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A Novel Picard-P Hybrid Iterative Scheme: Convergence Analysis and Applications

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Abstract: The trend of incorporating the Picard iteration into already established iterative processes has led to highly efficient convergence results, as observed in cases such as the Picard-Noor hybrid, Picard-S* hybrid, Picard-S hybrid, Picard-Ishikawa hybrid, Picard-Mann hybrid, and several others. Using this technique, in this paper, we have introduced a new hybrid iterative scheme by merging Picard iterative scheme with P iterative scheme. We used numerical example to show the efficiency new Picard-P hybrid iteration process. By using Picard-P hybrid iterative we proved strong and weak fixed point convergence results for Suzuki Generalized Nonexpansive Mapping (SGNEM). As an application we established the solution of delay differential equations using Picard-P hybrid iteration scheme.

***Keywords*:** Picard-P iteration, Suzuki Generalized Nonexpansive Mapping (SGNEM), fixed point, polynomiography

MSC: 47H09, 47H10

1. Introduction

In numerous fields of applied science, certain problems are frequently too complex or even unsolvable by using the ordinary analytical techniques introduced in the present literature. In such cases, it is always necessary to obtain an approximate value of the desired solution [1]. Among various approaches, Fixed Point (F.P.) theory provides highly effective techniques for obtaining approximate values of such solutions. The desired approximate solution to such problems can be expressed as the fixed point of a suitable operator, that is, as the solution of an equivalent fixed point equation.

$$\vartheta = \mathscr{U}\vartheta,$$

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Here, the self-map $\mathscr{U}$ denotes a suitable operator defined on a subset of a given space. Some types of these operators are already established in the literature. In this work, we present a some of these operators. Suppose a self-map $\mathscr{U}$ of the given subset V of a Banach Space (BS) is given. Then $\mathscr{U}$ is known as a Banach contraction if, for all two points ϑ, θ in the set V, we have

$$\|\mathscr{U}\vartheta - \mathscr{U}\theta\| \leq \alpha\|\vartheta - \theta\|, \text{ for some fixed } \alpha \in [0,\ 1) \tag{1}$$

Notice that when (1) true for the value α exactly equal to 1 then $\mathscr{U}$ is called nonexpansive. As almost always, we will write $U_{\mathscr{F}}$ for the F.P set of $\mathscr{U}$, that is, $U_{\mathscr{F}} = \{\vartheta_0 \in E :\ \mathscr{U}\vartheta_0 = \vartheta_0\}$.

The existence of F.Ps for nonexpansive mappings in the setting of Banach spaces was studied independently by Browder [2], Gohde [3] and Kirk [4]. They proved that, if V is nonempty closed bounded and convex subset of a uniformly convex Banach space, then every nonexpansive mapping $\mathscr{U} :\ V \to V$ has at-least one F.P. A numbers of generalization of nonexpansive mappings have been considered by some authors in recent years.

In 2008, Suzuki [5] introduced a new class of mappings as follow. A selfmap $\mathscr{U}$ on a subset V of a Banach space $\mathscr{D}$ is said to satisfy (C) condition (also known as Suzuki mapping), if for all $\vartheta,\ \theta \in V$, we have:

$$\frac{1}{2}||\vartheta - \mathscr{U}\vartheta|| \leq ||\vartheta - \theta|| \Rightarrow ||\mathscr{U}\vartheta - \mathscr{U}\theta|| \leq ||\vartheta - \theta||. \tag{2}$$

As we know that the iterative schemes like Abbas and Picard-Noor converges faster as compare to P iterative scheme, but on the other hand we have proved that by adding Picard iteration to P iteration, then the Picard-P hybrid iteration converges faster as compare to all above mentioned iterative schemes. We use numerical examples to show the efficiency of Picard-P hybrid iterative scheme by comparing it with the above mentioned iterative schemes. We also proved some F.P convergence results using newly introduced Picard-P iterative scheme for Suzuki Generalized Nonexpansive Mapping (SGNEM). Some of well known iterations are Mann [6] , Ishikawa [7], Noor [8], Agarwal [9], Abbas [10], P [11] etc.

The remainder of this paper is structured as follows: Section 2 provides essential definitions for subsequent discussions. Section 3 presents the main results of the proposed iteration process. Section 4 includes numerical examples to analyze the convergence behavior of the proposed method. Section 5 explores real-world applications of the proposed iteration process. Section 6 introduces polynomiographs to enhance the understanding of our analysis. Finally, Section 7 presents concluding remarks.

2. Preliminaries

Definition 1 Consider a Banach Space (BS) $\mathscr{D}$ and $\emptyset \neq V \subseteq \mathscr{D}$. Select an element $t \in \mathscr{D}$ and choose a bounded sequence, namely, $\mathscr{D} \supseteq \{\vartheta_i\}$. We may set $r(V,\ \{\vartheta_i\})$ as

$$r(V,\ \{\vartheta_i\}) = \inf\{\limsup_{i\to\infty} ||\vartheta_i - t|| :\ t \in V\}$$

The asymptotic radius of the sequence $\{\vartheta_i\}$ connected with the set V is given as,

$$\mathscr{A}(V,\ \{\vartheta_i\}) = \{s \in V :\ \limsup_{i\to\infty} ||\vartheta_i - t|| = r(V,\ \vartheta_i)\}$$

Remark 1 According [12], $\mathscr{D}$ is represent Uniformly Convex Banach Space (UCBS), consequently it can be established that $\mathscr{A}(V, \{\vartheta_i\})$ includes a unique element. Further more it is mentioned that when V is weakly compact and convex, then $\mathscr{A}(V, \{\vartheta_i\})$ is convex. (see e.g., [13, 14] and others).

Definition 2 [15] A BS $\mathscr{D}$ is said to have Opial property if and only if for each weakly convergent sequance $\{\vartheta_i\} \subseteq \mathscr{D}$ with weak limit $t \in V$, we have,

$$\limsup_{i\to\infty} ||\vartheta_i - t|| < \limsup_{i\to\infty} ||\vartheta_i - e_0|| \; \forall e_0 \in \mathscr{D} \setminus \{t\}.$$

It noted that all BS satisfies Opial property.

Definition 3 [16] Let V be a subset of BS $\mathscr{D}$. A self mapping $\mathscr{U}$ of V is said to satisfy condition (I), if one can find a non decreasing function $k: [0, \infty) \to [0, \infty)$ s.t $k(0) = 0$, $k(\vartheta) > 0 \; \forall \; \vartheta \in [0, \infty) - \{0\}$ and $||\vartheta - \mathscr{U}\vartheta|| \geq k(d(\vartheta, F_{\mathscr{U}}))$ when $\vartheta \in V$. Here $d(\vartheta, F_{\mathscr{U}})$ is the distance of ϑ to $F_{\mathscr{U}}$.

Lemma 1 [14] Suppose $\mathscr{D}$ be a BS, s.t $\emptyset \neq V \subseteq D$ and $\mathscr{U}: V \to V$ be a self-mapp, then we have:

(i) If $\mathscr{U}$ is nonexpansive map then $\mathscr{U}$ is Suzuki Generalized Nonexpansive Mapping (SGNEM).

(ii) If $\mathscr{U}$ is a SGNEM with a F.P, it is quasi-nonexpansive map.

(iii) If $\mathscr{U}$ is SGNEM, then $\|\vartheta - \mathscr{U}\theta\| \leq 3\|\mathscr{U}\vartheta - \vartheta\| + \|\vartheta - \theta\|$ for all $\vartheta, \theta \in V$.

Lemma 2 [5] Let a BS $\mathscr{D}$ is said to have satisfy the Opial's property and $\mathscr{U}$ be a SGNEM on a subset V of $\mathscr{D}$. When $\{e_i\}$ converges weakly to z and $\lim_{i\to\infty} \|\mathscr{U}e_i - e_i\| = 0$, then $\mathscr{U}z = z$.

Lemma 3 [5] Let $\mathscr{D}$ be a uniformly convex BS and V be a weakly compact convex subset of $\mathscr{D}$. Suppose $\mathscr{U}$ be a SGNEM, then $\mathscr{U}$ has a F.P.

3. Main result

First we introduce our new iteration process, called the Picard-P I.P, which is defined as:

$$\begin{cases} \vartheta_0 \in V \\ b_i = (1-\alpha_i)\vartheta_i + \alpha_i U\vartheta_i \\ c_i = (1-\beta_i)b_i + \beta_i \mathscr{U} b_i \\ d_i = (1-\gamma_i)\mathscr{U} b_i + \gamma_i \mathscr{U} c_i \\ \vartheta_{i+1} = \mathscr{U} d_i \end{cases} \tag{3}$$

Now we study the F.P approximation for SGNEM using the Picard-P iterative scheme (3).

Lemma 4 Suppose $\mathscr{D}$ be a norm linear space and $V \neq \emptyset$ closed and convex subset of $\mathscr{D}$. If $\mathscr{U}: V \to V$ is SGNEM satisfying $F_{\mathscr{U}} \neq \emptyset$. Assume that $\{\vartheta_i\}$ is a sequence of Picard-P iterative process (3). Then for each $t \in F_{\mathscr{U}}$, it follows that, $\lim_{i\to\infty} ||\vartheta_i - t||$ exists.

Proof. Let $t \in F(\mathscr{U})$ and $z \in V$. Since $\mathscr{U}$ satisfies condition (C), so

$$\frac{1}{2}\|t - \mathscr{U}t\| = 0 \leq \|t - z\| \text{ implies that } \|\mathscr{U}t - \mathscr{U}z\| \leq \|t - z\|.$$

So by Lemma 1 (iii) with (3), we have

$$
\begin{aligned}
||b_i - t| &= ||(1-\alpha_i)\vartheta_i + \alpha_i \mathscr{U}\vartheta_i - t|| \\
&= ||(1-\alpha_i)\vartheta_i + \alpha_i t - \alpha_i t + \alpha_i \mathscr{U}\vartheta_i - t|| \\
&\leq (1-\alpha_i)||\vartheta_i - t|| + \alpha_i||\vartheta_i - t|| \\
&\leq ||\vartheta_i - t||.
\end{aligned} \tag{4}
$$

Similarly by Lemma 1 (iii) with (3), we have

$$
\begin{aligned}
||c_i - t| &= ||(1-\beta_i)b_i + \beta_i \mathscr{U} b_i - t|| \\
&= ||(1-\beta_i)b_i + \beta_i t - \beta_i t + \beta_i \mathscr{U} b_i - t|| \\
&\leq (1-\beta_i)||\vartheta_i - t|| + \beta_i||b_i - t|| \\
&\leq ||b_i - t||.
\end{aligned} \tag{5}
$$

Using (4) and Lemma 1 (iii), we have

$$
\begin{aligned}
||d_i - t| &= ||(1-\gamma_i)\mathscr{U} b_i + \gamma_i \mathscr{U} c_i - t|| \\
&= ||(1-\gamma_i)\mathscr{U} b_i + \gamma_i t - \gamma_i t + \gamma_i \mathscr{U} c_i - t|| \\
&\leq (1-\gamma_i)||b_i - t|| + \gamma_i||b_i - t|| \\
||d_i - t| &\leq ||b_i - t||,
\end{aligned} \tag{6}
$$

and

$$
||\vartheta_{i+1} - t|| = ||\mathscr{U} d_i - t|| \leq ||\vartheta_i - t||. \tag{7}
$$

It can be observed from (7) that $||\vartheta_{i+1} - t|| \leq ||\vartheta_i - t||$. It follows that $\{||\vartheta_i - t||\}$ is bounded and non-increasing. Thus $\lim_{i\to\infty} ||\vartheta_i - t||$ exists for each element t of $F_{\mathscr{U}}$. □

Now we proof the following theorem which is important for the existence of F.P.

Theorem 1 Suppose $V \neq \phi$ be a convex closed subset of a UCB space $\mathscr{D}$. Assume that $\mathscr{U} : V \to V$ be a SGNEM and $F_{\mathscr{U}} \neq \emptyset$. Let $\{\vartheta_i\}$ is a sequence of Picard-P iterative process (3). Consequently $\{\vartheta_i\}$ is bounded in $\mathscr{D}$ with the property $\lim_{i\to\infty} ||\vartheta_i - \mathscr{U}\vartheta_i|| = 0$.

Proof. Since $F_{\mathscr{U}} \neq \emptyset$. So we may choose any $t \in F_{\mathscr{U}}$, Lemma 2 indicates that $\{\vartheta_i\}$ is bounded and $\lim_{i\to\infty} ||\vartheta_i - t||$ exists. Consider

$$\lim_{i\to\infty} ||\vartheta_i - t|| = q. \tag{8}$$

Need to prove that $\lim_{i\to\infty} ||\vartheta_i - \mathscr{U}\vartheta_i|| = 0$. From (4) we have

$$\begin{aligned} &||d_i - t|| \leq ||\vartheta_i - t|| \\ &\Rightarrow \limsup_{i\to\infty} ||d_i - t|| \leq \limsup_{i\to\infty} ||d_i - t|| = q. \end{aligned} \tag{9}$$

Since $t \in F_{\mathscr{U}}$, we can apply Lemma 1 (iii) to get

$$\begin{aligned} &||\mathscr{U}\vartheta_i - t|| \leq ||\vartheta_i - t|| \\ &\Rightarrow \limsup_{i\to\infty} ||\mathscr{U}\vartheta_i - t|| \leq \limsup_{i\to\infty} ||\vartheta_i - t||. \end{aligned} \tag{10}$$

Now from (7), we have

$$||\vartheta_{i+1} - t|| \leq ||d_i - t||.$$

Using this together with (8), we obtain

$$q \leq \liminf_{i\to\infty} ||d_i - t||. \tag{11}$$

From (9) and (11), we obtain

$$\lim_{i\to\infty} ||d_i - t|| = q. \tag{12}$$

Since $||d_i - t|| = \lim_{i\to\infty} ||(1-\gamma_i)\vartheta_i + \gamma_i \mathscr{U}\vartheta_i - t||$.

$$q = \lim_{i\to\infty} (1-\gamma_i)||(\vartheta_i - t)|| + \gamma_i||(\mathscr{U}\vartheta_i - t)||. \tag{13}$$

Considering (8), (10) and (13) along with the Lemma 1 (iii), one gets

$$\lim_{i\to\infty} ||\vartheta_i - \mathscr{U}\vartheta_i|| = 0.$$

Conversely, assume that $\{\vartheta_i\}$ is bounded with the property $\lim_{i\to\infty} ||\vartheta_i - \mathscr{U}\vartheta_i|| = 0$. Need to prove that $F_{\mathscr{U}} \neq \emptyset$. Let $t \in \mathbb{A}(E, \{\vartheta_i\})$. By Lemma 1 (iii),

$$r(\mathscr{U}t, \{\vartheta_i\}) = \limsup_{i\to\infty} ||\vartheta_i - \mathscr{U}t|| \leq \limsup_{i\to\infty}(3||\mathscr{U}\vartheta_i - \vartheta_i|| + ||\vartheta_i - t||)$$

$$= \limsup_{i\to\infty} ||\vartheta_i - t||$$

$$= r(t, \{\vartheta_i\}).$$

Thus $\mathscr{U}t \in \mathscr{A}(E, \{\vartheta_i\}$. We have $\mathscr{U}t = t$. It is proved that $t \in F_{\mathscr{U}}$ i.e $F_{\mathscr{U}} \neq \emptyset$. □

Following is the weak convergence theorem.

Theorem 2 Suppose $\mathscr{D}$ be a UCB space with the Opial property and $V \neq \phi$ be a convex closed subset of $\mathscr{D}$. Assume that $\mathscr{U} : V \to V$ be a SGNEM and $F_{\mathscr{U}} \neq \emptyset$. Let $\{\vartheta_i\}$ is a sequence of Picard-P iterative process (3). Then $\{\vartheta_i\}$ converges weakly to an element of $F_{\mathscr{U}}$.

Proof. By Theorem 1, the sequence $\{\vartheta_i\}$ is bounded. Since $\mathscr{D}$ is UCB space. Thus a subsequence $\{o_{i_m}\}$ of $\{\vartheta_i\}$ exists. S.t $\{o_{i_m}\}$ converges weakly to some $t' \in V$. From Theorem 1, $\lim_{m\to\infty} ||\vartheta_{i_m} - \mathscr{U}p^*_{i_m}|| = 0$. It is suffice to show that the given sequence converges weakly to t'. Indeed if $\{\vartheta_i\}$ does not converges weakly to t'. Then there exist a subsequence $\{\vartheta_{i_s}\}$ of $\{\vartheta_i\}$, which converges weakly, namely, $t \neq t'$. From Theorem 1, it is annotated that $\lim_{s\to\infty} ||\vartheta_{i_s} - \mathscr{U}\vartheta_{i_s}|| = 0$. Applying Lemma 1, we get $p*' \in F_{\mathscr{U}}$. By Theorem 2 together with Opial's condition, we have

$$\lim_{i\to\infty} ||p_i - t'|| = \lim_{m\to\infty} ||\vartheta_{i_m} - t|| < \lim_{m\to\infty} ||\vartheta_{i_m} - t||$$

$$= \lim_{r\to\infty} ||\vartheta_i - t|| = \lim_{s\to\infty} ||\vartheta_{i_s} - t||$$

$$< \lim_{s\to\infty} ||\vartheta_{i_s} - t'|| = \lim_{i\to\infty} ||\vartheta_i - t'||.$$

Thus, we get $\lim_{i\to\infty} ||\vartheta_i - t'|| < \lim_{i\to\infty} ||\vartheta_i - t'||$, this is a contradiction. Hence proved. □

Finally we provide some strong convergence results defined by Picard-P iterative scheme (3) for the sequence $\{\vartheta_i\}$ with the help of SGNEM.

Theorem 3 Suppose $\mathscr{D}$ is any UCB space and $V \subseteq \mathscr{D}$ is non empty convex and compact. If $\mathscr{U} : V \to V$ is SGNEM with $F_{\mathscr{U}} \neq \emptyset$ and $\{\vartheta_i\}$ is a sequence of Picard-P iterates (3). Then consequently, $\{\vartheta_i\}$ converges strongly to a F.P of $F_{\mathscr{U}}$.

Proof. Since the domain V is convex and compact subset of $\mathscr{D}$ and $\{\vartheta_i\} \subseteq V$. It follows that a subsequence $\{\vartheta_{i_m}\}$ of $\{\vartheta_i\}$ with a strong limit $s^* \in V$ i.e $\lim_{i_m\to\infty} ||\vartheta_{i_m} - s^*|| = 0$. Then applying Lemma1 (iii) for $\vartheta = \vartheta_{i_m}$ and $\theta = s^*$,

$$||\vartheta_{i_m} - Hs^*|| \leq 3||H\vartheta_{i_m} - \vartheta_{i_m}|| + ||\vartheta_{i_m} - s^*|| \text{ for all } \vartheta, \ \theta \in V. \tag{14}$$

By Theorem 1, $\lim_{i_q\to\infty}||\vartheta_{i_q}-\mathscr{U}\vartheta_{i_q}||=0$ and also $\lim_{i_m\to\infty}||\vartheta_{i_m}-\mathscr{U}p'_{i_m}||=0$. Accordingly Lemma 4 provide $\lim_{i_m\to\infty}\mathscr{U}s^*=\mathscr{U}s^*\Rightarrow\mathscr{U}s^*=s^*$.

By Lemma 4 $\lim_{i\to\infty}\|\vartheta_i-s^*\|$ exist. Hence we have proved that $s^*\in F_{\mathscr{U}}$ and $\vartheta_i\to p^*$. □

The strong convergence theorem without the compactness assumption is established as follows.

Theorem 4 Let $\mathscr{V}\neq\emptyset$ be convex closed subset of UCB space $\mathscr{D}$ and $\mathscr{U}:V\to V$ be a SGNEM and $F_{\mathscr{U}}\neq\emptyset$. Assume that $\{\vartheta_i\}$ is a sequence of Picard-P iterative process (3). Then $\{\vartheta_i\}$ convergence strongly to an element $F_{\mathscr{U}}$. If and only if $\liminf_{i\to\infty}d(\vartheta_i, F_{\mathscr{U}})=0$.

Proof. By Using Lemma 4, one has $\lim_{i\to\infty}||\vartheta_i-s^*||$ exists for every $s^*\in F_{\mathscr{U}}$. It follows that $\liminf_{i\to\infty}d(\vartheta_i, F_{\mathscr{U}})$ exists. Accordingly $\liminf_{i\to\infty}d(\vartheta_i, F_{\mathscr{U}})=0$. The above limit provide us two subsequences of ϑ_i namely $\{\vartheta_{i_m}\}$ and $\{\vartheta_m\}$ exists in $F_{\mathscr{U}}$ with property $\|\vartheta_{i_m}-\vartheta_m\|\leq\frac{1}{2^m}$. We need to prove that $\{\vartheta_m\}$ is Cauchy in $F_{\mathscr{U}}$. By looking into the proof of Lemma 4, we can see that $\{\vartheta_i\}$ is nonincreasing. Therefore,

$$\|\vartheta_{m+1}-\vartheta_m\|\leq\|\vartheta_{m+1}-\vartheta_{i_{m+1}}\|+\|\vartheta_{i_{m+1}}-\vartheta_m\|\leq\frac{1}{2^{m+1}}+\frac{1}{2^m}.$$

Consequently, we obtain that $\lim_{m\to\infty}\|\vartheta_{m+1}-\vartheta_m\|=0$. Which shows that $\{\vartheta_m\}$ is cauchy in $F_{\mathscr{U}}$. According to the Lemma 3 we get that $F_{\mathscr{U}}$ is closed. Thus, $\{\vartheta_i\}$ converges to a point $q_0\in F_{\mathscr{U}}$. From Lemma 4, it follows that $\lim_{i\to\infty}||\vartheta_i-s^*||$ exists, so s^* is the strong limit of $\{\vartheta_i\}$. □

Following strong convergence results defined by Picard-P iterative scheme (3) using condition (I) [16].

Theorem 5 Suppose $\mathscr{V}\neq\emptyset$ closed convex subset of a UCBS $\mathscr{D}$ and suppose $\mathscr{U}:V\to V$ satisfies SGNEM and $F_{\mathscr{U}}\neq\emptyset$. If $\mathscr{U}$ is endowed with condition (I), then $\{\vartheta_i\}$ defined by (3) converges strongly to a point of $\mathscr{U}$.

Proof. From the proof of Theorem 4. From the Theorem 1, we have $\liminf_{i\to\infty}||\mathscr{U}p_i-p_i||=0$. Condition (I) for $\mathscr{U}$ provides $\liminf_{i\to\infty}d(\vartheta_i, F_{\mathscr{U}})=0$. Now all the requirements of the Theorem 4 are available, so we concludes that $\{\vartheta_i\}$ has a strong limit in $F_{\mathscr{U}}$. □

4. Numerical examples

With the help of numerical example, we observe that the Picard-P iteration process indubitably exhibit faster convergence rate as compare to other iteration process in the setting of mapping with SGNEM.

Example Suppose $V=[0, 5]\subset\mathscr{D}$ and norm on V be defined as $||\vartheta||=|\vartheta|\ \forall\ \vartheta\ \in V$. Defined function $\mathscr{U}:V\to V$ as

$$\mathscr{U}\vartheta=\begin{cases}\dfrac{\vartheta+4}{5} & \text{if } \vartheta\in[0, 3)\\[2ex] \dfrac{\vartheta+5}{4} & \text{if } \vartheta\in[3, 5].\end{cases} \tag{15}$$

Then, $\mathscr{U}$ is SGNEM. But not NEM.

Proof. We show that given mapping is not nonexpensive, having $\vartheta=2.9$ and $\theta=3$, then it is not nonexpensive mapping.

Now we show that given mapping is Suzuki generalized nonexpensive.

Case (I): When $\vartheta,\ \theta\in[0, 3)$, we take $\vartheta=0$ and $\theta=2.8$, then

$$||\mathscr{U}\vartheta - \mathscr{U}\theta|| = \left|\frac{4}{5} - \frac{6.8}{5}\right| = 0.56$$

$$||\vartheta - \theta|| = 2.8,$$

which shows that,

$$||\mathscr{U}\vartheta - \mathscr{U}\theta|| \leq ||\vartheta - \theta||$$

$$\frac{1}{2}||\vartheta - \mathscr{U}\vartheta|| = \frac{2}{5} = 0.4.$$

This implies that,

$$\frac{1}{2}||\vartheta - \mathscr{U}\vartheta|| \leq ||\vartheta - \theta|| \Rightarrow ||\mathscr{U}\vartheta - \mathscr{U}\theta|| \leq ||\vartheta - \theta||.$$

Case (II): If $\vartheta,\ \theta \in [3,\ 5]$, we take $\vartheta = 3$ and $\theta = 5$, then

$$||\mathscr{U}\vartheta - \mathscr{U}\theta|| = \left|2 - \frac{5}{2}\right| = 0.5$$

$$||\vartheta - \theta|| = 2$$

which shows that,

$$||\mathscr{U}\vartheta - \mathscr{U}\theta|| \leq ||\vartheta - \theta||$$

$$\frac{1}{2}||\vartheta - \mathscr{U}\vartheta|| = \frac{1}{2} = 0.5$$

This implies that,

$$\frac{1}{2}||\vartheta - \mathscr{U}\vartheta|| \leq ||\vartheta - \theta|| \Rightarrow ||\mathscr{U}\vartheta - \mathscr{U}\theta|| \leq ||\vartheta - \theta||.$$

Case (III): If $\vartheta \in [0,\ 3)$, and $\theta \in [3,\ 5]$ we take $\vartheta = 0$ and $\theta = 3$, then

$$||\mathscr{U}\vartheta - \mathscr{U}\theta|| = \left|\frac{4}{5} - 2\right| = 1.2$$

$$||\vartheta - \theta|| = 3.$$

which shows that,

$$||\mathscr{U}\vartheta - \mathscr{U}\theta|| \leq ||\vartheta - \theta||$$

$$\frac{1}{2}||\vartheta - \mathscr{U}\vartheta|| = \frac{2}{5} = 0.4.$$

This implies that,

$$\frac{1}{2}||\vartheta - \mathscr{U}\vartheta|| \leq ||\vartheta - \theta|| \Rightarrow ||\mathscr{U}\vartheta - \mathscr{U}\theta|| \leq ||\vartheta - \theta||.$$

Case (IV): If $\theta \in [0,\ 3)$, and $\vartheta \in [3,\ 5]$ we take $\vartheta = 5$ and $\theta = 2$, then

$$||\mathscr{U}\vartheta - \mathscr{U}\theta|| = \left|\frac{5}{2} - \frac{6}{5}\right| = 1.3$$

$$||\vartheta - \theta|| = 3.$$

which shows that

$$||\mathscr{U}\vartheta - \mathscr{U}\theta|| \leq ||\vartheta - \theta||$$

$$\frac{1}{2}||\vartheta - \mathscr{U}\vartheta|| = \frac{5}{4} = 1.25.$$

This implies that

$$\frac{1}{2}||\vartheta - \mathscr{U}\vartheta|| \leq ||\vartheta - \theta|| \Rightarrow ||\mathscr{U}\vartheta - \mathscr{U}\theta|| \leq ||\vartheta - \theta||.$$

Hence $\mathscr{U}$ is a SGNEM. □

The Table shows the tabular comparison of Picard-P iteration scheme with Abbas [17], Picard-Noor [18], P [11] iteration process for initial value 0.5 respectively with $\alpha_i = 0.22$, $\beta_i = 0.66$, $\gamma_i = 0.25$ and the graphical comparison is given in Figure 1, which shows that Picard-P iteration process moving fast to the f.p of $\mathscr{U}$ as compared to other iterations process.

Table 1. Numerical results produced by Picard-P, Abbas, Picard-Noor and P iterative schemes for $\mathscr{U}$ of the Example 1

n	Picard-P	Abbas	Picard-Noor	P
1	0.5	0.5	0.5	0.5
2	0.98569536	0.96885920	0.92003936	0.91369280
3	0.99959075	0.99806050	0.98721259	0.98510213
4	0.99998829	0.99987920	0.99795502	0.99742841
5	0.99999966	0.99999247	0.99967296	0.99955610
6	0.99999999	0.99999953	0.99994770	0.99992337
7	0.99999999	0.99999997	0.99999163	0.99998677
8	1	0.99999999	0.99999866	0.99999771
9	1	0.99999999	0.99999978	0.99999960
10	1	1	0.99999996	0.99999993

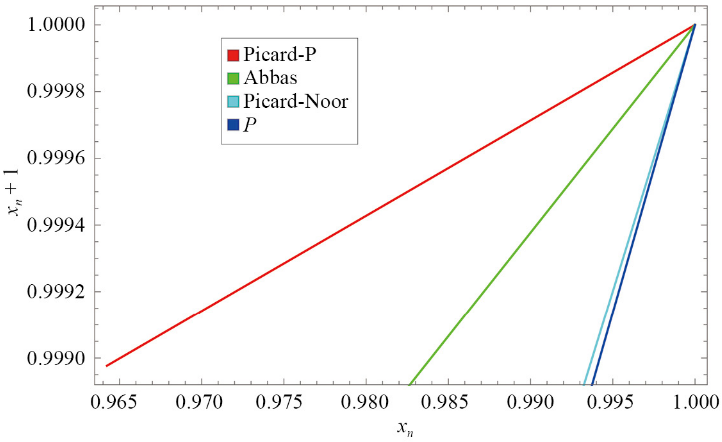


Figure 1. Graphical analysis of iteration schemes towards the fixed point of $\mathscr{U}$ in Example 1

We evaluate how closely the iterates x_n approach the fixed point 1 using arbitrary-precision calculations with different parameter sequences. The graphs display the logarithm of the error, $\log(|x_n - 1|)$, which highlights the magnitude of convergence. For instance, a value of $-d$ on the log scale indicates that the error is approximately 10^{-d}.

As shown in Figure 2, the Picard-P and Abbas iterations exhibit rapid convergence, with steep error reductions achieved within about 8-9 iterations. This observation is consistent with the iteration counts reported in Table 1. By contrast, the Picard-Noor and standard P iterations converge more slowly, as reflected by the gentler decline in their log-error curves, indicating comparatively weaker convergence rates.

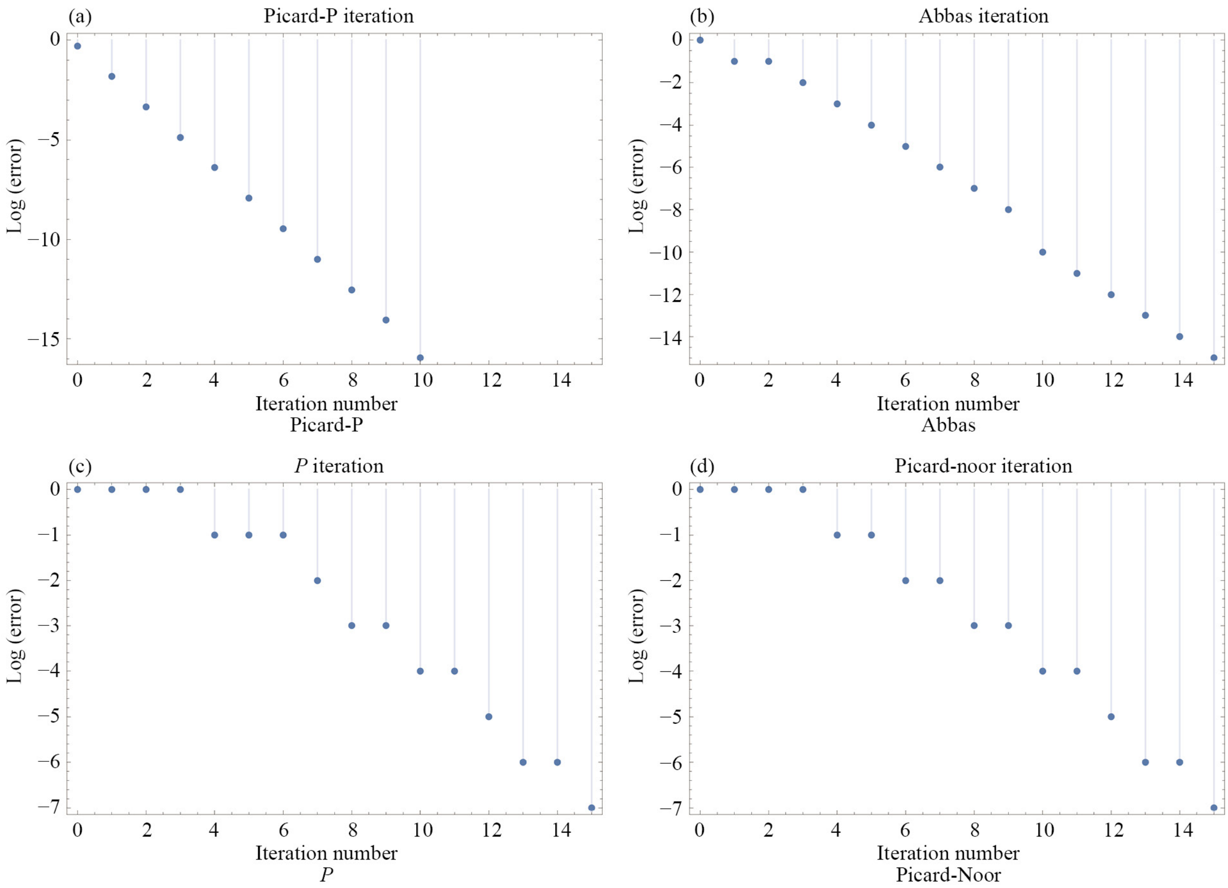


Figure 2. Error analysis of Picard-P, Abbas, Picard-Noor and P iterative schemes

5. Application

In this section, the space $\mathscr{D}([a, b])$, is taken to be the set of all continuous real-valued functions on $[a, b]$, with Chebyshev norm $\|e-\theta\|_{\infty} = \max_{u\in[a, b]} |e(u)-\theta(u)|$. Clearly $(\mathscr{D}([a, b]), \|\cdot\|_{\infty})$ is a BS, see [19]. Now, let a delay differential equation s.t

$$e'(u) = \mathscr{U}(u, e(u), e(u-\tau)), u \in [u_0, b], \tag{16}$$

with initial condition

$$\vartheta(u) = \rho(u), u \in [u_0-\tau, u_0]. \tag{17}$$

Some conditions are below:

(i) $u_0, b \in \mathbb{R}, \tau > 0$.

(ii) $\mathscr{U} \in \mathscr{D}\left([u_0, b] \times \mathbb{R}^2, \mathbb{R}\right)$.

(iii) $\rho \in \mathscr{D}([u_0-\tau, b], \mathbb{R})$.

(iv) There exist $L_{\mathscr{U}} > 0$, s.t

$$|\mathscr{U}(u,\ u_1,\ u_2)-\mathscr{U}(u,\ v_1,\ v_2)| \leq L_{\mathscr{U}}\left(\sum_{i=1}^{2}|u_i-v_i|\right),\ \forall u_i,\ v_i\in\mathbb{R},\ i=1,\ 2,\ u\in[u_0,\ b]. \tag{18}$$

(v) $2L_{\mathscr{U}}(b-u_0)<1$.

By a solution of problem (16)-(17) we means the function $p\in\mathscr{D}([u_0-\tau,\ b],\ n\mathbb{R})\cap\mathscr{D}^1([u_0,\ b],\ \mathbb{R})$.

The problem (16)-(17) can be reconstituted in the following form of integral equation:

$$e(u)=\begin{cases}\rho(u), & u\in[u_0-\tau,\ u_0]\\ \\ \rho(u_0)+\int_{u_0}^{u}\mathscr{U}(s,\ e(s),\ e(s-\tau))ds, & u\in[u_0,\ b].\end{cases} \tag{19}$$

We can now present the following result.

Theorem 6 Assume that conditions $(i)-(v)$ are satisfied. Then the problem (16)-(17) possesses a unique solution p, in $\mathscr{D}([u_0-\tau,\ b],\ \mathbb{R})\cap\mathscr{D}^1([u_0,\ b],\ \mathbb{R})$ and Picard-P I.P (3.1) with real sequence $\{\alpha_i\}_{i=0}^{\infty}$, $\{\beta_i\}_{i=0}^{\infty}$ and $\{\gamma_i\}_{i=0}^{\infty}$ in [0, 1] satisfying $\sum_{i=0}^{\infty}\alpha_i\beta_i\gamma_i=\infty$, converges to t.

Proof. Let $\{\vartheta_i\}_{i=0}^{\infty}$ be a iterative sequence generated by V I.P (2) for the operator $\mathscr{U}:\ \mathscr{D}([u_0-\tau,\ b],\ \mathbb{R})\to\mathscr{D}([u_0-\tau,\ b],\ \mathbb{R})$, defined by;

$$\mathscr{U}(e(u))\begin{cases}\rho(u), & u\in[u_0-\tau,\ u_0]\\ \\ \rho(u_0)+\int_{u_0}^{t}\mathscr{U}(s,\ e(s),\ e(s-\tau))ds, & u\in[u_0,\ b].\end{cases}$$

Denote t as a F.P of $\mathscr{U}$. We will prove that $\vartheta_i\to t$ as $i\to\infty$. For $u\in[u_0-\tau,\ u_0]$, clearly $\vartheta_i\to t$ as $i\to\infty$.

Next we show that $u\in[u_0,\ b]$, then

$$\begin{aligned}\|b_i-t\|_\infty &= \|(1-\alpha_i)\vartheta_i+\alpha_i\mathscr{U}\vartheta_n-\mathscr{U}t\|_\infty\\ &\leq(1-\alpha_i)\|\vartheta_i-t\|_\infty+\alpha_i\|\mathscr{U}\vartheta_i-\mathscr{U}t\|_\infty\\ &=(1-\alpha_i)\|\vartheta_i-t\|_\infty+\alpha_i\max_{u\in[u_0-\tau,\ b]}|\mathscr{U}\vartheta_i(u)-\mathscr{U}t(u)|\\ &=(1-\alpha_i)\|\vartheta_i-t\|_\infty+\alpha_i\max_{u\in[u_0-\tau,\ b]}\\ &\quad\left|\rho(u_0)+\int_{u_0}^{u}\mathscr{U}(s,\ \vartheta_i(s),\ \vartheta_i(s-\tau))ds-\rho(u_0)-\int_{u_0}^{t}\mathscr{U}(s,\ t(s),\ t(s-\tau))ds\right|\\ &=(1-\alpha_i)\|\vartheta_i-t\|_\infty+\alpha_i\max_{u\in[u_0-\tau,\ b]}\left|\int_{u_0}^{u}\mathscr{U}(s,\ \vartheta_i(s),\ \vartheta_i(s-\tau))-\mathscr{U}(s,\ t(s),\ t(s-\tau))ds\right|\end{aligned}$$

$$\leq (1-\alpha_i)\|\vartheta_i - t\|_\infty + \alpha_i \max_{u\in[u_0-\tau,\ b]} \int_{u_0}^{u} L_f\left(|\vartheta_i(s)-t(s)| + |\vartheta_i(s-\tau)-t(s-\tau)|\right) ds$$

$$\leq (1-\alpha_i)\|\vartheta_i - t\|_\infty + \alpha_i \int_{u_0}^{u} L_{\mathscr{U}}\left(\max_{s\in[u_0-\tau,\ b]}|\vartheta_i(s)-t(s)| + \max_{s\in[u_0-\tau,\ b]}|\vartheta_i(s-\tau)-t(s-\tau)|\right) ds$$

$$= (1-\alpha_i)\|\vartheta_i - t\|_\infty + \alpha_i \int_{u_0}^{t} L_{\mathscr{U}}\left(\|\vartheta_i - t\|_\infty + \|\vartheta_i - t\|_\infty\right) ds$$

$$\leq (1-\alpha_i)\|\vartheta_i - t\|_\infty + 2\alpha_i L_{\mathscr{U}}(b-t_0)\|\vartheta_i - t\|_\infty$$

$$= (1-\alpha_i(1-2L_{\mathscr{U}}(b-u_0)))\|\vartheta_i - t\|_\infty. \tag{20}$$

Also

$$\|c_i - t\|_\infty = \|(1-\beta_i)b_i + \beta_i \mathscr{U} b_i - \mathscr{U} t\|_\infty$$

$$\leq (1-\beta_i)\|b_i - t\|_\infty + \beta_i \|\mathscr{U} b_i - \mathscr{U} t\|_\infty$$

$$= (1-\beta_i)\|b_i - t\|_\infty + \beta_i \max_{u\in[u_0-\tau,\ b]} |\mathscr{U} b_i(u) - \mathscr{U} t(u)|$$

$$= (1-\beta_i)\|b_i - t\|_\infty + \beta_i \max_{u\in[u_0-\tau,\ b]} \left|\rho(u_0) + \int_{u_0}^{u} \mathscr{U}(s,\ b_i(s),\ b_i(s-\tau))ds - \rho(u_0) - \int_{u_0}^{t} \mathscr{U}(s,\ t(s),\ t(s-\tau))ds\right|$$

$$= (1-\beta_i)\|b_i - t\|_\infty + \beta_i \max_{u\in[u_0-\tau,\ b]} \left|\int_{u_0}^{u} \mathscr{U}(s,\ b_i(s),\ b_i(s-\tau)) - \mathscr{U}(s,\ t(s),\ t(s-\tau))ds\right|$$

$$\leq (1-\beta_i)\|b_i - t\|_\infty + \beta_i \max_{u\in[u_0-\tau,\ b]} \int_{u_0}^{u} L_f\left(|b_i(s)-t(s)| + |b_i(s-\tau)-t(s-\tau)|\right) ds$$

$$\leq (1-\beta_i)\|b_i - t\|_\infty + \beta_i \int_{u_0}^{u} L_{\mathscr{U}}\left(\max_{s\in[u_0-\tau,\ b]}|b_i(s)-t(s)| + \max_{s\in[u_0-\tau,\ b]}|b_i(s-\tau)-t(s-\tau)|\right) ds$$

$$= (1-\beta_i)\|b_i - t\|_\infty + \beta_i \int_{u_0}^{t} L_{\mathscr{U}}\left(\|b_i - t\|_\infty + \|b_i - t\|_\infty\right) ds$$

$$\leq (1-\beta_i)\|b_i - t\|_\infty + 2\beta_i L_{\mathscr{U}}(b-t_0)\|b_i - t\|_\infty$$

$$= (1-\beta_i(1-2L_{\mathscr{U}}(b-u_0)))\|b_i - t\|_\infty. \tag{21}$$

Similarly

$$
\begin{aligned}
\|d_i - t\|_\infty =& \|(1-\gamma_i)\mathscr{U} b_i + \gamma_i \mathscr{U} c_i - \mathscr{U} t\|_\infty \\
\leq& (1-\gamma_i)\|\mathscr{U} b_i - \mathscr{U} t\|_\infty + \gamma_i \|\mathscr{U} c_i - \mathscr{U} t\|_\infty \\
=& (1-\gamma_i)\max_{u\in[u_0-\tau,\, b]} |\mathscr{U} b_i(u) - \mathscr{U} t(u)| + \gamma_i \max_{u\in[u_0-\tau,\, b]} |\mathscr{U} c_i(u) - \mathscr{U} t(u)| \\
=& (1-\gamma_i)\max_{u\in[u_0-\tau,\, b]} \left|\rho(u_0) + \int_{u_0}^{u} \mathscr{U}(s,\, b_i(s),\, b_i(s-\tau))ds - \rho(u_0) - \int_{u_0}^{t} \mathscr{U}(s,\, t(s),\, t(s-\tau))ds\right| \\
&+ \gamma_i \max_{u\in[u_0-\tau,\, b]} \left|\rho(u_0) + \int_{u_0}^{u} \mathscr{U}(s,\, c_i(s),\, c_i(s-\tau))ds - \rho(u_0) - \int_{u_0}^{t} \mathscr{U}(s,\, t(s),\, t(s-\tau))ds\right| \\
=& (1-\gamma_i)\max_{u\in[u_0-\tau,\, b]} \left|\int_{u_0}^{u} \mathscr{U}(s,\, b_i(s),\, b_i(s-\tau)) - \mathscr{U}(s,\, t(s),\, t(s-\tau))ds\right| \\
&+ \gamma_i \max_{u\in[u_0-\tau,\, b]} \left|\int_{u_0}^{u} \mathscr{U}(s,\, c_i(s),\, c_i(s-\tau)) - \mathscr{U}(s,\, t(s),\, t(s-\tau))ds\right| \\
\leq& (1-\gamma_i)\int_{u_0}^{u} L_{\mathscr{U}}\left(\max_{s\in[u_0-\tau,\, b]} |b_i(s) - t(s)| + \max_{s\in[u_0-\tau,\, b]} |b_i(s-\tau) - t(s-\tau)|\right) ds \\
&+ \gamma_i \max_{u\in[u_0-\tau,\, b]} \int_{u_0}^{u} L_{\mathscr{U}}\left(|c_i(s) - t(s)| + |c_i(s-\tau) - t(s-\tau)|\right) ds \\
\leq& (1-\gamma_i)\|b_i - t\|_\infty + \gamma_i \int_{u_0}^{u} L_{\mathscr{U}}\left(\max_{s\in[u_0-\tau,\, b]} |c_i(s) - t(s)| + \max_{s\in[u_0-\tau,\, b]} |c_i(s-\tau) - t(s-\tau)|\right) ds \\
=& (1-\gamma_i)\int_{u_0}^{t} L_{\mathscr{U}}\left(\|b_i - t\|_\infty + \|b_i - t\|_\infty\right) ds + \gamma_i \int_{u_0}^{t} L_{\mathscr{U}}\left(\|c_i - t\|_\infty + \|c_i - t\|_\infty\right) ds \\
\leq& (1-\gamma_i)\|b_i - t\|_\infty + 2\gamma_i L_{\mathscr{U}}(b - t_0)\|c_i - t\|_\infty \\
=& (1-\gamma_i)(1 - 2L_{\mathscr{U}}(b - u_0)))\|c_i - t\|_\infty.
\end{aligned}
\tag{22}
$$

By using (20), (21), (22) together and also use assumption (v), we get,

$$
\|\vartheta_{i+1} - t\|_\infty \leq (1 - \alpha_i\beta_i\gamma_i(1 - 2L_{\mathscr{U}}(b - u_0)))\|\vartheta_i - t\|_\infty. \tag{23}
$$

Therefore, inductively

$$\|\vartheta_{i+1}-t\|_\infty \leq \prod_{k=0}^{i}(1-\alpha_i\beta_i\gamma_i(1-2L_{\mathscr{U}}(b-u_0)))\,\|\vartheta_0-t\|_\infty . \tag{24}$$

From assumption (v), it follows that $1-\alpha_i\beta_i\gamma_i(1-2L_{\mathscr{U}}(b-u_0))<1$. Since $1-\vartheta \leq e^{-\vartheta}$ for all $\vartheta \in [0,\ 1]$, (24) yields

$$\|\vartheta_{i+1}-t\|_\infty \leq \frac{\|\vartheta_0-t\|_\infty}{e^{(1-3L_{\mathscr{U}}(b-u_0))\sum_{k=0}^{i}\alpha_k\beta_k\gamma_k}}. \tag{25}$$

By taking limit (25) on both sides,we get, $\lim_{i\to\infty}\|\vartheta_i-t\|_\infty=0$, i.e. $\vartheta_i \to t$ for $i\to\infty$, hence V iterations converges to the solution of problem (16)-(17). □

6. Comparison via polynomiography

Bahman Kalantari devised polynomiography, a digital art form and visual analytic technique for root-finding [20, 21]. It involves the visualization of complicated polynomials, frequently with the use of mathematical concepts and iterative algorithms. The word polynomiography is a portmanteau of polynomial and graph with the emphasis on graphical representation of polynomial functions.

Numerous types of iteration processes are compared and analyzed using polynomiography techniques [22–26]. In polynomiography, convergence properties of a iteration process is used to generate a polynomiograph. The iteration function is use to approximate the root of a polynomial. One well-known root-finding algorithm is Newton's iteration method, also known as the Newton-Raphson method. Its definition is:

$$v_{n+1}=v_n-\frac{g(v_n)}{g'(v_n)}, \tag{26}$$

where $v_0\in\mathbb{C}$ is the starting point and g is a polynomial with complex coefficients. Now, writing (26) in terms of a fixed point iteration process as follows:

$$v_{n+1}=\mathscr{T}(v_n), \tag{27}$$

where $\mathscr{T}(v)=v-\dfrac{g(v)}{g'(v)}$. If the iteration process (27) converges to any fixed point $p\in\mathbb{C}$ of $\mathscr{T}$, then one has

$$p=\mathscr{T}(x)=p-\frac{g(p)}{g'(p)} \tag{28}$$

Thus, $\dfrac{g(p)}{g'(p)}=0$, which means that p is a root of g. Finding the fixed points of T is therefore equal to solving the problem of finding the roots of g. This enables us to use different fixed point iteration processes, such as the suggested Picard-S_n iteration.

Now, we apply the algorithm given as a pseudocode in Algorithm 1 to produce a polynomiograph. We color the points using the so called iteration coloring [21]. In this coloring technique, color assigning to each starting point is accomplished on the basis of number of iterations completed. As a result, the generated polynomiograph provides information about the speed of convergence of the iteration process, recorded as Average Number of Iterations (ANI) [27].

Algorithm 1 Creation of a polynomiograph.

Input: $g \in \mathbb{C}[Z]$, $\deg g \geq 2$-polynomial; L-iterative method; $S \subset \mathbb{C}$-region; K-maximum number of iterations; ε-precision; colors-color map.

Output: Polynomiograph of the complex-valued polynomial g over the region S.

```
for v_0 ∈ S do
    n = 0
    while |g(v_n)| > ε and n < K do
        v_{n+1} = L(v_n, g)
        n = n + 1
    Assign a color to n from the color map colors and apply it to v_0.
```

In the considered example, we use three sets of iterations' parameters

• $\alpha = 0.05$, $\beta = 0.05$, $\gamma = 0.05$;
• $\alpha = 0.5$, $\beta = 0.5$, $\gamma = 0.5$;
• $\alpha = 0.9$, $\beta = 0.9$, $\gamma = 0.9$.

For each of the three sets of iteration parameters, we created polynomiographs for the polynomial $g(v) = v^5 - 1$ using the Picard-P, Picard-Noor, Abbas, and P iterations found in the literature. We used the following parameters: area $S = [-5, 5]^2$, maximum number of iterations $K = 25$, $\varepsilon = 0.001$ and a color map given in Figure 3.

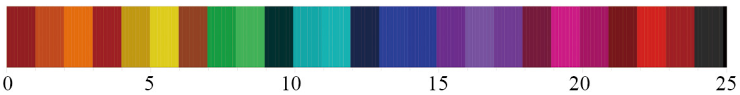


Figure 3. Colour map used in the examples

The produced polynomiographs for each of the three parameter values are shown in Figures 4-6 while the Table 2 contains the ANI values that were recorded from the polynomiographs. We can note distinct convergence rate for each iteration. Visual examination reveals that the proposed Picard-P iteration achieves the fastest speed of convergence, followed by the iterations of Picard-Noor, P, and Abbas. The ANI values in Table 2 support these observations. For the parameter values $\alpha = 0.05$, $\beta = 0.05$, $\gamma = 0.05$, we can note that the Picard-P iteration yields the lowest ANI value of 4.94981. The ANI values for other iterations are given as: Abbas (5.09126), Picard-Noor (7.35969) and P (8.84724).

Table 2. ANI values of the polynomiographs given in Figures 4-6

Iteration	$\alpha = \beta = \gamma = 0.05$	$\alpha = \beta = \gamma = 0.5$	$\alpha = \beta = \gamma = 0.9$
Picard-P	4.94981	3.63054	2.91260
Picard-Noor	7.35969	4.51131	3.23865
P	8.84724	5.64512	4.08255
Abbas	5.09126	4.65075	4.53038

The polynomiographs for the parameter setting $\alpha = 0.5$, $\beta = 0.5$, $\gamma = 0.5$ are shown in the Figure 5. We can notice that the P iteration yields the slowest speed of convergence. The Picard-P iteration method is the fastest of the other iterations that have been studied. The Picard-P iteration produces the lowest NAI value of 3.63054. In terms of convergence speed, the Picard-Noor iteration is the second best with the value 4.51131, followed by the Abbas (4.65075), and *P* (5.64512) iterations.

We use high values for the parameters in the third parameter setting. The fastest convergence rate is once more attained by the Picard-P iteration. We can see that unlike for the other two parameter choices, in the high parameter settings, all iterations require fewer iterations to required less number of iterations to reach the roots of the polynomial. The obtained NAI values from these generated polynomiographs for high parameter setting are shown in the Table 2.

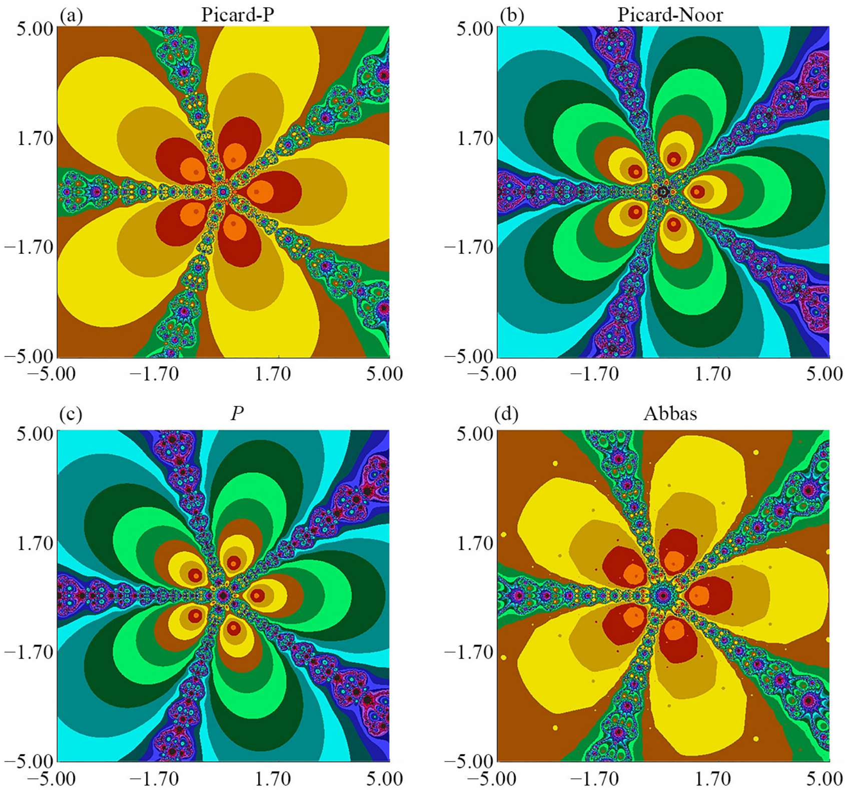
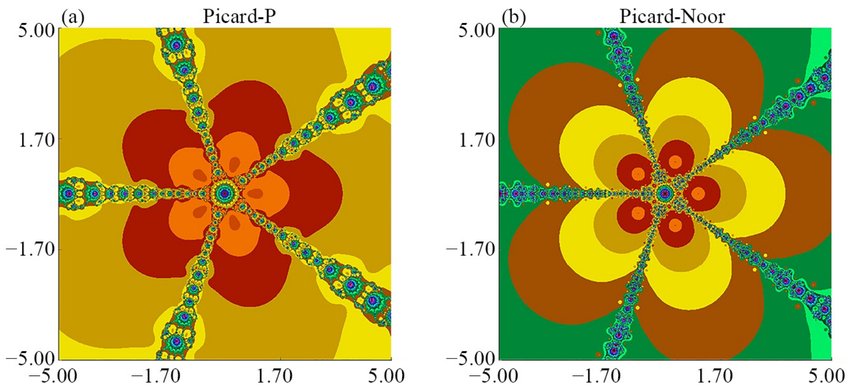


Figure 4. Polynomiographs generated by various iteration processes with the parameters $\alpha = 0.05$, $\beta = 0.05$, $\gamma = 0.05$



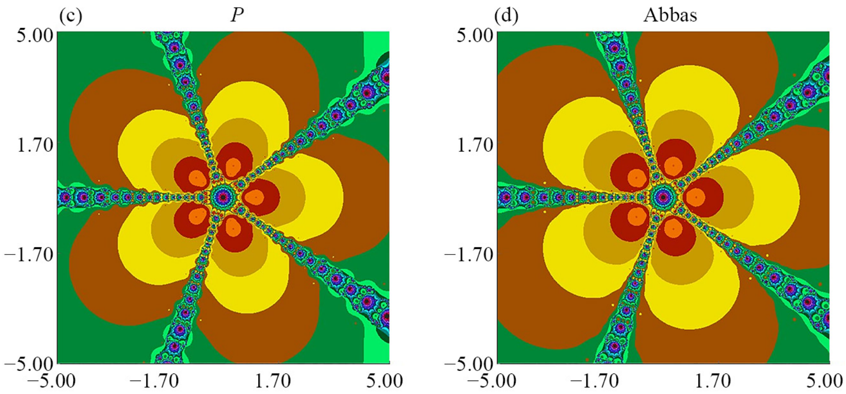


Figure 5. Polynomiographs generated by various iteration processes with the parameters $\alpha = 0.5$, $\beta = 0.5$, $\gamma = 0.5$

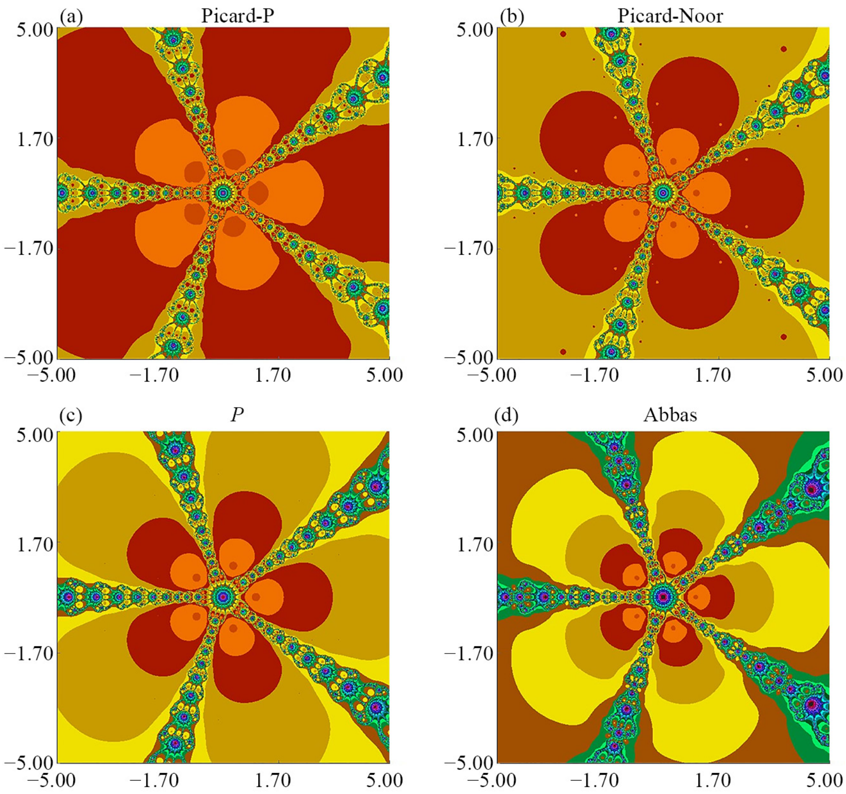


Figure 6. Polynomiographs generated by various iteration processes with the parameters $\alpha = 0.9$, $\beta = 0.9$, $\gamma = 0.9$

7. Conclusions

In view of the above discussion, we noted that the main theorems and outcome of this paper the convergence performance of Picard-P iterative scheme is examined using numerical tabulation and graphs in relationship with SGNE mappings. Strong and weak F.P convergence results using Picard-P iteration scheme for SGNE mappings are proved.

Moreover, iterative schemes find extensive applications beyond fixed-point theory, notably in the generation of fractals [28–30] and artistic patterns [27]. Consequently, a promising direction for future research lies in exploring the potential of the Picard-P iteration process for fractal formation and artistic design.

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Confilict of interest

The authors declare no conflict of interest.

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