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Research Article

Exponential and Non-Exponential Similarity Measures for Bipolar Complex Fuzzy Sets and Their Application in CRM Software

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Abstract: Throughout the customer lifecycle, organizations monitor and analyze relationships with their current and potential customers using an approach and set of procedures called Customer Relationship Management (CRM). The main objective of CRM is to raise customer happiness, retention, and loyalty while promoting corporate expansion and profitability. The utilization of positive and negative aspects is necessary in many real-life situations, for example, the attributes of CRM have both their effects and side effects and also contain extra information. In this regard, Bipolar Complex Fuzzy Set (BCFS) is the only structure that can model such information that has dual aspects and extra information. Thus, the purpose of this article is to analyze the study of Similarity Measures (SMs) in the environment of BCFS, such as exponential, non-exponential based SMs and weighted exponential, non-exponential based SMs. Afterward, we anticipate a technique of Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) by employing the invented SMs in the framework of BCFS and then we investigate an application "selection of finest CRM software" with the assistance of the diagnosed approach of TOPSIS. In the last part of this article, we anticipate the comparison of our devised theory with current theories to reveal the advantages and superiority.

***Keywords*:** customer relationship management software, bipolar complex fuzzy set, similarity measures, exponential-based and non-exponential-based Similarity Measures (SMs)

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1. Introduction

Customer Relationship Management (CRM) is an analysis of customer interactions and data across the customer lifecycle. CRM systems collect and maintain customer data, including contact data, past purchases, choices, and correspondence history. As a result of CRM software's centralization of client data into a single database, accessing and updating customer data is made simpler for staff members working in various departments. CRM systems frequently have automated functionality for jobs like arranging follow-up meetings, sending personalized marketing emails, and allocating leads or client inquiries to the relevant team members. CRM software enables companies to monitor and assess

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client contacts, including phone calls for sales, emails, website visits, and social media interactions. Gaining knowledge about consumer behavior and preferences can be done using this data. Businesses may design customized marketing efforts and sales pitches that are more likely to be relevant to certain customers by studying their needs and behaviors. Literature is rich about CRM Xu et al. [1] analyze the basic concept of CRM, elaborate on its characteristics, review its history, and address the current status of CRM. Winer [2] developed the framework for CRM. Payne and Frow [3] look at CRM's definitional element and find three different CRM perspectives. The importance of a cross-functional, process-oriented strategy that places the CRM at a strategic level has been emphasized. The study of the effects of CRM on customer relationships has been given in [4, 5]. Bose [6] studied the fundamental principle of CRM and critical aspects of the Information Technology (IT) development process. Reinartz et al. [7] discuss the measurement and impact of CRM on performance.

1.1 *Literature review*

The idea of Fuzzy Sets (FSs) was first given by Zadeh [8] to deal with uncertain problems in the real world. FS theory is a mathematical concept that allows for the representation of uncertainty and imprecision in decision-making problems. Unlike classical set theory, which defines membership in terms of a binary relationship, an FS allows for a Membership Degree (MD) belonging to [0, 1]. Due to the generalization of crisp set theory, FS has gained more attention from researchers, and later on, the idea of FS with fuzzy logic was introduced by Klir and Yuan [9]. Some fundamentals of FS were studied by Dubois and Prade [10]. FS is restricted due to considering the MD, while there are some situations where we have to discuss the Membership Grade (MG) as well as the Non-Membership Degree (NMD) in one structure. Atanassov [11] observed this limitation and introduced the notion of the Intuitionistic Fuzzy Set (IFS). The IFS set proved to be very useful because of considers MD and NMD as well. In IFS, the value of MD and NMD both lie in [0, 1] with the condition that the sum (MD, NMD) must lie in [0, 1]. After that, a lot of work has been introduced under the notion of IFS. The distance between the two IFSs was explained by Szmidt and Kacprzyk [12]. Furthermore, some operations like concentration, dilation, and normalization for IFSs were described by De et al. [13]. Seikh and Chatterjee [14] utilized the idea of interval valued Fermatean FS and identified the sustainable strategies for electronic waste management. Moreover, the idea of interval valued spherical fuzzy Method based on the Removal Effects of Criteria-VlseKriterijumska Optimizacija I Kompromisno Resenje (MEREC-VIKOR) approach has been utilized in [15] and used this developed approach for sustainable energies for electronic vehicle adoption.

A significant portion of human Decision-Making (DM) is based on dual or bipolar judgmental thinking on both positive and negative elements. For instance, the two sides of a choice or coordination are effect and a side effect, like and unlike, collaboration and competition, etc. To cover these situations, there was a need for a structure that can handle two-sided aspects of certain situations in real-life problems. So, based on this observation, the notion of Bipolar Fuzzy Set (BFS) was defined by Zhang [16]. In BFS, the MD has positive and negative aspects. The idea of BFS was admired by the researchers, as Akram [17] has used the notion of BFS in graph theory and delivered the notion of BF graph. Moreover, the study of Bipolar Fuzzy (BF) graphs has been delivered by Rashmanlou et al. [18]. Moreover, an extension of BFS, which is known as m-polar FSs, was explained by Chen et al. [19]. Aggregation Operators (AOs) are considered to be useful tools that can change the overall data into a single value. Based on this observation, Wei et al. [20] developed the notion of BF Hamacher AOs. Based on Dombi t-norm, t-conorm, and BFS, Jana et al. [21] developed the notion of BF Dombi AOs. Also, Riaz et al. [22] delivered the innovative BF sine trigonometric AOs and they have introduced the Superiority and Inferiority Ranking (SIR) technique for the medical tourism supply chain. Gul [23] proposed an extension of the VIKOR approach for Multi-Criteria Decision-Making (MCDM) using the BF rough set model. Moreover, Mahmood et al. [24] used the bipolar structure and proposed the idea of T-bipolar soft semigroup and their related results.

Complex fuzzy sets, classes, and logic play a significant part in applications such as periodic forecasting of events and advanced control systems, where multiple fuzzy variables connect in numerous ways that cannot be successfully conveyed using simple fuzzy rules. Two kinds of attempts have been made to define the notion of Complex Fuzzy Sets (CFSs). The theory of CFS is another extension of FS, first invented by Ramot et al. [25] in 2022 in polar form and the range of MD in this case was considered to be a unit circle in a complex plane. Also, Tamir et al. [26] invented the theory of CFS in Cartesian form and they used the range as a unit square in the complex plane. Based on CFS, some operations, properties, and equalities in CFSs were studied and explained by Zhang et al. [27]. Furthermore, the interval-valued complex fuzzy logic was explained by Greenfield et al. [28]. An overview of the theory and applications of CFS and CF logic has been delivered by Tamir et al. [29]. To discuss the positive and negative aspects of an object and two-dimensional data, the notion of Bipolar Complex Fuzzy Set (BCFS) was developed by Mahmood and Rehman [30]. It means that BCFS is a useful strategy for dealing with two-dimensional data as well as the positive and negative characteristics of an entity or element. In BCFSs, the degree of membership and non-membership are represented by $\varsigma_G^+(x)$ and $\varsigma_G^-(x)$ where Positive Membership Degree (PMD) and NMD are further defined by $\varsigma_G^+(x) = \varrho_G^+(x) + \imath\sigma_G^+(x)$ and $\varsigma_G^-(x) = \varrho_G^-(x) + \imath\sigma_G^-(x)$, where $\varsigma_G^+(x) : X \longrightarrow [0, 1] + \imath[0, 1]$ and $\varsigma_G^-(x) : X \longrightarrow [-1, 0] + \imath[-1, 0]$. BCFSs have found applications in decision-making, control theory, image processing, and pattern recognition. They are a powerful tool for representing uncertain and ambiguous information in a more flexible way than traditional fuzzy sets. Based on this newly developed structure, some new developments have been made. Mahmood et al. [31] introduced Aczel-Alsina AOs based on BCFSs and developed their applications. Mahmood and Rehman [32] developed some Maclaurin symmetric means AOs based on BCFSs and utilized this developed theory in multi-attribute decision-making scenarios.

Similarity Measures (SMs) are also a very important topic in the field of FS. SMs play a crucial role in comparing and quantifying the similarity between two FSs or fuzzy numbers. They help in determining how close or similar two fuzzy sets are to each other. The idea of SM between FSs was established by Liu [33]. Beg and Ashraf [34] developed some distance measures and SMs for FSs. Some researchers have also made some contributions to SM in the environment of BFS. Rajeshwari et al. [35] established a Distance Measure (DM) between two BFSs. For handling uncertainties in the study of the fuzzy Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) method provides a systematic approach to deal with the fuzzy membership degrees of alternatives and criteria. By incorporating fuzzy algebraic operations and fuzzy distance measures, it can effectively deal with uncertainties and ambiguities present in real-world decision scenarios. The method of TOPSIS for the selection of plant location under fuzzy information was defined and explained by Yong [36]. The bipolar fuzzy TOPSIS method was explained by Akram and Arshad [37]. Later on, the multi-criteria decision-making method was explained by Alghamdi et al. [38]. In the environment of CFS, the TOPSIS method was established by Barbat et al. [39]. Ijaz et al. [40] used the idea of complex q-rung orthopair fuzzy set and elaborated on dynamic aggregation operators for the selection of an optimal communication system.

Table 1. Different literature for SMs

Existing literature on similarity	Similarity measures	Applications	Structure	Nature of structure
Turkarslan et al. [41] theory	Cosine similarity measure	Medical diagnosis	Fuzzy structure	Discuss the membership grade
Ye [42] approach	Cosine similarity measure	Pattern recognition and medical diagnosis	Intuitionistic fuzzy environment	Discuss the membership and non-membership grade
Arora and Naithani [43] approach	Sine similarity measure	Decision-making problems	Pythagorean fuzzy environment	Discuss the membership and non-membership grades
Mahmood and Rehman [30] approach	Tangent similarity measure	Pattern recognition and medical diagnosis	Bipolar complex fuzzy environment	Discuss the complexity and bipolar nature of the aspects
Mahmood et al. [44] approach	Set-theoretic similarity measure	Pattern recognition and medical diagnosis	Complex hesitant fuzzy set	Discuss the Hesitant fuzzy information
Danish and Kumar [45] approach	Exponential entropy-based knowledge measure	Multi-criteria decision-making	Fuzzy environment	Discuss the membership grade
Ying [46] approach	Exponential similarity measure	Initial diagnosis of depression grades	Cubic q-rung orthopair hesitant fuzzy set	Discuss the membership and non-membership grade
Verma [47] approach	Generalized trigonometric similarity measure	Multi-Attribute Decision-Making (MADM)	q-rung orthopair fuzzy set	Discuss the membership and non-membership grade

Different researchers have developed SMs, and their applications are given in the Table 1.

1.2 *Motivation for the proposed work*

The reason for employing bipolar complex fuzzy sets is that they can simultaneously deal with the positive and negative aspects of human logic, which cannot be achieved using traditional fuzzy set theories. This duality plays a significant role in CRM software since customer comments tend to have contrasting feelings. The complex component can further handle the two-dimensional information effectively in uncertain and ambiguous situations. The existing fuzzy model can never discuss all these characteristics in one structure and has limitations. By integrating bipolarity and complex values, our model overcomes these limitations, providing a more realistic representation of customer behavior. This directly benefits CRM systems in prioritizing customer queries and customizing recommendations. The proposed similarity measures enhance clustering and matching accuracy in customer profiling. Thus, the extension offers both theoretical advancement and practical significance in customer relationship management.

The difference between the proposed similarity measure and other similarity measures has been discussed in Table 2. Here in Table 2, we have discussed their mathematical structure, sensitivity analysis, range, advantages, and application that will show why we have used the exponential and non-exponential similarity measures. The overall discussion is given by.

Table 2. Characteristic analysis of different similarity measures

Similarity measure	Mathematical nature	Sensitivity analysis	Range	Key advantages	Applications
Exponential similarity (proposed)	Based on an exponential function	High (amplifies small differences)	$[0, 1]$	Capture subtle variations	Pattern recognition, fuzzy systems
Non-exponential similarity (proposed)	Use linear or polynomial expressions	Moderate	$[0, 1]$	More interpretable	General fuzzy decision making
Cosine similarity	Vector-based measure angle cosine	Moderate to high	$[-1, 1]$	Capture directional similarity	Text mining
Set-theoretic similarity	Based on the intersection/Union ratio	Moderate	$[0, 1]$	Simple and intuitive	Classification
Tangent similarity	Based on the tangent function	High (non-linear behavior)	$[0, 1]$	Capture rapid changes	Advanced fuzzy logic
Sine similarity	Based on the sine function	Moderate	$[0, 1]$	Suitable for cycle	Signal processing

1.3 *Main contribution of the developed approach*

Based on the characteristics of BCFS that can handle positive and negative aspects of objects and two-dimensional variables in one structure, here in this article:

• We have introduced the structure of some SMs, like exponential and non-exponential based SMs.

• We have explored the TOPSIS technique based on the introduced SMs under the notion of BCFSs.

• For the utilization of the TOPSIS algorithm, we have introduced an illustrative example for the classification of CRM software.

• Furthermore, the comparative analysis of the developed theory shows the advantages and superiority of the delivered approach.

1.4 *Arrangement of the article*

The rest of the article is arranged as follows: In section 2, we have revised the basic notion of FSs, BFSs, CFSs, and BCFSs. Moreover, their basic operational rules are also defined here. Section 3 is devoted to defining the notion of similarity measures under the novel notion of BCFS. In section 4, we have delivered the idea of the TOPSIS method based on the introduced similarity measures and provided an illustrative example for the classification of CRM software. Section 5 discusses the comparative analysis of the delivered approach. Section 6 is about the concluding remarks.

2. Preliminaries

In this part of the article, we will examine and review some basic concepts of FSs, BFSs, CFSs, BCFSs, and their related properties.

Definition 1 [8] A fuzzy set G has the shape

$$G = \{(x,\ \varsigma_G(x))\,|x \in X\}$$

Where $\varsigma_G(x) \in [0,\ 1]$ and represents membership degree.

Definition 2 [8] Let G and H be two FSs $G = \{(x,\ \varsigma_G(x))\,|x \in X\}$ and $H = \{(x,\ \varsigma_H(x))\,|x \in X\}$. Their basic operations are stated below

1. $G \subseteq H$, if $\varsigma_G(x) \leq \varsigma_H(x)$, $\forall x \in X$,
2. $G \bigcup H = \{< x,\ \max(\varsigma_G(x),\ \varsigma_H(x)) > |x \in X\}$,
3. $G \bigcap H = \{< x,\ \min(\varsigma_G(x),\ \varsigma_H(x)) > |x \in X\}$,
4. $G^c = \{< x,\ (1 - \varsigma_G(x)) > |x \in X\}$.

Definition 3 [16] A bipolar fuzzy set G has the form

$$G = \left\{\left(x,\ \varsigma_G^+(x),\ \varsigma_G^-(x)\right)|x \in X\right\}$$

Where $\varsigma_G^+(x) \in [0,\ 1]$ and represents the Positive Membership Degree (PMD) and $\varsigma_G^-(x) \in [-1,\ 0]$ and represents the Negative Membership Degree (NMD).

Definition 4 [16] Let G and H be two BFs $G = \left\{\left(x,\ \varsigma_G^+(x),\ \varsigma_G^-(x)\right)|x \in X\right\}$, and $H = \left\{\left(x,\ \varsigma_H^+(x),\ \varsigma_H^-(x)|x \in X\right)\right\}$. Then

1. $G^C = \left\{\left(x,\ 1 - \varsigma_G^+(x),\ -1 - \varsigma_G^-(x)\right)\right\}$,
2. $G \bigcup H = \left\{\left(x,\ \max\left(\varsigma_G^+(x),\ \varsigma_H^+(x)\right),\ \min\left(\varsigma_G^-(x),\ \varsigma_H^-(x)\right)\right)\right\}$,
3. $G \bigcap H = \left\{\left(x,\ \min\left(\varsigma_G^+(x),\ \varsigma_H^+(x)\right),\ \max\left(\varsigma_G^-(x),\ \varsigma_H^-(x)\right)\right)\right\}$.

Definition 5 [26] A complex fuzzy set G has the form

$$G = \{(x,\ \varsigma_G(x))\,|x \in X\}$$

Where $\varsigma_G(x) = \varrho_G(x) + \imath\sigma_G(x)$ represents the complex-valued truth grade in the shape of a Cartesian coordinate, and $\varrho_G(x),\ \sigma_G(x) \in [0,\ 1]$. Furthermore, the pair $G = \{(x,\ \varrho_G(x) + \imath\sigma_G(x))\,|x \in X\}$ is called a complex fuzzy number.

Definition 6 [26] Let G and H be two CFSs $G = \{x,\ \varrho_G(x) + \imath\sigma_G(x)\}$ and $H = \{x,\ \varrho_H(x) + \imath\sigma_H(x)\}$. Then

1. $G^C = \{x,\ (1 - \varrho_G(x) + \imath(1 - \sigma_G(x)))\}$,
2. $G \bigcup H = \{x,\ \max(\varrho_G(x),\ \varrho_H(x)) + \imath(\max(\sigma_G(x),\ \sigma_H(x)))\}$,
3. $G \bigcap H = \{x,\ \min(\varrho_G(x),\ \varrho_H(x)) + \imath(\min(\sigma_G(x),\ \sigma_H(x)))\}$.

Definition 7 [30] A BCFS has the form

$$G=\left\{\left(x,\,\varsigma_G^+(x),\,\varsigma_G^-(x)\right)|x\in X\right\},$$

where $\varsigma_G^+(x)=\varrho_G^+(x)+\iota\sigma_G^+(x)$, $\varsigma_G^-(x)=\varrho_G^-(x)+\iota\sigma_G^-(x)$, $\varsigma_G^+(x):X\longrightarrow[0,\,1]+\iota[0,\,1]$ and $\varsigma_G^-(x):X\longrightarrow[-1,\,0]+\iota[-1,\,0]$ also $\varrho_G^+(x)$ and $\sigma_G^+(x)$ represent the real and imaginary parts of PMD and $\varrho_G^-(x)$ and $\sigma_G^-(x)$ represent the real and imaginary parts of NMD, respectively.

Definition 8 [30] Let $G=\left\{\left(x,\,\varsigma_G^+(x),\,\varsigma_G^-(x)\right)|x\in X\right\}$ and $H=\left\{\left(x,\,\varsigma_H^+(x),\,\varsigma_H^-(x)\right)|x\in X\right\}$ be two BCFSs. Then

1.

$$c(A)=G^c=\left\{\left(x,\,\varsigma_G^+(x),\,\varsigma_G^-(x)\right)|x\in X\right\}^c$$

$$=\left\{\left(x,\,\begin{pmatrix}\left[1-\varrho_G^+(x)\right]+\iota\left[1-\sigma_G^+(x)\right],\\ \left[-1-\varrho_G^-(x)\right]+\iota\left[-1-\sigma_G^-(x)\right],\end{pmatrix}\right)\middle|x\in X\right\}$$

2.

$$G\bigcup H=\left\{\left(x,\,\varsigma_{G\bigcup H}^+(x),\,\varsigma_{G\bigcup H}^-(x)\right)|x\in X\right\}$$

$$=\left\{\left(x,\,\begin{pmatrix}\max\left(\varrho_G^+(x),\,\varrho_H^+(x)\right)+\iota\max\left(\sigma_G^+(x),\,\sigma_H^+(x)\right),\\ \min\left(\varrho_G^-(x),\,\varrho_H^-(x)\right)+\iota\min\left(\sigma_G^-(x),\,\sigma_H^-(x)\right),\end{pmatrix}\right)\middle|x\in X\right\}$$

3.

$$G\bigcap H=\left\{\left(x,\,\varsigma_{G\bigcap H}^+(x),\,\varsigma_{G\bigcap H}^-(x)\right)|x\in X\right\}$$

$$=\left\{\left(x,\,\begin{pmatrix}\min\left(\varrho_G^+(x),\,\varrho_H^+(x)\right)+\iota\min\left(\sigma_G^+(x),\,\sigma_H^+(x)\right),\\ \max\left(\varrho_G^-(x),\,\varrho_H^-(x)\right)+\iota\max\left(\sigma_G^-(x),\,\sigma_H^-(x)\right),\end{pmatrix}\right)\middle|x\in X\right\}$$

3. Similarity measures for BCFS

In this part of the paper, we will discuss the SMs based on exponential functions. We will also discuss some SMs without exponential functions in the environment of BCFSs.

Definition 9 Let G and H be two BCFSs on X. Then the SM between G and H is identified by $S_c(G,\,H)$, which satisfies the conditions written below

1. $0\leq S_c(G,\,H)\leq 1$,
2. $S_c(G,\,H)=1\Longleftrightarrow G=H$,
3. $S_c(G,\,H)=S_c(H,\,G)$.

Definition 10 Let G and H be two BCFSs on X. Then the exponential-based similarity measure is described as follows.

$$S_c^1(G,H)=\left[\frac{1}{n}\sum_{k=1}^{n}\left[\left(2^{1-\left(\begin{array}{c}\left(\left|\varrho_G^+(x_k)-\varrho_H^+(x_k)\right|^\mu\vee\left|\varrho_G^-(x_k)-\varrho_H^-(x_k)\right|^\mu\right)\vee\\\left(\left|\sigma_G^+(x_k)-\sigma_H^+(x_k)\right|^\mu\vee\left|\sigma_G^-(x_k)-\sigma_H^-(x_k)\right|^\mu\right)\end{array}\right)}\right)-1\right]\right]^{\frac{1}{\mu}}$$

Where μ is positive and $\vee$ denotes the max operator.

Theorem 1 The SM $S_c^1(G, H)$ satisfies the conditions written below

1. $0 \leq S_c^1(G, H) \leq 1$,
2. $S_C^1(G, H) = 1 \Longleftrightarrow G = H$,
3. $S_C^1(G, H) = S_C^1(H, G)$.

Proof. As we have

$$S_c^1(G,H)=\left[\frac{1}{n}\sum_{k=1}^{n}\left[\left(2^{1-\left(\begin{array}{c}\left(\left|\varrho_G^+(x_k)-\varrho_H^+(x_k)\right|^\mu\vee\left|\varrho_G^-(x_k)-\varrho_H^-(x_k)\right|^\mu\right)\vee\\\left(\left|\sigma_G^+(x_k)-\sigma_H^+(x_k)\right|^\mu\vee\left|\sigma_G^-(x_k)-\sigma_H^-(x_k)\right|^\mu\right)\end{array}\right)}\right)-1\right]\right]^{\frac{1}{\mu}}$$

Since

$$\left|\varrho_G^+(x_k)-\varrho_H^+(x_k)\right|^\mu\in[0,1],\ \left|\varrho_G^-(x_k)-\varrho_H^-(x_k)\right|^\mu\in[-1,0],\ \left|\sigma_G^+(x_k)-\sigma_H^+(x_k)\right|^\mu\in[0,1]$$

and

$$\left|\sigma_G^-(x_k)-\sigma_H^-(x_k)\right|^\mu\in[-1,0]$$

this implies that

$$\left(\left|\varrho_G^+(x_k)-\varrho_H^+(x_k)\right|^\mu\vee\left|\varrho_G^-(x_k)-\varrho_H^-(x_k)\right|^\mu\right)\in[0,1]$$

and also

$$\left(\left|\sigma_G^+(x_k)-\sigma_H^+(x_k)\right|^\mu\vee\left|\sigma_G^-(x_k)-\sigma_H^-(x_k)\right|^\mu\right)\in[0,1]$$

so

$$\left(2^{1-\left(\begin{array}{c}\left(|\varrho_G^+(x_k)-\varrho_H^+(x_k)|^\mu \vee |\varrho_G^-(x_k)-\varrho_H^-(x_k)|^\mu\right)\vee\\ \left(|\sigma_G^+(x_k)-\sigma_H^+(x_k)|^\mu \vee |\sigma_G^-(x_k)-\sigma_H^-(x_k)|^\mu\right)\end{array}\right)}\right)\in[0,\,1]$$

$$\Longrightarrow\left(\left(2^{1-\left(\begin{array}{c}\left(|\varrho_G^+(x_k)-\varrho_H^+(x_k)|^\mu \vee |\varrho_G^-(x_k)-\varrho_H^-(x_k)|^\mu\right)\vee\\ \left(|\sigma_G^+(x_k)-\sigma_H^+(x_k)|^\mu \vee |\sigma_G^-(x_k)-\sigma_H^-(x_k)|^\mu\right)\end{array}\right)}\right)-1\right)\in[0,\,1]$$

For $k=1$, we have

$$\Longrightarrow\left(\left(2^{1-\left(\begin{array}{c}\left(|\varrho_G^+(x_1)-\varrho_H^+(x_1)|^\mu \vee |\varrho_G^-(x_1)-\varrho_H^-(x_1)|^\mu\right)\vee\\ \left(|\sigma_G^+(x_1)-\sigma_H^+(x_1)|^\mu \vee |\sigma_G^-(x_1)-\sigma_H^-(x_1)|^\mu\right)\end{array}\right)}\right)-1\right)\in[0,\,1]$$

For $k=2$

$$\Longrightarrow\left(\left(2^{1-\left(\begin{array}{c}\left(|\varrho_G^+(x_2)-\varrho_H^+(x_2)|^\mu \vee |\varrho_G^-(x_2)-\varrho_H^-(x_2)|^\mu\right)\vee\\ \left(|\sigma_G^+(x_2)-\sigma_H^+(x_2)|^\mu \vee |\sigma_G^-(x_2)-\sigma_H^-(x_2)|^\mu\right)\end{array}\right)}\right)-1\right)\in[0,\,1]$$

Continuing in this way we get

$$\sum_{k=1}^{n}\left[\left(2^{1-\left(\begin{array}{c}\left(|\varrho_G^+(x_k)-\varrho_H^+(x_k)|^\mu \vee |\varrho_G^-(x_k)-\varrho_H^-(x_k)|^\mu\right)\vee\\ \left(|\sigma_G^+(x_k)-\sigma_H^+(x_k)|^\mu \vee |\sigma_G^-(x_k)-\sigma_H^-(x_k)|^\mu\right)\end{array}\right)}\right)-1\right]\in n[0,\,1]$$

$$\Longrightarrow 0\leq\sum_{k=1}^{n}\left[\left(2^{1-\left(\begin{array}{c}\left(|\varrho_G^+(x_k)-\varrho_H^+(x_k)|^\mu \vee |\varrho_G^-(x_k)-\varrho_H^-(x_k)|^\mu\right)\vee\\ \left(|\sigma_G^+(x_k)-\sigma_H^+(x_k)|^\mu \vee |\sigma_G^-(x_k)-\sigma_H^-(x_k)|^\mu\right)\end{array}\right)}\right)-1\right]\leq n$$

$$\Longrightarrow 0\leq\frac{1}{n}\sum_{k=1}^{n}\left[\left(2^{1-\left(\begin{array}{c}\left(|\varrho_G^+(x_k)-\varrho_H^+(x_k)|^\mu \vee |\varrho_G^-(x_k)-\varrho_H^-(x_k)|^\mu\right)\vee\\ \left(|\sigma_G^+(x_k)-\sigma_H^+(x_k)|^\mu \vee |\sigma_G^-(x_k)-\sigma_H^-(x_k)|^\mu\right)\end{array}\right)}\right)-1\right]\leq 1$$

$$\Longrightarrow 0 \leq \left[\frac{1}{n}\sum_{k=1}^{n}\left[\left(2^{1-\left(\begin{pmatrix}|\varrho_G^+(x_k)-\varrho_H^+(x_k)|^\mu \vee |\varrho_G^-(x_k)-\varrho_H^-(x_k)|^\mu\end{pmatrix}\vee \\ \begin{pmatrix}|\sigma_G^+(x_k)-\sigma_H^+(x_k)|^\mu \vee |\sigma_G^-(x_k)-\sigma_H^-(x_k)|^\mu\end{pmatrix}\right)}\right)-1\right]\right]^{\frac{1}{\mu}} \leq 1$$

$$\Longrightarrow 0 \leq S_c^1(G, H) \leq 1$$

2. We have

$$S_c^1(G, H) = \left[\frac{1}{n}\sum_{k=1}^{n}\left[\left(2^{1-\left(\begin{pmatrix}|\varrho_G^+(x_k)-\varrho_H^+(x_k)|^\mu \vee |\varrho_G^-(x_k)-\varrho_H^-(x_k)|^\mu\end{pmatrix}\vee \\ \begin{pmatrix}|\sigma_G^+(x_k)-\sigma_H^+(x_k)|^\mu \vee |\sigma_G^-(x_k)-\sigma_H^-(x_k)|^\mu\end{pmatrix}\right)}\right)-1\right]\right]^{\frac{1}{\mu}}$$

$$\Leftrightarrow S_c^1(G, H) = \left[\frac{1}{n}\left[\left(2^{1-\left(\begin{pmatrix}|\varrho_G^+(x_1)-\varrho_H^+(x_1)|^\mu \vee |\varrho_G^-(x_1)-\varrho_H^-(x_1)|^\mu\end{pmatrix}\vee \\ \begin{pmatrix}|\sigma_G^+(x_1)-\sigma_H^+(x_1)|^\mu \vee |\sigma_G^-(x_1)-\sigma_H^-(x_1)|^\mu\end{pmatrix}\right)}\right)-1\right]\right.$$

$$+\left[\left(2^{1-\left(\begin{pmatrix}|\varrho_G^+(x_2)-\varrho_H^+(x_2)|^\mu \vee |\varrho_G^-(x_2)-\varrho_H^-(x_2)|^\mu\end{pmatrix}\vee \\ \begin{pmatrix}|\sigma_G^+(x_2)-\sigma_H^+(x_2)|^\mu \vee |\sigma_G^-(x_2)-\sigma_H^-(x_2)|^\mu\end{pmatrix}\right)}\right)-1\right]+\ldots$$

$$\left.+\left[\left(2^{1-\left(\begin{pmatrix}|\varrho_G^+(x_n)-\varrho_H^+(x_n)|^\mu \vee |\varrho_G^-(x_n)-\varrho_H^-(x_n)|^\mu\end{pmatrix}\vee \\ \begin{pmatrix}|\sigma_G^+(x_n)-\sigma_H^+(x_n)|^\mu \vee |\sigma_G^-(x_n)-\sigma_H^-(x_n)|^\mu\end{pmatrix}\right)}\right)-1\right]\right]^{\frac{1}{\mu}}$$

$\Longleftrightarrow$ Now for $G = H$ we have $\varsigma_G^+ = \varsigma_H^+$ and $\varsigma_G^- = \varsigma_H^-$ for the values of $k = 1, 2, 3, \ldots, n \Leftrightarrow \varrho_G^+(x_k) = \varrho_H^+(x_k)$ and $\varrho_G^-(x_k) = \varrho_H^-(x_k)$ also $\sigma_G^+(x_k) = \sigma_H^+(x_k)$, $\sigma_G^-(x_k) = \sigma_H^-(x_k)$ for $k = 1, 2, 3, \ldots, n$. Then

$$\Leftrightarrow S_c^1(G, H) = \left[\frac{1}{n}\left[2^{1-0}-1+2^{1-0}-1+2^{1-0}-1\ldots+2^{1-0}-1\right]\right]^{\frac{1}{\mu}}$$

$$\Leftrightarrow S_c^1(G, H) = 1$$

3. We have

$$S_c^1(G, H)=\left[\frac{1}{n}\sum_{k=1}^{n}\left[\left(2^{1-\left(\begin{array}{c}\left(\left|\varrho_G^+(x_k)-\varrho_H^+(x_k)\right|^\mu\vee\left|\varrho_G^-(x_k)-\varrho_H^-(x_k)\right|^\mu\right)\vee\\ \left(\left|\sigma_G^+(x_k)-\sigma_H^+(x_k)\right|^\mu\vee\left|\sigma_G^-(x_k)-\sigma_H^-(x_k)\right|^\mu\right)\end{array}\right)}\right)-1\right]\right]^{\frac{1}{\mu}}$$

$$S_c^1(G, H)=\left[\frac{1}{n}\sum_{k=1}^{n}\left[\left(2^{1-\left(\begin{array}{c}\left(\left|-\left(-\varrho_G^+(x_k)+\varrho_H^+(x_k)\right)\right|^\mu\vee\left|-\left(-\varrho_G^-(x_k)+\varrho_H^-(x_k)\right)\right|^\mu\right)\vee\\ \left(\left|-\left(-\sigma_G^+(x_k)+\sigma_H^+(x_k)\right)\right|^\mu\vee\left|-\left(-\sigma_G^-(x_k)+\sigma_H^-(x_k)\right)\right|^\mu\right)\end{array}\right)}\right)-1\right]\right]^{\frac{1}{\mu}}$$

$$=\left[\frac{1}{n}\sum_{k=1}^{n}\left[\left(2^{1-\left(\begin{array}{c}\left(\left|\left(-\varrho_G^+(x_k)+\varrho_H^+(x_k)\right)\right|^\mu\vee\left|\left(-\varrho_G^-(x_k)+\varrho_H^-(x_k)\right)\right|^\mu\right)\vee\\ \left(\left|\left(-\sigma_G^+(x_k)+\sigma_H^+(x_k)\right)\right|^\mu\vee\left|\left(-\sigma_G^-(x_k)+\sigma_H^-(x_k)\right)\right|^\mu\right)\end{array}\right)}\right)-1\right]\right]^{\frac{1}{\mu}}$$

$$=\left[\frac{1}{n}\sum_{k=1}^{n}\left[\left(2^{1-\left(\begin{array}{c}\left(\left|\varrho_H^+(x_k)-\varrho_G^+(x_k)\right|^\mu\vee\left|\varrho_H^-(x_k)-\varrho_G^-(x_k)\right|^\mu\right)\vee\\ \left(\left|\sigma_H^+(x_k)-\sigma_G^+(x_k)\right|^\mu\vee\left|\sigma_H^-(x_k)-\sigma_G^-(x_k)\right|^\mu\right)\end{array}\right)}\right)-1\right]\right]^{\frac{1}{\mu}}$$

$$=S_c^1(H, G)$$

Remark 1 If $\mu=1$ then the exponential-based similarity measure becomes

$$S_c^1(G, H)=\frac{1}{n}\sum_{k=1}^{n}\left[\left(2^{1-\left(\begin{array}{c}\left(\left|\varrho_G^+(x_k)-\varrho_H^+(x_k)\right|^\mu\vee\left|\varrho_G^-(x_k)-\varrho_H^-(x_k)\right|^\mu\right)\vee\\ \left(\left|\sigma_G^+(x_k)-\sigma_H^+(x_k)\right|^\mu\vee\left|\sigma_G^-(x_k)-\sigma_H^-(x_k)\right|^\mu\right)\end{array}\right)}\right)-1\right]$$

Let G and H be two BCFSs on X. The exponential-based similarity measure can also be calculated as

$$S_c^2(G, H)=\left[\frac{1}{n}\sum_{k=1}^{n}\left[\left(2^{1-\frac{1}{2}\left(\begin{array}{c}\left(\left|\varrho_G^+(x_k)-\varrho_H^+(x_k)\right|^\mu\vee\left|\varrho_G^-(x_k)-\varrho_H^-(x_k)\right|^\mu\right)+\\ \left(\left|\sigma_G^+(x_k)-\sigma_H^+(x_k)\right|^\mu\vee\left|\sigma_G^-(x_k)-\sigma_H^-(x_k)\right|^\mu\right)\end{array}\right)}\right)-1\right]\right]^{\frac{1}{\mu}}$$

Where $\mu>0$.

Theorem 2 The $S_c^2(G, H)$ satisfy the conditions written below

1. $0\leq S_C^2(G, H)\leq 1$,

2. $S_C^2(G, H)=1\Longleftrightarrow G=H$,

3. $S_C^2(G, H) = S_C^2(H, G)$.
Proof. As we have

$$S_c^2(G, H) = \left[\frac{1}{n}\sum_{k=1}^{n}\left[\left(2^{1-\frac{1}{2}\left(\begin{array}{c}\left(\left|\varrho_G^+(x_k) - \varrho_H^+(x_k)\right|^\mu \vee \left|\varrho_G^-(x_k) - \varrho_H^-(x_k)\right|^\mu\right) + \\ \left(\left|\sigma_G^+(x_k) - \sigma_H^+(x_k)\right|^\mu \vee \left|\sigma_G^-(x_k) - \sigma_H^-(x_k)\right|^\mu\right)\end{array}\right)}\right) - 1\right]\right]^{\frac{1}{\mu}}$$

Since

$$\left|\varrho_G^+(x_k) - \varrho_H^+(x_k)\right|^\mu \in [0, 1], \ \left|\varrho_G^-(x_k) - \varrho_H^-(x_k)\right|^\mu \in [-1, 0], \ \left|\sigma_G^+(x_k) - \sigma_H^+(x_k)\right|^\mu \in [0, 1]$$

and

$$\left|\sigma_G^-(x_k) - \sigma_H^-(x_k)\right|^\mu \in [-1, 0]$$

this implies that

$$\left(\left|\varrho_G^+(x_k) - \varrho_H^+(x_k)\right|^\mu \vee \left|\varrho_G^-(x_k) - \varrho_H^-(x_k)\right|^\mu\right) \in [0, 1]$$

and also

$$\left(\left|\sigma_G^+(x_k) - \sigma_H^+(x_k)\right|^\mu \vee \left|\sigma_G^-(x_k) - \sigma_H^-(x_k)\right|^\mu\right) \in [0, 1]$$

so

$$\Longrightarrow \left(\begin{array}{c}\left(\left|\varrho_G^+(x_k) - \varrho_H^+(x_k)\right|^\mu \vee \left|\varrho_G^-(x_k) - \varrho_H^-(x_k)\right|^\mu\right) + \\ \left(\left|\sigma_G^+(x_k) - \sigma_H^+(x_k)\right|^\mu \vee \left|\sigma_G^-(x_k) - \sigma_H^-(x_k)\right|^\mu\right)\end{array}\right) \in [0, 1]$$

$$\Longrightarrow \left(\begin{array}{c}\left(\left|\varrho_G^+(x_k) - \varrho_H^+(x_k)\right|^\mu \vee \left|\varrho_G^-(x_k) - \varrho_H^-(x_k)\right|^\mu\right) + \\ \left(\left|\sigma_G^+(x_k) - \sigma_H^+(x_k)\right|^\mu \vee \left|\sigma_G^-(x_k) - \sigma_H^-(x_k)\right|^\mu\right)\end{array}\right) \in [0, 1]$$

$$\Longrightarrow \left(2^{1-\frac{1}{2}\left(\begin{array}{c}\left(\left|\varrho_G^+(x_k) - \varrho_H^+(x_k)\right|^\mu \vee \left|\varrho_G^-(x_k) - \varrho_H^-(x_k)\right|^\mu\right) + \\ \left(\left|\sigma_G^+(x_k) - \sigma_H^+(x_k)\right|^\mu \vee \left|\sigma_G^-(x_k) - \sigma_H^-(x_k)\right|^\mu\right)\end{array}\right)}\right) \in [0, 1]$$

$$\Longrightarrow \left(\left(2^{1-\frac{1}{2}\left(\begin{array}{c} \left(\left| \varrho_G^+(x_k) - \varrho_H^+(x_k) \right|^\mu \vee \left| \varrho_G^-(x_k) - \varrho_H^-(x_k) \right|^\mu \right) + \\ \left(\left| \sigma_G^+(x_k) - \sigma_H^+(x_k) \right|^\mu \vee \left| \sigma_G^-(x_k) - \sigma_H^-(x_k) \right|^\mu \right) \end{array} \right)} \right) - 1 \right) \in [0, 1]$$

By applying this process

$$\Longrightarrow \sum_{k=1}^{n} \left[\left(2^{1-\frac{1}{2}\left(\begin{array}{c} \left(\left| \varrho_G^+(x_k) - \varrho_H^+(x_k) \right|^\mu \vee \left| \varrho_G^-(x_k) - \varrho_H^-(x_k) \right|^\mu \right) + \\ \left(\left| \sigma_G^+(x_k) - \sigma_H^+(x_k) \right|^\mu \vee \left| \sigma_G^-(x_k) - \sigma_H^-(x_k) \right|^\mu \right) \end{array} \right)} \right) - 1 \right] \in n[0, 1]$$

$$0 \leq \sum_{k=1}^{n} \left[\left(2^{1-\frac{1}{2}\left(\begin{array}{c} \left(\left| \varrho_G^+(x_k) - \varrho_H^+(x_k) \right|^\mu \vee \left| \varrho_G^-(x_k) - \varrho_H^-(x_k) \right|^\mu \right) + \\ \left(\left| \sigma_G^+(x_k) - \sigma_H^+(x_k) \right|^\mu \vee \left| \sigma_G^-(x_k) - \sigma_H^-(x_k) \right|^\mu \right) \end{array} \right)} \right) - 1 \right] \leq n$$

$$0 \leq \frac{1}{n} \sum_{k=1}^{n} \left[\left(2^{1-\frac{1}{2}\left(\begin{array}{c} \left(\left| \varrho_G^+(x_k) - \varrho_H^+(x_k) \right|^\mu \vee \left| \varrho_G^-(x_k) - \varrho_H^-(x_k) \right|^\mu \right) + \\ \left(\left| \sigma_G^+(x_k) - \sigma_H^+(x_k) \right|^\mu \vee \left| \sigma_G^-(x_k) - \sigma_H^-(x_k) \right|^\mu \right) \end{array} \right)} \right) - 1 \right] \leq 1$$

$$0 \leq \left[\frac{1}{n} \sum_{k=1}^{n} \left[\left(2^{1-\frac{1}{2}\left(\begin{array}{c} \left(\left| \varrho_G^+(x_k) - \varrho_H^+(x_k) \right|^\mu \vee \left| \varrho_G^-(x_k) - \varrho_H^-(x_k) \right|^\mu \right) + \\ \left(\left| \sigma_G^+(x_k) - \sigma_H^+(x_k) \right|^\mu \vee \left| \sigma_G^-(x_k) - \sigma_H^-(x_k) \right|^\mu \right) \end{array} \right)} \right) - 1 \right] \right]^{\frac{1}{\mu}} \leq 1$$

$$0 \leq S_c^2(G, H) \leq 1$$

2. We have

$$S_c^2(G, H) = \left[\frac{1}{n} \sum_{k=1}^{n} \left[\left(2^{1-\frac{1}{2}\left(\begin{array}{c} \left(\left| \varrho_G^+(x_k) - \varrho_H^+(x_k) \right|^\mu \vee \left| \varrho_G^-(x_k) - \varrho_H^-(x_k) \right|^\mu \right) + \\ \left(\left| \sigma_G^+(x_k) - \sigma_H^+(x_k) \right|^\mu \vee \left| \sigma_G^-(x_k) - \sigma_H^-(x_k) \right|^\mu \right) \end{array} \right)} \right) - 1 \right] \right]^{\frac{1}{\mu}}$$

$$\Leftrightarrow S_c^2(G,H) = \left[\frac{1}{n}\left[\left(2^{1-\frac{1}{2}\left(\begin{array}{l}\left(|\varrho_G^+(x_1)-\varrho_H^+(x_1)|^{\mu} \vee |\varrho_G^-(x_1)-\varrho_H^-(x_1)|^{\mu}\right)+\\ \left(|\sigma_G^+(x_1)-\sigma_H^+(x_1)|^{\mu} \vee |\sigma_G^-(x_1)-\sigma_H^-(x_1)|^{\mu}\right)\end{array}\right)}\right)-1\right]\right.$$

$$+\left[\left(2^{1-\frac{1}{2}\left(\begin{array}{l}\left(|\varrho_G^+(x_2)-\varrho_H^+(x_2)|^{\mu} \vee |\varrho_G^-(x_2)-\varrho_H^-(x_2)|^{\mu}\right)+\\ \left(|\sigma_G^+(x_2)-\sigma_H^+(x_2)|^{\mu} \vee |\sigma_G^-(x_2)-\sigma_H^-(x_2)|^{\mu}\right)\end{array}\right)}\right)-1\right]+\ldots$$

$$\left.+\left[\left(2^{1-\frac{1}{2}\left(\begin{array}{l}\left(|\varrho_G^+(x_n)-\varrho_H^+(x_n)|^{\mu} \vee |\varrho_G^-(x_n)-\varrho_H^-(x_n)|^{\mu}\right)+\\ \left(|\sigma_G^+(x_n)-\sigma_H^+(x_n)|^{\mu} \vee |\sigma_G^-(x_n)-\sigma_H^-(x_n)|^{\mu}\right)\end{array}\right)}\right)-1\right]\right]^{\frac{1}{\mu}}$$

$\Longleftrightarrow$ Now for $G = H$ we have $\varsigma_G^+ = \varsigma_H^+$ and $\varsigma_G^- = \varsigma_H^-$ for the values of $k = 1, 2, 3, \ldots, n \Leftrightarrow \varrho_G^+(x_k) = \varrho_H^+(x_k)$ and $\varrho_G^-(x_k) = \varrho_H^-(x_k)$ also $\sigma_G^+(x_k) = \sigma_H^+(x_k)$, $\sigma_G^-(x_k) = \sigma_H^-(x_k)$ for $k = 1, 2, 3, \ldots, n$. Then

$$\Leftrightarrow S_c^2(G,H) = \left[\frac{1}{n}\left[2^{1-0}-1+2^{1-0}-1+2^{1-0}-1\ldots+2^{1-0}-1\right]\right]^{\frac{1}{\mu}}$$

$$\Leftrightarrow S_c^2(G,H) = 1$$

3. We have

$$S_c^2(G,H) = \left[\frac{1}{n}\sum_{k=1}^{n}\left[\left(2^{1-\frac{1}{2}\left(\begin{array}{l}\left(|\varrho_G^+(x_k)-\varrho_H^+(x_k)|^{\mu} \vee |\varrho_G^-(x_k)-\varrho_H^-(x_k)|^{\mu}\right)+\\ \left(|\sigma_G^+(x_k)-\sigma_H^+(x_k)|^{\mu} \vee |\sigma_G^-(x_k)-\sigma_H^-(x_k)|^{\mu}\right)\end{array}\right)}\right)-1\right]\right]^{\frac{1}{\mu}}$$

$$= \left[\frac{1}{n}\sum_{k=1}^{n}\left[\left(2^{1-\frac{1}{2}\left(\begin{array}{l}\left(|-\left(-\varrho_G^+(x_k)+\varrho_H^+(x_k)\right)|^{\mu} \vee |-\left(-\varrho_G^-(x_k)+\varrho_H^-(x_k)\right)|^{\mu}\right)+\\ \left(|-\left(-\sigma_G^+(x_k)+\sigma_H^+(x_k)\right)|^{\mu} \vee |-\left(-\sigma_G^-(x_k)+\sigma_H^-(x_k)\right)|^{\mu}\right)\end{array}\right)}\right)-1\right]\right]^{\frac{1}{\mu}}$$

$$= \left[\frac{1}{n}\sum_{k=1}^{n}\left[\left(2^{1-\frac{1}{2}\left(\begin{array}{l}\left(\left|\left(-\varrho_G^{+}(x_k)+\varrho_H^{+}(x_k)\right)\right|^{\mu}\vee\left|\left(-\varrho_G^{-}(x_k)+\varrho_H^{-}(x_k)\right)\right|^{\mu}\right)+\\\left(\left|\left(-\sigma_G^{+}(x_k)+\sigma_H^{+}(x_k)\right)\right|^{\mu}\vee\left|\left(-\sigma_G^{-}(x_k)+\sigma_H^{-}(x_k)\right)\right|^{\mu}\right)\end{array}\right)}\right)-1\right]\right]^{\frac{1}{\mu}}$$

$$= \left[\frac{1}{n}\sum_{k=1}^{n}\left[\left(2^{1-\frac{1}{2}\left(\begin{array}{l}\left(\left|\varrho_H^{+}(x_k)-\varrho_G^{+}(x_k)\right|^{\mu}\vee\left|\varrho_H^{-}(x_k)-\varrho_G^{-}(x_k)\right|^{\mu}\right)+\\\left(\left|\sigma_H^{+}(x_k)-\sigma_G^{+}(x_k)\right|^{\mu}\vee\left|\sigma_H^{-}(x_k)-\sigma_G^{-}(x_k)\right|^{\mu}\right)\end{array}\right)}\right)-1\right]\right]^{\frac{1}{\mu}}$$

$$= S_c^2(H, G)$$

Remark 2 If we put $\mu = 1$, then the exponential-based similarity measure will become

$$S_c^2(G, H) = \frac{1}{n}\sum_{k=1}^{n}\left[\left(2^{1-\frac{1}{2}\left(\begin{array}{l}\left(\left|\varrho_G^{+}(x_k)-\varrho_H^{+}(x_k)\right|^{\mu}\vee\left|\varrho_G^{-}(x_k)-\varrho_H^{-}(x_k)\right|^{\mu}\right)+\\\left(\left|\sigma_G^{+}(x_k)-\sigma_H^{+}(x_k)\right|^{\mu}\vee\left|\sigma_G^{-}(x_k)-\sigma_H^{-}(x_k)\right|^{\mu}\right)\end{array}\right)}\right)-1\right]$$

3.1 *Non-exponential based SMs for BCFSs*

In this segment, we devise the non-exponential SMs under the setting of BCFSs.

Definition 11 Let G and H be two BCFSs on X. Then, some similarity measures without an exponential function are calculated as written below

$$S_c^3(G, H) = \left[\frac{1}{n}\sum_{k=1}^{n}\left[\frac{1-\left(\begin{array}{l}\left(\left|\varrho_G^{+}(x_k)-\varrho_H^{+}(x_k)\right|^{\mu}\vee\left|\varrho_G^{-}(x_k)-\varrho_H^{-}(x_k)\right|^{\mu}\right)\vee\\\left(\left|\sigma_G^{+}(x_k)-\sigma_H^{+}(x_k)\right|^{\mu}\vee\left|\sigma_G^{-}(x_k)-\sigma_H^{-}(x_k)\right|^{\mu}\right)\end{array}\right)}{1+\left(\begin{array}{l}\left(\left|\varrho_G^{+}(x_k)-\varrho_H^{+}(x_k)\right|^{\mu}\vee\left|\varrho_G^{-}(x_k)-\varrho_H^{-}(x_k)\right|^{\mu}\right)\vee\\\left(\left|\sigma_G^{+}(x_k)-\sigma_H^{+}(x_k)\right|^{\mu}\vee\left|\sigma_G^{-}(x_k)-\sigma_H^{-}(x_k)\right|^{\mu}\right)\end{array}\right)}\right]\right]^{\frac{1}{\mu}}$$

$$S_c^4(G, H) = \left[\frac{1}{n}\sum_{k=1}^{n}\left[\frac{\left(\begin{array}{l}\left(\left(\varrho_G^{+}(x_k)\wedge\varrho_H^{+}(x_k)^{\mu}\right)\vee\left(\varrho_G^{-}(x_k)\wedge\varrho_H^{-}(x_k)^{\mu}\right)\right)+\\\left(\left(\sigma_G^{+}(x_k)\wedge\sigma_H^{+}(x_k)\right)^{\mu}\vee\left(\sigma_G^{-}(x_k)\wedge\sigma_H^{-}(x_k)\right)^{\mu}\right)\end{array}\right)}{\left(\begin{array}{l}\left(\left(\varrho_G^{+}(x_k)\vee\varrho_H^{+}(x_k)^{\mu}\right)\vee\left(\varrho_G^{-}(x_k)\vee\varrho_H^{-}(x_k)^{\mu}\right)\right)+\\\left(\left(\sigma_G^{+}(x_k)\vee\sigma_H^{+}(x_k)\right)^{\mu}\vee\left(\sigma_G^{-}(x_k)\vee\sigma_H^{-}(x_k)\right)^{\mu}\right)\end{array}\right)}\right]\right]^{\frac{1}{\mu}}$$

$$S_C^5(G, H) = \left[\frac{1}{n} \sum_{k=1}^{n} \left(\Phi_{cc} \frac{\left(\left(\varrho_G^+ (x_k) \wedge \varrho_H^+ (x_k)^{\mu} \right) \vee \left(\varrho_G^- (x_k) \wedge \varrho_H^- (x_k)^{\mu} \right) \right)}{\left(\left(\varrho_G^+ (x_k) \vee \varrho_H^+ (x_k)^{\mu} \right) \vee \left(\varrho_G^- (x_k) \vee \varrho_H^- (x_k)^{\mu} \right) \right)} \right. \right.$$

$$\left. \left. + \Psi_{cc} \frac{\left(\left(\sigma_G^+ (x_k) \wedge \sigma_H^+ (x_k) \right)^{\mu} \vee \left(\sigma_G^- (x_k) \wedge \sigma_H^- (x_k) \right)^{\mu} \right)}{\left(\left(\sigma_G^+ (x_k) \vee \sigma_H^+ (x_k) \right)^{\mu} \vee \left(\sigma_G^- (x_k) \vee \sigma_H^- (x_k) \right)^{\mu} \right)} \right) \right]^{\frac{1}{\mu}}$$

Where $\mu > 0$ and $\Phi_{cc}, \Psi_{cc} \in [0, 1]$ such that $\Phi_{cc} + \Psi_{cc} = 1$.

Theorem 3 The similarity measure $S_C^3(G, H)$ satisfies the properties written below

1. $0 \leq S_C^3(G, H) \leq 1$,
2. $S_C^3(G, H) = 1 \Longleftrightarrow G = H$,
3. $S_C^3(G, H) = S_C^3(H, G)$.

Proof. As we have

$$S_c^3(G, H) = \left[\frac{1}{n} \sum_{k=1}^{n} \left[\frac{1 - \begin{pmatrix} \left(\left| \varrho_G^+ (x_k) - \varrho_H^+ (x_k) \right|^{\mu} \vee \left| \varrho_G^- (x_k) - \varrho_H^- (x_k) \right|^{\mu} \right) \vee \\ \left(\left| \sigma_G^+ (x_k) - \sigma_H^+ (x_k) \right|^{\mu} \vee \left| \sigma_G^- (x_k) - \sigma_H^- (x_k) \right|^{\mu} \right) \end{pmatrix}}{1 + \begin{pmatrix} \left(\left| \varrho_G^+ (x_k) - \varrho_H^+ (x_k) \right|^{\mu} \vee \left| \varrho_G^- (x_k) - \varrho_H^- (x_k) \right|^{\mu} \right) \vee \\ \left(\left| \sigma_G^+ (x_k) - \sigma_H^+ (x_k) \right|^{\mu} \vee \left| \sigma_G^- (x_k) - \sigma_H^- (x_k) \right|^{\mu} \right) \end{pmatrix}} \right] \right]^{\frac{1}{\mu}}$$

Since

$$\left| \varrho_G^+ (x_k) - \varrho_H^+ (x_k) \right|^{\mu} \in [0, 1], \quad \left| \varrho_G^- (x_k) - \varrho_H^- (x_k) \right|^{\mu} \in [-1, 0]$$

so

$$\left(\left| \varrho_G^+ (x_k) - \varrho_H^+ (x_k) \right|^{\mu} \vee \left| \varrho_G^- (x_k) - \varrho_H^- (x_k) \right|^{\mu} \right) \in [0, 1]$$

also

$$\left(\left| \sigma_G^+ (x_k) - \sigma_H^+ (x_k) \right|^{\mu} \vee \left| \sigma_G^- (x_k) - \sigma_H^- (x_k) \right|^{\mu} \right) \in [0, 1]$$

$$\Longrightarrow \begin{pmatrix} \left(\left| \varrho_G^+ (x_k) - \varrho_H^+ (x_k) \right|^{\mu} \vee \left| \varrho_G^- (x_k) - \varrho_H^- (x_k) \right|^{\mu} \right) \vee \\ \left(\left| \sigma_G^+ (x_k) - \sigma_H^+ (x_k) \right|^{\mu} \vee \left| \sigma_G^- (x_k) - \sigma_H^- (x_k) \right|^{\mu} \right) \end{pmatrix} \in [0, 1]$$

and the denominator will be greater than the numerator, so for $k = 1$ we have

$$\left[\frac{1-\left(\begin{array}{l}\left(\left|\varrho_G^+(x_1)-\varrho_H^+(x_1)\right|^\mu \vee \left|\varrho_G^-(x_1)-\varrho_H^-(x_1)\right|^\mu\right) \vee \\ \left(\left|\sigma_G^+(x_1)-\sigma_H^+(x_1)\right|^\mu \vee \left|\sigma_G^-(x_1)-\sigma_H^-(x_1)\right|^\mu\right)\end{array}\right)}{1+\left(\begin{array}{l}\left(\left|\varrho_G^+(x_1)-\varrho_H^+(x_1)\right|^\mu \vee \left|\varrho_G^-(x_1)-\varrho_H^-(x_1)\right|^\mu\right) \vee \\ \left(\left|\sigma_G^+(x_1)-\sigma_H^+(x_1)\right|^\mu \vee \left|\sigma_G^-(x_1)-\sigma_H^-(x_1)\right|^\mu\right)\end{array}\right)}\right] \in [0,\, 1]$$

For $k = 2$ we have

$$\left[\frac{1-\left(\begin{array}{l}\left(\left|\varrho_G^+(x_2)-\varrho_H^+(x_2)\right|^\mu \vee \left|\varrho_G^-(x_2)-\varrho_H^-(x_2)\right|^\mu\right) \vee \\ \left(\left|\sigma_G^+(x_2)-\sigma_H^+(x_2)\right|^\mu \vee \left|\sigma_G^-(x_2)-\sigma_H^-(x_2)\right|^\mu\right)\end{array}\right)}{1+\left(\begin{array}{l}\left(\left|\varrho_G^+(x_2)-\varrho_H^+(x_2)\right|^\mu \vee \left|\varrho_G^-(x_2)-\varrho_H^-(x_2)\right|^\mu\right) \vee \\ \left(\left|\sigma_G^+(x_2)-\sigma_H^+(x_2)\right|^\mu \vee \left|\sigma_G^-(x_2)-\sigma_H^-(x_2)\right|^\mu\right)\end{array}\right)}\right] \in [0,\, 1]$$

By continuing this way we get

$$\sum_{k=1}^{n}\left[\frac{1-\left(\begin{array}{l}\left(\left|\varrho_G^+(x_k)-\varrho_H^+(x_k)\right|^\mu \vee \left|\varrho_G^-(x_k)-\varrho_H^-(x_k)\right|^\mu\right) \vee \\ \left(\left|\sigma_G^+(x_k)-\sigma_H^+(x_k)\right|^\mu \vee \left|\sigma_G^-(x_k)-\sigma_H^-(x_k)\right|^\mu\right)\end{array}\right)}{1+\left(\begin{array}{l}\left(\left|\varrho_G^+(x_k)-\varrho_H^+(x_k)\right|^\mu \vee \left|\varrho_G^-(x_k)-\varrho_H^-(x_k)\right|^\mu\right) \vee \\ \left(\left|\sigma_G^+(x_k)-\sigma_H^+(x_k)\right|^\mu \vee \left|\sigma_G^-(x_k)-\sigma_H^-(x_k)\right|^\mu\right)\end{array}\right)}\right] \in n[0,\, 1]$$

$$\Longrightarrow 0 \leq \sum_{k=1}^{n}\left[\frac{1-\left(\begin{array}{l}\left(\left|\varrho_G^+(x_k)-\varrho_H^+(x_k)\right|^\mu \vee \left|\varrho_G^-(x_k)-\varrho_H^-(x_k)\right|^\mu\right) \vee \\ \left(\left|\sigma_G^+(x_k)-\sigma_H^+(x_k)\right|^\mu \vee \left|\sigma_G^-(x_k)-\sigma_H^-(x_k)\right|^\mu\right)\end{array}\right)}{1+\left(\begin{array}{l}\left(\left|\varrho_G^+(x_k)-\varrho_H^+(x_k)\right|^\mu \vee \left|\varrho_G^-(x_k)-\varrho_H^-(x_k)\right|^\mu\right) \vee \\ \left(\left|\sigma_G^+(x_k)-\sigma_H^+(x_k)\right|^\mu \vee \left|\sigma_G^-(x_k)-\sigma_H^-(x_k)\right|^\mu\right)\end{array}\right)}\right] \leq n$$

$$\Longrightarrow 0 \leq \frac{1}{n}\sum_{k=1}^{n}\left[\frac{1-\left(\begin{array}{l}\left(\left|\varrho_G^+(x_k)-\varrho_H^+(x_k)\right|^\mu \vee \left|\varrho_G^-(x_k)-\varrho_H^-(x_k)\right|^\mu\right) \vee \\ \left(\left|\sigma_G^+(x_k)-\sigma_H^+(x_k)\right|^\mu \vee \left|\sigma_G^-(x_k)-\sigma_H^-(x_k)\right|^\mu\right)\end{array}\right)}{1+\left(\begin{array}{l}\left(\left|\varrho_G^+(x_k)-\varrho_H^+(x_k)\right|^\mu \vee \left|\varrho_G^-(x_k)-\varrho_H^-(x_k)\right|^\mu\right) \vee \\ \left(\left|\sigma_G^+(x_k)-\sigma_H^+(x_k)\right|^\mu \vee \left|\sigma_G^-(x_k)-\sigma_H^-(x_k)\right|^\mu\right)\end{array}\right)}\right] \leq 1$$

$$\Longrightarrow 0 \leq \left[\frac{1}{n} \sum_{k=1}^{n} \left[\frac{1 - \begin{pmatrix} \left(\left| \varrho_G^+(x_k) - \varrho_H^+(x_k) \right|^\mu \vee \left| \varrho_G^-(x_k) - \varrho_H^-(x_k) \right|^\mu \right) \vee \\ \left(\left| \sigma_G^+(x_k) - \sigma_H^+(x_k) \right|^\mu \vee \left| \sigma_G^-(x_k) - \sigma_H^-(x_k) \right|^\mu \right) \end{pmatrix}}{1 + \begin{pmatrix} \left(\left| \varrho_G^+(x_k) - \varrho_H^+(x_k) \right|^\mu \vee \left| \varrho_G^-(x_k) - \varrho_H^-(x_k) \right|^\mu \right) \vee \\ \left(\left| \sigma_G^+(x_k) - \sigma_H^+(x_k) \right|^\mu \vee \left| \sigma_G^-(x_k) - \sigma_H^-(x_k) \right|^\mu \right) \end{pmatrix}} \right] \right]^{\frac{1}{\mu}} \leq 1$$

$$\Longrightarrow 0 \leq S_c^3(G, H) \leq 1$$

2. By definition we have

$$S_c^3(G, H) = \left[\frac{1}{n} \sum_{k=1}^{n} \left[\frac{1 - \begin{pmatrix} \left(\left| \varrho_G^+(x_k) - \varrho_H^+(x_k) \right|^\mu \vee \left| \varrho_G^-(x_k) - \varrho_H^-(x_k) \right|^\mu \right) \vee \\ \left(\left| \sigma_G^+(x_k) - \sigma_H^+(x_k) \right|^\mu \vee \left| \sigma_G^-(x_k) - \sigma_H^-(x_k) \right|^\mu \right) \end{pmatrix}}{1 + \begin{pmatrix} \left(\left| \varrho_G^+(x_k) - \varrho_H^+(x_k) \right|^\mu \vee \left| \varrho_G^-(x_k) - \varrho_H^-(x_k) \right|^\mu \right) \vee \\ \left(\left| \sigma_G^+(x_k) - \sigma_H^+(x_k) \right|^\mu \vee \left| \sigma_G^-(x_k) - \sigma_H^-(x_k) \right|^\mu \right) \end{pmatrix}} \right] \right]^{\frac{1}{\mu}}$$

$$\Longleftrightarrow S_c^3(G, H) = \left[\frac{1}{n} \left[\frac{1 - \begin{pmatrix} \left(\left| \varrho_G^+(x_1) - \varrho_H^+(x_1) \right|^\mu \vee \left| \varrho_G^-(x_1) - \varrho_H^-(x_1) \right|^\mu \right) \vee \\ \left(\left| \sigma_G^+(x_1) - \sigma_H^+(x_1) \right|^\mu \vee \left| \sigma_G^-(x_1) - \sigma_H^-(x_1) \right|^\mu \right) \end{pmatrix}}{1 + \begin{pmatrix} \left(\left| \varrho_G^+(x_1) - \varrho_H^+(x_1) \right|^\mu \vee \left| \varrho_G^-(x_1) - \varrho_H^-(x_1) \right|^\mu \right) \vee \\ \left(\left| \sigma_G^+(x_1) - \sigma_H^+(x_1) \right|^\mu \vee \left| \sigma_G^-(x_1) - \sigma_H^-(x_1) \right|^\mu \right) \end{pmatrix}} \right. \right.$$

$$+ \frac{1 - \begin{pmatrix} \left(\left| \varrho_G^+(x_2) - \varrho_H^+(x_2) \right|^\mu \vee \left| \varrho_G^-(x_2) - \varrho_H^-(x_2) \right|^\mu \right) \vee \\ \left(\left| \sigma_G^+(x_2) - \sigma_H^+(x_2) \right|^\mu \vee \left| \sigma_G^-(x_2) - \sigma_H^-(x_2) \right|^\mu \right) \end{pmatrix}}{1 + \begin{pmatrix} \left(\left| \varrho_G^+(x_2) - \varrho_H^+(x_2) \right|^\mu \vee \left| \varrho_G^-(x_2) - \varrho_H^-(x_2) \right|^\mu \right) \vee \\ \left(\left| \sigma_G^+(x_2) - \sigma_H^+(x_2) \right|^\mu \vee \left| \sigma_G^-(x_2) - \sigma_H^-(x_2) \right|^\mu \right) \end{pmatrix}}$$

$$+ \ldots$$

$$\left. \left. + \frac{1 - \begin{pmatrix} \left(\left| \varrho_G^+(x_n) - \varrho_H^+(x_n) \right|^\mu \vee \left| \varrho_G^-(x_n) - \varrho_H^-(x_n) \right|^\mu \right) \vee \\ \left(\left| \sigma_G^+(x_n) - \sigma_H^+(x_n) \right|^\mu \vee \left| \sigma_G^-(x_n) - \sigma_H^-(x_n) \right|^\mu \right) \end{pmatrix}}{1 + \begin{pmatrix} \left(\left| \varrho_G^+(x_n) - \varrho_H^+(x_n) \right|^\mu \vee \left| \varrho_G^-(x_n) - \varrho_H^-(x_n) \right|^\mu \right) \vee \\ \left(\left| \sigma_G^+(x_n) - \sigma_H^+(x_n) \right|^\mu \vee \left| \sigma_G^-(x_n) - \sigma_H^-(x_n) \right|^\mu \right) \end{pmatrix}} \right] \right]^{\frac{1}{\mu}}$$

$\Longleftrightarrow$ Now for $G = H$ we have $\varsigma_G^+ = \varsigma_H^+$ and $\varsigma_G^- = \varsigma_H^-$ for the values of $k = 1, 2, 3, \ldots, n \Leftrightarrow \varrho_G^+(x_k) = \varrho_H^+(x_k)$ and $\varrho_G^-(x_k) = \varrho_H^-(x_k)$ also $\sigma_G^+(x_k) = \sigma_H^+(x_k)$, $\sigma_G^-(x_k) = \sigma_H^-(x_k)$ for $k = 1, 2, 3, \ldots, n$. Then

$$\Longleftrightarrow S_c^3(G, H) = \left[\frac{1}{n}(1+1+\ldots+1)\right]^{\frac{1}{\mu}}$$

$$\Longleftrightarrow S_c^3(G, H) = 1$$

3. We have

$$S_c^3(G, H) = \left[\frac{1}{n}\sum_{k=1}^{n}\left[\frac{1-\begin{pmatrix}\left(\left|\varrho_G^+(x_k)-\varrho_H^+(x_k)\right|^\mu \vee \left|\varrho_G^-(x_k)-\varrho_H^-(x_k)\right|^\mu\right)\vee \\ \left(\left|\sigma_G^+(x_k)-\sigma_H^+(x_k)\right|^\mu \vee \left|\sigma_G^-(x_k)-\sigma_H^-(x_k)\right|^\mu\right)\end{pmatrix}}{1+\begin{pmatrix}\left(\left|\varrho_G^+(x_k)-\varrho_H^+(x_k)\right|^\mu \vee \left|\varrho_G^-(x_k)-\varrho_H^-(x_k)\right|^\mu\right)\vee \\ \left(\left|\sigma_G^+(x_k)-\sigma_H^+(x_k)\right|^\mu \vee \left|\sigma_G^-(x_k)-\sigma_H^-(x_k)\right|^\mu\right)\end{pmatrix}}\right]\right]^{\frac{1}{\mu}}$$

$$= \left[\frac{1}{n}\sum_{k=1}^{n}\left[\frac{1-\begin{pmatrix}\left(\left|-\left(-\varrho_G^+(x_k)+\varrho_H^+(x_k)\right)\right|^\mu \vee \left|-\left(-\varrho_G^-(x_k)+\varrho_H^-(x_k)\right)\right|^\mu\right)\vee \\ \left(\left|-\left(-\sigma_G^+(x_k)+\sigma_H^+(x_k)\right)\right|^\mu \vee \left|-\left(-\sigma_G^-(x_k)+\sigma_H^-(x_k)\right)\right|^\mu\right)\end{pmatrix}}{1+\begin{pmatrix}\left(\left|-\left(-\varrho_G^+(x_k)+\varrho_H^+(x_k)\right)\right|^\mu \vee \left|-\left(-\varrho_G^-(x_k)+\varrho_H^-(x_k)\right)\right|^\mu\right)\vee \\ \left(\left|-\left(-\sigma_G^+(x_k)+\sigma_H^+(x_k)\right)\right|^\mu \vee \left|-\left(-\sigma_G^-(x_k)+\sigma_H^-(x_k)\right)\right|^\mu\right)\end{pmatrix}}\right]\right]^{\frac{1}{\mu}}$$

$$= \left[\frac{1}{n}\sum_{k=1}^{n}\left[\frac{1-\begin{pmatrix}\left(\left|\varrho_H^+(x_k)-\varrho_G^+(x_k)\right|^\mu \vee \left|\varrho_H^-(x_k)-\varrho_G^-(x_k)\right|^\mu\right)\vee \\ \left(\left|\sigma_H^+(x_k)-\sigma_G^+(x_k)\right|^\mu \vee \left|\sigma_H^-(x_k)-\sigma_G^-(x_k)\right|^\mu\right)\end{pmatrix}}{1+\begin{pmatrix}\left(\left|\varrho_H^+(x_k)-\varrho_G^+(x_k)\right|^\mu \vee \left|\varrho_H^-(x_k)-\varrho_G^-(x_k)\right|^\mu\right)\vee \\ \left(\left|\sigma_H^+(x_k)-\sigma_G^+(x_k)\right|^\mu \vee \left|\sigma_H^-(x_k)-\sigma_G^-(x_k)\right|^\mu\right)\end{pmatrix}}\right]\right]^{\frac{1}{\mu}}$$

$$= S_c^3(H, G)$$

Theorem 4 The similarity measure $S_C^4(G, H)$ satisfies the properties written below
1. $0 \leq S_C^4(G, H) \leq 1$,
2. $S_C^4(G, H) = 1 \Longleftrightarrow G = H$,
3. $S_C^4(G, H) = S_C^4(H, G)$.

Proof. As we have

$$S_c^4(G, H) = \left[\frac{1}{n}\sum_{k=1}^{n}\left[\frac{\begin{pmatrix}\left(\left(\varrho_G^+(x_k)\wedge\varrho_H^+(x_k)\right)^\mu \vee \left(\varrho_G^-(x_k)\wedge\varrho_H^-(x_k)\right)^\mu\right)+ \\ \left(\left(\sigma_G^+(x_k)\wedge\sigma_H^+(x_k)\right)^\mu \vee \left(\sigma_G^-(x_k)\wedge\sigma_H^-(x_k)\right)^\mu\right)\end{pmatrix}}{\begin{pmatrix}\left(\left(\varrho_G^+(x_k)\vee\varrho_H^+(x_k)\right)^\mu \vee \left(\varrho_G^-(x_k)\vee\varrho_H^-(x_k)\right)^\mu\right)+ \\ \left(\left(\sigma_G^+(x_k)\vee\sigma_H^+(x_k)\right)^\mu \vee \left(\sigma_G^-(x_k)\vee\sigma_H^-(x_k)\right)^\mu\right)\end{pmatrix}}\right]\right]^{\frac{1}{\mu}}$$

Since

$$\left(\varrho_G^+(x_k) \wedge \varrho_H^+(x_k)\right)^\mu \in [0, 1], \quad \left(\varrho_G^-(x_k) \wedge \varrho_H^-(x_k)\right)^\mu \in [-1, 0]$$

so

$$\left(\left(\varrho_G^+(x_k) \wedge \varrho_H^+(x_k)\right)^\mu \vee \left(\varrho_G^-(x_k) \wedge \varrho_H^-(x_k)\right)^\mu\right) \in [0, 1]$$

also

$$\left(\left(\sigma_G^+(x_k) \wedge \sigma_H^+(x_k)\right)^\mu \vee \left(\sigma_G^-(x_k) \wedge \sigma_H^-(x_k)\right)^\mu\right) \in [0, 1]$$

$$\Longrightarrow \begin{pmatrix} \left(\left(\varrho_G^+(x_k) \wedge \varrho_H^+(x_k)\right)^\mu \vee \left(\varrho_G^-(x_k) \wedge \varrho_H^-(x_k)\right)^\mu\right) + \\ \left(\left(\sigma_G^+(x_k) \wedge \sigma_H^+(x_k)\right)^\mu \vee \left(\sigma_G^-(x_k) \wedge \sigma_H^-(x_k)\right)^\mu\right) \end{pmatrix} \in [0, 1]$$

and the denominator will be always greater than the numerator, so for $k = 1$ we have

$$\frac{1}{n}\sum_{k=1}^{n}\left[\frac{\begin{pmatrix} \left(\left(\varrho_G^+(x_1) \wedge \varrho_H^+(x_1)\right)^\mu \vee \left(\varrho_G^-(x_1) \wedge \varrho_H^-(x_1)\right)^\mu\right) + \\ \left(\left(\sigma_G^+(x_1) \wedge \sigma_H^+(x_1)\right)^\mu \vee \left(\sigma_G^-(x_1) \wedge \sigma_H^-(x_1)\right)^\mu\right) \end{pmatrix}}{\begin{pmatrix} \left(\left(\varrho_G^+(x_1) \vee \varrho_H^+(x_1)\right)^\mu \vee \left(\varrho_G^-(x_1) \vee \varrho_H^-(x_1)\right)^\mu\right) + \\ \left(\left(\sigma_G^+(x_1) \vee \sigma_H^+(x_1)\right)^\mu \vee \left(\sigma_G^-(x_1) \vee \sigma_H^-(x_1)\right)^\mu\right) \end{pmatrix}}\right] \in [0, 1]$$

For $k = 2$, we have

$$\frac{1}{n}\sum_{k=1}^{n}\left[\frac{\begin{pmatrix} \left(\left(\varrho_G^+(x_2) \wedge \varrho_H^+(x_2)\right)^\mu \vee \left(\varrho_G^-(x_2) \wedge \varrho_H^-(x_2)\right)^\mu\right) + \\ \left(\left(\sigma_G^+(x_2) \wedge \sigma_H^+(x_2)\right)^\mu \vee \left(\sigma_G^-(x_2) \wedge \sigma_H^-(x_2)\right)^\mu\right) \end{pmatrix}}{\begin{pmatrix} \left(\left(\varrho_G^+(x_2) \vee \varrho_H^+(x_2)\right)^\mu \vee \left(\varrho_G^-(x_2) \vee \varrho_H^-(x_2)\right)^\mu\right) + \\ \left(\left(\sigma_G^+(x_2) \vee \sigma_H^+(x_2)\right)^\mu \vee \left(\sigma_G^-(x_2) \vee \sigma_H^-(x_2)\right)^\mu\right) \end{pmatrix}}\right] \in [0, 1]$$

By continuing this way we get

$$\sum_{k=1}^{n}\left[\frac{\begin{pmatrix}\left(\left(\varrho_G^+(x_k)\wedge\varrho_H^+(x_k)\right)^{\mu}\vee\left(\varrho_G^-(x_k)\wedge\varrho_H^-(x_k)\right)^{\mu}\right)+\\\left(\left(\sigma_G^+(x_k)\wedge\sigma_H^+(x_k)\right)^{\mu}\vee\left(\sigma_G^-(x_k)\wedge\sigma_H^-(x_k)\right)^{\mu}\right)\end{pmatrix}}{\begin{pmatrix}\left(\left(\varrho_G^+(x_k)\vee\varrho_H^+(x_k)\right)^{\mu}\vee\left(\varrho_G^-(x_k)\vee\varrho_H^-(x_k)\right)^{\mu}\right)+\\\left(\left(\sigma_G^+(x_k)\vee\sigma_H^+(x_k)\right)^{\mu}\vee\left(\sigma_G^-(x_k)\vee\sigma_H^-(x_k)\right)^{\mu}\right)\end{pmatrix}}\right]\in n[0,\,1]$$

$$\Longrightarrow 0\leq\sum_{k=1}^{n}\left[\frac{\begin{pmatrix}\left(\left(\varrho_G^+(x_k)\wedge\varrho_H^+(x_k)\right)^{\mu}\vee\left(\varrho_G^-(x_k)\wedge\varrho_H^-(x_k)\right)^{\mu}\right)+\\\left(\left(\sigma_G^+(x_k)\wedge\sigma_H^+(x_k)\right)^{\mu}\vee\left(\sigma_G^-(x_k)\wedge\sigma_H^-(x_k)\right)^{\mu}\right)\end{pmatrix}}{\begin{pmatrix}\left(\left(\varrho_G^+(x_k)\vee\varrho_H^+(x_k)\right)^{\mu}\vee\left(\varrho_G^-(x_k)\vee\varrho_H^-(x_k)\right)^{\mu}\right)+\\\left(\left(\sigma_G^+(x_k)\vee\sigma_H^+(x_k)\right)^{\mu}\vee\left(\sigma_G^-(x_k)\vee\sigma_H^-(x_k)\right)^{\mu}\right)\end{pmatrix}}\right]\leq n$$

$$\Longrightarrow 0\leq\frac{1}{n}\sum_{k=1}^{n}\left[\frac{\begin{pmatrix}\left(\left(\varrho_G^+(x_k)\wedge\varrho_H^+(x_k)\right)^{\mu}\vee\left(\varrho_G^-(x_k)\wedge\varrho_H^-(x_k)\right)^{\mu}\right)+\\\left(\left(\sigma_G^+(x_k)\wedge\sigma_H^+(x_k)\right)^{\mu}\vee\left(\sigma_G^-(x_k)\wedge\sigma_H^-(x_k)\right)^{\mu}\right)\end{pmatrix}}{\begin{pmatrix}\left(\left(\varrho_G^+(x_k)\vee\varrho_H^+(x_k)\right)^{\mu}\vee\left(\varrho_G^-(x_k)\vee\varrho_H^-(x_k)\right)^{\mu}\right)+\\\left(\left(\sigma_G^+(x_k)\vee\sigma_H^+(x_k)\right)^{\mu}\vee\left(\sigma_G^-(x_k)\vee\sigma_H^-(x_k)\right)^{\mu}\right)\end{pmatrix}}\right]\leq 1$$

$$\Longrightarrow 0\leq\left[\frac{1}{n}\sum_{k=1}^{n}\left[\frac{\begin{pmatrix}\left(\left(\varrho_G^+(x_k)\wedge\varrho_H^+(x_k)\right)^{\mu}\vee\left(\varrho_G^-(x_k)\wedge\varrho_H^-(x_k)\right)^{\mu}\right)+\\\left(\left(\sigma_G^+(x_k)\wedge\sigma_H^+(x_k)\right)^{\mu}\vee\left(\sigma_G^-(x_k)\wedge\sigma_H^-(x_k)\right)^{\mu}\right)\end{pmatrix}}{\begin{pmatrix}\left(\left(\varrho_G^+(x_k)\vee\varrho_H^+(x_k)\right)^{\mu}\vee\left(\varrho_G^-(x_k)\vee\varrho_H^-(x_k)\right)^{\mu}\right)+\\\left(\left(\sigma_G^+(x_k)\vee\sigma_H^+(x_k)\right)^{\mu}\vee\left(\sigma_G^-(x_k)\vee\sigma_H^-(x_k)\right)^{\mu}\right)\end{pmatrix}}\right]\right]^{\frac{1}{\mu}}\leq 1$$

$$\Longrightarrow 0\leq S_c^4(G,\,H)\leq 1$$

2. By definition we have

$$S_c^4(G,\,H)=\left[\frac{1}{n}\sum_{k=1}^{n}\left[\frac{\begin{pmatrix}\left(\left(\varrho_G^+(x_k)\wedge\varrho_H^+(x_k)\right)^{\mu}\vee\left(\varrho_G^-(x_k)\wedge\varrho_H^-(x_k)\right)^{\mu}\right)+\\\left(\left(\sigma_G^+(x_k)\wedge\sigma_H^+(x_k)\right)^{\mu}\vee\left(\sigma_G^-(x_k)\wedge\sigma_H^-(x_k)\right)^{\mu}\right)\end{pmatrix}}{\begin{pmatrix}\left(\left(\varrho_G^+(x_k)\vee\varrho_H^+(x_k)\right)^{\mu}\vee\left(\varrho_G^-(x_k)\vee\varrho_H^-(x_k)\right)^{\mu}\right)+\\\left(\left(\sigma_G^+(x_k)\vee\sigma_H^+(x_k)\right)^{\mu}\vee\left(\sigma_G^-(x_k)\vee\sigma_H^-(x_k)\right)^{\mu}\right)\end{pmatrix}}\right]\right]^{\frac{1}{\mu}}$$

$$\Longleftrightarrow S_c^4(G,H)=\left[\frac{1}{n}\left[\frac{\begin{pmatrix}\left(\left(\varrho_G^+(x_1)\wedge\varrho_H^+(x_1)\right)^\mu\vee\left(\varrho_G^-(x_1)\wedge\varrho_H^-(x_1)\right)^\mu\right)+\\\left(\left(\sigma_G^+(x_1)\wedge\sigma_H^+(x_1)\right)^\mu\vee\left(\sigma_G^-(x_1)\wedge\sigma_H^-(x_1)\right)^\mu\right)\end{pmatrix}}{\begin{pmatrix}\left(\left(\varrho_G^+(x_1)\vee\varrho_H^+(x_1)\right)^\mu\vee\left(\varrho_G^-(x_1)\vee\varrho_H^-(x_1)\right)^\mu\right)+\\\left(\left(\sigma_G^+(x_1)\vee\sigma_H^+(x_1)\right)^\mu\vee\left(\sigma_G^-(x_1)\vee\sigma_H^-(x_1)\right)^\mu\right)\end{pmatrix}}+\frac{\begin{pmatrix}\left(\left(\varrho_G^+(x_2)\wedge\varrho_H^+(x_2)\right)^\mu\vee\left(\varrho_G^-(x_2)\wedge\varrho_H^-(x_2)\right)^\mu\right)+\\\left(\left(\sigma_G^+(x_2)\wedge\sigma_H^+(x_2)\right)^\mu\vee\left(\sigma_G^-(x_2)\wedge\sigma_H^-(x_2)\right)^\mu\right)\end{pmatrix}}{\begin{pmatrix}\left(\left(\varrho_G^+(x_2)\vee\varrho_H^+(x_2)\right)^\mu\vee\left(\varrho_G^-(x_2)\vee\varrho_H^-(x_2)\right)^\mu\right)+\\\left(\left(\sigma_G^+(x_2)\vee\sigma_H^+(x_2)\right)^\mu\vee\left(\sigma_G^-(x_2)\vee\sigma_H^-(x_2)\right)^\mu\right)\end{pmatrix}}+\ldots\frac{\begin{pmatrix}\left(\left(\varrho_G^+(x_n)\wedge\varrho_H^+(x_n)\right)^\mu\vee\left(\varrho_G^-(x_n)\wedge\varrho_H^-(x_n)\right)^\mu\right)+\\\left(\left(\sigma_G^+(x_n)\wedge\sigma_H^+(x_n)\right)^\mu\vee\left(\sigma_G^-(x_n)\wedge\sigma_H^-(x_n)\right)^\mu\right)\end{pmatrix}}{\begin{pmatrix}\left(\left(\varrho_G^+(x_n)\vee\varrho_H^+(x_n)\right)^\mu\vee\left(\varrho_G^-(x_n)\vee\varrho_H^-(x_n)\right)^\mu\right)+\\\left(\left(\sigma_G^+(x_n)\vee\sigma_H^+(x_n)\right)^\mu\vee\left(\sigma_G^-(x_n)\vee\sigma_H^-(x_n)\right)^\mu\right)\end{pmatrix}}\right]\right]^{\frac{1}{\mu}}$$

$\Longleftrightarrow$ Now for $G=H$ we have $\varsigma_G^+=\varsigma_H^+$ and $\varsigma_G^-=\varsigma_H^-$ for the values of $k=1,2,3,\ldots,n\Leftrightarrow\varrho_G^+(x_k)=\varrho_H^+(x_k)$ and $\varrho_G^-(x_k)=\varrho_H^-(x_k)$ also $\sigma_G^+(x_k)=\sigma_H^+(x_k)$, $\sigma_G^-(x_k)=\sigma_H^-(x_k)$ for $k=1,2,3,\ldots,n$. Then

$$\Longleftrightarrow S_c^4(G,H)=\left[\frac{1}{n}(1+1+\ldots+1)\right]^{\frac{1}{\mu}}$$

$$\Longleftrightarrow S_c^4(G,H)=1$$

3. We have

$$S_c^4(G,H)=\left[\frac{1}{n}\sum_{k=1}^{n}\left[\frac{\begin{pmatrix}\left(\left(\varrho_G^+(x_k)\wedge\varrho_H^+(x_k)\right)^\mu\vee\left(\varrho_G^-(x_k)\wedge\varrho_H^-(x_k)\right)^\mu\right)+\\\left(\left(\sigma_G^+(x_k)\wedge\sigma_H^+(x_k)\right)^\mu\vee\left(\sigma_G^-(x_k)\wedge\sigma_H^-(x_k)\right)^\mu\right)\end{pmatrix}}{\begin{pmatrix}\left(\left(\varrho_G^+(x_k)\vee\varrho_H^+(x_k)\right)^\mu\vee\left(\varrho_G^-(x_k)\vee\varrho_H^-(x_k)\right)^\mu\right)+\\\left(\left(\sigma_G^+(x_k)\vee\sigma_H^+(x_k)\right)^\mu\vee\left(\sigma_G^-(x_k)\vee\sigma_H^-(x_k)\right)^\mu\right)\end{pmatrix}}\right]\right]^{\frac{1}{\mu}}$$

$$=\left[\frac{1}{n}\sum_{k=1}^{n}\left[\frac{\begin{pmatrix}\left(\left(\varrho_H^+(x_k)\wedge\varrho_G^+(x_k)\right)^\mu\vee\left(\varrho_H^-(x_k)\wedge\varrho_G^-(x_k)\right)^\mu\right)+\\\left(\left(\sigma_H^+(x_k)\wedge\sigma_G^+(x_k)\right)^\mu\vee\left(\sigma_H^-(x_k)\wedge\sigma_G^-(x_k)\right)^\mu\right)\end{pmatrix}}{\begin{pmatrix}\left(\left(\varrho_H^+(x_k)\vee\varrho_G^+(x_k)\right)^\mu\vee\left(\varrho_H^-(x_k)\vee\varrho_G^-(x_k)\right)^\mu\right)+\\\left(\left(\sigma_H^+(x_k)\vee\sigma_G^+(x_k)\right)^\mu\vee\left(\sigma_H^-(x_k)\vee\sigma_G^-(x_k)\right)^\mu\right)\end{pmatrix}}\right]\right]^{\frac{1}{\mu}}=S_c^4(H,G)$$

Theorem 5 The similarity measure $S_C^5(G, H)$ satisfies the properties written below
1. $0 \leq S_C^5(G, H) \leq 1$,
2. $S_C^5(G, H) = 1 \Longleftrightarrow G = H$,
3. $S_C^5(G, H) = S_C^5(H, G)$.

$$S_C^5(G, H) = \left[\frac{1}{n}\sum_{k=1}^{n}\left(\Phi_{cc}\frac{\left(\left(\varrho_G^+(x_k) \wedge \varrho_H^+(x_k)\right)^\mu \vee \left(\varrho_G^-(x_k) \wedge \varrho_H^-(x_k)\right)^\mu\right)}{\left(\left(\varrho_G^+(x_k) \vee \varrho_H^+(x_k)\right)^\mu \vee \left(\varrho_G^-(x_k) \vee \varrho_H^-(x_k)\right)^\mu\right)}\right.\right.$$

$$\left.\left. + \Psi_{cc}\frac{\left(\left(\sigma_G^+(x_k) \wedge \sigma_H^+(x_k)\right)^\mu \vee \left(\sigma_G^-(x_k) \wedge \sigma_H^-(x_k)\right)^\mu\right)}{\left(\left(\sigma_G^+(x_k) \vee \sigma_H^+(x_k)\right)^\mu \vee \left(\sigma_G^-(x_k) \vee \sigma_H^-(x_k)\right)^\mu\right)}\right)\right]^{\frac{1}{\mu}}$$

Proof. As we have

$$S_C^5(G, H) = \left[\frac{1}{n}\sum_{k=1}^{n}\left(\Phi_{cc}\frac{\left(\left(\varrho_G^+(x_k) \wedge \varrho_H^+(x_k)\right)^\mu \vee \left(\varrho_G^-(x_k) \wedge \varrho_H^-(x_k)\right)^\mu\right)}{\left(\left(\varrho_G^+(x_k) \vee \varrho_H^+(x_k)\right)^\mu \vee \left(\varrho_G^-(x_k) \vee \varrho_H^-(x_k)\right)^\mu\right)}\right.\right.$$

$$\left.\left. + \Psi_{cc}\frac{\left(\left(\sigma_G^+(x_k) \wedge \sigma_H^+(x_k)\right)^\mu \vee \left(\sigma_G^-(x_k) \wedge \sigma_H^-(x_k)\right)^\mu\right)}{\left(\left(\sigma_G^+(x_k) \vee \sigma_H^+(x_k)\right)^\mu \vee \left(\sigma_G^-(x_k) \vee \sigma_H^-(x_k)\right)^\mu\right)}\right)\right]^{\frac{1}{\mu}}$$

As

$$\left(\varrho_G^+(x_k) \wedge \varrho_H^+(x_k)\right)^\mu \in [0, 1]$$

and

$$\left(\varrho_G^-(x_k) \wedge \varrho_H^-(x_k)\right)^\mu \in [-1, 0]$$

$$\Longrightarrow \left(\left(\varrho_G^+(x_k) \wedge \varrho_H^+(x_k)\right)^\mu \vee \left(\varrho_G^-(x_k) \wedge \varrho_H^-(x_k)\right)^\mu\right) \in [0, 1]$$

also

$$\left(\varrho_G^+(x_k) \vee \varrho_H^+(x_k)\right)^\mu \in [0, 1]$$

and

$$\left(\varrho_G^-(x_k)\vee\varrho_H^-(x_k)\right)^\mu\in[-1,\,0]$$

$$\Longrightarrow\left(\left(\varrho_G^+(x_k)\vee\varrho_H^+(x_k)\right)^\mu\vee\left(\varrho_G^-(x_k)\vee\varrho_H^-(x_k)\right)^\mu\right)\in[0,\,1]$$

$$\Longrightarrow\Phi_{cc}\frac{\left(\left(\varrho_G^+(x_k)\wedge\varrho_H^+(x_k)\right)^\mu\vee\left(\varrho_G^-(x_k)\wedge\varrho_H^-(x_k)\right)^\mu\right)}{\left(\left(\varrho_G^+(x_k)\vee\varrho_H^+(x_k)\right)^\mu\vee\left(\varrho_G^-(x_k)\vee\varrho_H^-(x_k)\right)^\mu\right)}\in[0,\,1]$$

And similarly

$$\Psi_{cc}\frac{\left(\left(\sigma_G^+(x_k)\wedge\sigma_H^+(x_k)\right)^\mu\vee\left(\sigma_G^-(x_k)\wedge\sigma_H^-(x_k)\right)^\mu\right)}{\left(\left(\sigma_G^+(x_k)\vee\sigma_H^+(x_k)\right)^\mu\vee\left(\sigma_G^-(x_k)\vee\sigma_H^-(x_k)\right)^\mu\right)}\in[0,\,1].$$

Thus for $k=1$, we have

$$\frac{1}{n}\left(\left[\Phi_{cc}\frac{\left(\left(\varrho_G^+(x_1)\wedge\varrho_H^+(x_1)\right)^\mu\vee\left(\varrho_G^-(x_1)\wedge\varrho_H^-(x_1)\right)^\mu\right)}{\left(\left(\varrho_G^+(x_1)\vee\varrho_H^+(x_1)\right)^\mu\vee\left(\varrho_G^-(x_1)\vee\varrho_H^-(x_1)\right)^\mu\right)}\right.\right.$$

$$\left.\left.+\ \Psi_{cc}\frac{\left(\left(\sigma_G^+(x_1)\wedge\sigma_H^+(x_1)\right)^\mu\vee\left(\sigma_G^-(x_1)\wedge\sigma_H^-(x_1)\right)^\mu\right)}{\left(\left(\sigma_G^+(x_1)\vee\sigma_H^+(x_1)\right)^\mu\vee\left(\sigma_G^-(x_1)\vee\sigma_H^-(x_1)\right)^\mu\right)}\right]\right)\in[0,\,1]$$

For $k=2$, we have

$$\frac{1}{n}\left(\left[\Phi_{cc}\frac{\left(\left(\varrho_G^+(x_2)\wedge\varrho_H^+(x_2)\right)^\mu\vee\left(\varrho_G^-(x_2)\wedge\varrho_H^-(x_2)\right)^\mu\right)}{\left(\left(\varrho_G^+(x_2)\vee\varrho_H^+(x_2)\right)^\mu\vee\left(\varrho_G^-(x_2)\vee\varrho_H^-(x_2)\right)^\mu\right)}\right.\right.$$

$$\left.\left.+\ \Psi_{cc}\frac{\left(\left(\sigma_G^+(x_2)\wedge\sigma_H^+(x_2)\right)^\mu\vee\left(\sigma_G^-(x_2)\wedge\sigma_H^-(x_2)\right)^\mu\right)}{\left(\left(\sigma_G^+(x_2)\vee\sigma_H^+(x_2)\right)^\mu\vee\left(\sigma_G^-(x_2)\vee\sigma_H^-(x_2)\right)^\mu\right)}\right]\right)\in[0,\,1]$$

Continuing this way we get

$$\sum_{k=1}^{n}\left(\left[\Phi_{cc}\frac{\left(\left(\varrho_G^+(x_k)\wedge\varrho_H^+(x_k)\right)^{\mu}\vee\left(\varrho_G^-(x_k)\wedge\varrho_H^-(x_k)\right)^{\mu}\right)}{\left(\left(\varrho_G^+(x_k)\vee\varrho_H^+(x_k)\right)^{\mu}\vee\left(\varrho_G^-(x_k)\vee\varrho_H^-(x_k)\right)^{\mu}\right)}\right.\right.$$

$$\left.\left.+\Psi_{cc}\frac{\left(\left(\sigma_G^+(x_k)\wedge\sigma_H^+(x_k)\right)^{\mu}\vee\left(\sigma_G^-(x_k)\wedge\sigma_H^-(x_k)\right)^{\mu}\right)}{\left(\left(\sigma_G^+(x_k)\vee\sigma_H^+(x_k)\right)^{\mu}\vee\left(\sigma_G^-(x_k)\vee\sigma_H^-(x_k)\right)^{\mu}\right)}\right]\right)\in n[0,1]$$

$$\Longrightarrow 0\leq\frac{1}{n}\sum_{k=1}^{n}\left(\left[\Phi_{cc}\frac{\left(\left(\varrho_G^+(x_k)\wedge\varrho_H^+(x_k)\right)^{\mu}\vee\left(\varrho_G^-(x_k)\wedge\varrho_H^-(x_k)\right)^{\mu}\right)}{\left(\left(\varrho_G^+(x_k)\vee\varrho_H^+(x_k)\right)^{\mu}\vee\left(\varrho_G^-(x_k)\vee\varrho_H^-(x_k)\right)^{\mu}\right)}\right.\right.$$

$$\left.\left.+\Psi_{cc}\frac{\left(\left(\sigma_G^+(x_k)\wedge\sigma_H^+(x_k)\right)^{\mu}\vee\left(\sigma_G^-(x_k)\wedge\sigma_H^-(x_k)\right)^{\mu}\right)}{\left(\left(\sigma_G^+(x_k)\vee\sigma_H^+(x_k)\right)^{\mu}\vee\left(\sigma_G^-(x_k)\vee\sigma_H^-(x_k)\right)^{\mu}\right)}\right]\right)\in[0,1]$$

$$\Longrightarrow 0\leq\left[\frac{1}{n}\sum_{k=1}^{n}\left[\Phi_{cc}\frac{\left(\varrho_G^+(x_k)\wedge\varrho_H^+(x_k)\right)^{\mu}\vee\left(\varrho_G^-(x_k)\wedge\varrho_H^-(x_k)\right)^{\mu}}{\left(\varrho_G^+(x_k)\vee\varrho_H^+(x_k)\right)^{\mu}\vee\left(\varrho_G^-(x_k)\vee\varrho_H^-(x_k)\right)^{\mu}}\right.\right.$$

$$\left.\left.+\Psi_{cc}\frac{\left(\sigma_G^+(x_k)\wedge\sigma_H^+(x_k)\right)^{\mu}\vee\left(\sigma_G^-(x_k)\wedge\sigma_H^-(x_k)\right)^{\mu}}{\left(\sigma_G^+(x_k)\vee\sigma_H^+(x_k)\right)^{\mu}\vee\left(\sigma_G^-(x_k)\vee\sigma_H^-(x_k)\right)^{\mu}}\right]\right]^{\frac{1}{\mu}}\in[0,1]$$

$$\Longrightarrow 0\leq S_c^5(G,H)\leq 1$$

2. By definition of $S_C^5(G,H)$, we have

$$S_C^5(G,H)=\left[\frac{1}{n}\sum_{k=1}^{n}\left[\Phi_{cc}\frac{\left(\varrho_G^+(x_k)\wedge\varrho_H^+(x_k)\right)^{\mu}\vee\left(\varrho_G^-(x_k)\wedge\varrho_H^-(x_k)\right)^{\mu}}{\left(\varrho_G^+(x_k)\vee\varrho_H^+(x_k)\right)^{\mu}\vee\left(\varrho_G^-(x_k)\vee\varrho_H^-(x_k)\right)^{\mu}}\right.\right.$$

$$\left.\left.+\Psi_{cc}\frac{\left(\sigma_G^+(x_k)\wedge\sigma_H^+(x_k)\right)^{\mu}\vee\left(\sigma_G^-(x_k)\wedge\sigma_H^-(x_k)\right)^{\mu}}{\left(\sigma_G^+(x_k)\vee\sigma_H^+(x_k)\right)^{\mu}\vee\left(\sigma_G^-(x_k)\vee\sigma_H^-(x_k)\right)^{\mu}}\right]\right]^{\frac{1}{\mu}}$$

$$\Longleftrightarrow S_C^5(G, H) = \left[\frac{1}{n}\left(\begin{bmatrix}\Phi_{cc}\dfrac{\left(\left(\varrho_G^+(x_1)\wedge\varrho_H^+(x_1)\right)^\mu \vee \left(\varrho_G^-(x_1)\wedge\varrho_H^-(x_1)\right)^\mu\right)}{\left(\left(\varrho_G^+(x_1)\vee\varrho_H^+(x_1)\right)^\mu \vee \left(\varrho_G^-(x_1)\vee\varrho_H^-(x_1)\right)^\mu\right)} \\ +\Psi_{cc}\dfrac{\left(\left(\sigma_G^+(x_1)\wedge\sigma_H^+(x_1)\right)^\mu \vee \left(\sigma_G^-(x_1)\wedge\sigma_H^-(x_1)\right)^\mu\right)}{\left(\left(\sigma_G^+(x_1)\vee\sigma_H^+(x_1)\right)^\mu \vee \left(\sigma_G^-(x_1)\vee\sigma_H^-(x_1)\right)^\mu\right)}\end{bmatrix} + \begin{bmatrix}\Phi_{cc}\dfrac{\left(\left(\varrho_G^+(x_2)\wedge\varrho_H^+(x_2)\right)^\mu \vee \left(\varrho_G^-(x_2)\wedge\varrho_H^-(x_2)\right)^\mu\right)}{\left(\left(\varrho_G^+(x_2)\vee\varrho_H^+(x_2)\right)^\mu \vee \left(\varrho_G^-(x_2)\vee\varrho_H^-(x_2)\right)^\mu\right)} \\ +\Psi_{cc}\dfrac{\left(\left(\sigma_G^+(x_2)\wedge\sigma_H^+(x_2)\right)^\mu \vee \left(\sigma_G^-(x_2)\wedge\sigma_H^-(x_2)\right)^\mu\right)}{\left(\left(\sigma_G^+(x_2)\vee\sigma_H^+(x_2)\right)^\mu \vee \left(\sigma_G^-(x_2)\vee\sigma_H^-(x_2)\right)^\mu\right)}\end{bmatrix} + \ldots \begin{bmatrix}\Phi_{cc}\dfrac{\left(\left(\varrho_G^+(x_n)\wedge\varrho_H^+(x_n)\right)^\mu \vee \left(\varrho_G^-(x_n)\wedge\varrho_H^-(x_n)\right)^\mu\right)}{\left(\left(\varrho_G^+(x_n)\vee\varrho_H^+(x_n)\right)^\mu \vee \left(\varrho_G^-(x_n)\vee\varrho_H^-(x_n)\right)^\mu\right)} \\ +\Psi_{cc}\dfrac{\left(\left(\sigma_G^+(x_n)\wedge\sigma_H^+(x_n)\right)^\mu \vee \left(\sigma_G^-(x_n)\wedge\sigma_H^-(x_n)\right)^\mu\right)}{\left(\left(\sigma_G^+(x_n)\vee\sigma_H^+(x_n)\right)^\mu \vee \left(\sigma_G^-(x_n)\vee\sigma_H^-(x_n)\right)^\mu\right)}\end{bmatrix}\right)\right]^{\frac{1}{\mu}}$$

$\Longleftrightarrow$ Now for $G = H$ we have $\varsigma_G^+ = \varsigma_H^+$ and $\varsigma_G^- = \varsigma_H^-$ for the values of $k = 1, 2, 3, \ldots, n \Leftrightarrow \varrho_G^+(x_k) = \varrho_H^+(x_k)$ and $\varrho_G^-(x_k) = \varrho_H^-(x_k)$ also $\sigma_G^+(x_k) = \sigma_H^+(x_k)$, $\sigma_G^-(x_k) = \sigma_H^-(x_k)$ for $S_C^5(G, H)$ and $\Phi_{cc}, \Psi_{cc} \in [0, 1]$ such that $\Phi_{cc} + \Psi_{cc} = 1$. Then

$$\Longleftrightarrow S_c^5(G, H) = \left[\frac{1}{n}(1+1+\ldots+1)\right]^{\frac{1}{\mu}}$$

$$\Longleftrightarrow S_c^5(G, H) = 1$$

3. We have

$$S_C^5(G,H)=\left[\left[\frac{1}{n}\sum_{k=1}^{n}\left(\left[\begin{array}{l}\Phi_{cc}\dfrac{\left(\left(\varrho_G^+(x_k)\wedge\varrho_H^+(x_k)\right)^{\mu}\vee\left(\varrho_G^-(x_k)\wedge\varrho_H^-(x_k)\right)^{\mu}\right)}{\left(\left(\varrho_G^+(x_k)\vee\varrho_H^+(x_k)\right)^{\mu}\vee\left(\varrho_G^-(x_k)\vee\varrho_H^-(x_k)\right)^{\mu}\right)}\\ +\Psi_{cc}\dfrac{\left(\left(\sigma_G^+(x_k)\wedge\sigma_H^+(x_k)\right)^{\mu}\vee\left(\sigma_G^-(x_k)\wedge\sigma_H^-(x_k)\right)^{\mu}\right)}{\left(\left(\sigma_G^+(x_k)\vee\sigma_H^+(x_k)\right)^{\mu}\vee\left(\sigma_G^-(x_k)\vee\sigma_H^-(x_k)\right)^{\mu}\right)}\end{array}\right]\right)\right]\right]^{\frac{1}{\mu}}$$

$$=\left[\frac{1}{n}\sum_{k=1}^{n}\left(\left[\begin{array}{l}\Phi_{cc}\dfrac{\left(\left(\varrho_H^+(x_k)\wedge\varrho_G^+(x_k)\right)^{\mu}\vee\left(\varrho_H^-(x_k)\wedge\varrho_G^-(x_k)\right)^{\mu}\right)}{\left(\left(\varrho_H^+(x_k)\vee\varrho_G^+(x_k)\right)^{\mu}\vee\left(\varrho_H^-(x_k)\vee\varrho_G^-(x_k)\right)^{\mu}\right)}\\ +\Psi_{cc}\dfrac{\left(\left(\sigma_H^+(x_k)\wedge\sigma_G^+(x_k)\right)^{\mu}\vee\left(\sigma_H^-(x_k)\wedge\sigma_G^-(x_k)\right)^{\mu}\right)}{\left(\left(\sigma_H^+(x_k)\vee\sigma_G^+(x_k)\right)^{\mu}\vee\left(\sigma_H^-(x_k)\vee\sigma_G^-(x_k)\right)^{\mu}\right)}\end{array}\right]\right)\right]^{\frac{1}{\mu}}$$

$$=S_C^5(H,G)$$

Remark 3 If we keep the value of $\mu=1$ then similarity measures without exponential based will become

$$S_c^3(G,H)=\frac{1}{n}\sum_{k=1}^{n}\left[\frac{1-\left(\begin{array}{l}\left(\left|\varrho_G^+(x_k)-\varrho_H^+(x_k)\right|^{\mu}\vee\left|\varrho_G^-(x_k)-\varrho_H^-(x_k)\right|^{\mu}\right)\vee\\ \left(\left|\sigma_G^+(x_k)-\sigma_H^+(x_k)\right|^{\mu}\vee\left|\sigma_G^-(x_k)-\sigma_H^-(x_k)\right|^{\mu}\right)\end{array}\right)}{1+\left(\begin{array}{l}\left(\left|\varrho_G^+(x_k)-\varrho_H^+(x_k)\right|^{\mu}\vee\left|\varrho_G^-(x_k)-\varrho_H^-(x_k)\right|^{\mu}\right)\vee\\ \left(\left|\sigma_G^+(x_k)-\sigma_H^+(x_k)\right|^{\mu}\vee\left|\sigma_G^-(x_k)-\sigma_H^-(x_k)\right|^{\mu}\right)\end{array}\right)}\right]$$

$$S_c^4(G,H)=\frac{1}{n}\sum_{k=1}^{n}\left[\frac{\left(\begin{array}{l}\left(\left(\varrho_G^+(x_k)\wedge\varrho_H^+(x_k)\right)^{\mu}\vee\left(\varrho_G^-(x_k)\wedge\varrho_H^-(x_k)\right)^{\mu}\right)+\\ \left(\left(\sigma_G^+(x_k)\wedge\sigma_H^+(x_k)\right)^{\mu}\vee\left(\sigma_G^-(x_k)\wedge\sigma_H^-(x_k)\right)^{\mu}\right)\end{array}\right)}{\left(\begin{array}{l}\left(\left(\varrho_G^+(x_k)\vee\varrho_H^+(x_k)\right)^{\mu}\vee\left(\varrho_G^-(x_k)\vee\varrho_H^-(x_k)\right)^{\mu}\right)+\\ \left(\left(\sigma_G^+(x_k)\vee\sigma_H^+(x_k)\right)^{\mu}\vee\left(\sigma_G^-(x_k)\vee\sigma_H^-(x_k)\right)^{\mu}\right)\end{array}\right)}\right]$$

$$S_C^5(G,H)=\frac{1}{n}\sum_{k=1}^{n}\left(\left[\begin{array}{l}\Phi_{cc}\dfrac{\left(\left(\varrho_G^+(x_k)\wedge\varrho_H^+(x_k)\right)^{\mu}\vee\left(\varrho_G^-(x_k)\wedge\varrho_H^-(x_k)\right)^{\mu}\right)}{\left(\left(\varrho_G^+(x_k)\vee\varrho_H^+(x_k)\right)^{\mu}\vee\left(\varrho_G^-(x_k)\vee\varrho_H^-(x_k)\right)^{\mu}\right)}\\ +\Psi_{cc}\dfrac{\left(\left(\sigma_G^+(x_k)\wedge\sigma_H^+(x_k)\right)^{\mu}\vee\left(\sigma_G^-(x_k)\wedge\sigma_H^-(x_k)\right)^{\mu}\right)}{\left(\left(\sigma_G^+(x_k)\vee\sigma_H^+(x_k)\right)^{\mu}\vee\left(\sigma_G^-(x_k)\vee\sigma_H^-(x_k)\right)^{\mu}\right)}\end{array}\right]\right)$$

3.2 *Weighted exponential and non-exponential based SMs for BCFSs*

Now we define exponential-based Weighted Generalized Similarity Measure (WGSM) and non-exponential-based WGSM.

Definition 12 Let G and H be two BCFs on X. Then GWSM based on the exponential function is given as

$$S^1_{cw}(G, H) = \left[\sum_{k=1}^{n} w_k \left[\left(2^{1-\left(\begin{array}{c} \left(|\rho^+_G(x_k) - \rho^+_H(x_k)|^\mu \vee |\rho^-_G(x_k) - \rho^-_H(x_k)|^\mu \right) \vee \\ \left(|\sigma^+_G(x_k) - \sigma^+_H(x_k)|^\mu \vee |\sigma^-_G(x_k) - \sigma^-_H(x_k)|^\mu \right) \end{array} \right)} \right) - 1 \right] \right]^{\frac{1}{\mu}}$$

Definition 13 Let G and H be two BCFs on X. Then exponential-based weighted similarity measure can also be computed as

$$S^2_{cw}(G, H) = \left[\sum_{k=1}^{n} w_c \left[\left(2^{1-\frac{1}{2}\left(\begin{array}{c} \left(|\rho^+_G(x_k) - \rho^+_H(x_k)|^\mu \vee |\rho^-_G(x_k) - \rho^-_H(x_k)|^\mu \right) + \\ \left(|\sigma^+_G(x_k) - \sigma^+_H(x_k)|^\mu \vee |\sigma^-_G(x_k) - \sigma^-_H(x_k)|^\mu \right) \end{array} \right)} \right) - 1 \right] \right]^{\frac{1}{\mu}}$$

Definition 14 Let G and H be two BCFSs on X. Then weighted SMs without exponential-based are calculated as follows

$$S^3_{cw}(G, H) = \left[\sum_{k=1}^{n} w_c \left[\left[\frac{1 - \left(\begin{array}{c} \left(|\rho^+_G(x_k) - \rho^+_H(x_k)|^\mu \vee |\rho^-_G(x_k) - \rho^-_H(x_k)|^\mu \right) \vee \\ \left(|\sigma^+_G(x_k) - \sigma^+_H(x_k)|^\mu \vee |\sigma^-_G(x_k) - \sigma^-_H(x_k)|^\mu \right) \end{array} \right)}{1 + \left(\begin{array}{c} \left(|\rho^+_G(x_k) - \rho^+_H(x_k)|^\mu \vee |\rho^-_G(x_k) - \rho^-_H(x_k)|^\mu \right) \vee \\ \left(|\sigma^+_G(x_k) - \sigma^+_H(x_k)|^\mu \vee |\sigma^-_G(x_k) - \sigma^-_H(x_k)|^\mu \right) \end{array} \right)} \right] \right] \right]^{\frac{1}{\mu}}$$

$$S^4_{cw}(G, H) = \left[\sum_{k=1}^{n} w_c \left[\frac{\left(\begin{array}{c} \left((\rho^+_G(x_k) \wedge \rho^+_H(x_k))^\mu \vee (\rho^-_G(x_k) \wedge \rho^-_H(x_k))^\mu \right) + \\ \left((\sigma^+_G(x_k) \wedge \sigma^+_H(x_k))^\mu \vee (\sigma^-_G(x_k) \wedge \sigma^-_H(x_k))^\mu \right) \end{array} \right)}{\left(\begin{array}{c} \left((\rho^+_G(x_k) \vee \rho^+_H(x_k))^\mu \vee (\rho^-_G(x_k) \vee \rho^-_H(x_k))^\mu \right) + \\ \left((\sigma^+_G(x_k) \vee \sigma^+_H(x_k))^\mu \vee (\sigma^-_G(x_k) \vee \sigma^-_H(x_k))^\mu \right) \end{array} \right)} \right] \right]^{\frac{1}{\mu}}$$

$$S^5_{Cw}(G, H) = \left[\sum_{k=1}^{n} w_c \left(\left[\begin{array}{l} \Phi_{cc} \frac{\left((\rho^+_G(x_k) \wedge \rho^+_H(x_k))^\mu \vee (\rho^-_G(x_k) \wedge \rho^-_H(x_k))^\mu \right)}{\left((\rho^+_G(x_k) \vee \rho^+_H(x_k))^\mu \vee (\rho^-_G(x_k) \vee \rho^-_H(x_k))^\mu \right)} \\ + \Psi_{cc} \frac{\left((\sigma^+_G(x_k) \wedge \sigma^+_H(x_k))^\mu \vee (\sigma^-_G(x_k) \wedge \sigma^-_H(x_k))^\mu \right)}{\left((\sigma^+_G(x_k) \vee \sigma^+_H(x_k))^\mu \vee (\sigma^-_G(x_k) \vee \sigma^-_H(x_k))^\mu \right)} \end{array} \right] \right) \right]^{\frac{1}{\mu}}$$

Where $\mu > 0$ and $\Phi_{cc}, \Psi_{cc} \in [0, 1]$ such that $\Phi_{cc} + \Psi_{cc} = 1$.
Remark 4 If $\mu = 1$ then generalized similarity measures without exponential based are calculated as

$$S_{cw}^{3}(G, H) = \sum_{k=1}^{n} w_c \left[\left[\frac{1 - \begin{pmatrix} \left(\left|\rho_G^+(x_k) - \rho_H^+(x_k)\right|^{\mu} \vee \left|\rho_G^-(x_k) - \rho_H^-(x_k)\right|^{\mu} \right) \vee \\ \left(\left|\sigma_G^+(x_k) - \sigma_H^+(x_k)\right|^{\mu} \vee \left|\sigma_G^-(x_k) - \sigma_H^-(x_k)\right|^{\mu} \right) \end{pmatrix}}{1 + \begin{pmatrix} \left(\left|\rho_G^+(x_k) - \rho_H^+(x_k)\right|^{\mu} \vee \left|\rho_G^-(x_k) - \rho_H^-(x_k)\right|^{\mu} \right) \vee \\ \left(\left|\sigma_G^+(x_k) - \sigma_H^+(x_k)\right|^{\mu} \vee \left|\sigma_G^-(x_k) - \sigma_H^-(x_k)\right|^{\mu} \right) \end{pmatrix}} \right] \right]$$

$$S_{cw}^{4}(G, H) = \sum_{k=1}^{n} w_c \left[\frac{\begin{pmatrix} \left(\left(\rho_G^+(x_k) \wedge \rho_H^+(x_k)\right)^{\mu} \vee \left(\rho_G^-(x_k) \wedge \rho_H^-(x_k)\right)^{\mu} \right) + \\ \left(\left(\sigma_G^+(x_k) \wedge \sigma_H^+(x_k)\right)^{\mu} \vee \left(\sigma_G^-(x_k) \wedge \sigma_H^-(x_k)\right)^{\mu} \right) \end{pmatrix}}{\begin{pmatrix} \left(\left(\rho_G^+(x_k) \vee \rho_H^+(x_k)\right)^{\mu} \vee \left(\rho_G^-(x_k) \vee \rho_H^-(x_k)\right)^{\mu} \right) + \\ \left(\left(\sigma_G^+(x_k) \vee \sigma_H^+(x_k)\right)^{\mu} \vee \left(\sigma_G^-(x_k) \vee \sigma_H^-(x_k)\right)^{\mu} \right) \end{pmatrix}} \right]$$

$$S_{Cw}^{5}(G, H) = \sum_{k=1}^{n} w_c \left(\left[\begin{array}{l} \Phi_{cc} \dfrac{\left(\left(\rho_G^+(x_k) \wedge \rho_H^+(x_k)\right)^{\mu} \vee \left(\rho_G^-(x_k) \wedge \rho_H^-(x_k)\right)^{\mu} \right)}{\left(\left(\rho_G^+(x_k) \vee \rho_H^+(x_k)\right)^{\mu} \vee \left(\rho_G^-(x_k) \vee \rho_H^-(x_k)\right)^{\mu} \right)} \\ \\ +\Psi_{cc} \dfrac{\left(\left(\sigma_G^+(x_k) \wedge \sigma_H^+(x_k)\right)^{\mu} \vee \left(\sigma_G^-(x_k) \wedge \sigma_H^-(x_k)\right)^{\mu} \right)}{\left(\left(\sigma_G^+(x_k) \vee \sigma_H^+(x_k)\right)^{\mu} \vee \left(\sigma_G^-(x_k) \vee \sigma_H^-(x_k)\right)^{\mu} \right)} \end{array} \right] \right)$$

4. TOPSIS method in the environment of BCFSs

In this section, we will discuss the notion of the TOPSIS Method in the environment of BCFs. The TOPSIS method provides a systematic approach to deal with the fuzzy membership degrees of alternatives and criteria. By incorporating fuzzy algebraic operations and fuzzy distance measures, it can effectively deal with uncertainties and ambiguities present in real-world decision scenarios.

4.1 *Algorithm*

In this part, we examine the BCF-TOPSIS approach and afterward analyze the mathematical model "selection of CRM software" with the help of the explored BCF-TOPSIS approach. Suppose that there is a set of η alternatives $\{A_1, A_2, A_3, \ldots, A_\eta\}$ and κ criteria $\{C_1, C_2, C_3, \ldots, C_\kappa\}$. Now we express the weights of criteria by a weight vector $W = (w_1, w_2, w_3, \ldots, w_\kappa)$. The assessment arguments of the attributes based on the criteria would be in the model of BCFS. To determine the finest alternative, we have the following steps:

Step 1: Now we construct a bipolar complex fuzzy decision matrix containing the assessment arguments.

$$= \begin{bmatrix} (\mu_{11}+\iota\varphi_{11},\, \nu_{11}+\iota\theta_{11}) & (\mu_{12}+\iota\varphi_{12},\, \nu_{12}+\iota\theta_{12})\ldots & (\mu_{1\kappa}+\iota\varphi_{1\kappa},\, \nu_{1\kappa}+\iota\theta_{1\kappa}) \\ (\mu_{21}+\iota\varphi_{21},\, \nu_{21}+\iota\theta_{21}) & (\mu_{22}+\iota\varphi_{22},\, \nu_{22}+\iota\theta_{22})\ldots & (\mu_{2\kappa}+\iota\varphi_{2\kappa},\, \nu_{2\kappa}+\iota\theta_{2\kappa}) \\ & \vdots & \\ (\mu_{\eta 1}+\iota\varphi_{\eta 1},\, \nu_{\eta 1}+\iota\theta_{\eta 1}) & (\mu_{\eta 2}+\iota\varphi_{\eta 2},\, \nu_{\eta 2}+\iota\theta_{\eta 2})\ldots & (\mu_{\eta\kappa}+\iota\varphi_{\eta\kappa},\, \nu_{\eta\kappa}+\iota\theta_{\eta\kappa}) \end{bmatrix}$$

Step 2: Now we construct the weighted bipolar complex fuzzy decision matrix by multiplying the weight matrix with the BCF decision matrix

$$M = [(s_{ij}+\iota\psi_{ij},\, t_{ij}+\omega_{ij})]_{n\times m}$$

$$\mu_{ij}^{w}+\iota\varphi_{ij}^{w} = w_j\,(\mu_{ij}+\iota\varphi_{ij}),\quad \nu_{ij}^{w}+\iota\theta_{ij}^{w} = w_j\,(\nu_{ij}+\iota\theta_{ij})$$

Step 3: Now we find the Bipolar Complex Fuzzy Positive Ideal Solution (BCFPIS) and Bipolar Complex Fuzzy Negative Ideal Solution (BCFNIS) as follows

$$\text{BCFPIS} = \left[\left(\mu_1^{+}+\iota\varphi_1^{+},\, \nu_1^{+}+\iota\theta_1^{+}\right)\left(\mu_2^{+}+\iota\varphi_2^{+},\, \nu_2^{+}+\iota\theta_2^{+}\right)\ldots\left(\mu_\kappa^{+}+\iota\varphi_\kappa^{+},\, \nu_\kappa^{+}+\iota\theta_\kappa^{+}\right)\right]$$

$$\text{BCFNIS} = \left[\left(\mu_1^{-}+\iota\varphi_1^{-},\, \nu_1^{-}+\iota\theta_1^{-}\right)\left(\mu_2^{-}+\iota\varphi_2^{-},\, \nu_2^{-}+\iota\theta_2^{-}\right)\ldots\left(\mu_\kappa^{-}+\iota\varphi_\kappa^{-},\, \nu_\kappa^{-}+\iota\theta_\kappa^{-}\right)\right]$$

Where

$$\mu_j^{+} = \max\left\{s_{ij}\right\},\ \varphi_j^{+} = \max\left\{\psi_{ij}\right\},\ \nu_j^{+} = \max\left\{t_{ij}\right\},\ \theta_j^{+} = \max\left\{\varphi_{ij}\right\},\ \mu_j^{-} = \min\left\{s_{ij}\right\},\ \varphi_j^{-} = \min\left\{\psi_{ij}\right\},$$

and $\nu_j^{-} = \min\left\{t_{ij}\right\},\ \theta_j^{-} = \min\left\{\varphi_{ij}\right\}$ $(j = 1,\, 2,\, 3,\, \ldots\kappa)$. Now, the distance of each alternative A_i $(1,\, 2,\, 3\ldots\eta)$ from BCFPIS and BCFNIS will be calculated as

$$D(A_i,\, \text{BCFPIS}) = 1 - \left[\frac{1}{k}\sum_{j=1}^{k}\left[\left(2^{1-\left(\begin{array}{c}\left(\left|s_{ij}-\mu_j^{+}\right|^{\mu}\vee\left|t_{ij}-\nu_j^{+}\right|^{\mu}\right)\vee \\ \left(\left|\psi_{ij}-\varphi_j^{+}\right|^{\mu}\vee\left|\omega_{ij}-\theta_j^{+}\right|^{\mu}\right)\end{array}\right)}\right)-1\right]\right]^{\frac{1}{\mu}}$$

$$D(A_i,\, \text{BCFNIS}) = 1 - \left[\frac{1}{k}\sum_{j=1}^{k}\left[\left(2^{1-\left(\begin{array}{c}\left(\left|s_{ij}-\mu_j^{-}\right|^{\mu}\vee\left|t_{ij}-\nu_j^{-}\right|^{\mu}\right)\vee \\ \left(\left|\psi_{ij}-\varphi_j^{-}\right|^{\mu}\vee\left|\omega_{ij}-\theta_j^{-}\right|^{\mu}\right)\end{array}\right)}\right)-1\right]\right]^{\frac{1}{\mu}}$$

Step 4: Now we find the relative closeness degree of alternative A_i concerning BCFPIS as given below

$$\mathfrak{C}_i = \frac{D(A_i,\ \text{BCFNIS})}{D(A_i,\ \text{BCFPIS}) + D(A_i,\ \text{BCFNIS})}, \quad i = 1, 2, 3, \ldots, \eta$$

The alternative that has the highest relative closeness is the best.

4.2 *Utilization of the TOPSIS technique over other techniques*

In this subsection, we have delivered the reasons why we have used the TOPSIS technique, and we have analyzed the characteristics of the TOPSIS technique as compared to the MARCOS or VIKOR method. The following Table 3 is established to show the characteristic analysis of the proposed TOPSIS method.

Table 3. Characteristic of TOPSIS technique

Methods	Key characteristics	Why TOPSIS is preferred
TOPSIS method	Ranks alternatives based on closeness to the ideal and farthest from the anti-ideal.	Provides a clear geometric interpretation; well-suited for handling complex fuzzy values; computationally efficient.
MARCOS Method	Focuses on ranking and selecting from a set of alternatives with conflicting criteria.	Emphasizes compromise solutions but is more sensitive to the choice of decision strategy; less intuitive in fuzzy environments.
VIKOR method	Compares alternatives with both ideal and anti-ideal solutions using utility functions.	Recently developed method; requires more extensive normalization and interpretation; less studied with bipolar complex fuzzy sets.
Justification	TOPSIS offers a balance between simplicity and interpretability in fuzzy decision-making. It directly uses the distance-based approach, aligning well with the similarity measures proposed. Existing literature supports its integration with fuzzy extensions.	Hence, TOPSIS was selected as it ensures consistency with our similarity modeling framework and facilitates easier implementation in CRM systems.

4.3 *Application of the TOPSIS method*

In this section, we will apply our stated TOPSIS method to choose the best Customer Relationship Management (CRM) software.

In order to better client interactions, increase productivity, and raise general customer happiness, a company is looking to adopt new CRM software. Since there are five different CRM software that is A_1, A_2, A_3, A_4, and A_5 are shortlisted by the company and have to determine the optimal CRM software that fits the company's specific requirements. Four essential attributes:

C_1: User-Friendliness: This attribute assesses how user-friendly and intuitive the CRM software is for our staff.

C_2: Features and Functionality: This attribute evaluates the breadth and depth of functionality the CRM software provides to match our unique company demands.

C_3: Customization Options: This attribute gauges the degree of personalization and adaptability the CRM software offers to fit our particular procedures and workflows.

C_4: Cost-effectiveness: This attribute takes into account all associated costs, such as those related to license, training, and support, for installing and maintaining the CRM software.

Each with a different critical weight will be used to make the decision. The objective is to identify the CRM system that will benefit the business the best. The goal is to use the invented technique of TOPSIS in the setting of BCFS to evaluate the available CRM software alternatives based on how well they perform in the stated qualities. Thus, we are able to choose the CRM software that most closely matches the needs of the company and optimizes the total advantages of putting the selected CRM system into practice. The evaluated values of these CRM software alternatives based on the considered attributes are in the structure of BCFS.

Step 1: The BCF decision matrix containing the assessment values of CRM software is given in Table 4.

Table 4. The assessment values of CRM software alternatives

	C_1	C_2	C_3	C_4
A_1	$\begin{pmatrix} 0.4+\imath 0.25, \\ -0.21-0.45 \end{pmatrix}$	$\begin{pmatrix} 0.39+\imath 0.41, \\ -0.71-\imath 0.15 \end{pmatrix}$	$\begin{pmatrix} 0.24+\imath 0.62, \\ -0.35-\imath 0.24 \end{pmatrix}$	$\begin{pmatrix} 0.52+\imath 0.34, \\ -0.14-\imath 0.36 \end{pmatrix}$
A_2	$\begin{pmatrix} 0.5+\imath 0.35, \\ -0.31-\imath 0.55 \end{pmatrix}$	$\begin{pmatrix} 0.49+\imath 0.21, \\ -0.51-\imath 0.25 \end{pmatrix}$	$\begin{pmatrix} 0.34+\imath 0.32, \\ -0.15-\imath 0.44 \end{pmatrix}$	$\begin{pmatrix} 0.42+\imath 0.14, \\ -0.24-\imath 0.46 \end{pmatrix}$
A_3	$\begin{pmatrix} 0.35+\imath 0.15, \\ -0.41-0.29 \end{pmatrix}$	$\begin{pmatrix} 0.47+\imath 0.21, \\ -0.42-\imath 0.34 \end{pmatrix}$	$\begin{pmatrix} 0.24+\imath 0.54, \\ -0.14-\imath 0.52 \end{pmatrix}$	$\begin{pmatrix} 0.48+\imath 0.24, \\ -0.2-\imath 0.54 \end{pmatrix}$
A_4	$\begin{pmatrix} 0.35+\imath 0.25, \\ -0.51-\imath 0.32 \end{pmatrix}$	$\begin{pmatrix} 0.19+\imath 0.51, \\ -0.64-\imath 0.22 \end{pmatrix}$	$\begin{pmatrix} 0.56+\imath 0.32, \\ -0.14-\imath 0.46 \end{pmatrix}$	$\begin{pmatrix} 0.31+\imath 0.37, \\ -0.54-\imath 0.16 \end{pmatrix}$
A_5	$\begin{pmatrix} 0.18+\imath 0.32, \\ -0.41-\imath 0.32 \end{pmatrix}$	$\begin{pmatrix} 0.24+\imath 0.35, \\ -0.41-\imath 0.25 \end{pmatrix}$	$\begin{pmatrix} 0.32+\imath 0.45, \\ -0.11-\imath 0.44 \end{pmatrix}$	$\begin{pmatrix} 0.32+\imath 0.14, \\ -0.52-\imath 0.14 \end{pmatrix}$

Step 2: The selection expert will allocate the weight that is $W = [0.25\ 0.35\ 0.23\ 0.17]$ to every attribute.

Step 3: The weighted BCF decision matrix is interpreted in Table 5.

Table 5. The weighted BCF decision matrix

	C_1	C_2	C_3	C_4
A_1	$\begin{pmatrix} 0.1198+\imath 0.0693, \\ -0.6770-\imath 0.8190 \end{pmatrix}$	$\begin{pmatrix} 0.1162+\imath 0.1236, \\ -0.9180-\imath 0.6223 \end{pmatrix}$	$\begin{pmatrix} 0.0663+\imath 0.2149, \\ -0.7692-\imath 0.6999 \end{pmatrix}$	$\begin{pmatrix} 0.1676+\imath 0.0987, \\ -0.6117-\imath 0.7746 \end{pmatrix}$
A_2	$\begin{pmatrix} 0.2154+\imath 0.1399, \\ -0.8194-\imath 0.9033 \end{pmatrix}$	$\begin{pmatrix} 0.2099+\imath 0.0791, \\ -0.8918-\imath 0.7900 \end{pmatrix}$	$\begin{pmatrix} 0.1353+\imath 0.1262, \\ -0.7243-\imath 0.8697 \end{pmatrix}$	$\begin{pmatrix} 0.1736+\imath 0.0514, \\ -0.7846-\imath 0.8763 \end{pmatrix}$
A_3	$\begin{pmatrix} 0.0943+\imath 0.0640, \\ -0.8565-\imath 0.7695 \end{pmatrix}$	$\begin{pmatrix} 0.0473+\imath 0.1513, \\ -0.9024-\imath 0.7059 \end{pmatrix}$	$\begin{pmatrix} 0.1720+\imath 0.0849, \\ -0.6362-\imath 0.8364 \end{pmatrix}$	$\begin{pmatrix} 0.0818+\imath 0.1008, \\ -0.8678-\imath 0.6561 \end{pmatrix}$
A_4	$\begin{pmatrix} 0.0331+\imath 0.0635, \\ -0.8594-\imath 0.8240 \end{pmatrix}$	$\begin{pmatrix} 0.4558+\imath 0.0706, \\ -0.8594-\imath 0.7900 \end{pmatrix}$	$\begin{pmatrix} 0.0634+\imath 0.0966, \\ -0.6871-\imath 0.8697 \end{pmatrix}$	$\begin{pmatrix} 0.0635+\imath 0.0253, \\ -0.8947-\imath 0.7158 \end{pmatrix}$
A_5	$\begin{pmatrix} 0.0484+\imath 0.0919, \\ -0.8002-\imath 0.7521 \end{pmatrix}$	$\begin{pmatrix} 0.0663+\imath 0.1020, \\ -0.8002-\imath 0.7071 \end{pmatrix}$	$\begin{pmatrix} 0.0919+\imath 0.1388, \\ -0.5759-\imath 0.8144 \end{pmatrix}$	$\begin{pmatrix} 0.0919+\imath 0.037, \\ -0.8492-\imath 0.6117 \end{pmatrix}$

Step 4: In this step, we get BCFPIS and BCFNIS

$$\text{BCFPIS} = \begin{bmatrix} \begin{pmatrix} 0.2154 + \imath 0.1399, \\ -0.677 - \imath 0.7521 \end{pmatrix}, & \begin{pmatrix} 0.4558 + \imath 0.1513, \\ -0.8002 - \imath 0.6223 \end{pmatrix}, \\ \begin{pmatrix} 0.1720 + \imath 0.2149, \\ -0.5759 - \imath 0.6999 \end{pmatrix}, & \begin{pmatrix} 0.1736 + \imath 0.1008, \\ -0.6117 - \imath 0.6117 \end{pmatrix} \end{bmatrix}$$

$$\text{BCFNIS} = \begin{bmatrix} \begin{pmatrix} 0.0331 + \imath 0.0635, \\ -0.8594 - \imath 0.9033 \end{pmatrix}, & \begin{pmatrix} 0.0473 + \imath 0.0706, \\ -0.9180 - \imath 0.79 \end{pmatrix}, \\ \begin{pmatrix} 0.0634 + \imath 0.0849, \\ -0.7692 - \imath 0.8697 \end{pmatrix}, & \begin{pmatrix} 0.0635 + \imath 0.0253, \\ -0.8947 - \imath 0.8763 \end{pmatrix} \end{bmatrix}$$

Step 5: By applying the formulas of $D(A_i, \text{BCFPIS})$ and $D(A_i, \text{BCFNIS})$. We get the following results

$$D(A_1, \text{BCFPIS}) = 0.1979 \qquad D(A_1, \text{BCFNIS}) = 0.2008$$

$$D(A_2, \text{BCFPIS}) = 0.208 \qquad D(A_2, \text{BCFNIS}) = 0.1318$$

$$D(A_3, \text{BCFPIS}) = 0.2453 \qquad D(A_3, \text{BCFNIS}) = 0.1428$$

$$D(A_4, \text{BCFPIS}) = 0.2008 \qquad D(A_4, \text{BCFNIS}) = 0.1826$$

$$D(A_5, \text{BCFPIS}) = 0.2273 \qquad D(A_5, \text{BCFNIS}) = 0.1818$$

Step 6: The relative closeness degree is given by $\mathfrak{C}_1 = 0.0503$, $\mathfrak{C}_2 = 0.3878$, $\mathfrak{C}_3 = 0.3679$, $\mathfrak{C}_4 = 0.4762$, $\mathfrak{C}_5 = 0.4443$. Now we order the CRM software alternatives according to relative closeness degrees, and we get $A_4 > A_5 > A_2 > A_3 > A_1$.

5. Comparison

In this portion, we will make a comparison of our stated TOPSIS method with some existing methods to show some usefulness and advantages of our described method. Here we compare our proposed TOPSIS method with the existing ones. Yong [36] proposed the Fuzzy-TOPSIS method, which only deals with the membership degree, and it neither deals with two-dimensional information nor with the negative information like we have in BCFSs. So, this method fails to calculate the degree of closeness of the information given in the structure of BCFS. Now we consider the complex fuzzy TOPSIS invented by Barbat et al. [39]. We compare our proposed method with this method, but we observe that it only deals with two-dimensional information and does not handle the positive and negative types of information like we have in BCFSs, so it fails to calculate the required value. Moreover, Alghamdi et al. [38] proposed the idea of the Bipolar Fuzzy-TOPSIS (BF-TOPSIS) method. We observed that this method only deals with negative and positive types of information, but does not handle two-dimensional data as we have in BCFSs. So, this method fails to calculate the closeness degree of the information that is in the structure of BCFS. But our proposed method can handle two-dimensional data as well as positive and negative types of data. Thus, the invented BCF TOPSIS is more generalized and efficient than the prevailing ones. The comparison is revealed in Table 6.

Table 6. The relative closeness degrees, given by the existing and invented TOPSIS methods for the BCF information presented in Table 4

Source	Method	Relative closeness degree	Ranking
Yong [36]	F-TOPSIS	$\chi\chi\chi\chi\chi$	$\chi\chi\chi\chi$
Barbat et al. [39]	CF-TOPSIS	$\chi\chi\chi\chi$	$\chi\chi\chi\chi$
Alghamdi et al. [38]	BF-TOPSIS	$\chi\chi\chi\chi$	$\chi\chi\chi\chi$
Invented method	BCF-TOPSIS	$\mathfrak{C}_1 = 0.0503$, $\mathfrak{C}_2 = 0.3878$, $\mathfrak{C}_3 = 0.3679$, $\mathfrak{C}_4 = 0.4762$, $\mathfrak{C}_5 = 0.4443$	$A_4 > A_5 > A_2 > A_3 > A_1$

Further, if we consider the information that is stated in Table 4, then we observe that if we remove the imaginary parts from the positive membership degree and negative membership degree, then it will be transformed into the model of BCF information, which is demonstrated in Table 7.

Table 7. The values in the model of BFS

	C_1	C_2	C_3	C_4
A_1	$(0.4, -0.21)$	$(0.39, -0.71)$	$(0.24, -0.35)$	$(0.52, -0.14)$
A_2	$(0.5, -0.31)$	$(0.49, -0.51)$	$(0.34, -0.15)$	$(0.42, -0.24)$
A_3	$(0.35, -0.41)$	$(0.47, -0.42)$	$(0.24, -0.14)$	$(0.48, -0.2)$
A_4	$(0.35, -0.51)$	$(0.19, -0.64)$	$(0.56, -0.14))$	$(0.31, -0.54)$
A_5	$(0.18, -0.41)$	$(0.24, -0.41)$	$(0.32, -0.11)$	$(0.32, -0.52)$

Now, we would employ the invented BCF-TOPSIS method to get the finest alternative.
Step 3: In this part, we suppose the associated weight is

$$W = [0.25\ 0.35\ 0.23\ 0.17]$$

Step 4: The weighted bipolar fuzzy decision matrix is constructed in Table 8.

Table 8. The weighted BF decision matrix

	C_1	C_2	C_3	C_4
A_1	$(0.1198, -0.6770)$	$(0.1162, -0.9180)$	$(0.0663, -0.7692)$	$(0.1676, -0.6117)$
A_2	$(0.2154, -0.819)$	$(0.2099, -0.892)$	$(0.1353, -0.7243)$	$(0.1736, -0.7846)$
A_3	$(0.0943, -0.8565)$	$(0.0473, -0.9024)$	$(0.1720, -0.6362)$	$(0.0818, -0.8678)$
A_4	$(0.0331, -0.8594)$	$(0.4558, -0.8594)$	$(0.0634, -0.6871)$	$(0.0635, -0.8947)$
A_5	$(0.0484, -0.8002)$	$(0.0663, -0.8002)$	$(0.0919, -0.5759)$	$(0.0919, -0.8492)$

Step 5: The positive and negative Ideal solutions concerning BF information is illustrated below

$$\text{BFPIS} = \left\{ \begin{array}{l} (0.2154, -0.6770), (0.4558, -0.8002), \\ (0.1720, -0.5759), (0.1736, -0.6117) \end{array} \right\}$$

$$\text{BFNIS} = \left\{ \begin{array}{l} (0.0331, -0.8594), (0.0473, -0.9180), \\ (0.0634, -0.7692), (0.0635, -0.8947) \end{array} \right\}$$

Step 6: The relative closeness degree of alternatives from BFPIS and BFNIS are interpreted below

$$D(A_1, \text{BFPIS} = 0.2004) \qquad D(A_1, \text{BFNIS} = 0.2615)$$

$$D(A_2, \text{BFPIS} = 0.2024) \qquad D(A_2, \text{BFNIS} = 0.2167)$$

$$D(A_3, \text{BFPIS} = 0.2842) \qquad D(A_3, \text{BFNIS} = 0.1314)$$

$$D(A_4, \text{BFPIS} = 0.3267) \qquad D(A_4, \text{BFNIS} = 0.0713)$$

$$D(A_5, \text{BFPIS} = 0.2581) \qquad D(A_5, \text{BFNIS} = 0.1719)$$

The relative closeness degrees are given by $\mathfrak{C}_1 = 0.5661$, $\mathfrak{C}_2 = 0.5170$, $\mathfrak{C}_3 = 0.3162$, $\mathfrak{C}_4 = 0.1791$, $\mathfrak{C}_5 = 0.3997$. Now we order the alternatives according to relative closeness degrees, and we get $A_1 > A_2 > A_5 > A_3 > A_4$.

Now, if we consider only the positive membership degree in the model of BCFS and neglect the negative membership degree, then the information in Table 4 would be transformed in the model of CFS, as demonstrated in Table 9.

Table 9. The values for CFS are given below

	C_1	C_2	C_3	C_4
A_1	$(0.4 + \imath 0.25)$	$(0.39 + \imath 0.41)$	$(0.24 + \imath 0.62)$	$(0.52 + \imath 0.34)$
A_2	$(0.5 + \imath 0.35)$	$(0.49 + \imath 0.21)$	$(0.34 + \imath 0.32)$	$(0.42 + \imath 0.14)$
A_3	$(0.35 + \imath 0.15)$	$(0.47 + \imath 0.21)$	$(0.24 + \imath 0.54)$	$(0.48 + \imath 0.24)$
A_4	$(0.35 + \imath 0.25)$	$(0.19 + \imath 0.51)$	$(0.56 + \imath 0.32)$	$(0.31 + \imath 0.37)$
A_5	$(0.18 + \imath 0.32)$	$(0.24 + \imath 0.35)$	$(0.32 + \imath 0.45)$	$(0.32 + \imath 0.14)$

After applying the invented BCF-TOPSIS, we interpreted the distance of each alternative from positive and negative ideal solutions, which is revealed as follows.

$$D(A_1, \text{CFPIS}) = 0.1 \qquad D(A_1, \text{CFNIS}) = 0.17$$

$$D(A_2, \text{CFPIS}) = 0.175 \qquad D(A_2, \text{CFNIS}) = 0.195$$

$$D(A_3, \text{CFPIS}) = 0.155 \qquad D(A_3, \text{CFNIS}) = 0.2225$$

$$D(A_4, \text{CFPIS}) = 0.225 \qquad D(A_4, \text{CFNIS}) = 0.1825$$

$$D(A_5, \text{CFPIS}) = 0.1825 \qquad D(A_5, \text{CFNIS}) = 0.1625$$

The relative closeness degree is given by $\mathfrak{C}_1 = 0.6296$, $\mathfrak{C}_2 = 0.5270$, $\mathfrak{C}_3 = 0.5855$, $\mathfrak{C}_4 = 0.4478$, $\mathfrak{C}_5 = 0.4710$. Now we order the alternatives according to relative closeness degrees, and we get $A_1 > A_3 > A_2 > A_5 > A_4$.

Finally, if the imaginary part of the positive membership degree in the model of BCFS is ignored and also ignores the negative membership degree in the model of BCFS, then, the information in Table 4 would be transformed in the model of FS, demonstrated in Table 10.

Table 10. The values for FS are given below

	C_1	C_2	C_3	C_4
A_1	(0.4)	(0.39)	(0.24)	(0.52)
A_2	(0.5)	(0.49)	(0.34)	(0.42)
A_3	(0.35)	(0.47)	(0.24)	(0.48)
A_4	(0.35)	(0.19)	(0.56)	(0.31)
A_5	(0.18)	(0.24)	(0.32)	(0.32)

$$D(A_1, \text{FPIS}) = 0.1 \qquad D(A_1, \text{FNIS}) = 0.165$$

$$D(A_2, \text{FPIS}) = 0.1 \qquad D(A_2, \text{FNIS}) = 0.125$$

$$D(A_3, \text{FPIS}) = 0.1225 \qquad D(A_3, \text{FNIS}) = 0.1225$$

$$D(A_4, \text{FPIS}) = 0.145 \qquad D(A_4, \text{FNIS}) = 0.07$$

$$D(A_5, \text{FPIS}) = 0.1375 \qquad D(A_5, \text{FNIS}) = 0.1625$$

The relative closeness degree is given by $\mathfrak{C}_1 = 0.6226$, $\mathfrak{C}_2 = 0.5555$, $\mathfrak{C}_3 = 0.5$, $\mathfrak{C}_4 = 0.3255$, $\mathfrak{C}_5 = 0.5416$. Now we order the alternatives according to the relative closeness degrees we get $A_1 > A_2 > A_5 > A_3 > A_4$.

So, in the end, we conclude that the TOPSIS method that we defined for BCFSs is more generalized than the prevailing TOPSIS methods of BFS, CFS, and FS.

6. Conclusion

In this paper, we introduced and examined exponential and non-exponential similarity measures specially designed for bipolar complex fuzzy sets, providing a novel framework for dealing with dual-sided and uncertain data in CRM systems. Combining bipolarity and complex values allows for a richer and more expressive modeling of customer preferences and feelings, which cannot be precisely represented by conventional fuzzy models. The similarity measures developed here prove useful in distinguishing between customer profiles with conflicting or uncertain attributes, thus enhancing decision-making precision in applications of CRM. The results indicate that exponential measures are sensitive to slight differences, whereas non-exponential measures provide stability and interpretability. One of the main contributions of this work is the bi-directional framework of similarity measures that reconciles the trade-offs between precision and flexibility, adaptable to different CRM scenarios. The framework can also be applied to other real-world applications that contain fuzzy, contradictory, and uncertain information, like medical diagnosis, risk evaluation, and intelligent recommendatory systems.

6.1 *Limitations and future study*

The developed approach is based on a bipolar complex fuzzy set. When decision makers try to use a more generalized structure of bipolar complex fuzzy rough set, then the developed approach fails to hold in these situations because the introduced work can never discuss the upper and lower approximation, and the developed approach cannot classify the data. In this situation, the chance of data loss increases. Moreover, the suggested work, although providing a new combination of non-exponential and exponential similarity measures for bipolar complex fuzzy sets, has some limitations. To begin with, the weights of the attributes applied in the analysis were hypothetical rather than based on actual data or expert opinion, which can impinge on the generalizability of the findings.

In the future, we aim to expand this work in various frameworks such as bipolar complex fuzzy soft set [48], bipolar complex spherical fuzzy set [49], and aggregation theory [50, 51]. We can develop new decision-making approaches and apply these approaches in the area as discussed in [52, 53]. Moreover, we can propose the Stepwise Weight Assessment Ratio Analysis (SWARA) and Assessment based on Data Aggregation Method (ADAM) techniques as proposed in [54]. We can propose the application of the proposed work in atmospheric water harvesting as used in [55].

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Data availability

The data will be available on reasonable request to corresponding author.

Ethics declaration statement

The authors state that this is their original work and it is neither submitted nor under consideration in any other journal simultaneously.

Human and animal participants

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflict of interest

The authors declare that they have no conflict of interest.

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