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Research Article

# Extension of $(\eta, \varphi)$-Derivation on the CSL Subalgebra of von Neumann Algebra

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**Abstract:** This article aims to demonstrate the following: consider $\mathscr{V}$, a Commutative Subspace Lattice (CSL) subalgebra of a von Neumann algebra acting on a Hilbert space $\mathscr{H}$. Suppose that $\mathscr{G}, \mathscr{F} : \mathscr{V} \to \mathscr{V}$ are two linear mappings that satisfy some certain functional identities. Then $\mathscr{G}$ is a generalized $(\eta, \varphi)$-derivation with associated $(\eta, \varphi)$-derivation $\mathscr{F}$ in $\mathscr{V}$, where $\eta$ and $\varphi$ are automorphisms in $\mathscr{V}$.

***Keywords*:** von Neumann algebra, semiprime rings, generalized $(\eta, \varphi)$-derivation and linear mappings

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## 1. Introduction

Throughout this article, $\mathscr{H}$ represents a complex Hilbert space. We denote the identity operator on $\mathscr{H}$ by $K$, and the collection of all bounded linear operators on $\mathscr{H}$ by $\mathscr{B}(\mathscr{H})$. In this context, the words "projection" and "subspace" refer specifically to "orthogonal projection" and "norm closed linear manifold," respectively. To simplify, a subspace is represented by its corresponding projection. If $\mathscr{S}_\alpha$ denotes a set of subspaces of $\mathscr{H}$, then $\mho\mathscr{S}_\alpha$ signifies the minimal subspace encompassing every $\mathscr{S}_\alpha$, while $\Omega\mathscr{S}_\alpha$ denotes the maximal subspace encompassing each $\mathscr{S}_\alpha$. If $\Gamma$ includes both 0 and $K$, then $\Gamma$ is a strongly closed lattice of normed closed subspaces (or orthogonal projections) that is invariant with respect to the standard lattice operations $\Omega$ and $\mho$. The set of all bounded operators on $\mathscr{B}(\mathscr{H})$ that maintain invariance for each subspace contained in $\Gamma$ is represented as $Alg\,\Gamma$.

The lattice of all closed subspaces left invariant for any operator in $\mathscr{V}$ is denoted as $Lat\ \mathscr{V}$ for a subalgebra $\mathscr{V}$ of $\mathscr{B}(\mathscr{H})$; that is, $Lat\ \mathscr{V} = \{Q : Q \text{ is a subspace of } \mathscr{H} \text{ and } AQ = QAQ \text{ for every } Q \in \mathscr{V}\}$. A subspace lattice $\Gamma$ is reflexive if $\Gamma = Lat\ Alg\ \Gamma$. A subspace lattice $\Gamma$ is referred to as a commutative, or CSL for short, if every orthogonal projection in it commutes pairwise. The structure denoted as $Alg\ \Gamma$ is alternatively designated as a CSL algebra. If $\Gamma$ is a CSL acting on a Hilbert space $\mathscr{H}$, and its orthogonal projections contained in a von Neumann algebra $S$, then $\mathscr{V} = S \cap Alg\ \Gamma$ is known as a CSL subalgebra of von Neumann algebra $S$.

A mapping $\mathscr{F}$ from $\mathscr{V}$ to itself is designated as a derivation if it fulfills the condition $\mathscr{F}(VW) = \mathscr{F}(V)W + V\mathscr{F}(W)$ for each instance of $V, W \in \mathscr{V}$. Furthermore, $\mathscr{F}$ is termed a Jordan derivation when it meets the requirement $\mathscr{F}(V^2) =$


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$\mathscr{F}(V)V+V\mathscr{F}(V)$ for all $V\in\mathscr{V}$. A mapping $\mathscr{G}$, which is additive and takes elements from $\mathscr{V}$ to $\mathscr{V}$, is called generalized derivation if there exists a derivation $\mathscr{F}$ such that for every $V,\ W\in\mathscr{V}$, $\mathscr{G}(VW)=\mathscr{G}(V)W+V\mathscr{F}(W)$ is satisfied. A mapping $\mathscr{G}:\mathscr{V}\to\mathscr{V}$ is said to be a generalized Jordan derivation if there exists a Jordan derivation $\mathscr{F}:\mathscr{V}\to\mathscr{V}$ that satisfies $\mathscr{G}(V^2)=\mathscr{G}(V)V+V\mathscr{F}(V)$ for any $V\in\mathscr{V}$. It is readily verifiable that, while all generalized derivations are generalized Jordan derivations, the contrary statement is not always the case.

Let $\eta$ and $\varphi$ represent a pair of automorphisms within $\mathscr{V}$. An additive map $\mathscr{F}$ from $\mathscr{V}$ to itself is called an $(\eta,\ \varphi)$-derivation (or a Jordan $(\eta,\ \varphi)$-derivation) if it satisfies the condition $\mathscr{F}(VW)=\mathscr{F}(V)\eta(W)+\varphi(V)\mathscr{F}(W)$ (or $\mathscr{F}(V^2)=\mathscr{F}(V)\eta(V)+\varphi(V)\mathscr{F}(V)$, respectively) for all $V,\ W\in\mathscr{V}$. Any $(\eta,\ \varphi)$-derivation also qualifies as a Jordan $(\eta,\ \varphi)$-derivation, but the reverse implication does not always hold. An additive map $\mathscr{G}:\mathscr{V}\longrightarrow\mathscr{V}$ is termed a generalized $(\eta,\ \varphi)$-derivation if there exists an $(\eta,\ \varphi)$-derivation $\mathscr{F}$ from $\mathscr{V}$ to itself such that for all $V,\ W\in\mathscr{V}$, it satisfies $\mathscr{G}(VW)=\mathscr{G}(V)\eta(W)+\varphi(V)\mathscr{F}(W)$. An additive map $\mathscr{G}$ from $\mathscr{V}$ to itself is termed a generalized Jordan $(\eta,\ \varphi)$-derivation if there exists a Jordan $(\eta,\ \varphi)$-derivation $\mathscr{F}$ from $\mathscr{V}$ to itself such that $\mathscr{G}(V^2)=\mathscr{G}(V)\eta(V)+\varphi(V)\mathscr{F}(V)$, for all $V\in\mathscr{V}$.

Every generalized $(\eta,\ \varphi)$-derivation can be identified as a generalized Jordan $(\eta,\ \varphi)$-derivation; however, the reverse implication is not generally true. Specifically, when $\mathscr{F}$ is a zero derivation, $\mathscr{G}$ is referred to as the left $\eta$-centralizer (or the left Jordan $\eta$-centralizer, respectively). For simplicity, see the following:

Generalized $(\eta,\ \varphi)$-derivation $\Longrightarrow$ Generalized Jordan $(\eta,\ \varphi)$-derivation;

Generalized derivation $\Longrightarrow$ Generalized Jordan derivation;

Derivation $\Longrightarrow$ Jordan derivation.

**Example 1** Consider $R'$ be a ring such that square of each element in $R'$ is zero but the product of some elements in $R'$ is non zero and $\mathscr{R}=\left\{\begin{bmatrix}0 & a & b\\ 0 & 0 & a\\ 0 & 0 & 0\end{bmatrix}\mid a,\ b\in R'\right\}$. Define mappings $\mathscr{G},\ \mathscr{F},\ \eta,\ \varphi:\mathscr{R}\longrightarrow\mathscr{R}$ such that

$$\mathscr{G}\left(\begin{bmatrix}0 & a & b\\ 0 & 0 & a\\ 0 & 0 & 0\end{bmatrix}\right)=\begin{bmatrix}0 & 0 & b\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},$$

$$\mathscr{F}\left(\begin{bmatrix}0 & a & b\\ 0 & 0 & a\\ 0 & 0 & 0\end{bmatrix}\right)=\begin{bmatrix}0 & a & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},$$

$$\eta\left(\begin{bmatrix}0 & a & b\\ 0 & 0 & a\\ 0 & 0 & 0\end{bmatrix}\right)=\begin{bmatrix}0 & -a & b\\ 0 & 0 & -a\\ 0 & 0 & 0\end{bmatrix},$$

and

$$\phi\left(\begin{bmatrix}0 & a & b\\ 0 & 0 & a\\ 0 & 0 & 0\end{bmatrix}\right)=\begin{bmatrix}0 & -a & -b\\ 0 & 0 & -a\\ 0 & 0 & 0\end{bmatrix}.$$

Then it easy to check that $\mathscr{G}$ is generalized Jordan $(\eta,\ \phi)$-derivation with associated Jordan $(\eta,\ \phi)$-derivation $\mathscr{F}$ but not generalized $(\eta,\ \phi)$-derivation.

Some remarkable interpretation found in [1, 2].

Author in [3] demonstrated that a Jordan derivation in a CSL algebra indeed acts as a derivation. A further extension noted in [4] indicating that a Jordan derivation in a CSL subalgebra of the von Neumann algebra also fulfills the role of a derivation. Furthermore, in [5] it was proved that a Jordan $\eta$-derivation is an $\eta$-derivation on $\mathscr{V}$. Also a Jordan $(\eta,\ \varphi)$-derivation on $\mathscr{V}$ is an $(\eta,\ \varphi)$-derivation, where $\eta$ and $\varphi$ are automorphisms on $\mathscr{V}$. Building upon the aforementioned line of inquiry, this article aims to explore the following:

Given a generalized Jordan $(\eta,\ \varphi)$-derivation $\mathscr{G}$ that corresponds to a Jordan $(\eta,\ \varphi)$-derivation $\mathscr{F}$ within $\mathscr{V}$, the algebraic identities

$$\mathscr{G}(V^{2q})=\mathscr{G}(V^{q})\eta(V^{q})+\varphi(V^{q})\mathscr{F}(V^{q}) \tag{1}$$

$$\mathscr{G}(V^{3m})=\mathscr{G}(V^{m})\eta(V^{2m})+\varphi(V^{m})\mathscr{F}(V^{m})\eta(V^{m})+\varphi(V^{2m})\mathscr{F}(V^{m}), \tag{2}$$

are valid for every $V\in\mathscr{V}$. However, the reverse implication is not valid in every case. Consequently, these identities characterize weaker conditions than those defining a generalized Jordan $(\eta,\ \varphi)$-derivation, as well as a generalized $(\eta,\ \varphi)$-derivation. In this article, we study under what condition in $\mathscr{V}$, $\mathscr{G}$ is a generalized $(\eta,\ \varphi)$-derivation associated with an $(\eta,\ \varphi)$-derivation $\mathscr{F}$ if it fulfills the algebraic equation (1) and (2). This article aims to address the question previously mentioned in the context where $\mathscr{V}$ is a CSL subalgebra of a von Neumann algebra operating on a Hilbert space.

Regarding the complementary work performed by the authors in [6–8], the author of this study relaxes the torsion constraint on $\mathscr{V}$ using Vandermonde determinant tools. The proof of Theorem 1 is established using appropriate arguments and significant modifications.

In order to finalize the proof of the principal theorems, the following result is necessary:

**Lemma 1** [5] Consider $S$ as a von Neumann algebra operating on a Hilbert space $\mathscr{H}$, and let $\Gamma$ be a CSL with orthogonal projections contained in $S$. Define $\mathscr{V}$ as the CSL subalgebra of the von Neumann algebra $S$, specifically $\mathscr{V}=S\cap Alg\,\Gamma$. If $\eta$ and $\varphi$ are two automorphisms in $\mathscr{V}$, then a Jordan $(\eta,\ \varphi)$-derivation on $\mathscr{V}$ is actually an $(\eta,\ \varphi)$-derivation.

## 2. Main theorems

We shall begin with the following theorems:

**Theorem 1** Let $q\geq 1$ be any fixed integer and $\mathscr{V}=S\cap \operatorname{Alg}\Gamma$ be a CSL subalgebra of the von Neumann algebra $S$. Let $\mathscr{G},\ \mathscr{F}:\mathscr{V}\to\mathscr{V}$ be two linear mappings satisfying the following algebraic identity

$$\mathscr{G}\left(V^{2q}\right)=\mathscr{G}(V^{q})\eta(V^{q})+\varphi(V^{q})\mathscr{F}(V^{q})\text{ for all }V\in\mathscr{V}, \tag{3}$$

where $\eta$ and $\varphi$ are automorphisms in $\mathscr{V}$. Then $\mathscr{G}$ is a generalized $(\eta,\ \varphi)$-derivation with associated $(\eta,\ \varphi)$-derivation $\mathscr{F}$ in $\mathscr{V}$.

**Proof.** Replacing $V$ by $V+nW$ in Equation (3), we find

$$\mathscr{G}\left(V^{2q}+\binom{2q}{1}V^{2q-1}nW+\binom{2q}{2}V^{2q-2}n^2W^2+\cdots+n^{2q}W^{2q}\right)$$

$$=\mathscr{G}\left(V^q+\binom{q}{1}V^{q-1}nW+\binom{q}{2}V^{q-2}n^2W^2+\cdots+n^qW^q\right)$$

$$\left(\eta(V^q)+\binom{q}{1}\eta(V^{q-1}nW)+\binom{q}{2}\eta(V^{q-2}n^2W^2)+\cdots\eta(n^qW^q)\right)$$

$$+\left(\varphi(V^q)+\binom{q}{1}\varphi(V^{q-1}nW)+\binom{q}{2}\varphi(V^{q-2}n^2W^2)+\cdots+\varphi(n^qW^q)\right)$$

$$\mathscr{F}\left(V^q+\binom{q}{1}V^{q-1}nW+\binom{q}{2}V^{q-2}n^2W^2+\cdots+n^qW^q\right),\ \text{for all } V,\ W\in\mathscr{V},$$

that is,

$$n\left[\binom{2q}{1}\mathscr{G}(V^{2q-1}W)-\binom{q}{1}\mathscr{G}(V^q)\eta(V^{q-1}W)-\binom{q}{1}\mathscr{G}(V^{q-1}W)\eta(V^q)\right.$$

$$\left.-\binom{q}{1}\varphi(V^q)\mathscr{F}(V^{q-1}W)-\binom{q}{1}\varphi(V^{q-1}W)\mathscr{F}(V^q)\right]+n^2\left[\binom{2q}{2}\mathscr{G}(V^{2q-2}W^2)\right.$$

$$-\binom{q}{2}\mathscr{G}(V^q)\eta(V^{q-2}W^2)-\binom{q}{1}\binom{q}{1}\mathscr{G}(V^{q-1}W)\eta(V^{q-1}W)-\binom{q}{2}\mathscr{G}(V^{q-2}W^2)\eta(V^q)$$

$$\left.-\binom{q}{2}\mathscr{F}(V^{q-2}W^2)-\binom{q}{1}\binom{q}{1}\varphi(V^{q-1}W)\mathscr{F}(V^{q-1}W)-\binom{q}{2}\varphi(V^{q-2}W^2)\mathscr{F}(V^q)\right]+\cdots$$

$$+n^{2q}\left[\mathscr{G}(V^{2q})-\mathscr{G}(V^q)\eta(V^q)-\varphi(V^q)\mathscr{F}(V^q)\right]=0\ \text{for all } V,\ W\in\mathscr{V}.$$

Utilize (3) to rewrite the expression mentioned above as

$$\sum_{i=1}^{2q-1} f_i(V,\ W)n^i=0,$$

where $f_i(V,\ W)$ stand for the coefficients of $n^i$'s for all $i=1,\ 2,\ \ldots,\ (2q-1)$. Upon substituting $i$ with $1,\ 2,\ \ldots,\ (2q-1)$, we obtain a system comprising $(2q-1)$ homogeneous equations, resulting in a Vandermonde matrix.

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^{2q-1} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ (2q-1) & (2q-1)^2 & \cdots & (2q-1)^{2q-1} \end{bmatrix},$$

Since, the determinant of the matrix is equal to the product of positive integers, each of which is less than $(2q-1)$, it follows immediately that $f_i(V, W) = 0$ for all $V, W \in \mathscr{V}$ and for all $i = 1, 2, \ldots, (2q-1)$. Particularly, we can express the term

$$f_1(V, W) = \binom{2q}{1}\mathscr{G}\left(V^{2q-1}W\right) - \binom{q}{1}\mathscr{G}(V^q)\eta(V^{q-1}W) - \binom{q}{1}\mathscr{G}(V^{q-1}W)\eta(V^q)$$

$$-\binom{q}{1}\varphi(V^q)\mathscr{F}(V^{q-1}W) - \binom{q}{1}\varphi(V^{q-1}W)\mathscr{F}(V^q) = 0 \text{ for all } V, W \in \mathscr{V}.$$

Put $V = K$ and making use of $\mathscr{F}(K) = 0$ and $\eta(K) = \varphi(K) = K$ to appear $2q\mathscr{G}(W) = q\mathscr{G}(K)\eta(W) + q\mathscr{G}(W) + q\mathscr{F}(W)$. Therefore, we observe that

$$\mathscr{G}(W) = \mathscr{G}(K)\eta(W) + \mathscr{F}(W) \text{ for all } W \in \mathscr{V}. \tag{4}$$

Next, explore the term

$$f_2(V, W) = \binom{2q}{2}\mathscr{G}(V^{2q-2}W^2) - \binom{q}{2}\mathscr{G}(V^q)\eta(V^{q-2}W^2) - \binom{q}{1}\binom{q}{1}\mathscr{G}(V^{q-1}W)\eta(V^{q-1}W)$$

$$-\binom{q}{2}\mathscr{G}(V^{q-2}W^2)\eta(V^q) - \binom{q}{2}\mathscr{F}(V^{q-2}W^2) - \binom{q}{1}\binom{q}{1}\varphi(V^{q-1}W)\mathscr{F}(V^{q-1}W)$$

$$-\binom{q}{2}\varphi(V^{q-2}W^2)\mathscr{F}(V^q) = 0 \text{ for all } V, W \in \mathscr{V}.$$

Rewrite the above expression by substituting $K$ for $V$ to obtain

$$\binom{2q}{2}\mathscr{G}(W^2) = \binom{q}{2}\mathscr{G}(K)\eta(W^2) + \binom{q}{1}\binom{q}{1}\mathscr{G}(W)\eta(W) + \binom{q}{2}\mathscr{G}(W^2)$$

$$+\binom{q}{2}\mathscr{F}(W^2) + \binom{q}{1}\binom{q}{1}\varphi(W)\mathscr{F}(W) \text{ for all } W \in \mathscr{V}.$$

This implies that

$$\frac{2q(2q-1)}{2}\mathscr{G}(W^2) = \frac{q(q-1)}{2}\mathscr{G}(K)\eta(W^2) + q^2\mathscr{G}(W)\eta(W) + \frac{q(q-1)}{2}\mathscr{G}(W^2)$$

$$+ \frac{q(q-1)}{2}\mathscr{F}(W^2) + q^2\varphi(W)\mathscr{F}(W).$$

Also, we have

$$2(2q-1)\mathscr{G}(W^2) = (q-1)\mathscr{G}(K)\eta(W^2) + 2q\mathscr{G}(W)\eta(W) + q(q-1)\mathscr{G}(W^2)$$

$$+ (q-1)\mathscr{F}(W^2) + 2q\varphi(W)\mathscr{F}(W).$$

A simple manipulation give us

$$(3q-1)\mathscr{G}(W^2) = (q-1)\mathscr{G}(K)\eta(W^2) + 2q\mathscr{G}(W)\eta(W) + (q-1)\mathscr{F}(W^2) + 2q\varphi(W)\mathscr{F}(W).$$

An application of (4) yields that

$$(3q-1)\left[\mathscr{G}(K)\eta(W^2) + \mathscr{F}(W^2)\right] = (q-1)\mathscr{G}(K)\eta(W^2) + 2q\left[\mathscr{G}(K)\eta(W) + \mathscr{F}(W)\right]\eta(W)$$

$$+ (q-1)\mathscr{F}(W^2) + 2q\varphi(W)\mathscr{F}(W).$$

On simplifying above expression, we obtain

$$2q\mathscr{F}(W^2) = 2q\mathscr{F}(W)\eta(W) + 2q\varphi(W)\mathscr{F}(W) \text{ for all } W \in \mathscr{V}.$$

That implies $\mathscr{F}(W^2) = \mathscr{F}(W)\eta(W) + \varphi(W)\mathscr{F}(W)$. Hence $\mathscr{F}$ is a Jordan $(\eta,\ \varphi)$-derivation. Use Lemma 1 to get that $\mathscr{F}$ is an $(\eta,\ \varphi)$-derivation on $\mathscr{V}$. Consider (4) once again, so that

$$\mathscr{G}(VW) = \mathscr{G}(K)\eta(V)\eta(W) + \mathscr{F}(VW)$$

$$= \mathscr{G}(K)\eta(V)\eta(W) + \mathscr{F}(V)\eta(W) + \varphi(V)\mathscr{F}(W)$$

$$= \left[\mathscr{G}(K)\eta(V) + \mathscr{F}(V)\right]\eta(W) + \varphi(V)\mathscr{F}(W)$$

$$= \mathscr{G}(V)\eta(W) + \varphi(V)\mathscr{F}(W).$$

Thus, $\mathscr{G}$ acts as a generalized $(\eta,\ \varphi)$-derivation on $\mathscr{V}$, associated with an $(\eta,\ \varphi)$-derivation $\mathscr{F}$, which is the desired conclusion.

**Theorem 2** Let $m \geq 1$ be any fixed integer and $\mathscr{V} = S \cap \text{Alg}\ \Gamma$ represents a CSL subalgebra of the von Neumann algebra $S$. Suppose that $\mathscr{G},\ \mathscr{F} : \mathscr{V} \to \mathscr{V}$ are two linear mappings that satisfy the algebraic identity $\mathscr{G}(V^{3m}) = \mathscr{G}(V^m)\eta(V^{2m}) + \varphi(V^m)\mathscr{F}(V^m)\eta(V^m) + \varphi(V^{2m})\mathscr{F}(V^m)$, where $\eta$ and $\varphi$ are automorphisms in $\mathscr{V}$. Then $\mathscr{G}$ is a generalized $(\eta,\ \varphi)$-derivation, accompanied by the $(\eta,\ \varphi)$-derivation $\mathscr{F}$ in $\mathscr{V}$.

**Proof.** Given that

$$\mathscr{G}(V^{3m}) = \mathscr{G}(V^m)\eta(V^{2m}) + \varphi(V^m)\mathscr{F}(V^m)\eta(V^m) + \varphi(V^{2m})\mathscr{F}(V^m) \text{ for all } V \in \mathscr{V}. \tag{5}$$

Replacing $V$ by $K$, we obtain $\mathscr{F}(K) = 0$. Continuing from condition (5), we substitute $V + qW$ for $V$ to obtain

$$\mathscr{G}\left(V^{3m} + \binom{3m}{1}(V^{3m})qW + \cdots + \binom{3m}{3m-2}V^2q^{3m-2}W^{3m-2} + \binom{3m}{3m-1}Vq^{3m-1}W^{3m-1} + q^{3m}W^{3m}\right)$$

$$= \mathscr{G}\left(V^m + \binom{m}{1}V^{m-1}qW + \cdots + \binom{m}{m-2}V^2q^{m-2}W^{m-2} + \binom{m}{m-1}Vq^{m-1}W^{m-1} + q^mW^m\right)$$

$$\eta\left(V^{2m} + \binom{2m}{1}V^{2m-1}qW + \cdots + \binom{2m}{2m-2}V^2q^{2m-2}W^{2m-2} + \binom{2m}{2m-1}Vq^{2m-1}W^{2m-1} + q^{2m}W^{2m}\right)$$

$$+ \varphi\left(V^m + \binom{m}{1}V^{m-1}qW + \cdots + \binom{m}{m-2}V^2q^{m-2}W^{m-2} + \binom{m}{m-1}Vq^{m-1}W^{m-1} + q^mW^m\right)$$

$$\mathscr{F}\left(V^m + \binom{m}{1}V^{m-1}qW + \cdots + \binom{m}{m-2}V^2q^{m-2}W^{m-2} + \binom{m}{m-1}q^{m-1}W^{m-1} + q^mW^m\right)$$

$$\eta\left(V^m + \binom{m}{1}V^{m-1}qW + \cdots + \binom{m}{m-2}V^2q^{m-2}W^{m-2} + \binom{m}{m-1}q^{m-1}W^{m-1} + q^mW^m\right)$$

$$+ \varphi\left(V^{2m} + \binom{2m}{1}(V^{2m-1})qW + \cdots + \binom{2m}{2m-2}V^2q^{2m-2}W^{2m-2} + \binom{2m}{2m-1}Vq^{2m-1}W^{2m-1} + q^{2m}W^{2m}\right)$$

$$\mathscr{F}\left(V^m + \binom{m}{1}V^{m-1}qW + \cdots + \binom{m}{m-2}V^2q^{m-2}W^{m-2} + \binom{m}{m-1}Vq^{m-1}W^{m-1} + q^mW^m\right),$$

for all $V,\ W \in \mathscr{V}$ and $q \geq 1$.

Rewrite the above expression using (5) as

$$\sum_{i=1}^{3m-1} q^i \mathscr{P}_i(V,\ W) = 0,$$

where $\mathscr{P}_i(V, W)$ represents the same meaning as in the last theorem. This yields $\mathscr{P}_i(V, W) = 0$ for all $V, W \in \mathscr{V}$ and for $i = 1, 2, \ldots, (3m-1)$. In particular, $\mathscr{P}_{3m-1}(V, W) = 0$ implies that

$$\begin{aligned}\binom{3m}{3m-1}\mathscr{G}(VW) = &\binom{2m}{2m-1}\mathscr{G}(W)\eta(V) + \binom{m}{m-1}\mathscr{G}(V) + \binom{m}{m-1}\mathscr{F}(W)\eta(V)\\ &+\binom{m}{m-1}\mathscr{F}(V) + \binom{m}{m-1}\mathscr{F}(K)\eta(V) + \binom{m}{m-1}\mathscr{F}(V)\\ &+\binom{2m}{2m-1}\varphi(V)\mathscr{F}(K),\end{aligned}$$

for all $V \in \mathscr{G}$. By simplifying the last relation by substituting $K$ for $W$, we obtain $2m\mathscr{G}(V) = 2m\mathscr{G}(K)\eta(V) + 2m\mathscr{F}(V)$ for all $V \in \mathscr{V}$. A hypothesis enable us to write

$$\mathscr{G}(V) = \mathscr{G}(K)\eta(V) + \mathscr{F}(V), \text{ for all } V \in \mathscr{V}. \tag{6}$$

Now consider $\mathscr{P}_{3m-2}(V, K) = 0$ and using the fact that $\eta(K) = \varphi(K) = K$ and $\mathscr{F}(K) = 0$, we obtain

$$\begin{aligned}\binom{3m}{3m-2}\mathscr{G}(V^2) = &\binom{2m}{2m-2}\mathscr{G}(K)\eta(V^2) + \binom{m}{m-1}\binom{2m}{2m-1}\mathscr{G}(V)\eta(V) + \binom{m}{m-2}\mathscr{G}(V^2)\\ &+\binom{m}{m-1}\binom{m}{m-1}\mathscr{F}(V)\eta(V) + \binom{m}{m-2}\mathscr{F}(V^2) + \binom{m}{m-1}\binom{m}{m-1}\varphi(V)\mathscr{F}(V)\\ &+\binom{m}{m-2}\mathscr{F}(V^2) + \binom{2m}{2m-1}\binom{m}{m-1}\varphi(V)\mathscr{F}(V).\end{aligned}$$

On simplification, we find that

$$\begin{aligned}3m(3m-1)\mathscr{G}(V^2) = &\,2m(2m-1)\mathscr{G}(K)\eta(V^2) + 4m^2\mathscr{G}(V)\eta(V)\\ &+m(m-1)\mathscr{G}(V^2)2m^2\mathscr{F}(V)\varphi(V) + m(m-1)\mathscr{F}(V^2)\\ &+2m^2\varphi(V)\mathscr{F}(V) + m(m-1)\mathscr{F}(V^2) + 4m^2\varphi(V)\mathscr{F}(V).\end{aligned}$$

This implies that

$$[3m(3m-1)-m(m-1)]\mathscr{G}(V^2)=2m(2m-1)\left[\mathscr{G}(V^2)-\mathscr{F}(V^2)\right]$$

$$+4m^2\mathscr{G}(V)\eta(V)+6m^2\varphi(V)\mathscr{F}(V)$$

$$+(-m^2-m)\mathscr{F}(V^2)+2m^2\mathscr{F}(V)\eta(V).$$

Encounter the last two expression together to find

$$[3m(3m-1)-2m(2m-1)-m(m-1)]\mathscr{G}(V^2)=-2m^2\mathscr{F}(V^2)+4m^2\mathscr{G}(V)\eta(V)$$

$$+6m^2\varphi(V)\mathscr{F}(V)+2m^2\mathscr{F}(V)\eta(V).$$

Which enable us to obtain

$$4m^2\mathscr{G}\left(V^2\right)=-2m^2\mathscr{F}\left(V^2\right)+4m^2\mathscr{G}(V)\eta(V)+6m^2\varphi(V)\mathscr{F}(V)+2m^2\mathscr{F}(V)\eta(V). \tag{7}$$

Replacing $V$ by $V^2$ in (6), we obtain

$$\mathscr{G}(V^2)=\mathscr{G}(K)\eta(V^2)+\mathscr{F}(V^2), \text{ for all } V\in\mathscr{V}. \tag{8}$$

In view of (7) and (8), we get

$$4m^2\left[\mathscr{G}(K)\eta(V^2)+\mathscr{F}(V^2)\right]=-2m^2\mathscr{F}(V^2)+4m^2\mathscr{G}(V)\eta(V)+6m^2\varphi(V)\mathscr{F}(V)+2m^2\mathscr{F}(V)\eta(V).$$

This entails the following expression

$$2\mathscr{G}(K)\eta(V^2)+\mathscr{F}(V^2)=-\mathscr{F}(V^2)+2\mathscr{G}(V)\eta(V)+3\varphi(V)\mathscr{F}(V)+\mathscr{F}(V)\eta(V).$$

Which implies that

$$2\mathscr{G}(K)\eta(V^2)+2\mathscr{F}(V^2)=-\mathscr{F}(V^2)+2\mathscr{G}(V)\eta(V)+3\varphi(V)\mathscr{F}(V)+\mathscr{F}(V)\eta(V).$$

Arranging the terms of the above, we find

$$3\mathscr{F}(V^2)=2\mathscr{G}(V)\eta(V)-2\mathscr{G}(K)\eta(V^2)+3\varphi(V)\mathscr{F}(V)+\mathscr{F}(V)\eta(V).$$

This implies that

$$3\mathscr{F}(V^2) = 2[\mathscr{G}(V) - \mathscr{G}(K)\eta(V)]\eta(V) + +3\varphi(V)\mathscr{F}(V) + \mathscr{F}(V)\eta(V).$$

Using the value of $\mathscr{F}(V)$ from (6), we have

$$3\mathscr{F}(V^2) = 2\mathscr{F}(V)\eta(V) + 3\varphi(V)\mathscr{F}(V) + \mathscr{F}(V)\eta(V) = 3\mathscr{F}(V)\eta(V) + 3\varphi(V)\mathscr{F}(V).$$

This yields that $\mathscr{F}(V^2) = \mathscr{F}(V)\eta(V) + \varphi(V)\mathscr{F}(V)$ for all $V \in \mathscr{V}$. Thus, $\mathscr{F}$ is a Jordan $(\eta,\ \varphi)$-derivation of $\mathscr{V}$. Using Lemma 1, $\mathscr{F}$ will be an $(\eta,\ \varphi)$-derivation on $\mathscr{V}$. Next, applying the similar technique as in last theorem, $\mathscr{G}$ serves as a generalized $(\eta,\ \varphi)$-derivation on $\mathscr{V}$ associated with an $(\eta,\ \varphi)$-derivation $\mathscr{F}$. Hence, we obtain the desired conclusion.

The theorem outlined above leads to immediate implications. Proofs of corollaries follow analogously from Theorem 1.

**Corollary 1** Let $q \geq 1$ be any fixed integer and let $\mathscr{V} = S\cap$ Alg $\Gamma$ represent a CSL subalgebra of the von Neumann algebra $S$. Consider two linear maps $\mathscr{G},\ \mathscr{F} : \mathscr{V} \to \mathscr{V}$ that fulfill the algebraic equation $\mathscr{G}(V^{2q}) = \mathscr{G}(V^q)\eta(V^q)$ for every $V \in \mathscr{V}$, where $\eta$ is an automorphism of $\mathscr{V}$. In this context, $\mathscr{G}$ can be classified as a left $\eta$-centralizer on $\mathscr{V}$.

**Corollary 2** Consider any fixed integer $q \geq 1$, and let $\mathscr{V} = S \cap$ Alg $\Gamma$ represents a CSL subalgebra within the von Neumann algebra $S$. Consider $\mathscr{F} : \mathscr{V} \to \mathscr{V}$ is an additive mapping that satisfies the algebraic identity $\mathscr{F}(V^{2q}) = \mathscr{F}(V^q)\eta(V^q) + \varphi(V^q)\mathscr{F}(V^q)$ for all $V \in \mathscr{V}$, where $\eta$ and $\varphi$ are automorphisms on $\mathscr{V}$. Then $\mathscr{F}$ is an $(\eta,\ \varphi)$-derivation on $\mathscr{V}$.

**Corollary 3** Let $q \geq 1$ be any fixed integer, and let $\mathscr{V} = S\cap$Alg $\Gamma$ represents a CSL subalgebra of the von Neumann algebra $S$. Suppose $\mathscr{G} : \mathscr{V} \to \mathscr{V}$ is an additive mapping such that $\mathscr{G}(V^{2q}) = \mathscr{G}(V^q)V^q$ holds for every $V \in \mathscr{V}$. Then $\mathscr{G}$ is a left centralizer in $\mathscr{V}$.

**Corollary 4** Let a fixed integer $q \geq 1$, and $\mathscr{V} = S \cap$ Alg $\Gamma$ denotes a CSL subalgebra of the von Neumann algebra $S$. If $\mathscr{F} : \mathscr{V} \to \mathscr{V}$ is an additive mapping that satisfies the identity $\mathscr{F}(V^{2q}) = \mathscr{F}(V^q)V^q + V^q\mathscr{F}(V^q)$ for all $V \in \mathscr{V}$, then $\mathscr{F}$ is a derivation of $\mathscr{V}$.

## 3. On semiprime rings

The study of Jordan $(\eta,\ \varphi)$-derivations in both algebras and rings covers a broad array of topics. It is evident that every $(\eta,\ \varphi)$-derivation is a Jordan $(\eta,\ \varphi)$-derivation. Ashraf et al. [9] and Lanski [10] provide insightful counterexamples that demonstrate that the reverse is not universally valid. Nevertheless, Bresar and Vukman's established result [11] confirms that within a 2-torsion-free prime ring, a Jordan $(\eta,\ \varphi)$-derivation actually functions as an $(\eta,\ \varphi)$-derivation. Lanski [10] further generalized this finding to 2-torsion-free semiprime rings. The meaning of 2 torsion free is, a ring with the condition that if $2x$ vanishes, then $x$ vanishes for all $x$ in the ring. A ring is prime (respectively, semiprime) if $R\mathscr{R}S = 0$ implies $R = 0$ or $S = 0$ (respectively, $R\mathscr{R}R = 0$ implies $R = 0$).

Investigating our subsequent central issue within the domain of pure ring theory is anticipated to substantially augment its intellectual appeal. It is observed that algebraic concepts are being applied in this context. Motivated by the aforementioned line of inquiry, we hereby propose the ensuing result:

**Theorem 3** Let $m \geq 1$ as any fixed integer, and $\mathscr{R}$ be a $(3m-1)!$-torsion-free semiprime ring with unity. Suppose that $\mathscr{G},\ \mathscr{F} : \mathscr{R} \to \mathscr{R}$ are two additive mappings that satisfy the algebraic identity $\mathscr{G}(R^{3m}) = \mathscr{G}(R^m)\eta(R^{2m}) + \varphi(R^m)\mathscr{F}(R^m)\eta(R^m) + \varphi(R^{2m})\mathscr{F}(R^m)$, where $\eta$ and $\varphi$ are automorphisms in $\mathscr{R}$. Thus, $\mathscr{G}$ is a generalized $(\eta,\ \varphi)$-derivation, accompanied by the $(\eta,\ \varphi)$-derivation $\mathscr{F}$ in $\mathscr{R}$.

**Proof.** The majority of the steps closely mirror those within the proof of Theorem 2; consequently, they have been omitted here for the reader's completion and comprehension. Moreover, it should be observed that the Vandermonde determinant within the $3m$-power expansion constitutes a product of distinct integers extending up to $(3m-1)$, and is therefore invertible under the condition of $(3m-1)!$-torsion-freeness.

## 4. Conclusion

The exploration of generalized $(\eta, \varphi)$-derivations on rings, alongside the intricate $(\eta, \varphi)$-derivation processes on CSL subalgebras of von Neumann algebras emerges as a compelling area of scholarly inquiry. We conclude that the two linear mappings $\mathscr{G}$, $\mathscr{F}$ satisfying the algebraic identities (given in Theorem 1 and 2) involved with automorphisms $\eta$ and $\varphi$ acting as a generalized $(\eta, \varphi)$-derivation, accompanied by the $(\eta, \varphi)$-derivation $\mathscr{F}$. Moreover, we emphasize the novelty through extending the $(\eta, \varphi)$-derivation frame work from CSL subalgebra to semiprime rings.

Future research could focus on continuity theorems across algebraic structures such as Banach algebra, semi-simple Banach algebra, Lie algebra, and C* algebra. Readers can explore functional identities related to specific derivations, including generalized $(\eta, \varphi)$-derivations on semiprime rings with involution and innovative generalized $(\eta, \varphi)$ higher derivations. Various additive maps for rings and their subsets are expressed through pure algebraic methods, revealing mathematical elegance and depth.


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## Conflict of interests

The authors declare that they have no conflicts of interest.

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