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Research Article

A Study of Higher Order Recurrence Relations: Symmetries, Periodicity and Stability Analysis

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Abstract: This study investigates higher-order recurrence relations by examining the behavior of their solutions through Lie symmetry analysis, periodicity, and stability. Using Lie symmetry techniques, the work in this paper uncovers invariant transformations and structural properties of the equations. The analysis also identifies conditions under which solutions display periodic behavior and evaluates the stability of equilibrium points. To support the theoretical results, graphical representations of the solutions are presented, illustrating the predicted dynamics. These findings offer valuable insights into the qualitative behavior of complex difference equations and considerably extend some existing findings in the literature.

Keywords: difference equation, symmetry, reduction, group invariant solutions

MSC: 39A10, 39A99, 39A13

1. Introduction

The study of difference equations and the available methods for solving them has received considerable attention over the years. One of the effective approaches for solving these equations is through Lie symmetry analysis, which involves finding an invariant that reduces the order of the equation. Various notable researchers like Maeda [1] and Hydon [2] have significantly advanced the use of Lie symmetry analysis in this area of focus. Difference equations are commonly applied in various real-world scenarios, such as disease modeling, loan amortization, meteorology, and image processing.

Recurrence equations of a general order have been thoroughly investigated by scholars like Gümüs and Abo-Zeid [3], who have explored the exact solutions, as well as the stability, oscillatory behavior and periodic properties of

$$x_{n+1} = \frac{ax_{n-2k-1}}{b - c\prod_{l=0}^{k} x_{n-2l-1}},\ n = 0,\ 1,\ 2,\ \ldots, \tag{1}$$

where a, b, c are non-negative real numbers. The initial conditions x_{-2k-1}, x_{-2k}, $\ldots$, x_{-1}, x_0 are real numbers, and k, n are non-negative integers. They found periodic solutions and determined the conditions under which solutions are stable.

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This work is motivated by the findings in [3]. In their work, they presented exact solutions of a class of higher-order difference equations (1), and it serves as a foundation for the current study, which aims to build upon and extend their results using alternative methods and perspectives. Equation (1) can be generalized as

$$x_{n+1} = \frac{x_{n-2k-1}}{\tilde{A}_n + \tilde{B}_n \prod_{l=0}^{k} x_{n-2l-1}}, \tag{2}$$

where $\tilde{A}_n$ and $\tilde{B}_n$ are arbitrary real sequences, provided that the denominator is not zero. For related work, please see [4–11]. Much work still needs to be done to determine exact formulas for solutions of general difference equations. Lie symmetry analysis, which has been widely applied to differential equations, has recently been utilized in the study of difference equations and fractional differential equations to gain insights into their structures and solutions [12–16]. For solutions of differential and fractional differential equations from different approaches, please see [17, 18].

This work provides a comprehensive analysis of higher-order recurrence relations through the lens of Lie symmetry methods. A key novelty lies in the application of continuous symmetry techniques to (2), offering a unified framework for analyzing the dynamic behavior of these higher-order difference equations. The main contributions include the identification of symmetries that ease the reduction and solution of (2), an exploration of the periodic nature of their solutions, and a detailed stability analysis of equilibrium points. The advantage of this Lie symmetry method is that it provides a convenient change of variables without the need for guesswork.

For definiteness, we study the equivalent recurrence relation given by

$$x_{n+2k+2} = \frac{x_n}{A_n + B_n \prod_{l=0}^{k} x_{n+2l}}, \ n = 0, \ 1, \ 2, \ \ldots, \tag{3}$$

for some arbitrary convenient real sequences $A_n = \tilde{A}_{n+2k+1}$ and $B_n = \tilde{B}_{n+2k+1}$ where $x_0, \ x_1, \ \ldots, \ x_{2k+1}$ are initial values and k is a non-negative integer.

The paper is organized as follows. Section 2 presents the foundational theory required for determining symmetries of difference equations and performing order reduction. It also includes a brief review of classical results concerning the stability of equilibrium points. In Section 3, we derive symmetries and solutions of equation (3), with a more in-depth analysis provided for certain special cases. Section 4 focuses primarily on the stability of equilibrium points and the periodic behavior of the solutions of (3). Lastly, in Section 5, we demonstrate that several known results from the literature can be recovered as particular cases of our work.

2. Preliminaries

Most of the definitions used in this paper are drawn from [2, 19], and most of the notation follows the conventions they adopted.

Definition 1 Let G be a local group of transformations acting on a manifold M. A subset $\mathscr{S} \subset M$ is called G-invariant, and G is called symmetry group of $\mathscr{S}$, if whenever $x \in \mathscr{S}$, and $g \in G$ is such that $g \cdot x$ is defined, then $g \cdot x \in \mathscr{S}$ [20].

Definition 2 Let G be a connected group of transformations acting on a manifold M [20]. A smooth real-valued function $\mathscr{V} : M \to \mathbb{R}$ is an invariant function for G if and only if

$$X(\mathscr{V}) = 0$$

for all $x \in M$, and every infinitesimal generator X of G [2].

Definition 3 A parameterized set of point transformations,

$$\Gamma_\varepsilon : x \mapsto \hat{x}(x;\ \varepsilon), \tag{4}$$

where $x = x_i,\ i = 1,\ \ldots,\ p$ are continuous variables, is a one-parameter local Lie group of transformations if the following conditions are satisfied:

(1) Γ_0 is the identity map if $\hat{x} = x$ when $\varepsilon = 0$.

(2) $\Gamma_a \Gamma_b = \Gamma_{a+b}$ for every a and b sufficiently close to 0.

(3) Each $\hat{x}_i$ can be represented as a Taylor series (in a neighborhood of $\varepsilon = 0$ that is determined by x), and therefore

$$\hat{x}_i(x : \varepsilon) = x_i + \varepsilon \xi_i(x) + O(\varepsilon^2),\ i = 1,\ \ldots,\ p. \tag{5}$$

Let us consider the following difference equation:

$$x_{n+2k+2} = G(n,\ x_n,\ x_{n+2},\ x_{n+6},\ \ldots,\ x_{n+2k+1}), \tag{6}$$

where G is a smooth function such that its partial derivation of G with respect to x_n is not zero. Symmetry groups are useful in determining infinitesimal transformations. Suppose

$$\hat{x}_n = x_n + \varepsilon \xi(n, x_n) + O(\varepsilon^2), \tag{7}$$

represents a one-parameter Lie group of transformations for equation (6), with the associated generator given by

$$V = \xi(n,\ x_n)\frac{\partial}{\partial x_n}. \tag{8}$$

It is crucial to recognize that finding the characteristic $\xi = \xi(n,\ x_n)$ requires knowledge of the $(2k)$th prolongation of V:

$$V^{[2k]} = \xi\frac{\partial}{\partial x_n} + (S^2\xi)\frac{\partial}{\partial x_{n+2}} + \cdots + (S^{2k}\xi)\frac{\partial}{\partial x_{n+2k}}, \tag{9}$$

where $S^i : n \to n+i$ is the forward shift operator. The expression in (7) forms a symmetry group if and only if the following condition is satisfied:

$$S^{2k+2}\xi(n,\ x_n) - V^{[2k]}(G) = 0\Big|_{x_{n+2k+2}=G(n,\ x_n,\ x_{n+2},\ \ldots,\ x_{n+2k})}. \tag{10}$$

The characteristic function is obtained by solving the functional equation (10) and using the canonical coordinate [21]

$$c_n = \int \frac{dx_n}{\xi(n,\, x_n)}, \tag{11}$$

which helps derive the invariants that can reduce the order of the difference equations. As shown in [21], selecting the canonical coordinate (11) guarantees that the recurrence equation can be transformed into the form $c_{n+1} - c_n = d_n$, with the solution expressed as

$$c_n = \sum_{k=n_0}^{n} d_k + w_1, \tag{12}$$

where w_1 is a constant. Based on (12), it is straightforward to find the solution in terms of the original variables. In this paper, we derive the solution using the canonical coordinate through a different approach.

The following definitions and theorem from [19] are useful for analyzing the local and global stability properties of the equilibrium point.

Theorem 4 The equilibrium point $\bar{x}$ of equation (6) is considered locally stable if for any $\varepsilon > 0$ [19], there exists $\delta > 0$ such that

$$|x_0 - \bar{x}| + |x_1 - \bar{x}| + \cdots + |x_{4k-2} - \bar{x}| + |x_{2k+1} - \bar{x}| < \delta, \tag{13}$$

implies

$$|x_n - \bar{x}| < \varepsilon \text{ for all } n \geq 0. \tag{14}$$

Theorem 5 The equilibrium point $\bar{x}$ of equation (6) is considered a global attractor if for any solution $\{x_n\}_{n=0}^{\infty}$ of (6) [19],

$$\lim_{n \to \infty} x_n = \bar{x}.$$

Theorem 6 The equilibrium point $\bar{x}$ of equation (6) is globally asymptotically stable if $\bar{x}$ is both locally stable and a global attractor of (6) [19]. Let

$$p_i = \frac{\partial G}{\partial x_{n+i}}(\bar{x},\, \ldots,\, \bar{x}), \;\; \text{for } i = 2r,\; r = 0,\; 1,\; \ldots,\; k.$$

Then, the characteristic equation corresponding to equation (6) near the equilibrium point $\bar{x}$ is

$$\lambda^{2k+2} - p_{2k}\lambda^{2k} - \cdots - p_2\lambda^2 - p_0 = 0. \tag{15}$$

Theorem 7 Suppose f is a smooth function defined in an open neighborhood around $\bar{x}$ [19]. The following statements hold:

(1) The equilibrium point $\bar{x}$ is locally asymptotically stable if all the roots of the characteristic equation (15) have absolute values less than one.

(2) The equilibrium point $\bar{x}$ is unstable if at least one root of the characteristic equation has an absolute value greater than one.

Theorem 8 The equilibrium point $\bar{x}$ of equation (6) is referred to as non-hyperbolic if there exists a root of equation (15) with an absolute value equal to one [19].

3. Main results

To determine the characteristic function ξ associated with equation (3), we apply the symmetry condition from equation (10) to equation (3), leading to

$$\xi(n+2k,\, x_{n+2k}) - \sum_{l=0}^{k} \frac{\partial G}{\partial x_{n+2l}} \xi(n+2l,\, x_{n+2l}) = 0, \tag{16}$$

where G is the right hand side expression in (3). We now apply the differential operator $\dfrac{\partial}{\partial x_{n+2}} - \dfrac{x_{n+4}}{x_{n+2}} \dfrac{\partial}{\partial x_{n+4}}$ to equation (16). This results in:

$$\begin{aligned} &-\frac{\partial G}{\partial x_{n+2}} \xi'(n+2,\, x_{n+2}) - \sum_{l=0}^{k} \frac{\partial^2 G}{\partial x_{n+2} \partial x_{n+2l}} \xi(n+2l,\, x_{n+2l}) \\ &- \frac{x_{n+4}}{x_{n+2}} \left[\sum_{l=0}^{k} \frac{\partial^2 G}{\partial x_{n+4} \partial x_{n+2l}} \xi(n+2l,\, x_{n+2l}) + \frac{\partial G}{\partial x_{n+4}} \xi'(n+4,\, x_{n+4}) \right] = 0, \end{aligned} \tag{17}$$

that is,

$$\begin{aligned} &-\frac{\partial G}{\partial x_{n+2}} \xi'(n+2,\, x_{n+2}) + \frac{x_{n+4}}{x_{n+2}} \frac{\partial G}{\partial x_{n+4}} \xi'(n+4,\, x_{n+4}) + \left[-\frac{\partial^2 G}{\partial x_n x_{n+2}} + \frac{x_{n+4}}{x_{n+2}} \frac{\partial^2 G}{\partial x_n x_{n+4}} \right] \xi(n,\, x_n) \\ &+ \left[-\frac{\partial^2 G}{\partial x_{n+2}^2} + \frac{x_{n+4}}{x_{n+2}} \frac{\partial^2 G}{\partial x_{n+2} x_{n+4}} \right] \xi(n+2,\, x_{n+2}) + \left[-\frac{\partial^2 G}{\partial x_{n+4} x_{n+2}} + \frac{x_{n+4}}{x_{n+2}} \frac{\partial^2 G}{\partial x_{n+4}^2} \right] \xi(n+4,\, x_{n+4}) \\ &+ \sum_{l \geq 3} \left(-\frac{\partial^2 G}{\partial x_{n+2} x_{n+2l}} + \frac{x_{n+4}}{x_{n+2}} \frac{\partial^2 G}{\partial x_{n+2l} x_{n+4}} \right) \xi(n+2l,\, x_{n+2l}) = 0. \end{aligned}$$

This can be simplified to:

$$x_{n+4} \xi'(n+2,\, x_{n+2}) - x_{n+4} \xi'(n+4,\, x_{n+4}) - \frac{x_{n+4}}{x_{n+2}} \left(\xi(n+2,\, x_{n+2}) + \xi(n+4,\, x_{n+4}) \right) = 0. \tag{18}$$

After a series of lengthy computations, we found the characteristic functions to be $Q(n,\ x_n) = \alpha_n x_n$, where α_n satisfy

$$\alpha_n + \alpha_{n+2} + \alpha_{n+4} + \cdots + \alpha_{n+2k-2} + \alpha_{n+2k} = 0, \tag{19}$$

that is,

$$Q(n,\ x_n) = \alpha_n x_n = \pm e^{\frac{m\pi}{k+1}i} x_n, \tag{20}$$

for $m = 1,\ \ldots,\ k$; $k \geq 1$ and $i^2 = -1$. If follows from (8) and (20) that the infinitesimal generators are

$$X_1(m) = e^{i\frac{nm\pi}{k+1}} x_n \frac{\partial}{\partial x_n},\ X_2(m) = (-1)^n e^{i\frac{nm\pi}{k+1}} \frac{\partial}{\partial x_n}, \tag{21}$$

$1 \leq m \leq k$. To linearize equation (3), we use the canonical coordinate given by

$$S_n = \int \frac{dx_n}{\alpha_n x_n} = \frac{1}{\alpha_n} \ln|x_n|, \tag{22}$$

where α_n satisfies equation (19). For convenience, we use the invariant

$$\begin{aligned} I_n =& \alpha_n S_n + \alpha_{n+2} S_{n+2} + \alpha_{n+4} S_{n+4} + \cdots + \alpha_{n+2k-2} S_{n+2k-2} + \alpha_{n+2k} S_{n+2k} \\ =& \ln|x_n x_{n+2} \ldots x_{n+2k-2}|. \end{aligned} \tag{23}$$

We examine how this invariant I_n behaves under the action of the vector fields $X_1^{[2k]}(m)$ and $X_2^{[2k]}(m)$. Applying $X_1^{[2k]}(m)$ to I_n gives:

$$\begin{aligned} X_1^{[2k]}(m) I_n =& \left(e^{i\frac{nm\pi}{k+1}} x_n \frac{\partial}{\partial x_n} + \cdots + e^{i\frac{(n+2k)m\pi}{k+1}} x_{n+2k} \frac{\partial}{\partial x_{n+2k}} \right) \ln|x_n x_{n+2} \ldots x_{n+2k}| \\ =& e^{i\frac{nm\pi}{k+1}} + \cdots + e^{i\frac{(n+2k)m\pi}{k+1}} \\ =& e^{i\frac{nm\pi}{k+1}} \left(\frac{1 - e^{i2m\pi}}{1 - e^{i\frac{nm\pi}{k+1}}} \right) \\ =& 0, \end{aligned} \tag{24}$$

and similarly for $X_2(m)$. The action of the vector fields $X_1(m)$ and $X_2(m)$ on the invariant I_n results in $X_r(m) I_n = 0$ for $r = 1,\ 2$. This shows that I_n is indeed invariant under the action of the above vector fields.

Letting $|r_n| = \exp\{-\ln|x_n x_{n+2} \dots x_{n+2k}|\}$, the following similarity variable can be obtained:

$$r_n = \frac{1}{x_n x_{n+2} \dots x_{n+2k-2}}. \tag{25}$$

On one hand, shifting equation (25) twice and substituting x_{n+2k+2} into the resulting equation gives

$$r_{n+2} = A_n r_n + B_n \tag{26}$$

which, upon iteration, leads to

$$r_{2n+l} = \left(\prod_{k_1=0}^{n-1} A_{l+2k_1}\right) r_l + \sum_{j=0}^{n-1} \left(\prod_{k_2=j+1}^{n-1} A_{l+2k_2}\right) B_{l+2j}, \tag{27}$$

for $l = 0,\ 1$. On the other hand, using (25), we obtain that

$$x_{n+2k+2} = \frac{r_n}{r_{n+2}} x_n \tag{28}$$

which, upon iteration, leads to

$$\begin{aligned} x_{(2k+2)n+l} &= \left(\prod_{m=0}^{n-1} \frac{r_{(2k+2)m+l}}{r_{(2k+2)m+l+2}}\right) x_l,\ l = 0,\ \dots,\ 2k+1, \\ &= \left(\prod_{m=0}^{n-1} \frac{r_{2[(k+1)m+\lfloor \frac{l}{2} \rfloor]+\tau(l)}}{r_{2[(k+1)m+\lfloor \frac{l}{2} \rfloor+1]+\tau(l)}}\right) x_l, \end{aligned} \tag{29}$$

where $\lfloor \cdot \rfloor$ is the floor function and $\tau(l)$ represents the remainder when l is divided by 2. Using (27) in (29), we have

$$\begin{aligned} x_{(2k+2)n+l} =& x_l \prod_{m=1}^{n-1} \frac{\left(\prod\limits_{k_1=0}^{(k+1)m+\lfloor \frac{l}{2} \rfloor-1} A_{2k_1+\tau(l)}\right) r_{\tau(l)} + \sum\limits_{j=0}^{(k+1)m+\lfloor \frac{l}{2} \rfloor-1} \left(\prod\limits_{k_2=j+1}^{(k+1)m+\lfloor \frac{l}{2} \rfloor-1} A_{2k_2+\tau(l)}\right) B_{2j+\tau(l)}}{\left(\prod\limits_{k_1=0}^{(k+1)m+\lfloor \frac{l}{2} \rfloor} A_{2k_1+\tau(l)}\right) r_{\tau(l)} + \sum\limits_{j=0}^{(k+1)m+\lfloor \frac{l}{2} \rfloor} \left(\prod\limits_{k_2=j+1}^{(k+1)m+\lfloor \frac{l}{2} \rfloor} A_{2k_2+\tau(l)}\right) B_{2j+\tau(l)}} \\ =& x_l \prod_{m=1}^{n-1} \frac{\left(\prod\limits_{k_1=0}^{(k+1)m+\lfloor \frac{l}{2} \rfloor-1} A_{2k_1+\tau(l)}\right) + \sum\limits_{j=0}^{(k+1)m+\lfloor \frac{l}{2} \rfloor-1} \left(\prod\limits_{k_2=j+1}^{(k+1)m+\lfloor \frac{l}{2} \rfloor-1} A_{2k_2+\tau(l)}\right) \frac{B_{2j+\tau(l)}}{r_{\tau(l)}}}{\left(\prod\limits_{k_1=0}^{(k+1)m+\lfloor \frac{l}{2} \rfloor} A_{2k_1+\tau(l)}\right) + \sum\limits_{j=0}^{(k+1)m+\lfloor \frac{l}{2} \rfloor} \left(\prod\limits_{k_2=j+1}^{(k+1)m+\lfloor \frac{l}{2} \rfloor} A_{2k_2+\tau(l)}\right) \frac{B_{2j+\tau(l)}}{r_{\tau(l)}}}. \end{aligned} \tag{30}$$

Equivalently, by back-shifting (30) $2k+1$ times, we obtain the solution of (2) as follows:

$$x_{(2k+2)n+l-2k-1} = x_{l-2k-1} \prod_{m=1}^{n-1} \frac{\left(\prod_{k_1=0}^{(k+1)m+\lfloor \frac{l}{2} \rfloor -1} \tilde{A}_{2k_1+\tau(l)} \right) + \sum_{j=0}^{(k+1)m+\lfloor \frac{l}{2} \rfloor -1} \left(\prod_{k_2=j+1}^{(k+1)m+\lfloor \frac{l}{2} \rfloor -1} \tilde{A}_{2k_2+\tau(l)} \right) \frac{\tilde{B}_{2j+\tau(l)}}{r_{\tau(l)-2k-1}}}{\left(\prod_{k_1=0}^{(k+1)m+\lfloor \frac{l}{2} \rfloor} \tilde{A}_{2k_1+\tau(l)} \right) + \sum_{j=0}^{(k+1)m+\lfloor \frac{l}{2} \rfloor} \left(\prod_{k_2=j+1}^{(k+1)m+\lfloor \frac{l}{2} \rfloor} \tilde{A}_{2k_2+\tau(l)} \right) \frac{\tilde{B}_{2j+\tau(l)}}{r_{\tau(l)-2k-1}}}. \tag{31}$$

If $A_n = \tilde{A} = A$ and $B_n = \tilde{B} = B$, where $A,\ B$ are constant, we simplify (30) and (31) to more manageable expressions:

$$x_{(2k+2)n+i} = x_i \prod_{s=0}^{n-1} \frac{A^{(k+1)s+\lfloor \frac{i}{2} \rfloor} + \frac{B}{r_{\tau(i)}} \sum_{l=0}^{(k+1)s+\lfloor \frac{i}{2} \rfloor -1} A^l}{A^{(k+1)s+\lfloor \frac{i}{2} \rfloor +1} + \frac{B}{r_{\tau(i)}} \sum_{l=0}^{(k+1)s+\lfloor \frac{i}{2} \rfloor} A^l}, \tag{32}$$

$$x_{(2k+2)n+i-2k-1} = x_{i-2k-1} \prod_{s=0}^{n-1} \frac{A^{(k+1)s+\lfloor \frac{i}{2} \rfloor} + \frac{B}{r_{\tau(i)-2k-1}} \sum_{l=0}^{(k+1)s+\lfloor \frac{i}{2} \rfloor -1} A^l}{A^{(k+1)s+\lfloor \frac{i}{2} \rfloor +1} + \frac{B}{r_{\tau(i)-2k-1}} \sum_{l=0}^{(k+1)s+\lfloor \frac{i}{2} \rfloor} A^l}, \tag{33}$$

for $i = 0,\ \ldots,\ 2k+1$, respectively.

4. Behavior and periodic nature of the solutions

Theorem 9 Let $(x_n)_{n\geq 0}$ be a solution to

$$x_{n+2k+2} = \frac{x_n}{A + B \prod_{i=0}^{k} x_{n+2i}}, \tag{34}$$

for some non-zero constants $A \neq \pm 1$ and B, such that the initial conditions $x_j,\ j = 0,\ \ldots,\ 2k+1$, satisfy $\tilde{r}_{-(2k+\tau(j))} = 1/r_{-(2k+\tau(j))} = \frac{1-A}{B}$. Then $(x_n)_{n>0}$ is periodic with period $2k+2$.

Proof. Assuming $\tilde{r}_{-(2k+\tau(j))} = \frac{1-A}{B}$. Using (32), we have that

$$x_{(2k+2)n+i} = x_i \prod_{s=0}^{n-1} \frac{A^{(k+1)s+\lfloor \frac{i}{2} \rfloor} + B\tilde{r}_{-(2k+\tau(j))} \sum_{l=0}^{(k+1)s+\lfloor \frac{i}{2} \rfloor -1} A^l}{A^{(k+1)s+\lfloor \frac{i}{2} \rfloor +1} + B\tilde{r}_{-(2k+\tau(j))} \sum_{l=0}^{(k+1)s+\lfloor \frac{i}{2} \rfloor} A^l} \tag{35}$$

$$=x_i \prod_{s=0}^{n-1} \frac{A^{(k+1)s+\lfloor \frac{i}{2} \rfloor} + B\left(\frac{1-A}{B}\right)\left(\frac{1-A^{(k+1)s+\lfloor \frac{i}{2} \rfloor}}{1-A}\right)}{A^{(k+1)s+\lfloor \frac{i}{2} \rfloor+1} + B\left(\frac{1-A}{B}\right)\left(\frac{1-A^{(k+1)s+\lfloor \frac{i}{2} \rfloor+1}}{1-A}\right)}$$

$$=x_i,$$

for all $i = 0,\ 1,\ \ldots,\ 2k+1$.

To recover the result in Theorem 3.1 in [3], we substitute $A = \dfrac{b}{c}$ in (34).

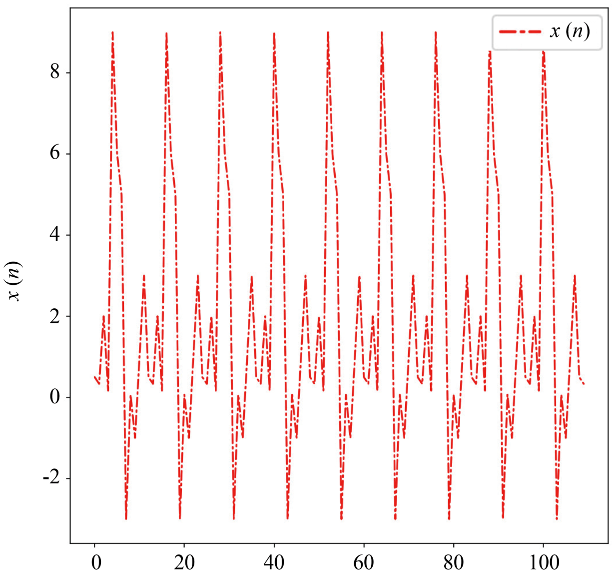

Figure 1. Plot of $x_{n+12} = \dfrac{x_n}{(4 - x_n x_{n+2} x_{n+4} x_{n+6} x_{n+8} x_{n+10})}$ with the initial conditions $x_0 = 1/2,\ x_1 = 1/3,\ x_2 = 2,\ x_3 = 1/6,\ x_4 = 9,\ x_5 = 6,\ x_6 = 5,\ x_7 = -3,\ x_8 = 1/15,\ x_9 = -1,\ x_{10} = 1,\ x_{11} = 3$ satisfying $x_0 x_2 x_4 x_6 x_8 x_{10} = x_1 x_3 x_5 x_7 x_9 x_{11} = (1-A)/B$

Figure 1 is an illustration of Theorem 9. The x_i's, $i = 0,\ \cdots,\ 2k+1$, satisfy the conditions in Theorem 9 and 12-periodic solutions were expected.

Theorem 10 Let $(x_n)_{n>0}$ be a solution to

$$x_{n+2k+2} = \frac{x_n}{-1 + B\prod_{i=0}^{k} x_{n+2i}}, \tag{36}$$

for some non-zero constant B. Then $(x_n)_{n>0}$ is periodic with period $4k+4$, provided that k is even.

Proof. Suppose that k is even. Using (32), we have that

$$x_{(2k+2)n+i} = x_i \prod_{s=0}^{n-1} \frac{(-1)^{(k+1)s+\lfloor \frac{i}{2} \rfloor} + B\tilde{r}_{\tau(i)} \sum_{l=0}^{(k+1)s+\lfloor \frac{i}{2} \rfloor - 1} (-1)^l}{(-1)^{(k+1)s+\lfloor \frac{i}{2} \rfloor + 1} + B\tilde{r}_{\tau(i)} \sum_{l=0}^{(k+1)s+\lfloor \frac{i}{2} \rfloor} (-1)^l} \tag{37}$$

$$= x_i \prod_{s=0}^{n-1} \frac{(-1)^{s+\lfloor \frac{i}{2} \rfloor} + B\tilde{r}_{\tau(i)} \left(\frac{1-(-1)^{s+\lfloor \frac{i}{2} \rfloor}}{2} \right)}{-(-1)^{s+\lfloor \frac{i}{2} \rfloor} + B\tilde{r}_{\tau(i)} \left(\frac{1+(-1)^{s+\lfloor \frac{i}{2} \rfloor}}{2} \right)}$$

$$= \begin{cases} x_i, & \text{if } n \text{ is even} \\ x_i \left[-1 + B\tilde{r}_{\tau(i)} \right]^{(-1)^{\lfloor \frac{i}{2} \rfloor + 1}}, & \text{if } n \text{ is odd,} \end{cases} \tag{38}$$

for all $i = 0, 1, \ldots, 2k+1$. A closer look at (38) shows that

$$x_{(4k+4)n+i} = x_i, \tag{39}$$

for $i = 0, 1, \ldots, 2k+1$, for all n.

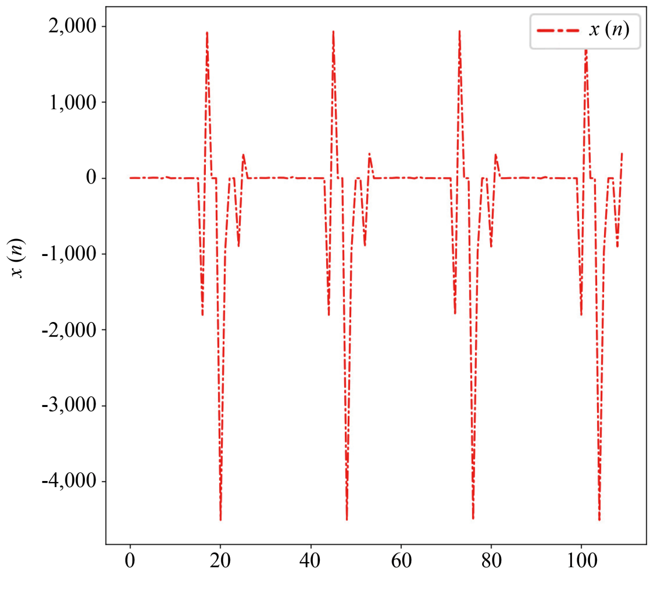

Figure 2. Plot of $x_{n+14} = x_n / (-1 + x_n x_{n+2} x_{n+4} x_{n+6} x_{n+8} x_{n+10} x_{n+12})$ with $x_0 = 2,\ x_1 = 3,\ x_2 = 3,\ x_3 = 6,\ x_4 = 3,\ x_5 = 6,\ x_6 = 5,\ x_7 = -3,\ x_8 = 15,\ x_9 = -1,\ x_{10} = 1,\ x_{11} = 1,\ x_{12} = -1,\ x_{13} = 1$ satisfying the conditions in Theorem 10

Figure 2 is an illustration of Theorem 10. The x_i's, $i = 0, \cdots, 2k+1$, satisfy the conditions in Theorem 10 and 28-periodic solutions were expected.

Theorem 11 Let $(x_n)_{n>0}$ be a solution to

$$x_{n+2k+2} = \frac{x_n}{1 + B \prod_{i=0}^{k} x_{n+2i}}, \tag{40}$$

for some non-zero constant B.

(i) The sole equilibrium point $\bar{x} = 0$ is non-hyperbolic.

(ii) For positive initial conditions x_i, $i = 0, \ldots, 2k+1$ and B, the solution $(x_n)_{n>0}$ converges to the equilibrium point $\bar{x} = 0$.

(iii) For positive initial conditions x_i's, $0 < i < 2k+1$ and B, the sole equilibrium point $\bar{x} = 0$ of (40) is (globally) asymptotically stable.

Proof. One can readily see that the sole equilibrium point of (40) is $x = 0$.

(i) If

$$f(x_n, x_{n+2}, \ldots, x_{n+2k}) = \frac{x_n}{1 + B \prod_{i=1}^{k} x_{n+2i}}, \tag{41}$$

we have

$$\left.\frac{df}{dx_n}\right|_{(0,\ 0,\ \ldots,\ 0)} = 1, \left.\frac{df}{dx_{n+2i}}\right|_{(0,\ 0,\ \ldots,\ 0)} = 0,\ i = 0,\ 1,\ 2,\ \ldots,\ k. \tag{42}$$

It follows that the characteristic equation of (34) about $\bar{x} = 0$ is given by $\lambda^{2k+2} - 1 = 0$ whose roots have magnitude equal to one. This proves that $\bar{x} = 0$ is non-hyperbolic.

(ii) If we assume that the initial conditions are all positive, then we have (from (32)) that:

$$\begin{aligned} x_{(2k+2)n+i} =& x_i \prod_{s=0}^{n-1} \frac{1 + B\tilde{r}_{\tau(i)}\left((k+1)s + \left\lfloor \frac{i}{2} \right\rfloor\right)}{1 + B\tilde{r}_{\tau(i)}\left((k+1)s + \left\lfloor \frac{i}{2} \right\rfloor + 1\right)} \\ =& x_i \prod_{s=0}^{n-1} \left(1 - \frac{B\tilde{r}_{\tau(i)}}{1 + B\tilde{r}_{\tau(i)}\left((k+1)s + \left\lfloor \frac{i}{2} \right\rfloor + 1\right)} \right) \\ =& u_i \prod_{s=0}^{n-1} \Xi(s). \end{aligned} \tag{43}$$

For $B > 0$, $0 < \Xi(s) < 1$, $s = 0,\ 1,\ \ldots,\ n-1$. It follows that $(x_n)_{n\geq 0} \to 0$, as $n \to \infty$.

(iii) Let $\varepsilon \geq 0$. Assume that the initial conditions $x_i's$, $i = 0, 1, \ldots, 2k+1$, are in such a manner that $|x_i| \leq \dfrac{\varepsilon}{(2k+1)(B+2)}$. It follows that

$$|x_0| + |x_1| + \cdots + |x_{2k+1}| \leq \frac{\varepsilon}{B+2}, \tag{44}$$

and we know from (45) that

$$x_{(2k+2)n+i} \leq x_i, \ i = 0, 1, \ldots, 2k+1, \tag{45}$$

for all n, as long as B is positive. In other words, for $|x_{(2k+2)n+i}| \leq |x_i| \leq \dfrac{\varepsilon}{(2k+1)(B+2)} \leq \varepsilon$, there exists $\delta = \varepsilon/(B+2)$ in such a manner that $|x_0| + |x_1| + \cdots + |x_{2k+1}| \leq \delta$. It follows that, the equilibrium point $\bar{x} = 0$ is locally stable. On the other hand (see (ii)), x_n tends to zero as n goes to infinity. The point $\bar{x} = 0$ is then a global attractor and locally stable: it is globally asymptotically stable.

To recover the result in Theorem 4.3 in [3], one can substitute $B = -\dfrac{c}{a}$ in (40). The result follows from part (ii).

Figure 3 is an illustration of Theorem 11.

Theorem 12 Let $(x_n)_{n>0}$ be a solution to Equation (34) for some constant A such that $|A| \neq 1$. The equilibrium point $\bar{x} = 0$ of (34) is asymptotically stable when $|A| > 1$ and unstable for $|A| < 1$. Additionally, any non-zero equilibrium point of (34) is non-hyperbolic.

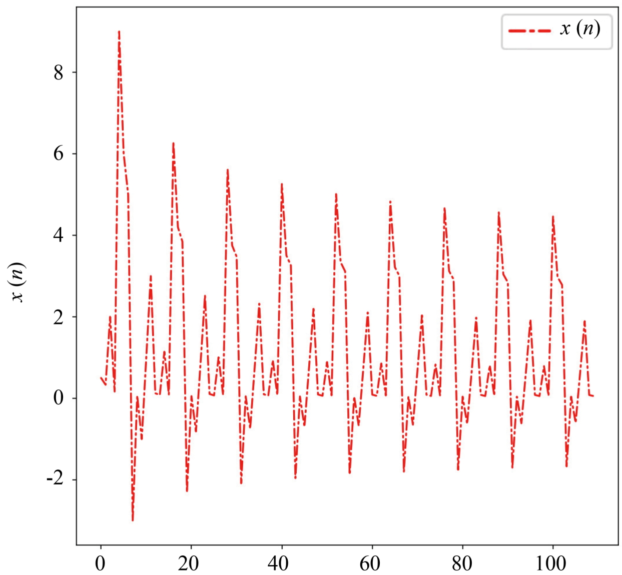

Figure 3. Plot of $x_{n+12} = x_n/(1 + x_n x_{n+2} x_{n+4} x_{n+6} x_{n+8} x_{n+10})$ with $x_0 = 1/2$, $x_1 = 1/3$, '$x_2 = 2$, $x_3 = 1/6$, $x_4 = 9$, $x_5 = 6$, $x_6 = 5$, $x_7 = -3$, $x_8 = 1/15$, $x_9 = -1$, $x_{10} = 1$, $x_{11} = 3$ satisfying the conditions in Theorem 11

Proof. The fixed points of (34) are found by solving the equation $\bar{x}(A + B\bar{x}^{k+1} - 1) = 0$. If

$$g(x_n,\ x_{n+2},\ \ldots,\ x_{n+2k}) = x_{n+2k+2} = \frac{x_n}{A + B\prod_{i=0}^{k} x_{n+2i}}, \tag{46}$$

we have

$$g_{,\ x_n} = \frac{A}{(A + B\prod_{i=0}^{k} x_{n+2i})^2}, \quad g_{,\ x_{n+2s}} = \frac{-Bx_n^2}{(A + B\prod_{i=0}^{k} x_{n+2i})^2}\prod_{\substack{i=1\\ i\neq s}}^{k} x_{n+2i}, \quad s = 1,\ 2,\ \ldots,\ k. \tag{47}$$

It follows that the characteristic equation associated with (34) is $\lambda^{2k+2} - 1/A = 0$ since $g_{,\ x_n}(0,\ 0,\ \ldots,\ 0) = 1/A$ and $g_{,\ x_{n+2s}}(0,\ 0,\ \ldots,\ 0) = 0$, $s = 1,\ 2,\ \ldots,\ k$. Clearly, $|\lambda| < 1$ when $|A| > 1$, which shows that the point $\bar{x} = 0$ is locally asymptotically stable. Similarly, $|\lambda| > 1$ when $|A| < 1$ and so, the point $\bar{x} = 0$ is unstable.

Any non-zero equilibrium point $\bar{x}$ satisfies $A + B\bar{x}^{k+1} - 1 = 0$. It follows that the characteristic equation associated with (34) is

$$\lambda^{2k+2} - (A-1)\lambda^{2k} - \cdots - (A-1)\lambda^2 - A = 0, \tag{48}$$

since $g_{,\ x_n}(\bar{x},\ \bar{x},\ \ldots,\ \bar{x}) = A$ and $g_{,\ x_{n+2s}}(\bar{x},\ \bar{x}, \ldots,\ \bar{x}) = A - 1$, $s = 1,\ 2,\ \ldots,\ k$. Equation (48) can take the form

$$(\lambda^2 - A)(\lambda^{2k} + \cdots + \lambda^2 + 1) = 0, \tag{49}$$

or simply $(\lambda^2 - A)(1 - \lambda^{2k+2})/(1 - \lambda^2) = 0$.

It follows that, when $A > 0$, the solutions of (49) are $\lambda = \pm\sqrt{A}$ or $\lambda_p = e^{i\frac{\pi s}{k+1}}$, $s = 1,\ 2,\ \ldots,\ k,\ k+2,\ \ldots,\ 2k+1$. Similarly, when $A < 0$, the solutions are $\lambda = \pm i\sqrt{|A|}$ or $\lambda_p = e^{i\frac{\pi s}{k+1}}$, $s = 1,\ 2,\ \ldots,\ k,\ k+2,\ \ldots,\ 2k+1$. This proves that any non-zero equilibrium point is non-hyperbolic since one can find a root of (48) such that its modulus is one.

5. Discussion and conclusion

This paper has investigated a higher-order difference equation through the lens of Lie symmetry analysis. By identifying the symmetry generators and utilizing canonical coordinates, we successfully derived invariants that simplify the equation, allowing us to express the solutions in a closed form. Furthermore, we analyzed the periodicity and stability of the solutions, particularly through a characteristic equation, thus highlighting the conditions under which the solutions exhibit periodic behavior and remain bounded. These findings contribute to a deeper understanding of the dynamic properties of such equations, reinforcing the results established in previous studies and providing a foundation for further exploration in the field of difference equations. This systematic method is applicable to most difference equations. A limitation of the method resides in the case where the characteristic functions are identically zero.

The results in this paper can be used to recover some known results in the literature by studying some special cases.

Setting $j = 2k + 1 - i$, it is easy to show that $\left\lfloor \frac{j}{2} \right\rfloor = k - \left\lfloor \frac{j}{2} \right\rfloor$ and $\tau(j) = 1 - \tau(i)$. Thus, equation (33) takes the form:

$$x_{(2k+2)n-j} = x_{-j} \prod_{s=0}^{n-1} \frac{A^{(k+1)s+k-\lfloor \frac{j}{2} \rfloor} + \dfrac{B}{r_{-(2k+\tau(j))}} \sum_{l=0}^{(k+1)s+k-\lfloor \frac{j}{2} \rfloor -1} A^l}{A^{(k+1)s+k+1-\lfloor \frac{j}{2} \rfloor} + \dfrac{B}{r_{-(2k+\tau(j))}} \sum_{l=0}^{(k+1)s+k-\lfloor \frac{j}{2} \rfloor} A^l}. \tag{50}$$

The case when $A_n = 1$ and $B_n = B$. Suppose B is a real constant. Using (33), the solution takes the form

$$x_{(2k+2)n-j} = x_{-j} \prod_{s=0}^{n-1} \frac{1 + B\tilde{r}_{-(2k+\tau(j))} \left((k+1)s + k - \left\lfloor \dfrac{j}{2} \right\rfloor \right)}{1 + B\tilde{r}_{-(2k+\tau(j))} \left((k+1)s + k - \left\lfloor \dfrac{j}{2} \right\rfloor + 1 \right)}, \tag{51}$$

$j = 0, \ldots, 2k+1$. Note that $\tilde{r}_j = 1/r_j$

To recover the results in Theorem 4.1 in [3], it is sufficient to substitute $B = -c/a$ in (51). Not only have we recovered the results in Theorem 4.1 in [3], we have also simplified the presentation of their results considerably.

Theorem 4.4 in [3] is obtained by setting $B = 1$ in (51).

The case when $A_n = A \neq 1$ and $B_n = B$ are constants. Here, thanks to (50), the solution becomes:

$$x_{(2k+2)n-j} = x_{-j} \prod_{s=0}^{n-1} \frac{A^{(k+1)s+k-\lfloor \frac{j}{2} \rfloor} + B\tilde{r}_{-(2k+\tau(j))} \left(\dfrac{1 - A^{(k+1)s+k-\lfloor \frac{j}{2} \rfloor}}{1-A} \right)}{A^{(k+1)s+k-\lfloor \frac{j}{2} \rfloor+1} + B\tilde{r}_{-(2k+\tau(j))} \left(\dfrac{1 - A^{(k+1)s+k-\lfloor \frac{j}{2} \rfloor+1}}{1-A} \right)}, \tag{52}$$

as in [3].

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Conflict of interest

The authors declare no competing financial interest.

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