Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

Research Article

Closed-Form Exact Solution of the Unified Boundary Value Problem for the Telegraph Equation on Intervals

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Received: 8 June 2025; **Revised:** 7 July 2025; **Accepted:** 8 July 2025

Abstract: This paper proposes a methodology for solving initial-boundary value problems for the one-dimensional telegraph equation with constant coefficients. The problem is addressed in its most general form, with the boundary conditions presented in a unified manner on an arbitrary bounded interval of the real line. Under the assumption that the data pertaining to the boundary value problem admit their Laplace transforms, it is demonstrated that there exists a unique solution in the Laplace domain. This exact operational solution can be obtained in a closed-form, facilitating the recovery of the explicit expression of the solution in the time domain, provided that the inverse Laplace transform is available as in tables of transforms. Some prior estimates are established using an integral representation of the solution, and the time domain solution is proven to be stable. As demonstrated in the illustrations, the operational solution is shown to exhibit a significantly higher degree of computational efficiency in comparison to both classical and generalized series solutions. The latter are hardly obtainable for the same problem using the Fourier decomposition approach. This enhancement can be attributed largely to the efficacy of the algorithms employed for the numerical inverse Laplace transform. Moreover, the exact closed-form operational solution can be extended to unbounded domains. The telegraph equation is fundamental in various fields of applied mathematics. Its exact solution in time or Laplace domains serves as a benchmark for numerical and semi-analytical methods.

***Keywords*:** telegraph equation, integral transforms, convolution and norms, stability analysis, exact closed-form expressions, existence and uniqueness, analytic Laplace inverse, numerical efficiency

MSC: 35C05, 35L20, 44A05

1. Introduction

The use of hyperbolic differential equations extends to a variety of disciplines, encompassing fields such as fluid mechanics, gas dynamics and the modeling of wave phenomena [1, 2]. In particular, the telegraph equation intervenes much in modeling the transmission and propagation of electrical signals [3], but also in the random walk theory as in [4], and in many other fields of industrial processes [5]. There are many papers on the different methods for solving the telegraph equation with classical boundary conditions on real intervals. The most common techniques are the numerical approaches, among which one can cite finite element, spectral method, spline approximations, modified cubic

DOI: https://doi.org/10.37256/cm.6420257402

B-splines collocation, hybrid numerical scheme, etc [6–9]. Semi-analytical methods like variational iterations, Adomian's decomposition and homotopy perturbations, have also been used as in [10–12]. They may sometimes lead to the exact solution of the telegraph equation in a closed form, but generally, they provide some infinite series solution as yielded also by the spectral methods [13]. Thus, closed-form exact solutions in time or in Laplace domains for the one-dimensional equation, even with constant coefficients, are scarce. Such a solution is presented in this paper.

The problem is exposed in Section 2 with brief comments on its physical significance. In Section 3, the data are subjected to some general assumptions, and the classical Fourier and Laplace integral transforms are used to obtain a representation of the solution in the form of integrals, which involves the convolution of certain functions with respect to the time variable. In Section 4, first, the time domain solution is shown to be stable when small changes are made to the input data. Then, the three classical boundary conditions (Dirichlet, Neumann, and Robin) that are considered together in a unified way, are taken into account in order to obtain the exact solution in the Laplace domain. The closed-form expression of the solution is obtained via a Cramer system of linear equations. This system guarantees the existence and uniqueness of the solution. This approach is analogous to that of the solution of the reaction-diffusion equation on bounded intervals, as previously discussed in the context of the work cited in [14]. The unified solution in the Laplace domain is first extended to semi-infinite and infinite real lines. Finally, several examples are given for specific hyperbolic equations. In some examples, analytic inverse can be expressed in a closed form through the utilization of tables of transforms as in [15]. However, numerical inverses in the time domain are always possible by using available algorithms, regardless of the complexity of the Laplace domain solutions. Finally, Section 5 outlines the conclusion.

2. The telegraph equation

The most general form of the telegraph equation with constant coefficients is considered on an arbitrary bounded interval, for space variable x, $l_1 < x < l_2$, and for time variable t, $0 < t < T$ if $T > 0$ or $t > 0$ when $T = \infty$. Following examples as in [16], the considered boundary value problem consists to determine $u(x, t)$ such that:

$$\frac{\partial^2 u}{\partial t^2} + k\frac{\partial u}{\partial t} - a^2\frac{\partial^2 u}{\partial x^2} + bu = f(x, t), \quad a > 0, \quad l_1 < x < l_2, \quad 0 < t < T, \tag{1}$$

subject to the initial conditions

$$u(x, 0) = \varphi(x), \quad \frac{\partial u}{\partial t}(x, 0) = \psi(x), \quad l_1 \leq x \leq l_2, \tag{2}$$

and to the boundary conditions for $0 \leq t \leq T$:

$$\alpha_1 u(l_1, t) + \beta_1 \frac{\partial u}{\partial x}(l_1, t) = g_1(t), \quad \alpha_1^2 + \beta_1^2 \neq 0, \tag{3}$$

$$\alpha_2 u(l_2, t) + \beta_2 \frac{\partial u}{\partial x}(l_2, t) = g_2(t), \quad \alpha_2^2 + \beta_2^2 \neq 0, \tag{4}$$

where α_1, β_1, α_2, β_2 are reals.

The unknown function $u(x, t)$ may represent displacements of the points from the equilibrium at location x and time t of a thin stretched string, subject to small transverse oscillations. The source term $f(x, t)$ is given and may represent the external forces as for instance, the weight of the string if it cannot be neglected. The term bu can be included in equation

(1) for a string undergoing vibration in an elastic medium. This occurs when the force $f(x, t)$ exerted on the string by the medium is proportional to the string's deflection. In situations where a string vibrates in a medium that creates friction, the force per unit length is equal to $-ku_t$. Here, u_t indicates the partial derivative of the function u with respect to time, and k is the coefficient of friction. For physical situations, it is required that both constant coefficients b and k be greater or equal to zero. The functions $\varphi(x)$ and $\psi(x)$, which are associated with the initial conditions, may be null or non-null. The Dirichlet, Neumann, and Robin boundary conditions are unified in equations (3) and (4) for some given expressions of the time-dependent functions $g_1(t)$ and $g_2(t)$. This allows for both homogeneous and nonhomogeneous forms of the boundary conditions to be considered. In order to ensure the physical coherence of the problem described by equations (1)-(4), it is necessary to impose certain restrictions on the sign of the coefficients appearing in the boundary conditions (3)-(4). In the majority of cases, physical limitations result in the restriction that α_1/β_1 is less than zero and α_2/β_2 is greater than zero. This is discussed in [16].

In the context of initial and boundary conditions, the concept of a well-posed problem, first introduced by J. Hadamard, is of particular significance. In accordance with this concept, a problem is designated as a well-posed problem if a theorem of existence and uniqueness of the solution exists for the corresponding problem. Moreover, the solution must depend continuously on the data of the problem. The concept of a class of correctness for initial conditions and boundary conditions emerges. This is the class of functions to which the initial and boundary conditions, as well as the source term, respectively belong. It is guaranteed that a unique solution to the respective problem can be found for these functions [17, 18]. For instance, the classical series solutions of the problem (1)-(4) are obtained through the application of the Fourier decomposition method, which involves the separation of variables, in conjunction with the Sturm-Liouville theory of eigenvalues and eigenfunctions. In the case where only bounded intervals are taken into account for the space variable x, it is sufficient to assume for initial data that the functions $\psi(x)$ and $\varphi(x)$ are twice continuously differentiable and their third-order derivatives are piecewise continuous on their domain of definition. Furthermore, it is assumed that the function $f(x, t)$ is continuous, while the functions $g_1(t)$ and $g_2(t)$ are twice continuously differentiable in order to employ the so-called auxiliary functions. The aforementioned hypotheses will result in a series solution that converges, as well as the series obtained by differentiating twice with respect to x or t [16, 18]. In this paper, the application of the classical Fourier and Laplace integral transforms to identify an operational solution, will necessitate fewer stringent assumptions.

3. Method of integral transforms

In this Section, we first briefly recall the basic definitions of Fourier and Laplace integral transforms. Then, a specific application of those classical transforms will lead to the reduction of equation (1) unto an integral equation.

3.1 *Basic properties and assumptions*

The Fourier Integral Transform (abbreviated FIT from now on), in relation to the space variable x, is defined for any Lebesgue integrable function $\phi(x)$ ($\int_{-\infty}^{\infty}|\phi(x)|dx < \infty$), as:

$$\Phi(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(x)\exp(-i\lambda x)dx, \tag{5}$$

where $i^2 = -1$, $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}$. The inverse Fourier transform of Φ is:

$$\phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Phi(\lambda)\exp(ix\lambda)d\lambda. \tag{6}$$

As a basic property, the FIT tends to 0 when $|\lambda|$ goes to ∞. In the context of piecewise continuous functions defined on $t \geq 0$, specifically when $f(t)$ is such a function, the Laplace Integral Transform (abbreviated as LIT henceforth) is expressed in the complex p-plane as follows [19]:

$$F(p) = \mathscr{L}\{f(t)\} = \int_0^{\infty} f(t)\mathrm{e}^{-pt}dt, \tag{7}$$

provided that the function $f(t)$ is of exponential order. That is to say, there exist constants C and σ such that $|f(t)| < C\mathrm{e}^{\sigma t}$, for sufficiently large values of t. The set $\mathscr{O}$ of all functions that satisfy the aforementioned properties and possess a Laplace transform is referred to as the set of original functions. The set $\mathscr{O}$ is a vector space and an algebra. The inversion, which originates from the Laplace domain p and is expressed in the time domain, is defined at each point t of continuity by the complex integral,

$$f(t) = \mathscr{L}^{-1}\{F(p)\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(p)\mathrm{e}^{pt}dp, \tag{8}$$

where $\gamma > \sigma$ is chosen so that $F(p)$ converges absolutely for all p such that $\mathrm{Re}(p) > \gamma$, and $F(p)$ is analytic at the right of the line $x = \gamma$. The Laplace transform establishes a one-to-one relationship between the set $\mathscr{O}$ of originals and the set of transforms. That is to say, if we are given a transform of an original, then this is the transform of a single original. Also, the LIT and its inverse can be obtained for many usual functions by using tables of transforms as in [15]. In the case when p is real as considered in the sequel, the inequality $p \geq \sigma$ needs to be satisfied. Relatively to the LIT, an important property is the convolution theorem [20]:

Let $f(t)$ and $g(t)$ be functions defined in $t \geq 0$. If $\mathscr{L}\{f(t)\} = F(p)$ and $\mathscr{L}\{g(t)\} = G(p)$, then

$$\mathscr{L}\{f(t) * g(t)\} = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\} = F(p)G(p)$$

where $f(t) * g(t)$ is called the convolution product of $f(t)$ and $g(t)$ and is defined by the integral

$$f(t) * g(t) = \int_0^t f(t-\eta)g(\eta)d\eta.$$

In the forthcoming method, derivatives of the data functions are not involved. The method involves only some integration of initial or boundary data functions, at most in convolution with integrable functions. Thus, we require the following minimal conditions for the present transform approach to remain valid: the two functions related to the initial conditions, $\varphi(x)$ and $\psi(x)$, must be Lebesgue measurable and integrable on the interval $[l_1, l_2]$, i.e., φ and ψ must belong to the space $L^1([l_1, l_2])$ equipped with the norm $\|\phi\|_1 = \int_{l_1}^{l_2} |\phi(x)|dx$. Note that this assumption includes all functions, $\psi(x)$ and $\varphi(x)$, that satisfy the Dirichlet conditions over their respective definition intervals. In other words, these functions are piecewise continuous or can be uniquely expressed as convergent series of eigenvalues and eigenfunctions of the Sturm-Liouville problem. Furthermore, the two time-dependent functions $g_1(t)$ and $g_2(t)$ related to the boundary conditions are assumed to be original functions or elements of $\mathscr{O}$, especially when $T = \infty$. As for the source term represented by $f(x, t)$, it can be considered as a function defined on $[l_1, l_2] \times [0, T]$. In this sense, it can be assumed that $f(x, t_0)$ is Lebesgue-measurable and integrable for any $t_0 \in [0, T]$ and, for any $x_0 \in [l_1, l_2]$, $f(x_0, t)$ is an original function, an element of $\mathscr{O}$. Thus, the acceptable source terms $f(x, t)$ include continuous functions and form a class of Lebesgue-integrable functions over $[l_1, l_2] \times [0, T]$ if T is finite. In light of these general hypotheses, we propose a novel methodology combining Fourier and Laplace integral transforms, with the aim of obtaining the closed-form operational solution of the initial-boundary-value problem (1)-(4). The function $u(x, t)$ to be determined can take either a classical or a generalized solution form, depending on the compatibility of the initial and boundary conditions. However, the current solution approach does not cover certain limiting cases, such as signals with super-exponential growth in the time or space variable and the presence of nonlinear terms in the system.

3.2 *Fourier and Laplace transforms*

Let $u(x, t)$ be a solution to the boundary value problem formed by the equation (1) together with the initial condition (2), and the boundary conditions (3) and (4). Assuming that the functions $u(x, t)$, $u_t(x, t)$, $u_{tt}(x, t)$, $u_x(x, t)$, and $u_{xx}(x, t)$ are integrable with respect to the variables x and t, they can be identified, without loss of generality, to their extension by 0 outside the rectangle $[l_1, l_2] \times [0, T]$. If $T < \infty$, $f(x, t)$, $\psi(x)$, $\varphi(x)$, $g_1(t)$, and $g_2(t)$ can be identified to their extension by 0 respectively outside the rectangle $[l_1, l_2] \times [0, T]$, and the intervals $[l_1, l_2]$ and $[0, T]$. The Fourier Integral Transform (FIT) with respect to the space variable x can be applied to $u(x, t)$ and will give:

$$\mathscr{F}(\lambda, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, t) \exp(-i\lambda x) dx = \frac{1}{\sqrt{2\pi}} \int_{l_1}^{l_2} u(x, t) \exp(-i\lambda x) dx.$$

The application of the FIT to the term $u_{tt}(x, t)$ leads to:

$$A(\lambda, t) = \frac{1}{\sqrt{2\pi}} \int_{l_1}^{l_2} \frac{\partial^2 u}{\partial t^2}(x, t) \exp(-i\lambda x) dx = \frac{1}{\sqrt{2\pi}} \frac{\partial^2}{\partial t^2} \int_{l_1}^{l_2} u(x, t) \exp(-i\lambda x) dx = \frac{\partial^2 \mathscr{F}}{\partial t^2}(\lambda, t).$$

Similarly, the FIT is also applied to the term $u_t(x, t)$ to give:

$$B(\lambda, t) = k \frac{1}{\sqrt{2\pi}} \int_{l_1}^{l_2} \frac{\partial u}{\partial t}(x, t) \exp(-i\lambda x) dx = k \frac{1}{\sqrt{2\pi}} \frac{\partial}{\partial t} \int_{l_1}^{l_2} u(x, t) \exp(-i\lambda x) dx = k \frac{\partial \mathscr{F}}{\partial t}(\lambda, t)$$

Concerning the diffusion term $-a^2 u_{xx}(x, t)$, the FIT is given by:

$$C(\lambda, t) = -\frac{a^2}{\sqrt{2\pi}} \int_{l_1}^{l_2} \frac{\partial^2 u}{\partial x^2}(x, t) \exp(-i\lambda x) dx.$$

By successive integration by parts over the finite interval $[l_1, l_2]$, $C(\lambda, t)$ is transformed into:

$$\begin{aligned} C(\lambda, t) = a^2 \lambda^2 \mathscr{F}(\lambda, t) - \frac{a^2}{\sqrt{2\pi}} \left[\frac{\partial u}{\partial x}(l_2, t) \exp(-i\lambda l_2) - \frac{\partial u}{\partial x}(l_1, t) \exp(-i\lambda l_1) \right] \\ + \frac{a^2}{\sqrt{2\pi}} \left[i\lambda u(l_2, t) \exp(-i\lambda l_2) - i\lambda u(l_1, t) \exp(-i\lambda l_1) \right]. \end{aligned} \tag{9}$$

The linear term $bu(x, t)$ is simply transform into

$$D(\lambda, t) = b\mathscr{F}(\lambda, t),$$

and the FIT of the source term $f(x, t)$ can be written as:

$$E(\lambda, t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x, t)\exp(-i\lambda x)dx=\frac{1}{\sqrt{2\pi}}\int_{l_1}^{l_2} f(x, t)\exp(-i\lambda x)dx.$$

Now, if $u(x, t)$ is solution of equation (1), then $\mathscr{F}(\lambda, t)$ is solution of the following equation: $A(\lambda, t)+B(\lambda, t)$ $+C(\lambda, t)+D(\lambda, t)=E(\lambda, t)$, i.e.,

$$\begin{aligned}&\frac{\partial^2\mathscr{F}}{\partial t^2}(\lambda, t)+k\frac{\partial\mathscr{F}}{\partial t}(\lambda, t)+(b+a^2\lambda^2)\mathscr{F}(\lambda, t)\\ &=\frac{a^2}{\sqrt{2\pi}}\times\left[\frac{\partial u}{\partial x}(l_2, t)\exp(-i\lambda l_2)-\frac{\partial u}{\partial x}(l_1, t)\exp(-i\lambda l_1)\right]\\ &\quad+\frac{a^2}{\sqrt{2\pi}}\left[i\lambda u(l_2, t)\exp(-i\lambda l_2)-i\lambda u(l_1, t)\exp(-i\lambda l_1)\right]+E(\lambda, t);\end{aligned}\tag{10}$$

for $\lambda\in\mathbb{R}$ and $0<t<T$.

Having in mind that the final analytical solution in the Laplace domain will also be satisfied even for $k=0$, it's necessary from here to assume that $k>0$. Dividing then by $k\neq 0$ and writing $a_1=a/\sqrt{k}$ and $b_1=b/k$, equation (10) is identical to:

$$\begin{aligned}&\frac{1}{k}\frac{\partial^2\mathscr{F}}{\partial t^2}(\lambda, t)+\frac{\partial\mathscr{F}}{\partial t}(\lambda, t)+(b_1+a_1^2\lambda^2)\mathscr{F}(\lambda, t)\\ &=\frac{a_1^2}{\sqrt{2\pi}}\times\left[\frac{\partial u}{\partial x}(l_2, t)\exp(-i\lambda l_2)-\frac{\partial u}{\partial x}(l_1, t)\exp(-i\lambda l_1)\right]\\ &\quad+\frac{a_1^2}{\sqrt{2\pi}}\left[i\lambda u(l_2, t)\exp(-i\lambda l_2)-i\lambda u(l_1, t)\exp(-i\lambda l_1)\right]+\frac{E(\lambda, t)}{k}.\end{aligned}\tag{11}$$

Then, by multiplying each member of equation (11) by $\exp[(b_1+a_1^2\lambda^2)t]$, we obtain:

$$\begin{aligned}&\frac{1}{k}\frac{\partial^2\mathscr{F}}{\partial t^2}(\lambda, t)\exp[(b_1+a_1^2\lambda^2)t]+\frac{\partial}{\partial t}(\mathscr{F}(\lambda, t)\exp[(b_1+a_1^2\lambda^2)t])\\ &=\frac{a_1^2}{\sqrt{2\pi}}\exp[(b_1+a_1^2\lambda^2)t]\times\left[\frac{\partial u}{\partial x}(l_2, t)\exp(-i\lambda l_2)-\frac{\partial u}{\partial x}(l_1, t)\exp(-i\lambda l_1)\right]\\ &\quad+\frac{a_1^2}{\sqrt{2\pi}}\exp[(b_1+a_1^2\lambda^2)t]\left[i\lambda u(l_2, t)\exp(-i\lambda l_2)-i\lambda u(l_1, t)\exp(-i\lambda l_1)\right]+\frac{E(\lambda, t)}{k}(b_1+a_1^2\lambda^2)t.\end{aligned}\tag{12}$$

We integrate equation (12) with respect to the time variable, from $\eta = 0$ to $\eta = t \leq T$, to have:

$$\begin{aligned}
&\frac{1}{k}\int_0^t \frac{\partial^2 \mathscr{F}}{\partial \eta^2}(\lambda,\, \eta)\exp[(b_1 + a_1^2\lambda^2)\eta]d\eta + \mathscr{F}(\lambda,\, t)\exp[(b_1 + a_1^2\lambda^2)t] - \mathscr{F}(\lambda,\, 0)\\
&= \frac{a_1^2}{\sqrt{2\pi}} \times \int_0^t \left[\frac{\partial u}{\partial x}(l_2,\, \eta)\exp(-i\lambda l_2) - \frac{\partial u}{\partial x}(l_1,\, \eta)\exp(-i\lambda l_1)\right]\exp[(b_1 + a_1^2\lambda^2)\eta]d\eta\\
&+ \frac{a_1^2}{\sqrt{2\pi}}\int_0^t [i\lambda u(l_2,\, \eta)\exp(-i\lambda l_2) - i\lambda u(l_1,\, \eta)\exp(-i\lambda l_1)]\exp[(b_1 + a_1^2\lambda^2)\eta]d\eta\\
&+ \frac{1}{k\sqrt{2\pi}}\int_{l_1}^{l_2}\int_0^t f(x,\, \eta)\exp(-i\lambda x)\exp[(b_1 + a_1^2\lambda^2)\eta]d\eta dx,
\end{aligned} \tag{13}$$

where the double integral has been permuted due to its assumed convergence. According to the initial conditions (2), $\mathscr{F}(\lambda,\, 0) = \mathscr{F}(\lambda,\, t = 0)$ is calculated as:

$$\mathscr{F}(\lambda,\, 0) = \frac{1}{\sqrt{2\pi}}\int_{l_1}^{l_2}\varphi(\xi)\exp(-i\lambda\xi)d\xi, \tag{14}$$

where the integrating variable is ξ instead of x. By dividing by $\exp[(b_1 + a_1^2\lambda^2)\eta]$, equation (13) is rewritten as:

$$\begin{aligned}
&\frac{1}{k}\int_0^t \frac{\partial^2 \mathscr{F}}{\partial \eta^2}(\lambda,\, \eta)\exp[-(b_1 + a_1^2\lambda^2)(t-\eta)]d\eta + \mathscr{F}(\lambda,\, t)\\
&= \frac{a_1^2}{\sqrt{2\pi}} \times \int_0^t \left[\frac{\partial u}{\partial x}(l_2,\, \eta)\exp(-i\lambda l_2) - \frac{\partial u}{\partial x}(l_1,\, \eta)\exp(-i\lambda l_1)\right]\exp[-(b_1 + a_1^2\lambda^2)(t-\eta)]d\eta\\
&+ \frac{a_1^2}{\sqrt{2\pi}}\int_0^t [i\lambda u(l_2,\, \eta)\exp(-i\lambda l_2) - i\lambda u(l_1,\, \eta)\exp(-i\lambda l_1)]\exp[-(b_1 + a_1^2\lambda^2)(t-\eta)]d\eta\\
&+ \frac{1}{k\sqrt{2\pi}}\int_{l_1}^{l_2}\int_0^t f(\xi,\, \eta)\exp(-i\lambda\xi)\exp[-(b_1 + a_1^2\lambda^2)(t-\eta)]d\eta d\xi\\
&+ \frac{1}{\sqrt{2\pi}}\exp[-(b_1 + a_1^2\lambda^2)t]\int_{l_1}^{l_2}\varphi(\xi)\exp(-i\lambda\xi)d\xi.
\end{aligned} \tag{15}$$

Choosing p as the Laplace variable, and using now the properties of convolution as specified in Section 3.1, and those of derivatives transforms, the Laplace Integral Transform (LIT) applied to equation (15) will lead to:

$$\frac{1}{k}\frac{p^2\mathscr{L}\mathscr{F}(\lambda,\,p)-p\mathscr{F}(\lambda,0)-\mathscr{F}_t(\lambda,0)}{b_1+a_1^2\lambda^2+p}+\mathscr{L}\mathscr{F}(\lambda,\,p)$$

$$\begin{aligned}
&=\frac{a_1^2}{\sqrt{2\pi}}\times\frac{U_x(l_2,\,p)\exp(-i\lambda l_2)-U_x(l_1,\,p)\exp(-i\lambda l_1)}{b_1+a_1^2\lambda^2+p}\\
&+\frac{a_1^2}{\sqrt{2\pi}}\frac{i\lambda U(l_2,\,p)\exp(-i\lambda l_2)-i\lambda U(l_1,\,p)\exp(-i\lambda l_1)}{b_1+a_1^2\lambda^2+p}\\
&+\frac{1}{k\sqrt{2\pi}}\frac{1}{b_1+a_1^2\lambda^2+p}\int_{l_1}^{l_2}F(\xi,\,p)\exp(-i\lambda\xi)d\xi\\
&+\frac{1}{\sqrt{2\pi}}\frac{1}{b_1+a_1^2\lambda^2+p}\int_{l_1}^{l_2}\varphi(\xi)\exp(-i\lambda\xi)d\xi;
\end{aligned}\tag{16}$$

where $\mathscr{L}\mathscr{F}(\lambda,\,p)$, $U_x(l_1,\,p)$, $U_x(l_2,\,p)$, $U(l_1,\,p)$, $U(l_2,\,p)$ and $F(\xi,\,p)$ are respectively the Laplace integral transforms of $\mathscr{F}(\lambda,t)$, $u_x(l_1,t)$, $u_x(l_2,t)$, $u(l_1,t)$, $u(l_2,t)$ and $f(\xi,t)$. The functions $\mathscr{F}(\lambda,0)$ and $\mathscr{F}_t(\lambda,0)$ are the Fourier integral transforms of the functions φ and ψ that are related to the initial conditions (2). Therefore, equation (16) can be reduced to:

$$\begin{aligned}
\mathscr{L}\mathscr{F}(\lambda,\,p)&=\frac{a^2}{\sqrt{2\pi}}\frac{U_x(l_2,\,p)\exp(-i\lambda l_2)-U_x(l_1,\,p)\exp(-i\lambda l_1)}{p^2+kp+a^2\lambda^2+b}\\
&+\frac{a^2}{\sqrt{2\pi}}\frac{i\lambda U(l_2,\,p)\exp(-i\lambda l_2)-i\lambda U(l_1,\,p)\exp(-i\lambda l_1)}{p^2+kp+a^2\lambda^2+b}\\
&+\frac{1}{\sqrt{2\pi}}\frac{1}{p^2+kp+a^2\lambda^2+b}\int_{l_1}^{l_2}F(\xi,\,p)\exp(-i\lambda\xi)d\xi\\
&+\frac{1}{\sqrt{2\pi}}\frac{k+p}{p^2+kp+a^2\lambda^2+b}\int_{l_1}^{l_2}\varphi(\xi)\exp(-i\lambda\xi)d\xi\\
&+\frac{1}{\sqrt{2\pi}}\frac{1}{p^2+kp+a^2\lambda^2+b}\int_{l_1}^{l_2}\psi(\xi)\exp(-i\lambda\xi)d\xi.
\end{aligned}\tag{17}$$

3.3 *Inverse transforms*

Returning to the original $\mathscr{F}(\lambda,t)$ of $\mathscr{L}\mathscr{F}(\lambda,\,p)$ from equation (17), the reverse into the time domain of that equation is derived as:

$$
\begin{aligned}
\mathscr{F}(\lambda, t) = {} & \frac{a^2}{\sqrt{2\pi}} \int_0^t \left[\frac{\partial u}{\partial x}(l_2, \eta) \exp(-i\lambda l_2) - \frac{\partial u}{\partial x}(l_1, \eta) \exp(-i\lambda l_1) \right] H_1(\lambda, t-\eta) d\eta \\
& + \frac{a^2}{\sqrt{2\pi}} \int_0^t [i\lambda u(l_2, \eta) \exp(-i\lambda l_2) - i\lambda u(l_1, \eta) \exp(-i\lambda l_1)] H_1(\lambda, t-\eta) d\eta \\
& + \frac{1}{\sqrt{2\pi}} \int_{l_1}^{l_2} \int_0^t f(\xi, \eta) \exp(-i\lambda \xi) H_1(\lambda, t-\eta) d\eta d\xi \\
& + \frac{1}{\sqrt{2\pi}} H_1(\lambda, t) \int_{l_1}^{l_2} \psi(\xi) \exp(-i\lambda \xi) d\xi \\
& + \frac{1}{\sqrt{2\pi}} H_2(\lambda, t) \int_{l_1}^{l_2} \varphi(\xi) \exp(-i\lambda \xi) d\xi,
\end{aligned} \tag{18}
$$

where

$$
H_1(\lambda, t) = 2 \frac{\exp\left(-\frac{1}{2} t k\right) \sinh\left(\frac{1}{2} t \sqrt{-4a^2\lambda^2 + k^2 - 4b}\right)}{\sqrt{-4a^2\lambda^2 + k^2 - 4b}}, \tag{19}
$$

is the inverse Laplace transform of

$$
L_1(\lambda, p) = \frac{1}{p^2 + kp + a^2\lambda^2 + b};
$$

and

$$
H_2(\lambda, t) = \exp\left(-\frac{1}{2} tk\right) \cosh\left(\frac{1}{2} t \sqrt{-4a^2\lambda^2 + k^2 - 4b}\right) + \frac{1}{2} k H_1(\lambda, t), \tag{20}
$$

is the inverse Laplace transform of

$$
L_2(\lambda, p) = \frac{p + k}{p^2 + kp + a^2\lambda^2 + b}.
$$

The function $u(x, t)$ is then obtained by applying the Fourier inversion formula (6) to the function $\mathscr{F}(\lambda, t)$ given by equation (18), λ (the Fourier variable) going from $-\infty$ to ∞. For this purpose, equation (18) can be written, due to the convolution property of the LIT: $\mathscr{F}(\lambda, t) = \mathscr{F}_1(\lambda, t) + \mathscr{F}_2(\lambda, t) + \mathscr{F}_3(\lambda, t) + \mathscr{F}_4(\lambda, t) + \mathscr{F}_5(\lambda, t)$, with respectively:

$$\mathscr{F}_1(\lambda, t) = \frac{a^2}{\sqrt{2\pi}} \int_0^t \left[\frac{\partial u}{\partial x}(l_2, t-\eta) \exp(-i\lambda l_2) - \frac{\partial u}{\partial x}(l_1, t-\eta) \exp(-i\lambda l_1) \right] H_1(\lambda, \eta) d\eta;$$

$$\mathscr{F}_2(\lambda, t) = \frac{a^2}{\sqrt{2\pi}} \int_0^t [i\lambda u(l_2, t-\eta) \exp(-i\lambda l_2) - i\lambda u(l_1, t-\eta) \exp(-i\lambda l_1)] H_1(\lambda, \eta);$$

$$\mathscr{F}_3 = \frac{1}{\sqrt{2\pi}} \int_{l_1}^{l_2} \int_0^t f(\xi, t-\eta) \exp(-i\lambda\xi) H_1(\lambda, \eta) d\eta d\xi;$$

$$\mathscr{F}_4(\lambda, t) = \frac{1}{\sqrt{2\pi}} H_1(\lambda, t) \int_{l_1}^{l_2} \psi(\xi) \exp(-i\lambda\xi) d\xi;$$

and

$$\mathscr{F}_5(\lambda, t) = \frac{1}{\sqrt{2\pi}} H_2(\lambda, t) \int_{l_1}^{l_2} \varphi(\xi) \exp(-i\lambda\xi) d\xi.$$

The order of integration can be changed since the integrals involved are convergent, and the inverse FIT I_i of each term $\mathscr{F}_i$, $1 \le i \le 5$, can be respectively expressed as:

$$I_1(x, t) = \frac{a^2}{2\pi} \int_0^t \frac{\partial u}{\partial x}(l_2, t-\eta) \int_{-\infty}^{\infty} \exp(-i\lambda l_2) \exp(i\lambda x) H_1(\lambda, \eta) d\lambda d\eta$$

$$- \frac{a^2}{2\pi} \int_0^t \frac{\partial u}{\partial x}(l_1, t-\eta) \int_{-\infty}^{\infty} \exp(-i\lambda l_1) \exp(i\lambda x) H_1(\lambda, \eta) d\lambda d\eta;$$

$$I_2(x, t) = \frac{a^2}{2\pi} \int_0^t u(l_2, t-\eta) \int_{-\infty}^{\infty} i\lambda \exp(-i\lambda l_2) \exp(i\lambda x) H_1(\lambda, \eta) d\lambda d\eta$$

$$- \frac{a^2}{2\pi} \int_0^t u(l_1, t-\eta) \int_{-\infty}^{\infty} i\lambda \exp(-i\lambda l_1) \exp(i\lambda x) H_1(\lambda, \eta) d\lambda d\eta;$$

$$I_3(x, t) = \frac{1}{\sqrt{2\pi}} \int_{l_1}^{l_2} \int_0^t \int_{-\infty}^{\infty} f(\xi, t-\eta) \exp(-i\lambda\xi) \exp(i\lambda x) H_1(\lambda, \eta) d\lambda d\eta d\xi;$$

$$I_4(x, t) = \frac{1}{2\pi} \int_{l_1}^{l_2} \int_{-\infty}^{\infty} \psi(\xi) H_1(\lambda, t) \exp(-i\lambda\xi) \exp(i\lambda x) d\lambda d\xi;$$

and

$$I_5(x, t) = \frac{1}{2\pi} \int_{l_1}^{l_2} \int_{-\infty}^{\infty} \varphi(\xi) H_2(\lambda, t) \exp(-i\lambda\xi) \exp(i\lambda x) d\lambda d\xi.$$

Therefore

$$u(x, t) = I_1(x, t) + I_2(x, t) + I_3(x, t) + I_4(x, t) + I_5(x, t). \tag{21}$$

We denote respectively by j_1 the integral:

$$j_1(x, \xi, t) = \frac{a^2}{2\pi} \int_{-\infty}^{\infty} H_1(\lambda, t) \exp(-i\lambda\xi) \exp(i\lambda x) d\lambda,$$

which LIT J is computed as:

$$J(x, \xi, p) = \int_{-\infty}^{\infty} \frac{\cos[(x-\xi)\lambda]}{p^2 + kp + a^2\lambda^2 + b} d\lambda = \frac{1}{2} \frac{a}{\sqrt{kp + p^2 + b}} \exp\left(-\frac{|x-\xi| \sqrt{kp + p^2 + b}}{a} \right). \tag{22}$$

And j_2 denotes the integral:

$$j_2(x, \xi, t) = \frac{a^2}{2\pi} \int_{-\infty}^{\infty} H_2(\lambda, t) \exp(-i\lambda\xi) \exp(i\lambda x) d\lambda,$$

which is transformed by the LIT to give:

$$K(x, \xi, p) = (p + k) J(x, \xi, p). \tag{23}$$

The LIT of the integral j_3:

$$j_3(x, \xi, t) = \frac{a^2}{2\pi} \int_{-\infty}^{\infty} H_1(\lambda, t) i\lambda \exp(-i\lambda\xi) \exp(i\lambda x) d\lambda,$$

can be expressed by using the partial differentiation of J with respect to x as:

$$\frac{\partial}{\partial x} J(x, \xi, p) = J_x(x, \xi, p) = \begin{cases} \frac{1}{2} \exp\left(-\frac{(\xi - x)\sqrt{kp + p^2 + b}}{a} \right) & \text{for} \quad l_1 \leq x < \xi, \\ -\frac{1}{2} \exp\left(-\frac{(x - \xi)\sqrt{kp + p^2 + b}}{a} \right) & \text{for} \quad \xi < x \leq l_2. \end{cases} \tag{24}$$

Note that this derivative is well defined when $x = \xi = l_1$ or when $x = \xi = l_2$, but possess a discontinuity jump only at $l_1 < \xi = x < l_2$. Using then equations (22)-(24), the application of the LIT to equation (21) results in the following relation:

$$U(x,\, p) = U_x(l_2,\, p)J(x,\, l_2,\, p) - U_x(l_1,\, p)J(x,\, l_1,\, p) + U(l_2,\, p)J_x(x,\, l_2,\, p) - U(l_1,\, p)J_x(x,\, l_1,\, p) + R(x,\, p) \tag{25}$$

where $U(x,\, p)$ is the LIT of the solution $u(x,\, t)$ to be determined, and

$$R(x,\, p) = \frac{1}{a^2}\int_{l_1}^{l_2} \left[(k+p)\,\varphi(\xi) + \psi(\xi) + F(\xi,\, p)\right] J(x,\, \xi,\, p)\, d\xi \tag{26}$$

stands for the LIT of $I_3(x,\, t) + I_4(x,\, t) + I_5(x,\, t)$. In order to give more explicit details on the relation (25), the Laplace Integral Transform (LIT) of the boundary relations (3) and (4) will now be considered.

4. Results and discussions

First, the representation (21) of the time domain solution is shown to be stable when small changes are made to the input data. Then, taking into account the relation (25), the unified solution is derived in the Laplace domain from a system of linear equations that serves to prove altogether its existence and uniqueness, then several examples are proposed for illustrating the results.

4.1 *Stability of the solution*

The stability of the solution $u(x, t)$, inverse of $U(x, p)$, with respect to small changes in the input data will be examined using formula (21). Note that, we shall keep the independent variables in the L^1 norm for more clarity. Applying absolute values to both sides of the first integral $I_1(x,\, t)$ results in the following inequality:

$$|I_1(x,\, t)| \le \frac{a^2}{2\pi}\left|\int_0^t \frac{\partial u}{\partial x}(l_2,\, t-\eta)\left(\int_{-\infty}^{\infty} \exp(-i\lambda l_2)\exp(i\lambda x)H_1(\lambda,\, t)d\lambda\right)d\eta\right|$$
$$+\frac{a^2}{2\pi}\left|\int_0^t \frac{\partial u}{\partial x}(l_1,\, t-\eta)\left(\int_{-\infty}^{\infty} \exp(-i\lambda l_1)\exp(i\lambda x)H_1(\lambda,\, t)d\lambda\right)d\eta\right|.$$

This implies, due to the standard bound of the convolution product:

$$\begin{aligned}\|I_1(x,\, t)\|_1 \le \frac{a^2}{2\pi}\left\|\frac{\partial u}{\partial x}(l_2,\, t)\right\|_1\left\|\int_{-\infty}^{\infty} \exp(-i\lambda l_2)\exp(i\lambda x)H_1(\lambda,\, t)d\lambda\right\|_1 \\ +\frac{a^2}{2\pi}\left\|\frac{\partial u}{\partial x}(l_1,\, t)\right\|_1\left\|\int_{-\infty}^{\infty} \exp(-i\lambda l_1)\exp(i\lambda x)H_1(\lambda,\, t)d\lambda\right\|_1;\end{aligned} \tag{27}$$

since, for $0 \le t \le T < \infty$, $\left|\int_{-\infty}^{\infty} \exp(-i\lambda l_1)\exp(i\lambda x)H_1(\lambda,\, t)d\lambda\right| = \frac{2\pi}{a^2}\,|j_1(x,\, \xi,\, t)|$ is Lebesgue integrable. In fact, to have a bounded (and here equally continuous) Laplace transform $J(x,\, \xi,\, p)$, which tends to zero when p tends to infinity as shown by Formula (22), $j_1(x, \xi, t)$ is assumed to be of exponential order. That is, there are positive constants M and σ such that, $|j_1(x, \xi, t)| \leq M\exp(\sigma T)$ on the bounded rectangle $l_1 \leq x, \xi \leq l_2$ times $0 \leq t \leq T$, and null everywhere else. Also, the Laplace transform of $j_3(x, \xi, t) = \frac{a^2}{2\pi}\int_{-\infty}^{\infty} H_1(\lambda, t) i\lambda \exp(-i\lambda\xi)\exp(i\lambda x)d\lambda$, that is $J_x(x, \xi, p)$, is bounded and tends almost surely to zero as p tends to infinity for all $l_1 \leq x, \xi \leq l_2$ (see Formula (24)). Therefore, similar arguments as used above will imply:

$$\begin{aligned} \|I_2(x, t)\|_1 \leq \frac{a^2}{2\pi}\|u(l_2, t)\|_1 \left\|\int_{-\infty}^{\infty} H_1(\lambda, t) i\lambda \exp(-i\lambda l_2)\exp(i\lambda x)d\lambda\right\|_1 \\ + \frac{a^2}{2\pi}\|u(l_1, t)\|_1 \left\|\int_{-\infty}^{\infty} H_1(\lambda, t) i\lambda \exp(-i\lambda l_1)\exp(i\lambda x)d\lambda\right\|_1. \end{aligned} \tag{28}$$

Knowing that $f(x, t)$, $\varphi(x)$ and $\psi(x)$ are assumed Lebesgue integrals, similar calculations as above lead to:

$$\|I_3(x, t)\|_1 \leq \frac{1}{\sqrt{2\pi}}\|f(x, t)\|_1 \left\|\int_{-\infty}^{\infty} \exp(-i\lambda\xi)\exp(i\lambda x)H_1(\lambda, t)d\lambda\right\|_1; \tag{29}$$

$$\|I_4(x, t)\|_1 \leq \frac{1}{2\pi}\|\varphi(x)\|_1 \left\|\int_{-\infty}^{\infty} \exp(-i\lambda\xi)\exp(i\lambda x)H_1(\lambda, t)d\lambda\right\|_1; \tag{30}$$

and

$$\|I_5(x, t)\|_1 \leq \frac{1}{2\pi}\|\psi(x)\|_1 \left\|\int_{-\infty}^{\infty} \exp(-i\lambda\xi)\exp(i\lambda x)H_2(\lambda, t)d\lambda\right\|_1. \tag{31}$$

Now, taking the L^1 norm for both parts of the relation (21) and using the inequalities (27)-(31), we get:

$$\begin{aligned} \|u(x, t)\|_1 \leq C_1 \left\|\frac{\partial u}{\partial x}(l_2, t)\right\|_1 + C_2 \left\|\frac{\partial u}{\partial x}(l_1, t)\right\|_1 + C_3\|u(l_2, t)\|_1 + C_4\|u(l_1, t)\|_1 \\ + C_5\|f(x, t)\|_1 + C_6\|\varphi(x)\|_1 + C_7\|\psi(x)\|_1; \end{aligned} \tag{32}$$

where C_i, $i = 1, \ldots, 7$ are positive constants given exactly by the inequalities (27)-(31), as for instance

$$C_1 = \frac{a^2}{2\pi}\left\|\int_{-\infty}^{\infty} \exp(-i\lambda l_2)\exp(i\lambda x)H_1(\lambda, t)d\lambda\right\|_1.$$

Considering the space $(L_1, \|.\|_1)$ of all Lebesgue measurable and integrable functions, the relation (32) guarantees the continuous dependence of the solution $u(x, t)$ on fluctuations in the input data. If we replace $(L_1, \|.\|_1)$ by $(L_p, \|.\|_p)$ for $1 < p \leq \infty$ in the assumptions formulated in Section 3.1, the inequality (32) will still be true by substituting the subscript 1 by p. This is because of the properties of the convolution operator. Note that,

$$\|\phi\|_p = \left(\int_{l_1}^{l_2} |\phi(x)|^p dx \right)^{1/p},$$

for $1 \leq p < \infty$, and for $p = \infty$, $(L^\infty, \|.\|_\infty)$ is defined as the set of all Lebesgue measurable and almost surely bounded functions equipped with the essential supremum norm. As, the domains of definition involved in the above transforms, namely $[l_1, l_2]$ and $[0, T]$ are both finite, the family of spaces L^p, each with the norm $\|.\|_p$, is decreasing with continuous injections: $1 \leq p \leq q \leq \infty \implies L^p \supset L^q$.

4.2 *The exact operational solution*

As assumed in Section 3.1, the functions $g_1(t)$ and $g_2(t)$ that are related to the boundary conditions are also original functions (of exponential order). Their LIT will respectively be denoted in the Laplace domain by $G_1(p)$ and $G_2(p)$. Thus, the analogue of the time domain problem (1)-(4) can be written in the Laplace domain as the following two-point boundary value problem:

$$(p^2 + kp + b)U(x, p) - a^2 \frac{d^2U}{dx^2}(x, p) = F(x, p) + (p+k)\varphi(x) + \psi(x), \tag{33}$$

$$\alpha_1 U(l_1, p) + \beta_1 \frac{dU}{dx}(l_1, p) = G_1(p), \quad \alpha_1, \beta_1 \in \mathbb{R}, \quad \alpha_1^2 + \beta_1^2 \neq 0, \tag{34}$$

$$\alpha_2 U(l_2, p) + \beta_2 \frac{dU}{dx}(l_2, p) = G_2(p), \quad \alpha_2, \beta_2 \in \mathbb{R}, \quad \alpha_2^2 + \beta_2^2 \neq 0. \tag{35}$$

It's noticeable that the functions φ and ψ that are related to the initial conditions are now incorporated in equation (33). The values of the spatial variable x at the boundaries, i.e., $x = l_1$ and $x = l_2$, can be substituted in the expression of $U(x, p)$ obtained in equation (25) to give the following equations:

$$\begin{aligned} R(l_1, p) = {} & U(l_1, p)\left[J_x(l_1, l_1, p) - 1\right] + U_x(l_1, p)J(l_1, l_1, p) \\ & - U(l_2, p)J_x(l_1, l_2, p) - U_x(l_2, p)J(l_1, l_2, p), \end{aligned} \tag{36}$$

and

$$\begin{aligned} R(l_2, p) = {} & U(l_1, p)J_x(l_2, l_1, p) + U_x(l_1, p)J(l_2, l_1, p) \\ & + U(l_2, p)\left[1 - J_x(l_2, l_2, p)\right] - U_x(l_2, p)J(x, l_2, p). \end{aligned} \tag{37}$$

The equations (34)-(37) form a system $(\mathscr{S})$ of four linear equations with four unknown functions that are $U(l_1, p)$, $U_x(l_1, p)$, $U(l_2, p)$ and $U(l_2, p)$. Therefore, $U(x, p)$ is solution of the analogue problem (33)-(35) in the Laplace domain, or equivalently, $u(x, t)$ is solution of the problem (1)-(4) in the time domain, if and only if, the system $(\mathscr{S})$ admits a solution. The determinant of $(\mathscr{S})$ is calculated as:

$$\det(\mathscr{S}) = \begin{vmatrix} \alpha_1 & \beta_1 & 0 & 0 \\ 0 & 0 & \alpha_2 & \beta_2 \\ \frac{1}{2} & \frac{a}{2\sqrt{kp+p^2+b}} & \frac{-\chi(l_2-l_1,\, p)}{2} & \frac{-a\chi(l_2-l_1,\, p)}{2\sqrt{kp+p^2+b}} \\ \frac{-\chi(l_2-l_1,\, p)}{2} & \frac{a\chi(l_2-l_1,\, p)}{2\sqrt{kp+p^2+b}} & \frac{1}{2} & \frac{-a}{2\sqrt{kp+p^2+b}} \end{vmatrix}; \tag{38}$$

where $\chi(x,\, p)$ represents the function $\chi(x,\, p) = \exp\left(\frac{-x\sqrt{kp+p^2+b}}{a}\right)$. The reduction of this determinant can be done by computations and will lead to:

$$\begin{aligned} \det(\mathscr{S}) = {} & \frac{a^2\alpha_1\alpha_2}{4}\left(\frac{1-\chi(2(l_2-l_1),\, p)}{kp+p^2+b}\right) + \frac{1}{2}a\frac{\alpha_1\beta_2-\alpha_2\beta_1}{\sqrt{kp+p^2+b}} \\ & + \frac{1}{4}\left(\chi(2(l_2-l_1),\, p)-1\right)\beta_1\beta_2. \end{aligned} \tag{39}$$

In consequence, the determinant of the system $(\mathscr{S})$ is null if and only if the coefficients in equation (39) are null for all functions in variable p. That is, since $a > 0$:

$$\begin{cases} \alpha_1\alpha_2 = 0, \\ \alpha_1\beta_2 - \alpha_2\beta_1 = 0, \\ \beta_1\beta_2 = 0. \end{cases} \tag{40}$$

The system (40) is equivalent to the two sub-systems:

$$\begin{cases} \alpha_1\alpha_2 = 0, \\ \alpha_1\beta_2 = 0, \\ \beta_1 = 0, \end{cases} \tag{41}$$

or

$$\begin{cases} \alpha_1\alpha_2 = 0, \\ \alpha_2\beta_1 = 0, \\ \beta_2 = 0. \end{cases} \tag{42}$$

Each sub-system will lead to a contradiction with the hypotheses on the coefficients α_1, β_1, α_2 and β_2, since it will imply that $\alpha_1 = \beta_1 = 0$ or $\alpha_2 = \beta_2 = 0$. Thus, in all cases $\det(\mathscr{S}) \neq 0$. Therefore, whenever $\alpha_1^2 + \beta_1^2 \neq 0$ and $\alpha_2^2 + \beta_2^2 \neq 0$, the problem (1)-(4) in the time domain, or equivalently the analogue problem (33)-(35) in the Laplace domain, admits a unique solution under the specified hypotheses. The function $U(x,\, p)$ given by equation (25) is the exact and unique solution in the Laplace domain of the problem (1)-(4). And, the functions $U(l_1,\, p)$, $U_x(l_1,\, p)$, $U(l_2,\, p)$ and $U_x(l_2,\, p)$ that are involved in formula (25), are respectively obtained by using determinants:

$$U(l_1,\, p) = \frac{\begin{vmatrix} G_1(p) & \beta_1 & 0 & 0 \\ G_2(p) & 0 & \alpha_2 & \beta_2 \\ R(l_1,\, p) & \dfrac{a}{2\sqrt{kp+p^2+b}} & \dfrac{-\chi(l_2-l_1,\, p)}{2} & \dfrac{-a\chi(l_2-l_1,\, p)}{2\sqrt{kp+p^2+b}} \\ R(l_2,\, p) & \dfrac{a\chi(l_2-l_1,\, p)}{2\sqrt{kp+p^2+b}} & \dfrac{1}{2} & \dfrac{-a}{2\sqrt{kp+p^2+b}} \end{vmatrix}}{\det(\mathscr{S})}; \tag{43}$$

$$U_x(l_1,\, p) = \frac{\begin{vmatrix} \alpha_1 & G_1(p) & 0 & 0 \\ 0 & G_2(p) & \alpha_2 & \beta_2 \\ \dfrac{1}{2} & R(l_1,\, p) & \dfrac{-\chi(l_2-l_1,\, p)}{2} & \dfrac{-a\chi(l_2-l_1,\, p)}{2\sqrt{kp+p^2+b}} \\ \dfrac{-\chi(l_2-l_1,\, p)}{2} & R(l_2,\, p) & \dfrac{1}{2} & \dfrac{-a}{2\sqrt{kp+p^2+b}} \end{vmatrix}}{\det(\mathscr{S})}; \tag{44}$$

$$U(l_2,\, p) = \frac{\begin{vmatrix} \alpha_1 & \beta_1 & G_1(p) & 0 \\ 0 & 0 & G_2(p) & \beta_2 \\ \dfrac{1}{2} & \dfrac{a}{2\sqrt{kp+p^2+b}} & R(l_1,\, p) & \dfrac{-a\chi(l_2-l_1,\, p)}{2\sqrt{kp+p^2+b}} \\ \dfrac{-\chi(l_2-l_1,\, p)}{2} & \dfrac{a\chi(l_2-l_1,\, p)}{2\sqrt{kp+p^2+b}} & R(l_2,\, p) & \dfrac{-a}{2\sqrt{kp+p^2+b}} \end{vmatrix}}{\det(\mathscr{S})}; \tag{45}$$

and

$$U_x(l_x,\, p) = \frac{\begin{vmatrix} \alpha_1 & \beta_1 & 0 & G_1(p) \\ 0 & 0 & \alpha_2 & G_2(p) \\ \dfrac{1}{2} & \dfrac{a}{2\sqrt{kp+p^2+b}} & \dfrac{-\chi(l_2-l_1,\, p)}{2} & R(l_1,\, p) \\ \dfrac{-\chi(l_2-l_1,\, p)}{2} & \dfrac{a\chi(l_2-l_1,\, p)}{2\sqrt{kp+p^2+b}} & \dfrac{1}{2} & R(l_2,\, p) \end{vmatrix}}{\det(\mathscr{S})}. \tag{46}$$

In brief, formulae (43)-(46) being determined, the Laplace domain solution $U(x,\, p)$ of the boundary value problem (1)-(4) is expressed in a unified way by equation (25). Infinite series in the time domain are well-known as exact solutions for such linear boundary value problems [16]. The operational solution $U(x,\, p)$ can thus be considered as the exact Laplace transform in the closed form of those infinite series solutions, performed by using the Fourier decomposition method via the Sturm-Liouville theory.

4.3 *Extension to unbounded intervals*

As indicated in the introduction, the exact operational solution (25) can be extended with respect to the space variable x to encompass the entire real line or semi-infinite intervals.

In the case where the entire real line is taken into account for the space variable x ($l_1 = -\infty$ and $l_2 = +\infty$), classical methods for obtaining solutions assume that the functions $f(x,\, t)$ and its partial derivative $f_x(x,\, t)$ are continuous on the product space $(l_1,\, l_2) \times (0,\, T)$. Meanwhile, the functions $\psi(x)$ and $\varphi(x)$ are, respectively, once and twice continuously differentiable (see [17] for further details). However, to find the extension of results from the unified solution, we only need to check two things:

• The related-to-initial condition function φ is continuous, while ψ is Lebesgue measurable and integrable on the entire real line.

• The source term is Lebesgue measurable and integrable with respect to the axis variable x. It is also of exponential order with respect to the time variable t.

As an example, when $k = b = 0$, $l_1 = -\infty$ and $l_2 = \infty$, (1) takes the form of the non-homogeneous wave equation on the real line:

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f(x,\, t), \quad -\infty < x < \infty, \quad 0 < t < T. \tag{47}$$

subject to the initial conditions (2) for $-\infty < x < \infty$. But, $J(x,\, \xi,\, p) = J_x(x,\, \xi,\, p) = 0$, for $\xi = \pm\infty$ and the p-domain analytic solution (25) can be simplified into:

$$U(x,\, p) = R(x,\, p) = U_1(x,\, p) + U_2(x,\, p) + U_3(x,\, p). \tag{48}$$

where:

$$U_1(x,\, p) = \frac{1}{2a}\int_{-\infty}^{\infty} \varphi(\xi)\exp\left(-\frac{p|\xi - x|}{a}\right) d\xi,$$

$$U_2(x,\, p) = \frac{1}{2a}\int_{-\infty}^{\infty} \psi(\xi)\frac{\exp\left(-\frac{p|\xi - x|}{a}\right)}{p} d\xi,$$

and

$$U_3(x,\, p) = \frac{1}{2a}\int_{-\infty}^{\infty} F(\xi,\, p)\frac{\exp\left(-\frac{p|\xi - x|}{a}\right)}{p} d\xi.$$

The expression of $U_1(x,\, p)$ can be split into the following parts:

$$U_1(x,\, p) = \frac{1}{2a}\left[\int_{-\infty}^{x} \varphi(\xi)\exp\left(\frac{p(\xi - x)}{a}\right) d\xi + \int_{x}^{\infty} \varphi(\xi)\exp\left(\frac{p(x - \xi)}{a}\right) d\xi\right],$$

and by using a change of variable $\eta = x - \xi$ in the first integral, and $\eta = \xi - x$ in the second, one obtains:

$$U_1(x,\, p) = \frac{1}{2a}\left[\int_{0}^{\infty} \exp\left(-\frac{p\eta}{a}\right)[\varphi(x-\eta) + \varphi(x+\eta)]\, d\eta\right],$$

which can be rewritten, due to the scaling property of the Laplace transform as:

$$U_1(x,\, p) = \frac{1}{2}\left(\mathscr{L}\varphi(x - at) + \mathscr{L}\varphi(x + at)\right),$$

where $\mathscr{L}\varphi(x \pm at)$ designates the Laplace transform of $\varphi(x \pm at)$. Thus, the inversion of $U_1(x,\, p)$ in the time domain will yield:

$$u_1(x,\, t) = \frac{1}{2}\left(\varphi(x - at) + \varphi(x + at)\right).$$

A similar calculus for $U_2(x,\, p)$ will lead to:

$$U_2(x,\, p) = \frac{1}{2p}\left(\mathscr{L}\psi(x - at) + \mathscr{L}\psi(x + at)\right).$$

So, the inverse in the time domain is:

$$u_2(x,\,t)=\frac{1}{2}\left(\int_0^t\psi(x-a\tau)d\tau+\int_0^t\psi(x+a\tau)d\tau\right),$$

which is reduced by appropriate change of variable ($\xi=x-a\tau$ and $\xi=x+a\tau$) into:

$$u_2(x,\,t)=\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi.$$

From the above calculations and due to the convolution property of the LIT, the third function $U_3(x,\,p)$ is inverted into the time domain as:

$$u_3(x,\,t)=\frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t+\tau)}f(\xi,\,\tau)d\xi d\tau.$$

Finally, the original in the time domain of the p-domain solution (48) is obtained as:

$$u(x,\,t)=\frac{1}{2}\left(\varphi(x-at)+\varphi(x+at)\right)+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi+\frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t+\tau)}f(\xi,\,\tau)d\xi d\tau. \tag{49}$$

That is the D'Alembert's formula of the Cauchy problem for non-homogeneous wave equation, as obtained in the formula (5.44) in [16].

For a particular illustration with a semi-infinite interval, we may consider as in the Example 4.3.6 reported by [20], the displacement of a semi-infinite string which is initially at rest in its equilibrium position. The problem consists to solve the wave equation:

$$\frac{\partial^2 u}{\partial t^2}-a^2\frac{\partial^2 u}{\partial x^2}=0,\quad 0<x<\infty,\quad t>0;$$

with the initial and boundary conditions

$$u\,(x,\,0)=u_t\,(x,\,0)=0,\quad 0\le x<\infty;$$

$$u\,(0,\,t)=Af_0(t),\quad t\ge 0;$$

and

$$u\,(x,\,t)\to 0\quad \text{as}\quad x\to\infty;$$

where A is a constant and $f_0(t)$ a defined function. After replacing in the problem (1)-(4) all the involved parameters by their corresponding values, i.e, $l_1 = 0$, $l_2 = \infty$, $k = b = 0$, $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = 0$, $\varphi(x) = \psi(x) = 0$, $f(x, t) = R(x, t) = 0$, $g_1(t) = A f_0(t)$, and $g_2(t) = 0$, the computations in the present case lead to the exact solution:

$$U(x, p) = A\mathscr{L} f_0(p)\mathrm{e}^{-\frac{xp}{a}};$$

where $\mathscr{L} f_0(p)$ designates the Laplace transform of $f_0(t)$ and e^x stands for $\exp(x)$. Up to the variable names, this solution is identical to that specified in the formula (4.3.49) in [20], whose inversion in the time domain gives:

$$u(x, t) = A f_0\left(t - \frac{x}{a}\right) H\left(t - \frac{x}{a}\right),$$

where H is the well-known Heaviside function.

Further, as a generalization of examples with semi-unbounded intervals, the problem (1)-(4) can be considered for $l_1 = 0$, and $l_2 = \infty$ and $k > 0$ or $b > 0$. Then, equation (1) is valid on $0 < x < \infty$ and $0 < t < T$, and is subject to the initial conditions (2) for $0 \le x < \infty$ and to the boundary condition (3) at $x = 0$ for $0 \le t \le T$. Since, it can be assumed that $U_x(l_2, p) = U_x(\infty, p) = 0$ for some bounded solution, $u(l_2, t)$ is to be considered as a constant and the condition (4) will read $u(l_2, t) = g_2$, g_2 being fixed. Furthermore, $J(x, l_2, p) = J_x(x, l_2, p) = 0$ will imply the reduction of the solution (25) to:

$$U(x, p) = -U_x(0, p)J(x, 0, p) - U(0, p)J_x(x, 0, p) + R(x, p),$$

which can provide an explicit formula in the Laplace domain, depending on the expressions given to the initial and boundary data. In brief, one of the advantages of the unified solution in the Laplace domain, as given by formula (25), is that both infinite and semi-infinite one-dimensional telegraph problems can be handled by that solution, for any positive or zero values given to the coefficients k and b.

4.4 *Closed-form solution on a finite interval*

We first consider the problem with non-homogeneous telegraph equation reported as Example 2 in [21]:

$$\begin{aligned} u_{tt}(x, t) + u_t(x, t) + u(x, t) - u_{xx}(x, t) = \left(-x^3 + x^2\right)\left(2\cos(2t) + \sin(2t) + \sin^2(t)\right) \\ + (6x - 2)\sin^2(t), \quad 0 < x < l, \quad 0 < t < T, \end{aligned} \tag{50}$$

subject to the homogeneous initial conditions:

$$u(x, 0) = 0, \quad u_t(x, 0) = 0, \quad 0 \le x \le l, \tag{51}$$

and to the Dirichlet homogeneous boundary conditions:

$$u(0, t) = 0, \quad u(l, t) = 0, \quad 0 \le t \le T. \tag{52}$$

According to the former notations, $a = b = k = 1$, $f(x,t) = \left(-x^3 + x^2\right)\left(2\cos(2t) + \sin(2t) + \sin^2(t)\right) + (6x-2)\sin^2(t)$, $l_1 = 0$, $l_2 = l > 0$, $\varphi(x) = \psi(x) = 0$, $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = 0$ and $g_1(t) = g_2(t) = 0$. The Laplace transform $F(x,\, p)$ of $f(x,\, t)$ is expressed as:

$$F(x,\, p) = -2\,\frac{p^2x^3 - p^2x^2 + px^3 - px^2 + x^3 - x^2 - 6x + 2}{(p^2+4)\,p}.$$

The p-domain function $R(x,\, p)$ obtained in equation (26) can be explicitly determined, and the expressions of this function at the boundary $x = 0$ and $x = l$ are respectively:

$$R(0,\, p) = \frac{\left(\sqrt{p^2+p+1}\,l^2 - \sqrt{p^2+p+1}\,l + 3\,l - 2\right) l\,\mathrm{e}^{-\sqrt{p^2+p+1}\,l}}{\sqrt{p^2+p+1}\,p\,(p^2+4)};$$

and

$$R(l,\, p) = -\frac{\left(\sqrt{p^2+p+1}\,l^2 - \sqrt{p^2+p+1}\,l - 3\,l + 2\right) l}{\sqrt{p^2+p+1}\,p\,(p^2+4)}.$$

Now, the determinant of the system $\mathscr{S}$ specified in equation (38) is reduced to:

$$\det(\mathscr{S}) = -\frac{1}{4}\,\frac{\left(\mathrm{e}^{-l\sqrt{p^2+p+1}}\right)^2 - 1}{p^2+p+1};$$

and the unique exact solution of the time domain problem (50)-(52) reads in the Laplace domain (see equation (25)):

$$U(x,\, p) = -2\,\frac{x^2\,(x-1)}{p\,(p^2+4)} + 2\,\frac{l^2\,(l-1)\sinh\left(x\sqrt{p^2+p+1}\right)}{p\,(p^2+4)\sinh\left(l\sqrt{p^2+p+1}\right)} \tag{53}$$

Note that the solution reported in [21], namely $u(x,t) = x^2\,(1-x)\,(\sin(t))^2$, valid only if $l = 1$, is the inverse Laplace transform of the first part of solution (53) since:

$$\mathscr{L}^{-1}\left(-2\,\frac{x^2\,(x-1)}{p\,(p^2+4)}\right) = x^2\,(1-x)\,(\sin(t))^2,$$

where $\mathscr{L}^{-1}$ denotes the inverse Laplace transform operator. It can be checked that all equations of the analog problem (33)-(35) are satisfied by formula (53), proving thus the exactness of the operational solution.

Now, if we let $l = 1$ and change the source term of equation (50) as

$$f(x,\, t) = \mathrm{e}^{-t}\left(2t^2 + \left(x^2 - x\right)\left(-t^2 + 2t - 2\right)\right),$$

while keeping the homogeneous initial and boundary conditions (51)-(52), similar calculus as above lead to the operational solution:

$$U(x,\, p) = -\frac{(-2+2x)\,x}{p^3+3\,p^2+3\,p+1} + 2\,\frac{\mathrm{e}^{-x\sqrt{p^2+p+1}}}{\sqrt{p^2+p+1}\,(p^3+3\,p^2+3\,p+1)\sinh\left(\sqrt{p^2+p+1}\right)}. \tag{54}$$

Again, it can be checked that all equations of the p-domain analog problem (33)-(35) are satisfied. The solution (54) corresponds to a particular case ($b = k = 1$) of Example 3 reported in [21], with the parameterized telegraph equation:

$$\begin{aligned} &u_{tt}\,(x,\, t) + ku_t\,(x,\, t) + bu\,(x,\, t) - u_{xx}\,(x,\, t) \\ &= \mathrm{e}^{-t} \times \left(2t^2 + \left(x^2 - x\right)\left(-2 - 2t\,(k-2) - t^2\,(b-k+1)\right)\right). \end{aligned} \tag{55}$$

Though the solution reported in [21], namely $u(x,\, t) = t^2\left(-x^2+x\right)\mathrm{e}^{-t}$, satisfies equation (55) together with the initial and boundary conditions (51)-(52) when $l = k = b = 1$, that solution still be the inverse Laplace transform of only a part of the complete p-domain solution (54). Indeed, one has:

$$\mathscr{L}^{-1}\left(-\frac{(-2+2x)\,x}{p^3+3\,p^2+3\,p+1}\right) = \mathscr{L}^{-1}\left(-\frac{(-2+2x)\,x}{(p+1)^3}\right) = t^2\left(-x^2+x\right)\mathrm{e}^{-t}.$$

The second fraction of the solution (54) has no evident analytic inverse, but as already mentioned, the numerical inversion into the time domain can always be performed.

Finally, the first Example reported in [21] consists of the following parameterized and non-homogeneous telegraph equation:

$$\begin{aligned} &u_{tt}\,(x,\, t) + k\,u_t\,(x,\, t) + b\,u\,(x,\, t) - u_{xx}\,(x,\, t) = b\,\cos(t)\sin(x) - k\,\sin(t)\sin(x), \\ &0 < x < l, \quad 0 < t < T, \end{aligned} \tag{56}$$

subject to the initial and boundary conditions:

$$u\,(x,\, 0) = \sin(x), \quad u_t\,(x,\, 0) = 0, \quad 0 \le x \le l, \tag{57}$$

and

$$u\,(0,\, t) = 0, \quad u\,(l,\, t) = \sin(l)\cos(t), \quad 0 \le t \le T. \tag{58}$$

Following the same steps as described in Section 4.1, the calculations of the unique operational solution yield:

$$U(x, p) = \frac{\sin(x)\, p}{p^2 + 1},$$

which corresponds well to the exact time domain solution: $u(x, t) = \sin(x)\cos(t)$ as reported in [21].

4.5 *Infinite series solution on a finite interval*

We return to the boundary value problem of the wave equation, but now on a finite interval, by considering at first the Example 5.4 reported in [16], i.e, the homogeneous wave equation:

$$u_{tt}(x, t) - a^2 u_{xx}(x, t) = 0, \quad 0 < x < l, \quad t > 0, \tag{59}$$

subject to the initial conditions:

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = 0, \quad 0 \le x \le l, \tag{60}$$

and to the homogeneous Dirichlet boundary conditions:

$$u(0, t) = 0, \quad u(l, t) = 0, \quad t \ge 0, \tag{61}$$

where:

$$\varphi(x) = \begin{cases} \dfrac{A}{x_0} x & \text{for} \quad 0 \le x \le x_0, \\[2ex] \dfrac{A(l - x)}{l - x_0} & \text{for} \quad x_0 < x \le l. \end{cases} \tag{62}$$

The equations (59)-(62) model the displacement from equilibrium of a uniform string of length l, fixed at its end points by shifting the point $x = x_0$ by distance A at time $t = 0$, and then releasing it with zero initial speed. All external forces are neglected. The function φ related to the initial conditions is not differentiable in the point $x = x_0$, and the generalized series solution in the time domain is reported as

$$u(x, t) = \frac{2Al^2}{\pi^2 x_0 (l - x_0)} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{n\pi x_0}{l} \sin\frac{n\pi x}{l} \cos\frac{n\pi a t}{l}.$$

Values are assigned to the parameters of the problem as follow: $l_1 = 0$, $l_2 = l$, $k = 0$, $b = 0$, $f(x, t) = 0$, $\psi(x) = 0$, $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = 0$, $g_1(t) = g_2(t) = 0$. The analogue solution (25) in the Laplace domain can be explicitly calculated, and for some specific values of the parameters, i.e., $a = 1$, $l = 100$, $x_0 = 25$, $A = 6$, it reads:

$$U(x,\, p) = -\frac{\left(3\,\mathrm{e}^{-150\,p} - 4\,\mathrm{e}^{-125\,p} + 3\,\mathrm{e}^{-100\,p} - 4\,\mathrm{e}^{-75\,p} + 3\,\mathrm{e}^{-50\,p} - 4\,\mathrm{e}^{-25\,p} + 3\right)\mathrm{e}^{-xp}}{25\,(\mathrm{e}^{-150\,p} + \mathrm{e}^{-100\,p} + \mathrm{e}^{-50\,p} + 1)\,p^2}$$
$$-\frac{\left(\mathrm{e}^{-200\,p} - 4\,\mathrm{e}^{-125\,p} + 4\,\mathrm{e}^{-75\,p} - 1\right)\mathrm{e}^{-(-x+100)p}}{25\,p^2\,(\mathrm{e}^{-200\,p} - 1)} + R(x,\, p); \tag{63}$$

where

$$R(x,\, p) = \begin{cases} \dfrac{1}{25}\,\dfrac{6xp + 3\,\mathrm{e}^{-xp} - 4\,\mathrm{e}^{p(x-25)} + \mathrm{e}^{(-100+x)p}}{p^2} & \text{for} \quad 0 \le x \le 25, \\ \\ \dfrac{1}{25}\,\dfrac{-2xp + 3\,\mathrm{e}^{-xp} - 4\,\mathrm{e}^{-p(x-25)} + 200\,p + \mathrm{e}^{(-100+x)p}}{p^2} & \text{for} \quad 25 < x \le 100. \end{cases}$$

For the values specified above, the function φ is rewritten as:

$$\varphi(x) = \begin{cases} \dfrac{6}{25}x & \text{for} \quad 0 \le x \le 25, \\ \\ \dfrac{-2}{25}x + 8 & \text{for} \quad 25 < x \le 100; \end{cases}$$

while the time-domain series solution takes the form:

$$u(x,\, t) = \frac{64}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{n\pi}{4} \sin\frac{n\pi x}{100} \cos\frac{n\pi t}{100}. \tag{64}$$

Figure 1 illustrates the solution to the problem (59)-(62) at different times t for the specific values assigned above. The two extremities of the uniform string (illustrated in the graphs as points of abscissa 0 and 100), are constrained under the conditions (61). The series solution provided by the Fourier decomposition method is truncated at the second, third, and fifth terms, respectively ($n = 2,\ 3,\ 5$). The inverse Laplace curves were generated by numerical inversion of the formula (63), using the invlap.m function. The invlap.m function is a robust, available, and ready-to-use tool implemented in Matlab by Hollenbeck in 1998. It has been shown to be highly precise and to extend the applicability of the Laplace transform technique (further details are given in [22]). If the energy doesn't decrease, the simulation is run until the time is the same as the main harmonic period. In this example, that's when $2l/a = 200$. Thus, T is chosen equal to 200 and the curves were obtained through the application of a mesh of equal space and time steps $\Delta x = \Delta t = 1$. The curves at $t = 0$ (see Figure 1a) provide an excellent illustration of the precision of the inverse Laplace curve, which aligns closely with the initial condition, as expressed by the relation $u(x,\, 0) = \phi(x)$. Unless they collapse together as in one line (see Figures 1b and 1d), it can be observed that as the order of truncation of the series solution increases, the resulting curves approach that of the inverse Laplace. This is evident in Figures 1a and 1c. Furthermore, the amount of Central Processing Unit (CPU) time required for computing the series solution is considerable, even at the lowest truncation order of $n = 2$, in comparison to that of the inverse Laplace solution, which is 35 times faster at the first execution of the script. This ratio increases significantly with an increase in the truncation order, n.

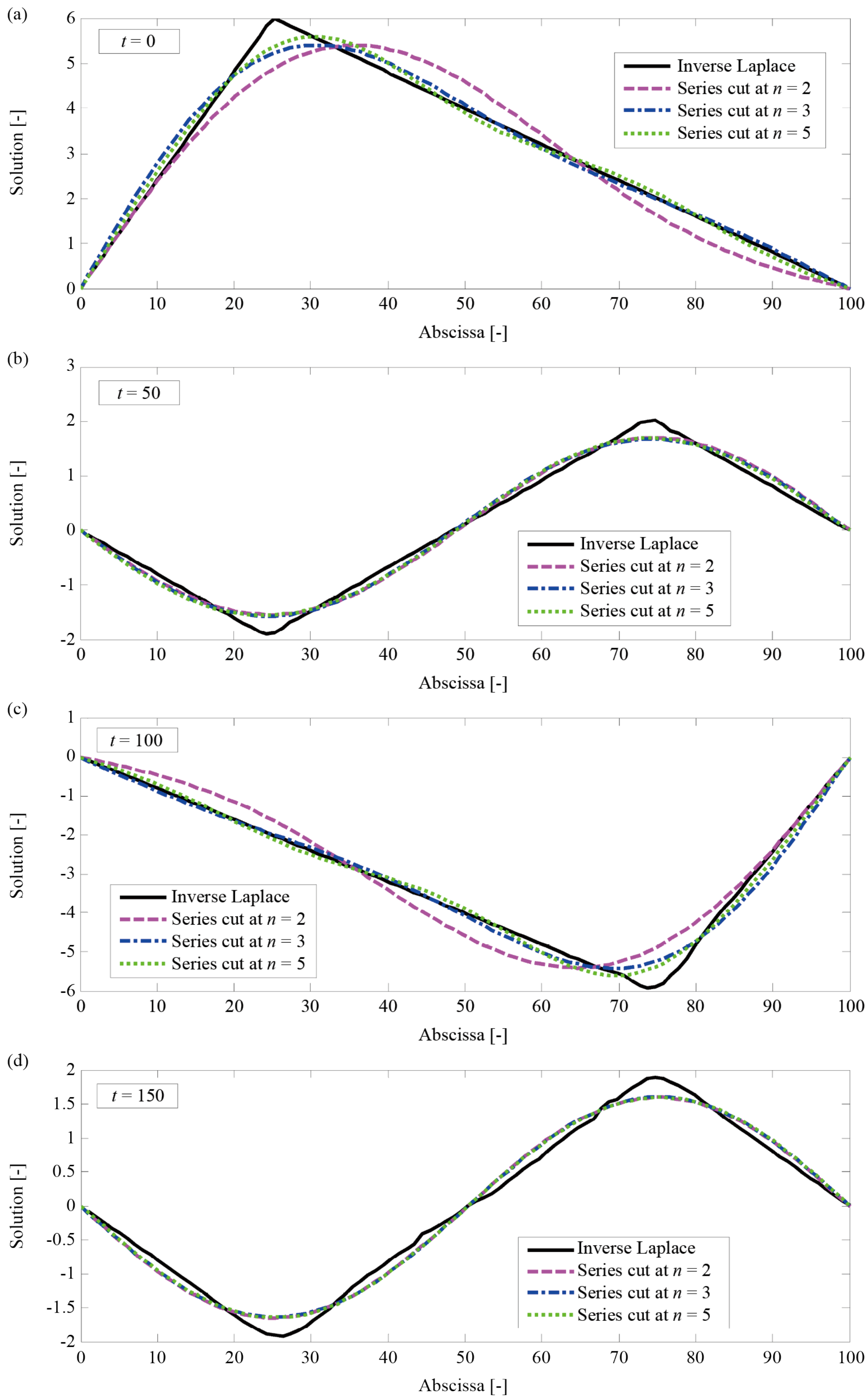

Figure 1. Solutions $u(x, t)$ at different times t in equations (63) and (64)

We proceed now to examine Example 5.7, as presented in [16]. The upper extremity of an elastic, homogeneous, heavy rod of length l is rigidly fixed to the ceiling of a freely falling elevator, which comes to a complete stop upon reaching a velocity of v_0. The boundary value problem for vibrations of the rod is governed by the equation:

$$u_{tt}(x, t) - a^2 u_{xx}(x, t) = -g, \quad 0 < x < l, \quad 0 < t < T, \tag{65}$$

subject to the initial conditions:

$$u(x, 0) = 0, \quad u_t(x, 0) = v_0, \quad 0 \le x \le l, \tag{66}$$

and to the boundary conditions of Dirichlet type at $x = 0$ (the fixed end) and of Neumann type at $x = l$ (the free end):

$$u(0, t) = 0, \quad u_x(l, t) = 0, \quad 0 \le t \le T, \tag{67}$$

where g denotes the constant acceleration due to the gravity, and $v_0 > 0$ the maximal speed to be reached by the free falling elevator. The exact series solution, as obtained by the Fourier decomposition method, is reported as:

$$\begin{aligned} u(x, t) = {} & \frac{8l}{a\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \\ & \times \left\{ \frac{-2gl}{a\pi(2n-1)^2} \left[1 - \cos\frac{(2n-1)a\pi t}{2l} \right] + v_0 \sin\frac{(2n-1)a\pi t}{2l} \right\} \sin\frac{(2n-1)a\pi x}{2l}. \end{aligned} \tag{68}$$

Assigning the corresponding values to the parameters in the initial problem (1)-(4), that is, $l_1 = 0$, $l_2 = l$, $k = 0$, $b = 0$, $f(x, t) = -g$, $\varphi(x) = 0$, $\psi(x) = v_0$, $\alpha_1 = \beta_2 = 1$, $\beta_1 = \alpha_2 = 0$, $g_1(t) = g_2(t) = 0$, the p-domain solution (25) is calculated as:

$$U(x, p) = \frac{-\mathrm{e}^{-\frac{xp}{a}} p v_0 + \mathrm{e}^{-2\frac{lp}{a}} p v_0 - \mathrm{e}^{-\frac{p(2l-x)}{a}} p v_0 + \mathrm{e}^{-\frac{xp}{a}} g - \mathrm{e}^{-2\frac{lp}{a}} g + \mathrm{e}^{-\frac{p(2l-x)}{a}} g + v_0 p - g}{p^3 \left(\mathrm{e}^{-2\frac{lp}{a}} + 1 \right)}. \tag{69}$$

Once again, it can be checked that $U(x, p)$, as expressed in equation (69), satisfies the analog system (33)-(35), and that the initial conditions (66) are also satisfied, since $pU(x, p) \to 0$ and $p^2 U(x, p) \to v_0$ when $p \to \infty$. Now, for the sample case when $g = 10$, $l = 1$, $v_0 = 5$, and $a = 1$, the solution (69) takes the form:

$$U(x, p) = -5 \frac{\mathrm{e}^{-xp} p + \mathrm{e}^{p(-2+x)} p - \mathrm{e}^{-2p} p - 2\mathrm{e}^{-xp} - 2\mathrm{e}^{p(-2+x)} + 2\mathrm{e}^{-2p} - p + 2}{p^3 (\mathrm{e}^{-2p} + 1)}; \tag{70}$$

while the reported series solution (62) reduces to:

$$u(x, t) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \left\{ \frac{-20}{\pi(2n-1)^2} \left[1 - \cos\frac{(2n-1)\pi t}{2} \right] + 5\sin\frac{(2n-1)\pi t}{2} \right\} \times \sin\frac{(2n-1)\pi x}{2}. \tag{71}$$

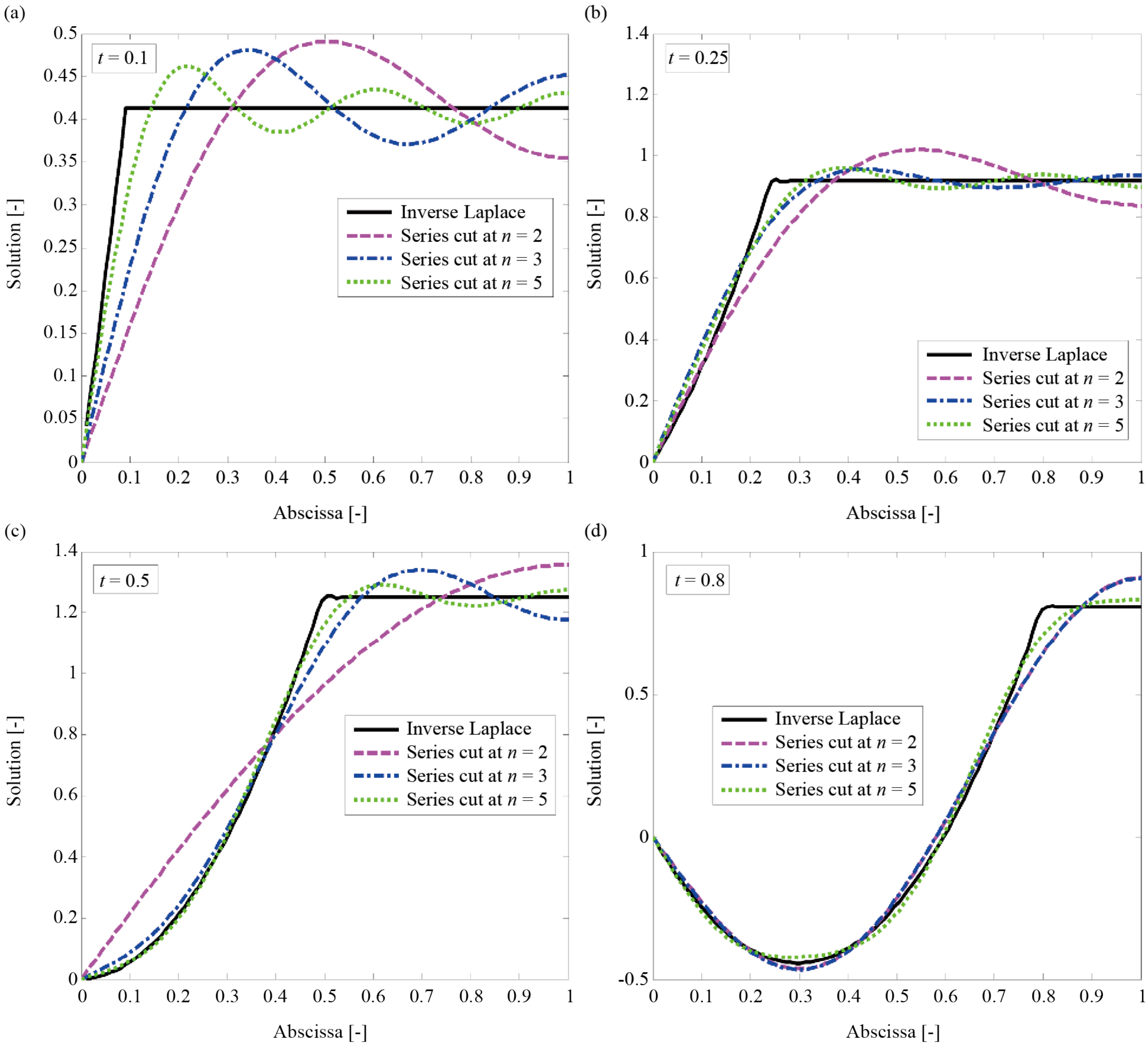

Figure 2. Solutions $u(x, t)$ at different times t in equations (70) and (71)

The curves on Figure 2 are obtained by choosing a mesh of space step $\Delta x = 0.01$ equal to that of time step $\Delta t = 0.01$. Figures 2a to 2d illustrate the position of the rod at successive time points, with the time interval increasing. From the moment the elevator is interrupted, the vibration propagates from the upper end of the heavy rod (fixed point with abcissa 0 on the graphs) to its lower extremity (variable point with abcissa 1). This evolution cannot be properly accounted for by truncated series solution curves, in contrast to the inverse Laplace curve. Nevertheless, as illustrated in Figure 1, the series solution curves appear to approach the inverse Laplace curve as the order of truncation, denoted by n, increases. Once more, the CPU time required by the inverse Laplace procedure is significantly less than that needed for the series solutions. In summary, the inverse Laplace transform of the closed-form operational solution (25) is notable not only for its precision and consistency, but also for its computational efficiency.

5. Conclusion

This paper has studied the most general form of the one-dimensional linear hyperbolic equation with constant coefficients. We have presented a solution in the Laplace domain that is both unified and exact. The parameterized form of the problem has been considered by specifying the telegraph equation on a generic bounded interval with the Neumann, Dirichlet, and Robin boundary conditions formulated in a unified manner. Employing suitable integral transforms, a time-domain representation of the solution in the form of integrals was proved to be stable when small changes are made to the input data. Then, the Laplace domain solution was derived in a closed analytic form, which can also be extended to unbounded intervals. Furthermore, the analytic inverse in the time domain can be recovered by means of Laplace inversion theorems. In all cases, highly efficient algorithms are available, and inverse Laplace transforms can be performed numerically into the time domain, regardless of the complexity of the operational solution. The unified solution offers a novel alternative to infinite series solutions in the time domain, which cannot always be expressed in closed forms. The efficacy of this approach was demonstrated through several illustrative examples. The potential applications of this approach are particularly pertinent in the fields of applied mathematics and engineering, especially with regard to enhancing the numerical efficiency of simulations in the study of hyperbolic equations for engineering calculations. In future work, the present solution-finding approach will be used to analyze one or multidimensional wave and telegraph equations with variable coefficients.

Acknowledgement

This research is dedicated in memory of Emeritus Professor Victor Kofi Seyelom Assiamoua (V.S.K. Assiamoua), whose mentorship and intellectual impact were of the utmost value. His mastery of teaching mathematics and his ability to explain new concepts in an easy-to-understand manner inspired us all to pursue our respective fields of research. I am Dr. Kwassi Anani, and I had the great honor of being one of his first doctoral students in the Department of Mathematics. I would like to express my sincere gratitude.

Conflict of interest

The authors declare no competing financial interest.

References

[1] Khankhasaev VN, Darmakheev EV. On certain applications of the hyperbolic heat transfer equation and methods for its solution. *Journal of Mathematical Sciences*. 2021; 254(5): 677-685. Available from: https://doi.org/10.1007/s10958-021-05332-3.

[2] Ahmad H, Seadawy AR, Khan TA. Study on numerical solution of dispersive water wave phenomena by using a reliable modification of variational iteration algorithm. *Mathematics and Computers in Simulation*. 2020; 177: 13-23. Available from: https://doi.org/10.1016/j.matcom.2020.04.005.

[3] Brio M, Webb GM, Zakharian AR. *Numerical Time-Dependent Partial Differential Equations for Scientists and Engineers*. Cambridge: Academic Press; 2010.

[4] Banasiak J, Mika JR. Singularly perturbed telegraph equations with applications in the random walk theory. *Journal of Applied Mathematics and Stochastic Analysis*. 1998; 11(1): 9-28. Available from: https://doi.org/10.1155/S1048953398000021.

[5] Roussy G, Pearce JA. *Foundations and Industrial Applications of Microwave and Radio Frequency Fields: Physical and Chemical Processes*. New York: Wiley and Sons; 1995.

[6] Mohanty RK. An unconditionally stable difference scheme for the one-space-dimensional linear hyperbolic equation. *Applied Mathematics Letters*. 2004; 17(1): 101-105. Available from: https://doi.org/10.1016/S0893-9659(04)90019-5.

[7] Abd-Elhameed WM, Doha EH, Youssri YH, Bassuony MA. New Tchebyshev-Galerkin operational matrix method for solving linear and nonlinear hyperbolic telegraph type equations. *Numerical Methods for Partial Differential Equations*. 2016; 32(6): 1553-1571. Available from: https://doi.org/10.1002/num.22074.

[8] Köksal ME. Recent developments of numerical methods for analyzing telegraph equations. *Archives of Computational Methods in Engineering*. 2023; 30(6): 3509-3527. Available from: https://doi.org/10.1007/s11831-023-09909-w.

[9] Singh BK, Shukla JP, Gupta M. Study of one dimensional hyperbolic telegraph equation via a hybrid cubic *B*-spline differential quadrature method. *International Journal of Applied and Computational Mathematics*. 2021; 7(1): 1-17. Available from: https://doi.org/10.1007/s40819-020-00939-7.

[10] Lin J, He Y, Reutskiy SY, Lu J. An effective semi-analytical method for solving telegraph equation with variable coefficients. *The European Physical Journal Plus*. 2018; 133: 1-11. Available from: https://doi.org/10.1140/epjp/i2018-12104-1.

[11] Sayed AY, Abdelgaber KM, Elmahdy AR, El-Kalla IL. Solution of the telegraph equation using adomian decomposition method with accelerated formula of adomian polynomials. *Information Sciences Letters*. 2021; 10(1): 39-46.

[12] Jena SR, Sahu I. A reliable method for voltage of telegraph equation in one and two space variables in electrical transmission: approximate and analytical approach. *Physics Written*. 2023; 98(10): 105216. Available from: https://doi.org/10.1088/1402-4896/acf538.

[13] Sari M, Gunay A, Gurarslan G. A solution to the telegraph equation by using DGJ method. *International Journal of Applied Nonlinear Science*. 2014; 17(1): 57-66.

[14] Anani K. Analytical approximations in short times of exact operational solutions to reaction-diffusion problems on bounded intervals. *Applications and Applied Mathematics*. 2023; 19(1): R2064.

[15] Poularikas AD. *Laplace Transforms, the Handbook of Formulas and Tables for Signal Processing*. Boca Raton: CRC Press; 2018.

[16] Henner V, Belozerov T, Nepomnyashchy A. One-dimensional hyperbolic equations. In: Boggess A, Rosen K. (eds.) *Partial Differential Equations: Analytical Methods and Applications*. Boca Raton: CRC Press; 2019. p.43-98.

[17] Marin M, Öhsner A. Hyperbolic equations. In: *Essentials of Partial Differential Equations*. Switzerland: Springer; 2018. p.201-224.

[18] Cioranescu D, Donato P, Roque MP. Hyperbolic equations. In: *An Introduction to Second Order Partial Differential Equations: Classical and Variational Solutions*. Singapore: World Scientific Publishing; 2018. p.91-112.

[19] Herron IH, Foster MR. Laplace transform methods. In: *Partial Differential Equations in Fluiddynamics*. Cambridge: Cambridge University Press; 2008. p.148-182.

[20] Debnath L, Bhatta D. Laplace transforms and their basic properties. In: *Integral Transforms and Their Applications*. Boca Raton: CRC press; 2014. p.143-196.

[21] Atta AG, Abd-Elhameed WM, Moatimid GM, Youssri YH. Advanced shifted sixth-kind Chebyshev tau approach for solving linear one-dimensional hyperbolic telegraph type problem. *Mathematical Sciences*. 2023; 17(4): 415-429. Available from: https://doi.org/10.1007/s40096-022-00460-6.

[22] De Hoog FR, Knight JH, Stokes AN. An improved method for numerical inversion of Laplace transforms. *SIAM Journal on Scientific and Statistical Computing*. 1982; 3(3): 357-366. Available from: https://doi.org/10.1137/0903022.