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Research Article

Riemannian Manifolds Isometric to a Sphere

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Abstract: To investigate the geometry of a Riemannian manifold (N^m, g), sometimes it is convenient to isometrically immerse it into the Euclidean space E^{m+n}, which is always possible through Nash's embedding Theorem provided that the codimension n is taken to be sufficiently high. The isometric immersion $\psi : (N^m, g) \to \mathbb{R}^{m+n}$ can be treated as the position vector of points of N^m in $\mathbb{R}^{m+n}$ and therefore can be expressed as $\psi = \xi + \psi^\perp$, where ξ is tangential to N^m, whereas $\psi^\perp$ is normal to N^m. The Ricci tensor Ric of the Riemannian manifold (N^m, g) yields a symmetric operator Q, known as the Ricci operator, which satisfies the relation Ric $(X, Y) = g(QX, Y)$. In this article, we consider the isometric immersion $\psi : (N^m, g) \to \mathbb{R}^{m+n}$ of a compact Riemannian manifold (N^m, g) and show that if the tangential vector field ξ on (N^m, g) is an eigenvector of Q with constant eigenvalue $\lambda \neq 0$, that is, $Q\xi = \lambda\xi$, and the Ricci curvature Ric (ξ, ξ) satisfies $\int_{N^m} R(\xi, \xi) \geq \frac{m-1}{m} \int_{N^m} (\text{div } \xi)^2$, then (N^m, g) is necessarily isometric to the Euclidean sphere $S^m(c)$ of constant curvature. Moreover, the converse also holds.

Keywords: Nash's embedding theorem, Nash's vector, Nash's function, isometric to sphere

MSC: 53C20, 53A50

1. Introduction

Let (N^m, g) denote an m-dimensional Riemannian manifold. Its geometry can be studied from two main aspects: intrinsic and extrinsic. The intrinsic aspect focuses on features such as distance functions, geodesics, and Jacobi fields that are determined by the manifold's own metric structure. These fundamental elements help derive global geometric properties of (N^m, g) (see [1, 2]). Another important direction in the study of intrinsic geometry involves exploring the presence of specific vector fields on (N^m, g), including Killing and conformal vector fields, which play a significant role in shaping both its geometric structure and topological characteristics (see [1, 3, 4]). Moreover, a key aspect of intrinsic geometry involves analyzing specific partial differential equations, including Obata's equation and the Fischer-Marsden equation (cf. [5, 6]).

Though the origin of differential geometry lies in studying curves and surfaces in the Euclidean space $\mathbb{R}^3$, after the discovery by Nash (cf. [7]) that an m-dimensional Riemannian manifold (N^m, g) can be isometrically embedded in a Euclidean space $\mathbb{R}^n$ for sufficiently large $n > m$, it paved the way for studying extrinsic geometry of (N^m, g), known as submanifold geometry (cf. [1, 3, 4, 8, 9]).

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The field of submanifold geometry is vast, covering a wide range of results-from global aspects like total absolute curvature (cf. [10–13]) to more localized findings in submanifold theory (cf. [14–21]). Several known obstructions exist to embedding the Riemannian manifold $(N^m,\ g)$ into a Euclidean space $\mathbb{R}^n$, and noteworthy developments on this topic are discussed in [22, 23]. In a relatively recent article (cf. [11]), the authors have classified m-dimensional Einstein submanifolds of a Euclidean space $\mathbb{R}^n$.

Note that for an m-dimensional Riemannian manifold $(N^m,\ g)$, by Nash's theorem, there exists an isometric immersion $\psi : (N^m,\ g) \to \mathbb{R}^{m+n}$, where the metric g is the pullback of the Euclidean metric $\langle,\ \rangle$ on the Euclidean space $\mathbb{R}^{m+n}$ such that:

$$\psi^*(\langle,\ \rangle) = g$$

holds. Viewing the immersion ψ as the position vector of N^m in $\mathbb{R}^{m+n}$, it can be represented as:

$$\psi = \xi + \psi^\perp$$

where $\xi \in \mathfrak{X}(N^m)$, the space of vector fields on N^m, and $\psi^\perp$ is normal to N^m. We call ξ the Nash vector field on $(N^m,\ g)$. Also, we have the function $\phi = \langle \psi^\perp,\ H\rangle$ called Nash's function, where H is the mean curvature vector field.

One of the interesting problems in extrinsic geometry of the Riemannian manifold $(N^m,\ g)$ is to use the Nash vector field ξ on $(N^m,\ g)$ and find conditions on ξ such that the Riemannian manifold is isometric to the m-sphere $S^m(c)$ of constant curvature c. In this article, we take on this question and prove the following:

Theorem 1 An m-dimensional compact and connected Riemannian manifold $(N^m,\ g)$, $m \geq 2$, with Nash vector field ξ satisfying $Q(\xi) = \lambda\xi$, for a constant $\lambda \neq 0$, where Q is the Ricci operator of $(N^m,\ g)$ and the Ricci curvature $\text{Ric}\ (\xi,\ \xi)$ satisfies:

$$\int_{N^m} \text{Ric}\ (\xi,\ \xi) \geq \frac{m-1}{m} \int_{N^m} (\text{div}\ \xi)^2,$$

if and only if $\lambda > 0$ and $(N^m,\ g)$ is isometric to $S^m(c)$, $c = \dfrac{\lambda}{m-1}$.

2. Preliminaries

Let $(N^m,\ g)$ be an m-dimensional Riemannian manifold and $\psi : (N^m,\ g) \to \mathbb{R}^{m+n}$ be the isometric immersion for sufficiently large n. Considering ψ as the position vector of N^m in $\mathbb{R}^{m+n}$, it can be written as:

$$\psi = \xi + \psi^\perp, \tag{1}$$

where $\xi \in \mathfrak{X}(N^m)$ and $\psi^\perp$ is normal to N^m, which gives that:

$$\langle X,\ \psi^\perp\rangle = 0, \quad X \in \mathfrak{X}(N^m).$$

We call the vector field ξ tangent to N^m the Nash vector field. We denote the Euclidean connection on $\mathbb{R}^{m+n}$ by $\bar{\nabla}$ and the Riemannian connection on $(N^m,\ g)$ by ∇. Then we have the following relations (cf. [8, 9]):

$$\bar{\nabla}_X Y = \nabla_X Y + \sigma(X,\ Y), \quad X,\ Y \in \mathfrak{X}(N^m), \tag{2}$$

$$\bar{\nabla}_X N = -T_N X + \nabla^{\perp}_X N, \quad X \in \mathfrak{X}(N^m),\ N \in \mathscr{V}, \tag{3}$$

where σ is the second fundamental form, T_N is the shape operator with respect to the normal vector field N, and $\nabla^{\perp}$ is the connection on the normal bundle $\mathscr{V}$. σ and T_N are related by:

$$\langle \sigma(X,\ Y),\ N \rangle = g(T_N X,\ Y). \tag{4}$$

Taking the covariant derivative in equation (1) with respect to $X \in \mathfrak{X}(N^m)$ and using equations (2) and (3), we arrive at:

$$X = \nabla_X \xi + \sigma(X,\ \xi) - T_{\psi^{\perp}} X + \nabla^{\perp}_X \psi^{\perp},$$

which yields to:

$$\nabla_X \xi = X + T_{\psi^{\perp}} X, \quad \nabla^{\perp}_X \psi^{\perp} = -\sigma(X,\ \xi). \tag{5}$$

The mean curvature H of the Riemannian manifold $(N^m,\ g)$ is given by:

$$mH = \sum_j \sigma(e_j,\ e_j), \tag{6}$$

where $\{e_1,\ \ldots,\ e_m\}$ denotes a locally defined orthonormal basis on $(N^m,\ g)$.

Next, we define a function ϕ on $(N^m,\ g)$ by:

$$\phi = \langle \psi^{\perp},\ H \rangle, \tag{7}$$

and call ϕ the Nash function of $(N^m,\ g)$. Using equation (5) equipped with a local orthonormal frame $\{e_1,\ \ldots,\ e_m\}$, we compute:

$$\operatorname{div} \xi = \sum_j g(\nabla_{e_j} \xi,\ e_j) = \sum_j g(e_j + T_{\psi^{\perp}} e_j,\ e_j) = m + \sum_j g(T_{\psi^{\perp}} e_j,\ e_j).$$

Using equations (4) and (6), we get:

$$\operatorname{div} \xi = m + \sum_j g(\sigma(e_j,\ e_j),\ \psi^{\perp}) = m + mg(H,\ \psi^{\perp}).$$

That is, on employing equation (7):

$$\operatorname{div} \xi = m(1+\phi). \tag{8}$$

Lemma 1 Let ϕ be the Nash function of an m-dimensional compact Riemannian manifold $(N^m,\ g)$. Then:

(i) $\displaystyle\int_{N^m} (1+\phi) = 0,$

(ii) $\displaystyle\int_{N^m} (1+\phi)^2 = \int_{N^m} (\phi^2 - 1).$

Proof. Integrating equation (8), we get (i). Now integrating $(1+\phi)^2$:

$$\int_{N^m} (1+\phi)^2 = \int_{N^m} (1+\phi^2) + 2\int_{N^m} \phi,$$

and using (i), we have:

$$\int_{N^m} (1+\phi)^2 = \int_{N^m} (\phi^2 - 1).$$

Using equation (5), we get:

$$\nabla_X \nabla_Y \xi = \nabla_X Y + \nabla_X T_{\psi^{\perp}} Y,$$

using this in the following expression of the curvature tensor of $(N^m,\ g)$:

$$R(X,\ Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,\ Y]}\xi,$$

we conclude:

$$R(X,\ Y)\xi = (\nabla_X T_{\psi^{\perp}})(Y) - (\nabla_Y T_{\psi^{\perp}})(X), \tag{9}$$

where:

$$(\nabla_X T_{\psi^\perp})(Y) = \nabla_X T_{\psi^\perp} Y - T_{\psi^\perp}(\nabla_X Y).$$

Observe that the shape operator $T_{\psi^\perp}$ is symmetric, and therefore:

$$g((\nabla_X T_{\psi^\perp})(Y),\ Z) = g(Y,\ (\nabla_X T_{\psi^\perp})(Z)),$$

holds. Using the above symmetry and a local orthonormal frame $\{e_1,\ \ldots,\ e_m\}$ in equation (9), to compute the Ricci curvature Ric $(Y,\ \xi)$, we have:

$$\text{Ric}\ (Y,\ \xi) = \sum_j g(R(e_j,\ Y)\xi,\ e_j) = \sum_j g(Y,\ (\nabla_{e_j} T_{\psi^\perp})(e_j)) - \sum_j g((\nabla_Y T_{\psi^\perp})(e_j),\ e_j). \tag{10}$$

Note that

$$\begin{aligned}
\sum_j g((\nabla_Y T_{\psi^\perp})e_j,\ e_j) &= \sum_j g(\nabla_Y T_{\psi^\perp} e_j,\ e_j) - \sum_j g(\nabla_Y e_j,\ T_{\psi^\perp} e_j) \\
&= \sum_j \Big[Y g(T_{\psi^\perp} e_j,\ e_j) - g(T_{\psi^\perp} e_j,\ \nabla_Y e_j) \Big] - \sum_j g(\nabla_Y e_j,\ T_{\psi^\perp} e_j) \\
&= \sum_j Y g(S(e_j,\ e_j),\ \psi^\perp) - 2\sum_j g(\nabla_Y e_j,\ T_{\psi^\perp} e_j) \\
&= mY(\phi) - 2\sum_j g(\nabla_Y e_j,\ T_{\psi^\perp} e_j).
\end{aligned} \tag{11}$$

However, as $\nabla_Y e_j = \sum_k w_j^k e_k$, where the connection forms w_j^k are skew-symmetric (i.e., $w_j^k = -w_k^j$) and $T_{\psi^\perp} e_j$ is symmetric (i.e., $g(T_{\varphi^*} e_i,\ e_j) = g(T_{\varphi^*} e_j,\ e_i)$), we conclude:

$$\sum_j g(\nabla_Y e_j,\ T_{\psi^\perp} e_j) = 0.$$

Thus, equation (11) becomes:

$$\sum_j g(\nabla_Y T_{\psi^\perp} e_j,\ e_j) = mY(\phi).$$

Inserting this result into equation (10), we conclude:

$$\text{Ric}\,(Y,\ \xi) = \sum_j g\left(Y,\ (\nabla_{e_j} T_{\psi^\perp})e_j\right) - mY(\phi).$$

That is, the Ricci operator Q has the following expression:

$$Q(\xi) = -m\nabla\phi + \sum_j (\nabla_{e_j} T_{\psi^\perp})(e_j). \tag{12}$$

where $\nabla\phi$ represents the gradient vector field associated with the Nash function ϕ.

3. Proof of the theorem

Suppose $(N^m,\ g)$ be a compact, connected m-dimensional Riemannian manifold of dimension $m \geq 2$, and assume it admits a Nash vector field ξ that satisfies:

$$Q(\xi) = \lambda\xi, \tag{13}$$

where λ is a constant and the Ricci curvature Ric $(\xi,\ \xi)$ satisfies

$$\int_{N^m} \text{Ric}\,(\xi,\ \xi) \geq \frac{m-1}{m} \int_{N^m} (\text{div}\ \xi)^2. \tag{14}$$

Using equation (5), we have

$$T_{\psi^\perp}X - \phi X = \nabla_X\xi - (1+\phi)X. \tag{15}$$

On taking a local orthonormal frame $\{e_1,\ \ldots,\ e_m\}$ and using the fact

$$\|T_{\psi^\perp} - \phi I\|^2 = \sum_j g(T_{\psi^\perp}e_j - \phi e_j,\ T_{\psi^\perp}e_j - \phi e_j),$$

with equation (15), we conclude

$$\|T_{\psi^\perp} - \phi I\|^2 = \|\nabla\xi\|^2 + m(1+\phi)^2 - 2(1+\phi)\text{div}\ \xi.$$

And treating it with equation (8), we have

$$\|T_{\psi^\perp} - \phi I\|^2 = \|\nabla\xi\|^2 - m(1+\phi)^2. \tag{16}$$

Note that we also have

$$\begin{aligned}\|T_{\psi^\perp} - \phi I\|^2 &= \|T_{\psi^\perp}\|^2 + m\phi^2 - 2\sum_j \phi g(T_{\psi^\perp} e_j,\ e_j),\\ &= \|T_{\psi^\perp}\|^2 - m\phi^2.\end{aligned} \tag{17}$$

Also, using equation (5), we get the following expression for the Lie-derivative $L_\xi g$ with respect to the Nash vector field:

$$(L_\xi g)(X,\ Y) = g(\nabla_X \xi,\ Y) + g(\nabla_Y \xi,\ X) = 2[g(X,\ Y) + g(T_{\psi^\perp} X,\ Y)],$$

where we have used the symmetry of the shape operator $T_{\psi^\perp}$. Thus, choosing a local orthonormal frame $\{e_1,\ \dots,\ e_m\}$, we have

$$|L_\xi g|^2 = \sum_{j,\ k} (L_\xi g)(e_j,\ e_k)^2 = 4[m + \|T_{\psi^\perp}\|^2 + 2\sum_j g(T_{\psi^\perp} e_j,\ e_j)] = 4[m + \|T_{\psi^\perp}\|^2 + 2m\phi].$$

Integrating the above equation while using Lemma 1, we have

$$\int_{N^m} \frac{1}{2} |L_\xi g|^2 = \int_{N^m} 2(\|T_{\psi^\perp}\|^2 - m). \tag{18}$$

Recall the following integral formula (cf. [24]):

$$\int_{N^m} \{\mathrm{Ric}\ (\xi,\ \xi) + \frac{1}{2} |L_\xi g|^2 - \|\nabla \xi\|^2 - (\mathrm{div},\ \xi)^2\} = 0.$$

We integrate equation (16) and use the above formula to arrive at

$$\int_{N^m} \|T_{\psi^\perp} - \phi I\|^2 = \int_{N^m} \{\mathrm{Ric}\ (\xi,\ \xi) + \frac{1}{2} |L_\xi g|^2 - (\mathrm{div}\ \xi)^2 - m(1+\phi)^2\}.$$

Inserting equations (8) and (18) into the above equation, we have

$$\int_{N^m} \|T_{\psi^\perp} - \phi I\|^2 = \int_{N^m} \{\mathrm{Ric}\ (\xi,\ \xi) + 2\|T_{\psi^\perp}\|^2 - m] - m^2(1+\phi)^2 - m(1+\phi)^2\},$$

where we used Lemma 1.

The above equation, in view of equation (17), takes the form

$$\int_{N^m} \|T_{\psi^\perp} - \phi I\|^2 = \int_{N^m} \{\mathrm{Ric}(\xi,\ \xi) + 2\|T_{\psi^\perp} - \phi I\|^2 - m^2(1+\phi)^2 - m(1+\phi)^2\}.$$

That is, on using equation (8), we arrive at

$$\int_{N^m} \|T_{\psi^\perp} - \phi I\|^2 = \int_{N^m} \left\{\frac{m-1}{m}(\mathrm{div}\ \xi)^2 - \mathrm{Ric}\ (\xi,\ \xi)\right\}.$$

Using inequality (14), in the above equation, we have the conclusion:

$$T_{\psi^\perp} = \phi I. \tag{19}$$

By applying the covariant derivative to the above equation, we obtain:

$$(\nabla_X T_{\psi^\perp})(Y) = X(\phi)Y,$$

and it yields

$$\sum_j (\nabla_{e_j} T_{\psi^\perp})(e_j) = \nabla\phi. \tag{20}$$

For a local orthonormal frame $\{e_1,\ \ldots,\ e_m\}$ on $(N^m,\ g)$, combining equations (12) and (20), we get:

$$Q(\xi) = -(m-1)\nabla\phi, \tag{21}$$

which in view of equation (19) implies

$$\nabla\phi = -\frac{\lambda}{m-1}\xi. \tag{22}$$

By computing the divergence of the above expression and referring to equation (8), we arrive at:

$$\Delta\phi = -\frac{m}{m-1}\lambda(1+\phi).$$

Rescaling the above equation with $f = 1+\phi$, we have:

$$\Delta f = -\frac{m}{m-1}\lambda f,$$

that is,

$$f\Delta f = -\frac{m\lambda}{m-1}f^2.$$

And integrating the above equation, we get:

$$\int_{N^m} \|\nabla f\|^2 = \frac{\lambda m}{m-1}\int_{N^m} f^2. \tag{23}$$

Observe that when f is constant, the function ϕ remains constant as well, which through equation (21) implies $Q(\xi) = 0$, contrary to the assumption that $\lambda \neq 0$. Hence, f is nonconstant, and equation (23) then implies $\lambda > 0$. The equation (22) gives $\nabla f = -\frac{\lambda}{m-1}\xi$, and taking the covariant derivative (That is, operating ∇_X on both sides of this equation.) gives:

$$\nabla_X \nabla f = -\frac{\lambda}{m-1}(X + T_{\psi^\perp}X) = -\frac{\lambda}{m-1}(1+\phi)X = -\frac{\lambda}{m-1}fX,$$

where we have used equation (19). The above equation is Obata's differential equation (cf. [7]). Hence, $(N^m,\ g)$ is isometric to $S^m(c)$, where $c = \frac{\lambda}{m-1}$.

The converse is evident, as if $(N^m,\ g)$ being isometric to $S^m(c)$, then we have a natural embedding:

$$\psi : S^m(c) \to \mathbb{R}^{m+1}$$

with Nash's vector $\xi = 0$ and $\psi^\perp = \psi$. Also, $Q(\xi) = (m-1)c\xi$, with $\lambda = (m-1)c \neq 0$, and the two conditions hold.

4. Conclusions

In this article, we examined an m-dimensional Riemannian manifold $(N^m,\ g)$ and employed the isometric embedding $\psi : (N^m,\ g) \to \mathbb{R}^{m+n}$ into Euclidean space $\mathbb{R}^{m+n}$, where n is sufficiently large, as guaranteed by the Nash embedding theorem (cf. [7]) and used the decomposition of the position vector $\psi = \xi + \psi^\perp$, which gives a vector field ξ on the Riemannian manifold $(N^m,\ g)$ and the normal vector $\psi^\perp$ to $(N^m,\ g)$. In the main result, we assumed that $(N^m,\ g)$ is compact and ξ is an eigenvector of the Ricci operator Q with nonzero constant eigenvalue λ, that is $Q(\xi) = \lambda\xi$, with constant $\lambda \neq 0$ and the integral inequality satisfied by the Ricci curvature $Ric(\xi,\ \xi)$ which amounts to $(N^m,\ g)$ being isometric to the sphere $S^m(c)$, where $\lambda = (m-1)c$. The restrictions used in this result are suitably chosen to match with the sphere $S^m(c)$. In particular, we required $\lambda \neq 0$. A natural quest could be what happens if $\lambda = 0$, that is, $Q(\xi) = 0$? In this situation, we need to drop the compactness of $(N^m,\ g)$ and only consider $(N^m,\ g)$ is complete Riemannian manifold. Indeed it will be an interesting question to proceed to find the conditions under which a complete Riemannian manifold $(N^m,\ g)$ satisfying $Q(\xi) = 0$ is isometric to the Euclidean space R^m.

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Conflict of interest

The author declares no competing financial interest.

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