducibility lies in the existence of small sets accessible from all states. This insight motivates our heuristic approach to generalize irreducibility concepts from general state spaces to weak irreducibility frameworks. In this work, we formalize the notion of weak irreducibility for Markov chains defined on both countable and general state spaces. Unlike classical irreducibility requiring mutual communication, weak irreducibility characterizes a one-sided accessibility structure. Our contribution unfolds in two dimensions.

(i) Weak irreducibility Markov chains retain the core stochastic stability properties of classical irreducible chains on countable spaces, thereby expanding the applicability of stability theory. A concrete example demonstrates this framework's utility in practical settings.

(ii) The simplicity of countable spaces provides an accessible platform for understanding stability concepts. This enables straightforward construction of illustrative examples-such as recurrent but non-Harris recurrent chains, or positive recurrent yet non-regular chains-that become analytically challenging in general spaces. By establishing these foundational connections, we provide a critical pathway for extending stability analysis from discrete to continuous-state systems.

The paper is organized as follows: Section 2 establishes the foundational framework by introducing key concepts, notations, and core stochastic stability theory for classical Markov chains. Section 3 formalizes the notion of weak irreducibility and systematically investigates its stochastic stability properties on countable state spaces, culminating in an illustrative example demonstrating practical applications. Section 4 extends the theoretical framework to general state spaces, addressing the challenges inherent in non-separable settings and establishing connections with classical ψ-irreducibility. Finally, section 5 contains the conclusion.

2. Preliminaries

In this section, we present various notations, basic definitions, and critical results of Markov Chains on countable state space that will be used throughout the paper. The important symbols used in this paper are listed in Table 1.

Table 1. Quick reference table for symbols and terminology

Symbol	Description
$C(k)$	Equivalent class
τ_k	First return time
$L(j,\ k)$	First return time probabilities
η_k	Occupation time
$U(i,\ k)$	Potential kernel
m_{ik}	Expected return times
ξ	Invariant measures

The classical theory of irreducible Markov chains provides foundational stability results that will be contrasted with weak irreducibility frameworks in subsequent sections. To establish this comparison, we introduce key probabilistic quantities:

$$L(j,\ k) := \mathbb{P}(\tau_k < \infty \mid Z_0 = j) = \mathbb{P}\left(\bigcup_{n=1}^{\infty}\{Z_n = k\} \,\middle|\, Z_0 = j\right),$$

where $\tau_k = \inf\{n \geq 1 : Z_n = k\}$ denotes the first return time to state k for $j,\ k \in Y$. The condition $L(j,\ k) > 0$ is equivalent to the existence of some $m > 0$ satisfying $P^m(j,\ k) > 0$. A Markov chain Z is termed irreducible if there exists an essential state $i \in Y$ whose essential class satisfies $C(i) = Y$.

For stability analysis, we define the potential kernel $U(i,\ k)$ and expected return times m_{ik} via:

$$U(i,\ k) := \sum_{n=1}^{\infty} P^n(i,\ k) = \mathbb{E}_i[\eta_k],\ m_{ik} := \mathbb{E}_i[\tau_k],$$

where $\eta_k = \sum_{n=1}^{\infty} \mathbf{1}_{\{Z_n = k\}}$ represents the occupation time at state k.

A state k is classified as:

• Recurrent if $U(k,\ k) = \infty$ (equivalently, $L(k,\ k) = 1$),

• Positive recurrent if $m_{kk} < \infty$,

• Null recurrent otherwise.

These properties extend to the chain level:

• Z is recurrent if there exists a recurrent state k with $C(k) = Y$,

• Z is positive recurrent if there exists a positive recurrent state k satisfying $C(k) = Y$.

The following results summarize classical stochastic stability theory, as established in classical references [18, 19].

Proposition 1 For a Markov chain Z on countable state space Y and $k \in Y$:
(i) If k is essential, then all $j \in C(k)$ are essential.
(ii) If k is recurrent, then k is essential and all $j \in C(k)$ are recurrent.
(iii) If k is positive recurrent, then k is recurrent (hence essential), and all $j \in C(k)$ are positive recurrent.

Proposition 2 (Ratio limit theorem) For a Markov chain Z on countable state space Y and states $i,\ m \in Y$, the following limit holds:

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} P^k(i,\ m)}{\sum_{k=0}^{n} P^k(m,\ m)} = L(i,\ m).$$

Proposition 3 For an irreducible Markov chain Z on countable state space Y, the following are equivalent:
(i) Z is recurrent.
(ii) There exists $k \in Y$ such that

$$\{j \in Y : U(j,\ k) = \infty\} = Y, \tag{1}$$

(iii) There exists $k \in Y$ such that

$$\{j \in Y : L(j,\ k) = 1\} = Y. \tag{2}$$

Proposition 4 For an irreducible Markov chain Z on countable state space Y, the following are equivalent:
(i) Z is positive recurrent.
(ii) There exists $k \in Y$ such that

$$\{j \in Y : m_{jk} = \infty\} = Y. \tag{3}$$

Among various stability concepts, invariant measures represent the strongest form of stability. A σ-finite measure $\xi = \{\xi(k)\}_{k\in Y}$ on $(Y,\ \Delta(Y))$ is termed an invariant measure if it satisfies the balance equation

$$\sum_{k\in Y} \xi(k)P(k,\ j) = \xi(j) \text{ for all } j \in Y.$$

When ξ is a probability measure, it is called a stationary distribution.

Proposition 5 If a Markov chain Z is recurrent, then it has a unique (up to constant multiples) invariant measure.

Proposition 6 If a Markov chain Z is positive recurrent, then it admits a unique stationary distribution ξ given by

$$\xi(j) = \frac{1}{\mathbb{E}_j[\tau_j]} \text{ for all } j \in Y.$$

3. The weak irreducibility Markov chain on countable state space

In this section, we discuss the weak irreducibility and other weak stochastic stability properties of Markov chains on a countable state space. In section 2, we reviewed the classical notions of irreducibility and related stochastic stability properties based on communication classes $C(i)$. We now generalize these concepts through accessible classes $\overline{C}(i) := \{j \in Y : j \to i\}$, which relax mutual communication requirements. By definition, $C(i) \subset \overline{C}(i)$ holds trivially for all $i \in Y$.

Definition 1 For a Markov chain Z on countable state space Y:

(i) Weak irreducibility: There exists an essential state $k \in Y$ such that $\overline{C}(k) = Y$.

(ii) Weak recurrence: There exists a recurrent state $k \in Y$ satisfying $\overline{C}(k) = Y$.

(iii) Weak positive recurrence: There exists a positive recurrent state $k \in Y$ with $\overline{C}(k) = Y$.

(iv) Weak Harris recurrence: There exists $k \in Y$ such that $\{j \in Y : L(j,\ k) = 1\} = Y$.

(v) Weak regularity: There exists $k \in Y$ satisfying $\{j \in Y : m_{jk} < \infty\} = Y$.

The connection between classical irreducibility and its weak counterpart becomes evident through the following result:

Theorem 1 If a Markov chain Z is irreducible, then it is weak irreducibility.

Proof. Let Z be irreducible. For any $i,\ j \in Y$, we have $i \leftrightarrow j$ by definition, implying $j \in \overline{C}(i)$. Thus, $\overline{C}(i) = Y$ for all i, establishing weak irreducibility.

The weak irreducibility of a Markov chain Z on countable space Y is characterized by the following structural decomposition:

Theorem 2 A Markov chain Z is weakly irreducible if and only if there exists an essential state $k \in Y$ such that Y admits the disjoint decomposition

$$Y = C(k) \cup D,$$

where $D = Y - C(k) = \{j \in Y : j \to k \text{ but } k \nrightarrow j\}$.

Proof. Necessity: Assume Z is weak irreducibility. By Definition 1(i), there exists $k \in Y$ such that $\{j \in Y : j \to k\} = Y$. This implies

$$Y = \{j \in Y : j \leftrightarrow k\} \cup \{j \in Y : j \to k \text{ but } k \nrightarrow j\} = C(k) \cup D.$$

Sufficiency: Conversely, suppose $Y = C(k) \cup D$ with D defined as above. For any $j \in D$, we have $j \to k$ by construction. Since $C(k) \subset \{j \in Y : j \to k\}$, it follows that $\{j \in Y : j \to k\} = Y$, establishing weak irreducibility.

These structural differences between classical irreducibility and weak irreducibility lead to distinct stochastic stability hierarchies, as summarized in Table 2.

Table 2. Structural differences table between classical and weak irreducibility

Irreducible	Weakly irreducibee
Harris recurrent $\Leftrightarrow$ Recurrent	Weak Harris recurrent $\Rightarrow$ Weak recurrent
Regular $\Leftrightarrow$ Positive recurrent	Weak regular $\Rightarrow$ Weak positive recurrent

Theorem 3 For a Markov chain Z on countable state space Y, the following are equivalent.

(i) The Markov chain Z is weak recurrence.

(ii) There exists $k \in Y$ satisfying

$$\{j \in Y : U(j,\ k) = \infty\} = Y.$$

Proof. (i)⇒ (ii): There exists a recurrent state $k \in Y$ with $\overline{C}(k) = Y$. For all $j \in Y$, we have $U(k,\ k) = \infty$ and $L(j,\ k) > 0$ by construction. By Proposition 2, this implies $U(j,\ k) = \infty$, (i) implies (ii) be proved.

(ii) ⇒ (i): Conversely, suppose condition (ii) holds for some $k \in Y$. Then $U(j,\ k) = \infty$ for all $j \in Y$, which implies $j \to k$ for all j. Thus, $\overline{C}(k) = Y$, and k is recurrent. This confirms weak recurrence.

Theorem 4 If a Markov chain Z is weak Harris recurrence, then it is weak recurrence.

Proof. Assume Z is weakly Harris recurrent. from the $L(j,\ k) > 0$ for all $j \in Y$ and some k, then we have $\bar{C}(k) = Y$. The $L(k,\ k) = 1$ implies $U(k,\ k) = \infty$. Thus $\{j \in Y : U(j,\ k) = \infty\} = Y$ from Proposition 2 and the result is proved.

Theorem 5 If a Markov chain Z is weak regularity, then it is weak positive recurrence.

Proof. Assume Z is Weak regularity. By Definition 1vi, there exists $k \in Y$ such that $\{j \in Y : m_{jk} < \infty\} = Y$. Since $m_{jk} < \infty$ for all $j \in Y$, it follows that k is positive recurrent. By Theorem 3, $\overline{C}(k) = Y$, establishing weak positive recurrence.

Remark 1 Comparing Theorem 3 and Proposition 3 reveals critical differences in stability criteria.

1. Formulations (1) and (2) are equivalent when Z is irreducible,namely $C(i) = Y$ is satisfied.

2. This equivalence breaks down for weak irreducibility chains. Specifically, (1) and (2) imply $\overline{C}(k) = Y$ but not vice versa.

3. The converse of Theorem 4 is false. A counter example is provided in Example 1, where weak recurrence does not guarantee Harris recurrence.

These distinctions highlight the necessity of stronger assumptions for classical stability properties to persist under weak irreducibility.

Remark 2 From Proposition 4, we observe that the equivalence between formulations (3) and weak positive recurrence holds, but fails for weak irreducibility chains. This structural divergence directly implies.

1. The converse of Theorem 5 does not hold in general.

2. Weak Harris positive recurrence does not necessarily imply weak regularity.

These observations are rigorously demonstrated through explicit counter examples. In particular, Examples 1-3 construct Markov chains with absorbing states (see Figure 2) that violate the converse statements of Theorems 4 and 5.

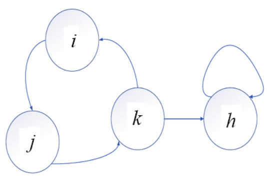

Figure 2. Markov chain with absorbing state h

Example 1 Let Z be an irreducible and transient Markov chain on countable state space Y with transition matrix P. Take $j_0 \in X$ and define the modified kernel $\tilde{P}$ via:

$$\widetilde{P}(j,\ k) = \begin{cases} P(j,\ k),\ j \neq j_0, \\ \\ 1,\ j = k = j_0. \end{cases} \tag{4}$$

The induced chain $\tilde{Z}$ has an absorbing state j_0. We claim that $\tilde{Z}$ is weak recurrence but not weakly Harris recurrence.

Proof. By construction, $\tilde{Z}$ and Z share identical transition dynamics until reaching j_0. This implies

$$\tilde{L}(k,\ j_0) = L(k,\ j_0) \text{ for all } k \neq j_0.$$

Since Z is irreducible and transient, and notice that $\widetilde{L}(j_0,\ j_0) = 1$, then $0 < L(j,\ k) < 1$ for all $j,\ j \in Y$ and $\{j \in Y : \widetilde{L}(j,\ j_0) > 0\} = Y$. Hence Z is weak recurrence.

To disprove weak Harris recurrence, observe that

$$\{j \in Y : \tilde{L}(j,\ J_0) = 1\} \neq Y.$$

and $k \neq j_0$, we have $\widetilde{L}(j,\ k) = L(j,\ k) < 1$ for $j \neq j_0$, and so $\{j \in Y : \widetilde{L}(j,\ k) = 1\} \neq Y$, confirming that $\tilde{Z}$ is not weakly Harris recurrent. The proof is complete.

Example 2 Let $\tilde{Z}$ be the Markov chain constructed in Example 1. Then $\tilde{Z}$ is weak positive recurrence but not weak regularity.

Proof. By Example 1, $\tilde{Z}$ is weak recurrence with absorbing state J_0. The result are obvious from $\widetilde{m}_{j_0 j_0} = 1$ and $\widetilde{m}_{i j_0} = m_{i j_0} = \infty$ for $i \neq j_0$, establishing weak positive recurrence and this violates the condition for weak regularity. The proof is complete.

Example 3 Suppose Z is irreducible and null recurrent in Example 1. Then the modified chain $\tilde{Z}$ is weak Harris recurrent but not weak regularity.

Proof. From the construction (4), we have $\widetilde{L}(k,\ j_0) = 1$ for $k \in Y$, that is $\{k \in Y : \widetilde{L}(k,\ j_0) = 1\} = Y$. So $\widetilde{Z}$ is weak Harris recurrent.

To disprove weak regularity,since $\widetilde{m}_{j_0 j_0} = 1 < \infty$ for $j = j_0$ and $\widetilde{m}_{j j_0} = m_{j j_0} = \infty$, $\widetilde{m}_{jj} = m_{jj} = \infty$ for $j \neq j_0$. It follows that $\{j \in Y : \widetilde{m}_{ji} < \infty\} \neq Y$ for all $j \in Y$. This prove that $\widetilde{\Phi}$ is not weak regularity. This completes the proof.

We now demonstrate the existence of invariant measures and stationary distributions under weak irreducibility, paralleling classical results for irreducible chains (see Theorems 6-7).

Theorem 6 If a Markov chain Z on countable state space Y is weak recurrence, then it admits a unique stationary distribution (unique up to a scalar multiple).

Proof. For all $i,\ j \in Y$, let

$$e_{ij}^n = \mathbb{P}_i(Z_n = j,\ \Phi_v \neq i,\ 0 < v < n),\ e_{ij} = \sum_{i=1}^{n} e_{ij}^n.$$

Easily we have the following property of e_{ij}^n from the above definition.

$$e_{ij}^1 = P(i,\ j),\ e_{ii} = L(i,\ i),\ e_{ij}^n = \sum_{k \neq i} e_{ik}^{n-1} P(k,\ j).$$

We obtain

$$\sum_k e_{ik}P(k,\ j) = e_{ii}P(i,\ j) + \sum_{k\neq i} e_{ik}P(k,\ j)$$

$$= L(i,\ i)P(i,\ j) + \sum_{k\neq i}\sum_{n=1}^{\infty} e_{ik}^{n}P(k,\ j)$$

$$= (L(i,\ i)-1)P(i,\ j) + e_{ij}^{1} + \sum_{n=1}^{\infty}\sum_{k\neq i} e_{ik}^{n}P(k,\ j)$$

$$= (L(i,\ i)-1)P(i,\ j) + e_{ij}.$$

Then we have that $L(i,\ i) = 1$ when i is recurrent state guaranteed from the definition of weak recurrent. So $\xi = \{e_{ik} : k \in Y\}$ is invariant measure. This complete the existence.

Suppose that $\mu = \{\mu(j) : j \in Y\}$ is an invariant measure. Then there exists the previous recurrent state i, such that $\mu(i) > 0$. In this case, write

$$\mu_0(j) = \frac{\mu(j)}{\mu(i)}(j \in X),\ \mu_0 = \{\mu_0(j) : j \in Y\},$$

we have that μ_0 is invariant measure and $\mu_0(i) = 1$. By combinating e_{ij}^{n} property, we inductively verify for all $j \in Y$ and $n \geqslant 1$,

$$\mu_0(j) \geqslant \sum_{m=1}^{n} e_{ij}^{m}.$$

Let $n \to \infty$, then $\mu_0(i) \geqslant e_{ij}$ for $j \in Y$. If $\mu_0 \neq \pi$, since $\mu_0(i) = e_{ii} = 1$, then $\mu_0(j) > \pi(j) = e_{ij}$ for some $j(\neq i)$. Since we have $j \to i$ from $\bar{C}(i) = X$, so $P^m(j,\ i) > 0$ for some $m > 0$. Hence we have

$$1 = \mu_0(i) = \sum_k \mu_0(k)P^m(k,\ i) > \sum_k e_{ik}P^m(k,\ i) = e_{ii} = 1.$$

It is simple that $\mu_0 = \xi$ from the above contradiction and the theorem is proved.

Theorem 7 If a Markov chain Z on countable state space Y is weak positive recurrence, then it admits a unique stationary distribution ξ satisfying

$$\xi(j) = \frac{1}{\mathbb{E}_j[\tau_j]} \text{ for all } j \in C(i).$$

Proof. Since Z is weak positive recurrence, weak positive recurrence implies the existence of a positive recurrent state k and weak irreducibility. Applying Theorem 2, the state space decomposes as $Y = C(k) \cup D$. Let μ denote the stationary distribution on $C(k)$ guaranteed by Proposition 6. Define

$$\xi(j) := \begin{cases} \mu(i), & j \in C(k), \\ 0, & j \in D. \end{cases}$$

This construction ensures that $\xi = \{\xi(j),\ j \in Y\}$ is stationary distribution of Markov Chain Z, and the corresponding balance equation are satisfied with

$$\xi(j) = \frac{1}{\mathbb{E}_j[\tau_j]} \text{ for all } j \in C(i).$$

The proof is complete.

The q-coloring Glauber dynamics introduced in [3] represent a fundamental model in computational complexity and graph theory. This framework provides a concrete example of weakly irreducible Markov chains through its stochastic stability properties. Specifically, the associated random walk on proper colorings exhibits weak irreducibility and weak positive recurrence (see Example 4), enabling rigorous analysis of mixing times and equilibrium distributions.

Example 4 Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E \subseteq \{\{v, w\} \subset V \mid v \neq w\}$. When $e = \{v, w\} \in E$, vertices v and w are said to be adjacent, and w is referred to as a neighbor of v (or equivalently, v is a neighbor of w).

For a fixed integer $q \geq 2$, define the color set $[q] := \{1, 2, \ldots, q\}$. A q-coloring of G, illustrated in Figure 3a, is an assignment of colors to vertices V, formally represented by a function $x : V \to [q]$. The set of all such functions is denoted $\Omega := [q]^V$, and each $x \in \Omega$ is called a configuration.

A proper q-coloring, exemplified in Figure 3b, is a configuration $x \in \Omega$ satisfying the constraint:

$$x(u) \neq x(v) \text{ for all adjacent vertices } u \sim v.$$

The set of proper q-colorings is denoted $\Omega_0 := \{x \in \Omega \mid x(u) \neq x(v) \text{ for all } u \sim v\}$.

The following two fundamental problems hold significant theoretical importance in q-coloring Glauber dynamic system.

(a) Formulate a computationally efficient algorithm for generating independent samples from the set Ω_0 of proper q-colorings.

(b) Develop a statistically rigorous framework to accurately approximate the cardinality $|\Omega_0|$, given access to random samples drawn from Ω_0.

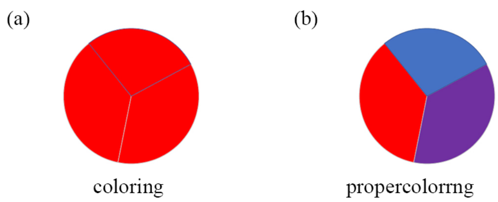

Figure 3. Coloring and proper coloring

We proceed by constructing a Markov chain defined over the set of all proper q-colorings of graph G, specifically designed to analyze the aforementioned questions.

Let $G = (V, E)$ be a graph with vertex set V and edge set E. Given a configuration $x \in \Omega$ and vertex $v \in V$, a color $k \in [q]$ is said to be available at v if $k \notin \{x(w) : w \sim v\}$. Denote the set of available colors at v by

$$A_v(x) := [q] \setminus \{x(w) : w \sim v\}.$$

For $x \in \Omega$, $v \in V$, and $j \in [q]$, define the single-vertex update $x_j^v \in \Omega$ via

$$x_j^v(w) = \begin{cases} x(w), & w \neq v, \\ j, & w = v. \end{cases}$$

Let $f : \Omega \times (V \times [q]) \to \Omega$ be the deterministic update rule given by

$$f(x, (v, j)) = \begin{cases} x_j^v, & j \in A_v(x), \\ x, & \text{otherwise}. \end{cases}$$

Construct a Markov chain $Z = \{Z_n\}_{n=0}^{\infty}$ on Ω as follows: Let $\{X_n\}_{n=1}^{\infty}$ be an i.i.d. sequence uniformly distributed over $V \times [q]$, and define

$$Z_n = f(Z_{n-1}, X_n) \text{ for } n \geq 1,$$

with initial state $Z_0 = x \in \Omega$. The transition probability is then

$$P(x, y) := \mathbb{P}\big(f(x, Z_1) = y\big) \text{ for } x, y \in \Omega.$$

Assume q is sufficiently large such that $\Omega_0 \neq \emptyset$. For improper configurations $y_1 \in \Omega \setminus \Omega_0$ and proper colorings $x_0 \in \Omega_0$, the following properties hold:

1. If y_1 and x_0 differ only at vertex $w \in V$, then

$$P(y_1, x_0) = \frac{1}{|V| \cdot |A_w(y_1)|}, \; P(x_0, y_1) = 0.$$

2. Any $y \in \Omega \setminus \Omega_0$ can reach x_0 in finite steps (as illustrated in Figure 4), implying $\mathbb{E}_{x_0}[\tau_{x_0}] < \infty$, where τ_{x_0} denotes the return time to x_0.

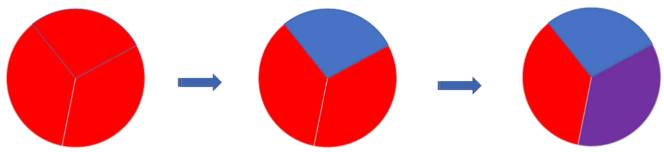

Figure 4. The process of proper coloring

By construction, all proper colorings $x_1 \in \Omega_0$ communicate with x_0 (i.e., $x_1 \leftrightarrow x_0$). Thus, the closed communicating class $\bar{C}(x_0) = \Omega$, which implies Z is weak irreducibility and weak positive recurrence. Applying Theorem 2, the state space decomposes as $\Omega = C(x) \cup D$, where $C(x) = \Omega_0$ is a closed communicating class and $D = \{y \in \Omega \setminus \Omega_0 : y \to x,\ x \not\to y\}$.

Moreover, the transition matrix satisfies $P(x,\ y) = P(y,\ x)$ for all $x,\ y \in \Omega_0$, establishing symmetry. Consequently, Theorem 7 guarantees the existence of a unique stationary distribution $\xi = \{\xi(x)\}_{x\in\Omega}$ with

$$\xi(x) = \frac{1}{|\Omega_0|} = \frac{1}{\mathbb{E}_x[\tau_x]} \text{ for all } x \in \Omega_0.$$

4. The weak irreducibility Markov chain on general state space

This section extends the theory of weak irreducibility to general state spaces Y, including non-separable settings. However, two fundamental challenges arise in this generalization:

1. The classical communicating relation $i \leftrightarrow j$ becomes ill-defined in uncountable spaces due to the lack of discrete topology.

2. While accessibility of sets $B \in \Delta(Y)$ can be characterized via $P^n(x,\ B) > 0$, accessibility of individual states $y \in Y$ requires additional structure to avoid measure-theoretic pathologies.

Weak irreducibility eliminates separability assumptions by focusing on set-level accessibility $\overline{C}(B) = Y$, where $\overline{C}(B)$ denotes the accessible class of B. This aligns with Nummelin's splitting technique [16], which constructs auxiliary atoms to recover discrete-space intuition.

We first revisit the classical concept of atoms as [17]: A measurable set $\alpha \in \Delta(Y)$ is called an atom if the transition kernel satisfies

$$P(x,\ B) = P(y,\ B) \text{ for all } x,\ y \in \alpha \text{ and } B \in \Delta(Y).$$

When α is an accessible atom (i.e., $\{x \in X : L(x,\ \alpha) > 0\} = Y$), the stability analysis parallels countable spaces. This is formalized through the condition

$$\{x \in Y : L(x,\ \alpha) > 0\} = Y. \tag{5}$$

While any singleton $\{x\}$ is trivially an atom, it rarely satisfies accessibility requirements (i.e., $\{x\}$ is not an accessible atom in most practical settings). Unlike Definition 1 for countable spaces, weak irreducibility on general spaces does not rely on (5). Instead, it leverages Nummelin's splitting technique [16], which constructs an auxiliary chain admitting an accessible atom under the Minorization Condition.

To construct the split chain $\check{Z}$, we first recall the Minorization Condition $M(B,\ \delta,\ \nu)$: There exists a measurable set $B \in \Delta(Y)$, constant $\delta > 0$, and probability measure ν with $\nu(B^c) = 0$ and $\nu(B) = 1$, such that

$$P(x,\ A) \geqslant \delta I_B(x)\nu(A) \text{ for all } x \in Y,\ A \in \Delta(Y). \tag{6}$$

The first step split the space Y. Writing $\check{Y} = Y \times \{0,\ 1\}$, the space $Y_0 = Y \times \{0\}$ and $Y_1 = Y \times \{1\}$ with respectively generated Borel σ-field $\Delta(Y_0)$ and $\Delta(Y_1)$. Let $\Delta(\check{Y})$ be the δ-field generated by $\Delta(Y_0),\ \Delta(Y_1), \Delta(\check{Y})$ is the smallest σ-filed containing sets as $B_0 = B \times \{0\},\ B_1 = B \times \{1\}$.

The next step split the measure. If λ is any measure on $\Delta(Y)$, then split the measure λ into two measure on Y_0 and Y_1 by defining the following measure λ^* on $\Delta(\check{Y})$

$$\begin{cases} \lambda^*(B_0) = \lambda(A \cap B)(1-\delta) + \lambda(A \cap B^c) \\ \\ \lambda^*(B_1) = \lambda(A \cap B)\delta, \end{cases} \tag{7}$$

where δ and B are the constant and the set in (6).

Now the third and the most subtle step is to split the chain Z to a chain $\check{Z}$ on $(\check{Z},\ \Delta(\check{Y}))$. Define the split kernel $\check{P}(x_i,\ A)$ for each $x_i \in \check{Y}$ and $A \in \Delta(\check{Y})$ by

$$\begin{cases} \check{P}(x_0,\ \cdot) = P(x,\ \cdot)^*,\ x_0 \in X_0 \setminus B_0 \\ \\ \check{P}(x_0,\ \cdot) = (1-\delta)^{-1}[P(x,\ \cdot)^* - \delta \nu^*(\cdot)],\ x_0 \in B_0 \\ \\ \check{P}(x_1,\ \cdot) = \nu^*(\cdot),\ x_1 \in X_1, \end{cases} \tag{8}$$

where $C,\ \delta$ and ν are the set, the constant and the measure in (6). From (7) and (8), by calculating we have

$$\lambda P^k(A) = \lambda^* \check{P}^k(A_0 \cup A_1),\ A \in \Delta(Y). \tag{9}$$

The set $\check{\alpha} = C_1$ is assured that is an atom of chain $\check{Z}$ by (8). The equivalence of the following (10) and (11) were promised by (9)

$$\{x \in Y : L(x,\ C) > 0\} = Y \tag{10}$$

$$\{x_i \in \check{Y} : \check{L}(x_i,\ C_1) > 0\} = \check{Y} \tag{11}$$

Since the weak irreducibility of chain $\check{Z}$ can be defined by (11), so the weak irreducibility of chain Z can be defined by (10) when the Minorization Condition is satisfied. In fact, the Minorization Condition can be weakened to the existence of ν_t-small set C, namely, there exists an $t > 0$ and a non-trivial measure ν_t on $\Delta(Y)$, such that for each $x \in C$ and $D \in \Delta(Y),\ P^t(x,\ D) \geqslant \nu_t(D)$.

Theorem 8 If there exists a ν_m-small set C and (10) is admitted, then there exists an h-skeleton chain $\Phi^h = \{\Phi_0,\ \Phi_h,\ \Phi_{2h},\ \cdots\}$ satisfy the Minorization Condition.

Proof. If C is ν_m-small, then

$$P^m(x, A) \geqslant I_C(x)\nu_m(A),\ A \in \Delta(Y). \tag{12}$$

We have $U(x, C) > 0$ for each $x \in A$ from (10), hence

$$0 < \int \nu_m(dx)U(x, C) = \sum_{n=1}^{\infty} \nu_m(dx)P^n(x, C).$$

For some r, then

$$\int \nu_m(dy)P^r(x, C) > 0. \tag{13}$$

From (12) and (13) and (C-K) equation, for all $x \in C$, we have

$$P^{m+r}(x, C) = \int P^m(x, dy)P^r(y, C)$$

$$\geqslant \int \nu_m(dy)P^r(y, C) > 0.$$

Let $h = m+r,\ \delta = \nu_m P^r(C) > 0,\ \nu(A) = \frac{1}{\delta}\nu_m P^r(A \cap C)$ and notice that the above formula, then

$$P^h(x, A) \geqslant \delta I_C(x)\nu(A),\ A \in \Delta(Y).$$

The theorem is proved.

Furthermore, the resolvent K_{a_ε} admits the Minorization Condition. In fact, since for each $x \in Y,\ A \in \Delta(Y)$ and $0 < \varepsilon < 1$, some $h > 0$

$$K_{a\varepsilon}(x, A) = (1-\varepsilon)\sum_{n=0}^{\infty} P^h(x, A)\varepsilon^h$$

$$\geqslant (1-\varepsilon)\varepsilon^h P^h(x, A) = (1-\varepsilon)\varepsilon^h \delta I_C(x)\nu(A).$$

Taking $\delta_\varepsilon = (1-\varepsilon)\varepsilon^h\delta$, then K_{a_ε} admits Minorization Condition $M(C,\ \delta_\varepsilon,\ \nu)$.

Building on the previous analysis, we now formalize weak irreducibility for Markov chains on general state spaces:

Definition 2 A Markov chain Z on general state space Y is weak irreducibility if there exists a ν_m-small set $C \in \Delta(Y)$ satisfying

$$\{x \in Y : L(x,\ C) > 0\} = Y.$$

The ν_m-small set C in Definition 2 can be replaced by the ν_a-petite C introduced by meyn and Tweedie [20]. Recall that a Markov chain Z is φ-irreducible [17] if there exists a nontrivial measure φ on $\Delta(Y)$ such that $\varphi(B) > 0$ implies $L(x,\ B) > 0$ for all $x \in Y$. This approach eliminates separability assumptions and provides a unified treatment of weak irreducibility across discrete and continuous state spaces. The relationship between weak irreducibility and φ-irreducibility is summarized in Theorem 9.

Theorem 9 If a Markov chain Z on general state space Y is weak irreducibility, then it is φ-irreducibility. The converse, if the state space is separable, then the φ-irreducibility imply the weak irreducibility.

Proof. If Z is weak irreducibility, then there exists a ν_m-small D such that $\{x \in Y : L(x,\ D) > 0\} = Y$. Hence $P^n(x,\ D) > 0$ for each $x \in Y$ and some n. Since D is ν_m-small, then $P^m(x,\ B) \geqslant \nu_m(B)$ for each $x \in Y$ and $B \in \Delta(Y)$. Taking $\varphi = \nu_m$, for every $x \in Y$ and $\varphi(B) > 0$, we have

$$P^{m+n}(x,\ B) = \int P^m(y,\ B)P^n(x,\ dy) \geqslant \int_C P^m(y,\ B)P^n(x,\ dy)$$

$$\geqslant \int_C \nu_m(B)P^n(x,\ dy) > 0.$$

From the proposition 4.2.1 in [17] Markov chain Z is φ-irreducibility.

Conversely, if the space is separable, and Z is φ-irreducibility, then for every $A \in \Delta^+(Y) := \{A \in \Delta(Y) : \varphi(A) > 0\}$, there exists $n \geqslant 1$ and a ν_n-small set $F \subseteq A$ such that $F \in \Delta^+(Y)$ and $\nu_n(F) > 0$ from theorem 5.2.2 in [17]. Again since Z is φ-irreducibility, without loss of generality , let φ is irreducible measure, then $\varphi(C) > 0$ from $\nu_m(C) > 0$. Hence $L(x,\ C) > 0$ for all $x \in X$. The weak irreducibility follows immediately.

The definition of weak irreducibility does not require separability assumptions and a reference measure, making it more general than classical φ-irreducibility frameworks. This distinction highlights the practicality of weak irreducibility, as demonstrated in Example 5 below.

Example 5 (Proposition 4.3.1 in [17]) Suppose that $Z = \{Z_n\}$ is a random walk on a half line with increment variable W(namely $Z_n = [Z_{n-1} + W_n]^+$) is φ-irreducible, with $\varphi(0,\ +\infty) = 0,\ \varphi(\{0\}) = 1$, if and only if $P(W < 0) > 0$.

Proof. The necessity is trivial. Conversely, since $\{0\}$ is single point, hence $\{0\}$ is small set. Since $P(W < 0) > 0$, then there exists $\delta > 0$ and $\varepsilon > 0$ such that $P(W < -\varepsilon) > \delta$. Hence $x/\varepsilon < n$ for $x \in [0,\ +\infty)$ and some n, we have $P^n(x,\ \{0\}) \geqslant \delta^n > 0$. It is obvious that $\{x \in X : L(x,\ \{0\}) > 0\} = Y$. Hence Z is φ-irreducibility from Theorem 9.

5. Conclusions

This work introduces the concept of weak irreducibility as a generalized framework for analyzing stochastic stability in both countable and general state spaces. By relaxing the classical communication requirements of irreducibility, weak irreducibility eliminates the need for separability assumptions and a priori reference measures, enabling a unified treatment of Markov chains in non-separable settings.

The framework can be extended to non-Markovian processes or infinite-dimensional state spaces, where separability is often absent. Further exploration of small/petite sets in data-driven contexts may enhance algorithmic convergence analysis.

By decoupling irreducibility from topological constraints, this work provides a robust foundation for studying stability in complex systems, offering both theoretical insights and practical tools for applications in statistical physics, computational complexity, and beyond.

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Conflict of interest

The authors declare no competing financial interest.

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