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Research Article

Bi-Semiderivations on Triangular Algebra

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Abstract: The intension of present study is to investigate the structure of bi-semiderivations on triangular algebra with the help of associated module homomorphism. Moreover, we prove that all bi-semiderivations φ on triangular algebra will act as inner bi-derivations under some specific conditions stated that $\varphi(e, e) = 0 = \varphi(\gamma, \gamma)$, where e represents unity and γ stands for the module homomorphism.

***Keywords*:** bi-semiderivation, module, triangular algebra

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1. Introduction

A left (right) module $\mathfrak{M}_{\mathfrak{A}}$ is called faithful if $a = 0$ is the only element in $\mathfrak{A}$ such that $a\mathfrak{M} = (0)$ $(\mathfrak{M}a = (0))$. Consider a ring R to be commutative with identity throughout. By a triangular algebra, we mean an algebra of the form

$$\mathfrak{K} = \begin{pmatrix} \mathfrak{A} & \mathfrak{M} \\ 0 & \mathfrak{B} \end{pmatrix} \tag{1}$$

where $\mathfrak{M}$ is an $(\mathfrak{A}, \mathfrak{B})-$bimodule which is faithful as a left $\mathfrak{A}-$module as well as a right $\mathfrak{B}-$module and $\mathfrak{A}$ and $\mathfrak{B}$ are unital algebras over R. The upper triangular matrix algebras and nest algebras are common examples of triangular algebra. To understand the concept well, we pose an example as below:

Example 1 Triangular matrix algebra over different rings

Let $\mathfrak{A} = R[\eta]$ be the polynomial ring over a commutative ring R, and let $\mathfrak{B} = R[\rho]$ be another polynomial ring over R. Define $\mathfrak{M}$ as an $(\mathfrak{A}, \mathfrak{B})$-bimodule where elements of $\mathfrak{M}$ are formal power series $R[\eta, \rho]$ with multiplication rules:

(i) Left multiplication by $\mathfrak{A}$ $(R[\eta])$ follows polynomial multiplication in η.

(ii) Right multiplication by $\mathfrak{B}$ $(R[\rho])$ follows polynomial multiplication in ρ.

Then, the triangular algebra is

$$\mathfrak{K} = \left\{ \begin{pmatrix} \gamma(\eta) & \xi(\eta, \rho) \\ 0 & h(\rho) \end{pmatrix} \middle| \; \gamma(\eta) \in \mathfrak{A}, \; h(\rho) \in \mathfrak{B}, \; \xi(\eta, \rho) \in \mathfrak{M} \right\}. \tag{2}$$

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The main properties of $\mathfrak{A}$ are given below:

• $\mathfrak{A} = R[\eta]$ is a unital algebra over R.

• $\mathfrak{B} = R[\rho]$ is also a unital algebra over R.

• $\mathfrak{M} = R[[\eta, \rho]]$ is an $(\mathfrak{A}, \mathfrak{B})$-bimodule.

• $\mathfrak{M}$ is faithful as a left $\mathfrak{A}$-module and as a right $\mathfrak{B}$-module because multiplication by a nonzero polynomial affects all elements of $\mathfrak{M}$.

For every $\nu, \eta \in \mathfrak{R}$, a mapping $\mathfrak{D} : \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$ is considered symmetric if $\mathfrak{D}(\nu, \eta) = \mathfrak{D}(\eta, \nu)$. If a mapping $\mathfrak{D} : \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$ is additive in both slots, it is often referred to as bi-additive.

We put here the conception of symmetric bi-derivations in a straightforward approach: A mapping $\mathfrak{D} : \mathfrak{R} \times \mathfrak{R} \longrightarrow \mathfrak{R}$, which is biadditive, is referred to as a bi-derivation if, for all $\nu \in \mathfrak{R}$, a mapping $\eta \mapsto \mathfrak{D}(\nu, \eta)$ and, for all $\eta \in \mathfrak{R}$, a mapping $\nu \mapsto \mathfrak{D}(\nu, \eta)$ is a derivation on $\mathfrak{R}$; which can be concluded that $\mathfrak{D}(\nu\eta, z) = \mathfrak{D}(\nu, z)\eta + \nu\mathfrak{D}(\eta, z)$ for all $\nu, \eta, z \in \mathfrak{R}$ and $\mathfrak{D}(\nu, \eta z) = \mathfrak{D}(\nu, \eta)z + \eta\mathfrak{D}(\nu, z)$ for all $z, \nu, \eta \in \mathfrak{R}$. One suggest to look in [1, 2] for details and the references included in it.

In Bergen [3], the concept of semiderivations is defined. Following [2] an additive mapping $f : R \longrightarrow R$ is known as a semi-derivation, if there is a function $g : R \longrightarrow R$ such that $f(\mu\omega) = f(\mu)g(\omega) + \mu f(\omega) = f(\mu)\omega + g(\mu)f(\omega)$, and $f(g(\mu)) = g(f(\mu))$ for any $\mu, \omega \in R$. All semiderivations corresponding to g are merely standard derivations if g is an identity map of R. Following [2], a bi-additive function that is symmetric and is called a symmetric bi-semiderivation associated with a function $g : R \longrightarrow R$ (or simply a symmetric bi-semiderivation of a ring R) if $\varphi : R \times R \longrightarrow R$, $\varphi(\mu\omega, c) = \varphi(\mu, c)g(\omega) + \mu\varphi(\omega, c) = \varphi(\mu, c)\omega + g(\mu)\varphi(\omega, c)$ and $d(g(\mu)) = g(d(\mu))$ and where $d : R \longrightarrow R$ for every $\mu, \omega, c \in R$, is the trace of φ of R.

Next, we define the bi-semiderivation on some commutative algebra as follows to understand the concept well.

Example 2 Consider a commutative algebra $\mathfrak{K}$ over a commutative ring

$$\mathfrak{K} = \begin{bmatrix} \mathfrak{A} & \mathfrak{M} \\ 0 & \mathfrak{B} \end{bmatrix}. \tag{3}$$

Define mappings $\varphi : \mathfrak{K} \times \mathfrak{K} \longrightarrow \mathfrak{K}$ and $\xi : \mathfrak{K} \longleftrightarrow \mathfrak{K}$ such that

$$\varphi(\eta, \rho) = \begin{bmatrix} 0 & \kappa_1\kappa_2 \\ 0 & 0 \end{bmatrix} \text{ and } \xi(\eta) = \begin{bmatrix} \varsigma_1 & \kappa_1 \\ 0 & \tau_1 \end{bmatrix} \text{respectively,} \tag{4}$$

where $\eta = \begin{bmatrix} \varsigma_1 & \kappa_1 \\ 0 & \tau_1 \end{bmatrix}$, and $\rho = \begin{bmatrix} \varsigma_2 & \kappa_2 \\ 0 & \tau_2 \end{bmatrix}$ and $\theta = \begin{bmatrix} \varsigma_3 & \kappa_3 \\ 0 & \tau_3 \end{bmatrix}$. We use the definition of bi-semiderivation and simplify that

$$\varphi(\eta\rho, \theta) = \varphi(\eta, \theta)\rho + \xi(\eta)\varphi(\rho, \theta) \tag{5}$$

$$\varphi\left(\begin{bmatrix} \varsigma_1 & \kappa_1 \\ 0 & \tau_1 \end{bmatrix}\begin{bmatrix} \varsigma_2 & \kappa_2 \\ 0 & \tau_2 \end{bmatrix}, \begin{bmatrix} \varsigma_3 & \kappa_3 \\ 0 & \tau_3 \end{bmatrix}\right) \tag{6}$$

$$= \varphi\left(\begin{bmatrix} \varsigma_1\varsigma_2 & \varsigma_1\kappa_2 + \kappa_1\tau_2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \varsigma_3 & \kappa_3 \\ 0 & \tau_3 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 0 & \kappa_3\varsigma_1\kappa_2 + \kappa_3\kappa_1\tau_2 \\ 0 & 0 \end{bmatrix}$$

$$\varphi\left(\begin{bmatrix} \varsigma_1 & \kappa_1 \\ 0 & \tau_1 \end{bmatrix}, \begin{bmatrix} \varsigma_3 & \kappa_3 \\ 0 & \tau_3 \end{bmatrix}\right)\begin{bmatrix} \varsigma_2 & \kappa_2 \\ 0 & \tau_2 \end{bmatrix} \tag{7}$$

$$+\xi\left(\begin{bmatrix} \varsigma_1 & \kappa_1 \\ 0 & \tau_1 \end{bmatrix}\right)\varphi\left(\begin{bmatrix} \varsigma_2 & \kappa_2 \\ 0 & \tau_2 \end{bmatrix}, \begin{bmatrix} \varsigma_3 & \kappa_3 \\ 0 & \tau_3 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 0 & \kappa_1\kappa_3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \varsigma_2 & \kappa_2 \\ 0 & \tau_2 \end{bmatrix} + \begin{bmatrix} \varsigma_1 & \kappa_1 \\ 0 & \tau_1 \end{bmatrix}\begin{bmatrix} 0 & \kappa_2\kappa_3 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \kappa_1\kappa_3\tau_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \varsigma_1\kappa_2\kappa_3 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \kappa_1\kappa_3\tau_2 + \varsigma_1\kappa_2\kappa_3 \\ 0 & 0 \end{bmatrix}.$$

Similarly, we can verify that

$$\varphi(\eta,\, \rho\theta) = \varphi(\eta,\, \rho)\theta + \xi(\rho)\varphi(\eta,\, \theta). \tag{8}$$

Hence φ is a bi-semiderivation associated with the mapping ξ on $\mathfrak{K}$.

Benkovic [4] presented some results on biderivations on triangular algebra. He showed that every bi-derivation on a triangular algebra will be the sum of an extremal bi-derivation and an inner bi-derivation if specific conditions imposed on the structure. In [5], authors presented the generalization of [4] and obtained the results on characterization of $(\alpha,\, \beta)-$ biderivations on triangular algebras.

In [6], the author proposed abstract triangular algebra linear mappings and established many remarkable results. He finds the structure of derivations, automorphisms, commuting mapping, and Lie derivations of triangular algebras. For a brief review, refer to [6, 7] and the references included within it.

Zhang and Yu [8] proved that every Jordan derivation on a triangular algebra acts like a derivation. Several studies contributed as a refinement of the previous study can be found in [9, 10] on triangular algebra and nest algebra. Some related research on the structural properties of related mappings on rings and their subsets is presented in [11].

Motivated by all the above literature reviews, we intend to investigate that form of bi-semiderivation on triangular algebra. Infact, we prove that every bi-semiderivation will act as inner bi-semiderivation under some conditions.

2. Main results

We start with the following lemmas.

Lemma 1 [4] Every bimodule homomorphism $\gamma : \mathfrak{M} \to \mathfrak{M}$ is of the standard form if every derivation of the triangular algebra $Tri(\mathfrak{A},\, \mathfrak{M},\, \mathfrak{B})$ is inner. That is, there exist $a_0 \in \mathfrak{A},\; b_0 \in \mathfrak{B}$ such that

$$\gamma(m) = a_0 m + m b_0. \tag{9}$$

Lemma 2 Let $\varphi : \mathfrak{K}\times\mathfrak{K}\to\mathfrak{K}$ be a bi-semiderivation linked with the surjective map ξ. Then

$$\varphi(\eta,\,\rho)[\mu,\,\rho] = [\eta,\,\rho]\varphi(\mu,\,\rho) \tag{10}$$

for every $\eta,\ \rho,\ \mu,\ \rho \in \mathfrak{K}$.

Proof. Consider $\varphi(\eta\mu,\ \rho\rho)$, since φ is a semi-derivation in both arguments, then we have

$$\varphi(\eta\mu,\,\rho\rho) = \varphi(\eta,\,\rho\rho)\mu + \xi(\eta)\varphi(\mu,\,\rho\rho), \tag{11}$$

and for every $\eta,\ \rho,\ \mu,\ \rho \in \mathfrak{K}$. On simplification, we find

$$\varphi(\eta\mu,\,\rho\rho) = \varphi(\eta,\,\rho)\rho\mu + \xi(\rho)\varphi(\eta,\,\rho)\mu + \xi(\eta)\varphi(\mu,\,\rho)\rho + \xi(\eta)\xi(\rho)\varphi(\mu,\,\rho). \tag{12}$$

Also simply (11) as below to get

$$\varphi(\eta\mu,\,\rho\rho) = \varphi(\eta,\,\rho)\mu\rho + \xi(\eta)\varphi(\mu,\,\rho)\rho + \xi(\rho)\varphi(\eta,\,\rho)\mu + \xi(\rho)\xi(\eta)\varphi(\mu,\,\rho). \tag{13}$$

Subtracting (12) and (13) to obtain

$$\varphi(\eta,\,\rho)[\mu,\,\rho] - [\xi(\eta),\,\xi(\rho)]\varphi(\mu,\,\rho) = 0. \tag{14}$$

Since ξ is a surjective map on $\mathfrak{A}$, for any elements $\eta,\ \rho \in \mathfrak{K}$, there exist corresponding elements $\eta,\ \rho \in \mathfrak{K}$ such that $\xi(\eta) = \eta$ and $\xi(\rho) = \rho$. Substituting these into the given relation, we obtain

$$\varphi(\eta,\,\rho)[\mu,\,\rho] = [\eta,\,\rho]\varphi(\mu,\,\rho), \tag{15}$$

for each $\eta,\ \rho,\ \mu,\ \rho \in \mathfrak{K}$. □

Lemma 3 Let $\mathfrak{K} = Tri(\mathfrak{A},\ \mathfrak{M},\ \mathfrak{B})$ be a triangular algebra and $\varphi : \mathfrak{K}\times\mathfrak{K}\to\mathfrak{K}$ be a bi-semiderivation associated with the surjective map ξ. If $\eta,\ \rho \in \mathfrak{K}$ such that $[\eta,\ \rho] = 0$ then $\varphi(\eta,\ \rho) = e\varphi(\eta,\ \rho)\gamma + \gamma\varphi(\eta,\ \rho)\gamma$.

Proof. Since we have $\varphi(\eta,\ \rho) \in \mathfrak{K}$, then

$$\varphi(\eta,\,\rho) = e\varphi(\eta,\,\rho)e + e\varphi(\eta,\,\rho)\gamma + \gamma\varphi(\eta,\,\rho)\gamma. \tag{16}$$

Using Lemma 2, we have

$$\varphi(\eta,\ \rho)[e,\ em\gamma] = [a_\eta,\ b_\rho]\varphi(e,\ em\gamma) = 0. \tag{17}$$

Since $m = em = em\gamma$

$$0 = \varphi(\eta,\ \rho)[e,\ em\gamma] = \varphi(\eta,\ \rho)em. \tag{18}$$

Hence

$$e\varphi(\eta,\ \rho)e\mathfrak{M} = 0. \tag{19}$$

However, we can observe that $\mathfrak{M}$ is a faithful left $\mathfrak{K}$ module.

$$e\varphi(\eta,\ \rho)e = 0. \tag{20}$$

Therefore, we reach out at

$$\varphi(\eta,\ \rho) = e\varphi(\eta,\ \rho)\gamma + \gamma\varphi(\eta,\ \rho)\gamma, \text{ for each } \eta,\ \rho,\ \mu,\ \rho \in \mathfrak{K}. \tag{21}$$

□

Lemma 4 Let $\varphi : \mathfrak{K} \times \mathfrak{K} \to \mathfrak{K}$ be a bi-semiderivation associated with the surjective map ξ. Then
1. $\varphi(\eta,\ 1) = 0 = \varphi(1,\ \eta)$ for all $\eta \in \mathfrak{K}$, if $\xi(1) = 1$;
2. $\varphi(\eta,\ 0) = 0 = \varphi(0,\ \eta)$ for all $\eta \in \mathfrak{K}$, if $\xi(0) = 0$.

Proof. To prove (i), consider $\varphi(\eta,\ 1 \cdot 1)$. Using the bi-semiderivation property, we have

$$\varphi(\eta,\ 1 \cdot 1) = \varphi(\eta,\ 1) \cdot 1 + \xi(1)\varphi(\eta,\ 1). \tag{22}$$

Since $1 \cdot 1 = 1$ and $\xi(1) = 1$, this simplifies to

$$\varphi(\eta,\ 1) = \varphi(\eta,\ 1) + \varphi(\eta,\ 1). \tag{23}$$

Subtracting $\varphi(\eta,\ 1)$ from both sides yields

$$\varphi(\eta,\ 1) = 0. \tag{24}$$

Similarly, using the bi-semiderivation property for $\varphi(1 \cdot 1,\ \eta)$, we have

$$\varphi(1\cdot 1,\ \eta) = \varphi(1,\ \eta)\cdot 1 + \xi(1)\cdot\varphi(1,\ \eta). \tag{25}$$

Since $\xi(1) = 1$, this simplifies to

$$\varphi(1,\ \eta) = \varphi(1,\ \eta) + \varphi(1,\ \eta), \tag{26}$$

which again implies

$$\varphi(1,\ \eta) = 0. \tag{27}$$

We can conclude that $\varphi(\eta,\ 1) = 0 = \varphi(1,\ \eta)$. Thus, (i) is proved.

To prove (ii), consider $\varphi(\eta,\ 0\cdot 0)$. Using the bi-semiderivation property, we have

$$\varphi(\eta,\ 0\cdot 0) = \varphi(\eta,\ 0)\cdot 0 + \xi(0)\varphi(\eta,\ 0). \tag{28}$$

Since $0\cdot 0 = 0$ and $\xi(0) = 0$, this simplifies to

$$\varphi(\eta,\ 0) = 0. \tag{29}$$

Similarly, using the definition of bi-semiderivation for $\varphi(0\cdot 0,\ \eta)$, we have

$$\varphi(0\cdot 0,\ \eta) = \varphi(0,\ \eta)\cdot 0 + \xi(0)\cdot\varphi(0,\ \eta). \tag{30}$$

Since $\xi(0) = 0$, this simplifies to

$$\varphi(0,\ \eta) = 0. \tag{31}$$

We can conclude that $\varphi(\eta,\ 0) = 0 = \varphi(0,\ \eta)$. Thus, (ii) is proved.

This completes the proof. □

Theorem 1 Consider a triangular algebra $\mathfrak{K} = Tri(\mathfrak{A},\ \mathfrak{M},\ \mathfrak{B})$. If the subsequent circumstances are met:

1. There exists two natural projections $\pi_{\mathfrak{A}} : \mathfrak{K} \longrightarrow \mathfrak{A}$ and $\pi_{\mathfrak{B}} : \mathfrak{K} \longrightarrow \mathfrak{B}$ such that $\pi_{\mathfrak{A}}(Z(\mathfrak{K})) = Z(\mathfrak{A})$ and $\pi_{\mathfrak{B}}(Z(\mathfrak{K})) = Z(\mathfrak{B})$,

2. $\mathfrak{A}$ or $\mathfrak{B}$ is non commutative,

3. for $\alpha \in Z(\mathfrak{K}),\ 0 \neq a \in \mathfrak{K}$, if $\alpha a = 0$, then $\alpha = 0$,

4. every derivation of $\mathfrak{A}$ is inner, then every bi-semiderivation $\varphi : \mathfrak{K}\times\mathfrak{K} \to \mathfrak{K}$ associated with the surjective function ξ that satisfies $\varphi(e,\ e) = 0 = \varphi(\gamma,\ \gamma)$ is an inner bi-derivation.

Proof. Let $\eta,\ \rho \in \mathfrak{A}$, we can write $\eta = \kappa + \varpi + b$ and $\rho = \kappa\prime + \varpi\prime + b\prime$, where $\kappa,\ \kappa\prime \in \mathfrak{A}, \varpi,\ \varpi\prime \in \mathfrak{M},\ b,\ b\prime \in \mathfrak{B}$.

From the bilinearity of φ, we reach out at

$$
\begin{aligned}
\varphi(\eta,\ \rho) &= \varphi(\kappa+\varpi+b,\ \kappa\prime+\varpi\prime+b\prime)\\
&= \varphi(\kappa,\ \kappa\prime)+\varphi(\kappa,\ \varpi\prime)+\varphi(\kappa,\ b\prime)+\varphi(\varpi,\ \kappa\prime)+\varphi(\varpi,\ \varpi\prime)\\
&\quad +\varphi(\varpi,\ b\prime)+\varphi(b,\ \kappa\prime)+\varphi(b,\ \varpi\prime)+\varphi(b,\ b\prime)
\end{aligned} \tag{32}
$$

for each $\eta,\ \rho\in\mathfrak{R}$. We will explore the all conditions of bi-semiderivation. Firstly, we find out the value of $\varphi(\kappa,\ b)$ and $\varphi(b,\ \kappa)$ for every $\kappa\in\mathfrak{A},\ b\in\mathfrak{B}$.

Lemma 3, enables us to write

$$
\begin{aligned}
\varphi(\kappa,\ b) &= e\varphi(\kappa,\ b)\gamma+\gamma\varphi(\kappa,\ b)\gamma\\
&= e\varphi(\kappa e,\ \gamma b)\gamma+\gamma\varphi(\kappa e,\ b)\gamma\\
&= e\varphi(\kappa,\ \gamma b)\gamma+e\xi(\kappa)\varphi(e,\ \gamma b)\gamma+\gamma\varphi(\kappa,\ b)e\gamma+\gamma\xi(\kappa)\varphi(e,\ b)\gamma\\
&= e\varphi(\kappa,\ \gamma)b\gamma+e\xi(\gamma)\varphi(\kappa,\ b)\gamma+e\xi(\kappa)\varphi(e,\ \gamma)b\gamma\\
&\quad +e\xi(\kappa)\xi(\gamma)\varphi(e,\ b)\gamma+\gamma\varphi(\kappa,\ b)e\gamma+\gamma\xi(\kappa)\varphi(e,\ b)\gamma
\end{aligned} \tag{33}
$$

We deduce that

$$
\varphi(\kappa,\ b)=0,\ \text{for all } \kappa\in\mathfrak{A},\ b\in\mathfrak{B}. \tag{34}
$$

Again, consider

$$
\begin{aligned}
\varphi(b,\ \kappa) &= e\varphi(b,\ \kappa)\gamma+\gamma\varphi(b,\ \kappa)\gamma\\
&= e\varphi(b,\ \kappa e)\gamma+\gamma\varphi(b,\ \kappa e)\gamma\\
&= e\varphi(b,\ \kappa)e\gamma+e\xi(\kappa)\varphi(b,\ e)\gamma+\gamma\varphi(b,\ \kappa)e\gamma+\gamma\xi(\kappa)\varphi(b,\ e)\gamma.
\end{aligned} \tag{35}
$$

We find that

$$
\varphi(b,\ \kappa)=0,\ \text{for every } \kappa\in\mathfrak{A},\ b\in\mathfrak{B}. \tag{36}
$$

Next claim is to compute the values of $\varphi(\kappa,\ \varpi)$ and $\varphi(\varpi,\ b)$, for every $\kappa \in \mathfrak{A},\ b \in \mathfrak{B},\ \varpi \in \mathfrak{M}$. Define a map as $h : \mathfrak{M} \to \mathfrak{M}$ as $h(\varpi) = \varphi(e,\ \varpi)$ for every $\varpi \in \mathfrak{M}$, and let $\xi(\eta) = \eta$ for all $\eta \in \mathfrak{A}$. In the case, h will seen as a bimodule homomorphism. For every $\kappa \in \mathfrak{A},\ b \in \mathfrak{B}$ and $\varpi \in \mathfrak{M}$, we notice that

$$\begin{aligned} h(\kappa\varpi) &= \varphi(e,\ \kappa\varpi) \\ &= \varphi(e,\ \kappa)\varpi + \xi(\kappa)\varphi(e,\ \varpi) = \kappa\varphi(e,\ \varpi) \end{aligned} \tag{37}$$

Since,

$$\begin{aligned} \varphi(e,\ \varpi) &= \varphi(e,\ e\varpi) \\ &= \varphi(e,\ e)\varpi + \xi(e)\varphi(e,\ \varpi) \\ &= e\varphi(e,\ \varpi). \end{aligned} \tag{38}$$

Thus, we have,

$$h(\kappa\varpi) = \kappa\varphi(e,\ \varpi) = \kappa h(\varpi) \tag{39}$$

and

$$\begin{aligned} h(\varpi b) &= \varphi(e,\ \varpi b) \\ &= \varphi(e,\ \varpi)b + \xi(\varpi)\varphi(e,\ b) \\ &= \varphi(e,\ \varpi)b \\ &= h(\varpi)b \end{aligned} \tag{40}$$

Make use of Lemma 1, the standard form of h look like as

$$h(\varpi) = \kappa_0\varpi + \varpi b_0,\ \kappa_0 \in Z(\mathfrak{A}),\ b_0 \in Z(\mathfrak{B}). \tag{41}$$

Next, we may simplify that

$$\varphi(e,\ \varpi) = h(\varpi) = (\kappa_0 + \tau^{-1}(b_0))\varpi = \alpha\varpi,\ \text{for all } \varpi \in \mathfrak{M} \tag{42}$$

where $\alpha = \kappa_0 + \tau^{-1}(b_0) \in \pi_{\mathfrak{A}}(Z(\mathfrak{A}))$. In similar way, a mapping $L : \mathfrak{M} \to \mathfrak{M}$ define such that $L(\varpi) = \varphi(\varpi,\ e)$ for every $\varpi \in \mathfrak{M}$, will be a bimodule homomorphism. There must exists a $\beta \in \pi_{\mathfrak{A}}(Z(\mathfrak{A}))$ such that

$$\varphi(\varpi,\ e) = \beta\varpi \text{ for every } \varpi \in \mathfrak{M}. \tag{43}$$

Next claim to prove that

$$\varphi(e,\ \varpi) = \alpha\varpi = -\varphi(\varpi,\ e). \tag{44}$$

To do this, to prove that,

$$\alpha + \beta = 0, \tag{45}$$

is sufficient. By taking the condition (ii), we assume that, $\mathfrak{A}$ is an algebra and is non-commutative.

Choose $\kappa,\ \kappa\prime \in \mathfrak{A}$ such that

$$[\kappa,\ \kappa\prime] \neq 0. \tag{46}$$

Since $\varphi(e,\ \varpi) = \alpha\varpi$ and $\varphi(\varpi,\ e) = \beta\varpi$, using Lemma 2, we have

$$\begin{aligned}\varphi(\kappa,\ \kappa\prime)[e,\ \varpi] &= [\kappa,\ \kappa\prime]\varphi(e,\ \varpi)\\ &= \alpha[\kappa,\ \kappa\prime]\varpi\end{aligned} \tag{47}$$

and

$$\begin{aligned}\varphi(\kappa,\ \kappa\prime)[\varpi,\ e] &= [\kappa,\ \kappa\prime]\varphi(e,\ \varpi)\\ &= \beta[\kappa,\ \kappa\prime]\varpi\end{aligned} \tag{48}$$

for all $\varpi \in \mathfrak{M}$. From equations (47) and (48), we arrive at

$$(\alpha + \beta)[\kappa,\ \kappa\prime]\mathfrak{M} = 0. \tag{49}$$

Since the left $\mathfrak{A}$ module $\mathfrak{M}$ is faithful, which indicates that

$$(\alpha + \beta)[\kappa,\ \kappa\prime] = 0. \tag{50}$$

Moreover, $[\kappa,\ \kappa\prime] \neq 0$, we draw the conclusion after putting condition (iii) that

$$\alpha + \beta = 0 \tag{51}$$

So we have,

$$\varphi(e,\ \varpi) = \alpha\varpi = -\varphi(\varpi,\ e). \tag{52}$$

Now, in case $\mathfrak{B}$ is non-commutative, then

$$\begin{aligned} e[e,\ \varpi]\varphi(b,\ b\prime) &= e\varphi(e,\ \varpi)[b,\ b\prime] \\ &= \varphi(e,\ \varpi)[b,\ b\prime] \\ &= \alpha\varpi[b,\ b\prime] \end{aligned} \tag{53}$$

Also,

$$\begin{aligned} e[e,\ \varpi]\varphi(b,\ b\prime) &= e\varphi(e,\ \varpi)[b,\ b\prime] \\ &= \varphi(e,\ \varpi)[b,\ b\prime] \\ &= \beta\varpi[b,\ b\prime] \end{aligned} \tag{54}$$

Making use of equation (53) and (54), we get

$$\mathfrak{M}(\alpha + \beta)[b,\ b\prime] = 0. \tag{55}$$

By the faithful property of right $\mathfrak{B}$ module of $\mathfrak{M}$, we have

$$(\alpha + \beta)[b,\ b\prime] = 0. \tag{56}$$

Again making use of condition (iii), we obtain

$$(\alpha + \beta) = 0. \tag{57}$$

We show that in a similar pattern that

$$\varphi(\gamma,\ \varpi) = \alpha\varpi = -\varphi(\varpi,\ \gamma). \tag{58}$$

Let $\kappa \in \mathfrak{A}$ and $\varpi \in \mathfrak{M}$ are arbitrary elements, then we may reword the expression

$$\begin{aligned} \varphi(\kappa,\ \varpi) &= \varphi(\kappa e,\ \varpi) \\ &= \varphi(\kappa,\ \varpi)e + \xi(\kappa)\varphi(e,\ \varpi) \\ &= \varphi(\kappa,\ \varpi)e + \kappa\varphi(e,\ \varpi) \end{aligned} \tag{59}$$

Since the inequality $\varphi(\kappa,\ \gamma) = 0$ implies

$$\begin{aligned} \varphi(\kappa,\ \varpi)e &= \varphi(\kappa,\ \varpi\gamma)e \\ &= \varphi(\kappa,\ \varpi)\gamma e + \xi(\varpi)\varphi(\kappa,\ \gamma)e \end{aligned} \tag{60}$$

By equation (59)

$$\begin{aligned} \varphi(\kappa,\ \varpi) &= \varphi(e,\ \varpi) \\ &= \kappa\varphi(e,\ \varpi) \\ &= \alpha\kappa\varpi \end{aligned} \tag{61}$$

Similarly, we can prove that

$$\varphi(\varpi,\ b) = -\varphi(b,\ \varpi) = \alpha\varpi b. \tag{62}$$

Now, compute the values of $\varphi(\kappa,\ \kappa\prime)$ and $\varphi(b,\ b\prime)$.
Since we utilize

$$\begin{aligned} \varphi(\kappa,\ e) &= \varphi(\kappa,\ 1-\gamma) \\ &= \varphi(\kappa,\ 1) - \varphi(\kappa,\ \gamma) \\ &= 0 \end{aligned} \tag{63}$$

We have,

$$\begin{aligned}\varphi(\kappa,\ e) &= \varphi(e\kappa,\ e)\\ &= \varphi(e,\ e)\kappa + \xi(e)\varphi(\kappa,\ e)\\ &= 0 = \varphi(e,\ \kappa).\end{aligned} \tag{64}$$

Now,

$$\begin{aligned}\varphi(\kappa,\ \kappa\prime) &= \varphi(e(\kappa e),\ \kappa\prime)\\ &= \varphi(e,\ \kappa\prime)\kappa e + \xi(e)\varphi(\kappa e,\ \kappa\prime)\\ &= \varphi(e,\ \kappa\prime)\kappa e + \xi(e)\xi(\kappa)\varphi(e,\ \kappa\prime) + \xi(e)\varphi(\kappa,\ \kappa\prime)e\\ &= e\varphi(\kappa,\ \kappa\prime)e \in e\mathfrak{K}e = \mathfrak{A}\end{aligned} \tag{65}$$

Which implies that

$$\varphi(\kappa,\ \kappa\prime) \in \mathfrak{A}. \tag{66}$$

By Lemma 3, we have

$$\begin{aligned}\varphi(\kappa,\ \kappa\prime)[e,\ \varpi] &= [\kappa,\ \kappa\prime]\varphi(e,\ \varpi)\\ &= \alpha[\kappa,\ \kappa\prime]\varpi\end{aligned} \tag{67}$$

Which gives that

$$(\varphi(\kappa,\ \kappa\prime) - \alpha[\kappa,\ \kappa\prime])\mathfrak{M} = 0, \tag{68}$$

and by using the faithful property of the left $\mathfrak{A}$ module $\mathfrak{M}$, we obtain

$$\varphi(\kappa,\ \kappa\prime) = \alpha[\kappa,\ \kappa\prime] \text{ for each } \kappa,\ \kappa\prime \in \mathfrak{A}. \tag{69}$$

Similarly, $\varphi(b,\ b\prime) \in \gamma\mathfrak{K}\gamma = \mathfrak{B}$ and

$$
\begin{aligned}
e[e,\ \varpi]\varphi(b,\ b\prime) &= e\varphi(e,\ \varpi)[b,\ b\prime] \\
&= \varphi(e,\ \varpi)[b,\ b\prime] \\
&= \alpha\varpi[b,\ b\prime]
\end{aligned}
\tag{70}
$$

for all $\varpi \in \mathfrak{M}$.

Hence $\mathfrak{M}(\varphi(b,\ b\prime) - \tau(\alpha)[b,\ b\prime]) = 0$. The faithfulness of right $\mathfrak{B}$ module $\mathfrak{M}$ implies

$$
\varphi(b,\ b\prime) = \tau(\alpha)[b,\ b\prime] \tag{71}
$$

Now, the only remaining part is to find the value of $\varphi(\varpi,\ n)$

$$
\begin{aligned}
\varphi(\varpi,\ n) &= \varphi(e\varpi,\ n) \\
&= \varphi(e,\ n)\varpi + \xi(e)\varphi(\varpi,\ n) \\
&= \alpha n\varpi + 0 \\
&= 0
\end{aligned}
\tag{72}
$$

This implies that $\varphi(\varpi,\ n) = 0$.

At the end, let $\lambda = \alpha + \tau(\alpha) \in Z(\mathfrak{A})$. Putting all the simplified values in equation (32)

$$
\begin{aligned}
\varphi(\eta,\ \rho) &= \alpha[\kappa,\ \kappa\prime] + \alpha\kappa\varpi\prime - \alpha\kappa\prime\varpi + \alpha\varpi b\prime - \alpha\varpi\prime b + \tau(\alpha)[b,\ b\prime] \\
&= \begin{pmatrix} \alpha & 0 \\ 0 & \tau(\alpha) \end{pmatrix} \left[\begin{pmatrix} \kappa & \varpi \\ & b \end{pmatrix},\ \begin{pmatrix} \kappa\prime & \varpi\prime \\ & b\prime \end{pmatrix} \right] \\
&= \lambda[\eta,\ \rho]
\end{aligned}
\tag{73}
$$

for all $\eta,\ \rho \in \mathfrak{K}$. To verify above notation, we use the matrix representation of η and ρ, we compute the commutator $[\eta,\ \rho]$

$$
[\eta,\ \rho] = \begin{pmatrix} [\kappa,\ \kappa'] & \kappa\varpi' - \kappa'\varpi + \varpi b' - \varpi' b \\ 0 & [b,\ b'] \end{pmatrix}. \tag{74}
$$

Substituting into $\phi(\eta,\ \rho)$, we obtain

$$\phi(\eta, \rho) = \begin{pmatrix} \alpha[\kappa, \kappa'] & \alpha(\kappa\varpi' - \kappa'\varpi + \varpi b' - \varpi' b) \\ 0 & \tau(\alpha)[b, b'] \end{pmatrix}. \tag{75}$$

Let $\lambda = \begin{bmatrix} \alpha & 0 \\ 0 & \tau(\alpha) \end{bmatrix}$. Then, we conclude that

$$\begin{aligned} \phi(\eta, \rho) &= \begin{bmatrix} \alpha & 0 \\ 0 & \tau(\alpha) \end{bmatrix} \left[\begin{pmatrix} \kappa & \varpi \\ 0 & b \end{pmatrix}, \begin{pmatrix} \kappa' & \varpi' \\ 0 & b' \end{pmatrix} \right] \\ &= \lambda[\eta, \rho] \end{aligned} \tag{76}$$

Hence every bi-semiderivation φ will be of the form of an inner biderivation provided that $\varphi(e, e) = 0 = \varphi(\gamma, \gamma)$.□

3. Conclusion

The aim of the present research is to characterize the structure of bi-semiderivation on triangular algebra. As an application, we obtain the result that every bi-semiderivation on triangular algebra will be of the form of inner derivation. For further study, it will be fascinating to observe the action of such mappings on triangular matrix algebra, nest algebra, block matrices (upper triangular), etc. Our results are also applied to generalized triangular matrix algebra, ternary mapping, and related topics.

Use of AI tools declaration

The authors affirm that no Artificial Intelligence (AI) techniques were utilized in the writing of this paper.

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Conflict of interest

The authors declared that there are no conflicts of interest.

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