Contemporary Mathematics
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UNIVERSAL WISER PUBLISHER

Research Article

Applications of Symmetric Quantum Calculus to Multivalent Functions in Geometric Function Theory

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Received: 11 June 2025; **Revised:** 8 July 2025; **Accepted:** 11 July 2025

Abstract: This paper investigates multivalent analytic functions through the lens of symmetric quantum calculus. Using a generalized symmetric operator, we present novel classes of multivalent starlike functions in the framework of symmetric q-calculus linked with Janowski-type functions. We establish inclusion relationships among these classes and derive an adequate condition on coefficients for class membership. Towards the end of the paper, we develop several results on differential subordination, superordination, and sandwich-type theorems involving the same operator framework. These results include sharp bounds and identification of extremal dominant and subordinant functions. The work highlights the versatility of symmetric q-analytical approaches in geometric function analysis and provides a unified approach that extends several known existing contributions in the field.

***Keywords*:** analytic functions, multivalent functions, Janowski functions, symmetric q-calculus, symmetric q-differential operator, differential subordination, differential superordination

MSC: 30C45, 30C50, 30C80

1. Background and motivation

The field of symmetric q-calculus has garnered significant growing focus in the past few years in light of its widespread use over various mathematical disciplines, encompassing fractional calculus, Geometric Function Theory (GFT), mathematical physics, and quantum mechanics [1]. A central reason for this rising interest is the symmetric q-derivative's ability to approximate classical derivatives of differentiable functions, often demonstrating optimized convergence behavior versus the standard q-derivative, despite the fact that this advantage remains under ongoing investigation.

As an extension of classical q-calculus, symmetric q-calculus introduces enhanced operator symmetry, making it particularly well-suited for exploring function classes with inherent symmetries. The primary roots of q-calculus trace back to the work of Jackson [2], who pioneered the theory of quantum-calculus counterparts to differential and integral operators in the early 20th century. In 1990, building on this groundwork, Ismail and collaborators [3] introduced the notion of q-starlikeness, a key influence on the development of modern GFT.

Over time, this concept has evolved, giving rise to a diverse range of results characterizing the structure and behavior of q-starlike and related function classes. For example, Arif et al. [4] defined the Noor integral operator integral operator in

DOI: https://doi.org/10.37256/cm.6420257431

the q-calculus setting using convolution techniques to enrich the theory through new analytic function subclasses. Parallel studies, such as [5], examined generalized differential operators in the q-calculus setting and their application to newly defined analytic families. Srivastava et al. [6] further advanced the field by defining families of q-symmetric harmonic functions constructed through q-differentiation. Wongsaijai and Sukantamala [7] conducted a comprehensive analysis of well-defined subclasses of functions exhibiting q-starlikeness, approaching the topic from varied theoretical angles, also offering valuable perspectives on their analytic features. Srivastava et al. ([8, 9]) utilizing q-calculus and Janowski functions, introduced and investigated three distinct subclasses of q-starlike functions. A number of other researchers (see [10–15]) have also investigated and expanded the theory of q-starlike functions, approaching the topic from diverse analytical frameworks and perspectives.

Several significant advancements have arisen from applying symmetrized quantum calculus approach to geometric function theory. For instance, Kanas et al. [16] proposed symmetric q-extensions of classical operators of differentiation, using them to develop new families of functions with starlikeness and convexity properties, establishing a foundation for subsequent extensions.

Expanding on this, Khan et al. [17] revisited the generalized symmetrically structured conic region, incorporating a symmetric variant of the q-difference calculus to refine its structure. Their work led to the formulation of a new family of symmetric q-starlike analytic mappings, contributing substantially to the understanding of symmetric q-operators, a variety of subclasses of symmetric q-deformed starlike and convex functions, along with novel approaches for enlarging conic structures within the unit disk. In related developments, Sabir et al. [18] provided a systematic investigation of bi-univalent functions with m-fold symmetry employing symmetric form of quantum difference calculus methods, deriving essential structural results.

In the latest developments, Khan et al. [19] applied symmetrized q-derivative operators to multivalent functions, while another study [20], employed these tools in defining generalized symmetric conic domains and new function subclasses throughout the unit disk. These findings deepen theoretical understanding and contribute to new geometric interpretations within GFT. For a survey of recent work, the reader is referred to [21–24].

The purpose of this paper is twofold. First, we define and investigate new subclasses concerning multivalent starlike functions constructed through a symmetric q-differentiation framework that generalizes the classical Al-Oboudi operator, associated with Janowski-type functions. For these classes, we derive inclusion results and obtain a sufficient coefficient condition for class membership. Second, we develop a unified operator-based framework for establishing differential subordination and superordination results. This includes the identification of best dominants and subordinants, as well as a sandwich-type theorem.

The results presented herein generalize several classical findings in the literature (see [7, 10, 11, 19, 25–28]) and illustrate the role of symmetric quantum calculus in the study of multivalent theory of analytic functions.

Differential subordination and superordination are foundational to GFT, offering powerful tools for comparing analytic functions via their response to differential operators. These theories trace back to Lindelöf's concept of subordination in 1909 and were rigorously formalized by Miller and Mocanu ([29, 30]) in the late 20th century. During 2003, they introduced the complementary concept of differential superordination, leading to the formulation of sandwich-type theorems.

Recent research has extended these theories to higher-order cases and q-analogues of classical operators. This has opened new research avenues, particularly in symmetric function theory and geometric interpretations of q-operators (see, e.g. [31, 32]). Applications extend beyond pure mathematics, impacting fields like signal processing, system identification, fluid dynamics, and approximation theory. For a comprehensive overview of recent developments, see [33–35].

2. Basic concepts and terminology

Let $\mathscr{H}(U)$ denote the set of all complex-valued mappings possessing analyticity in the unit disk

$$U = \{\omega \in \mathbb{C}: |\omega| < 1\}.$$

Within this general class, we define the subclass $\mathscr{H}(\upsilon, \tau) \subset \mathscr{H}(U)$ to consist of all functions that admit an expansion in powers of ω written as

$$\Upsilon(\omega) = \upsilon + \upsilon_\tau \omega^\tau + \upsilon_{\tau+1}\omega^{\tau+1} + \ldots, \, \omega \in U \; ,$$

where $\upsilon \in \mathbb{C}$ and $\tau \in \mathbb{N} = \{1, 2, 3....\}$. Furthermore, we denote by $\mathscr{A}_\tau$ as a particular subclass of $\mathscr{H}(U)$ containing functions normalized in a manner that

$$\mathscr{A}_\tau = \{\Upsilon \in \mathscr{H}(U): \Upsilon(\omega) = \omega^\tau + \upsilon_{\tau+1}\omega^{\tau+1} + \ldots, \, \omega \in U\}.$$

The case $\tau = 1$ corresponds to the classical set of normalized holomorphic functions, which we denote by $\mathscr{A} = \mathscr{A}_1$. Let us consider S, a subfamily of $\mathscr{A}_\tau$ that comprises injective holomorphic mappings on the unit disk U. The notation $\mathscr{S}^*(\tau)$ is used to represent the set of all normalized multivalent starlike functions with valency τ. Let $\Upsilon \in \mathscr{A}_\tau$; the function is termed starlike if the following criterion is fulfilled:

$$Re\left\{\frac{\omega\Upsilon'(\omega)}{\Upsilon(\omega)}\right\} > 0, \;\; \omega \in U.$$

We note that $\mathscr{S}^*(1) = \mathscr{S}^*$, the subclass of analytic mappings with starlike behavior in U. We proceed by recalling that Janowski introduced a generalization of the starlike function class $\mathscr{S}^*$in the following manner:

Definition 1 [36] We say that $\Upsilon \in \mathscr{A}$ lies in the subclass $\mathscr{S}^*(\Lambda, X)$ precisely when the following condition holds:

$$\mathrm{Re}\left(\frac{(X-1)\frac{\omega\Upsilon'(\omega)}{\Upsilon(\omega)} - (\Lambda-1)}{(X+1)\frac{\omega\Upsilon'(\omega)}{\Upsilon(\omega)} - (\Lambda+1)}\right) \geq 0, \; -1 \leq X < \Lambda \leq 1.$$

We remind the reader that differential subordination involves finding conditions under which a holomorphic function $\Upsilon(\omega)$ is dominated by $T(\omega)$ in the sense of subordination, denoted $\Upsilon \prec T$, if a function μ, analytic in U and satisfying the Schwarz conditions, exists, such that

$$\mu(0) = 0,$$

$$|\mu(\omega)| < 1, \;\text{ for all } \omega \in U,$$

$$\text{and } \Upsilon(\omega) = T(\mu(\omega)), \text{ for all } \omega \in U.$$

If, in addition, T is univalent (i.e., one-to-one) in the open unit disk, then the relation $\Upsilon \prec T$ can be characterized by $\Upsilon(0) = T(0)$ and $\Upsilon(U) \subset T(U)$.

Let Φ: $\mathbb{C}^3 \times U \to \mathbb{C}$ be a given function and denote by P a one-to-one holomorphic function on the unit disk. Let s be holomorphic in the open unit disk and assume it fulfills a differential subordination of second order having the form

$$\Phi(s(\omega),\, \omega s'(\omega),\, \omega^2 s''(\omega);\, \omega) \prec P(\omega),\, \omega \in U, \tag{1}$$

hence s is considered a solution of the second-order subordination relation involving derivatives, defined by (1). A function u, univalent in U is regarded as the dominant function for every solution of (1), provided that

$$s \prec u, \text{ for all } s \text{ satisfying (1).}$$

Moreover, if u is the smallest (in the subordination sense) among all such dominants—i.e., if

$$u(\omega) \prec v(\omega), \text{ for all dominants } v \text{ satisfying (1),}$$

then u is termed the sharpest dominant. The extremal dominant is unique aside from rotational symmetry in the unit disk.

Consider a function Φ: $\mathbb{C}^3 \times U \to \mathbb{C}$ and assume that P is a holomorphic function in U. Provided that a function s is analytic in U, and both s and

$$\Phi\left(s(\omega),\, \omega s'(\omega),\, \omega^2 s''(\omega);\, \omega\right)$$

are one-to-one in U and assuming that the inequality

$$P(\omega) \prec \Phi(s(\omega),\, \omega s'(\omega),\, \omega^2 s''(\omega);\, \omega),\, \omega \in U \tag{2}$$

holds, it follows that s satisfies the second-order differential inclusion (in the sense of superordination) (2). Here, if $\Upsilon \prec T$, then T is referred to as being superordinate to Υ (see [29]). In this context, function u, being analytic, qualifies as a subordinant provided that

$$u \prec s, \text{ given any } s \text{ meeting (2).}$$

If u is the greatest (with respect to subordination) among all such subordinants—i.e.

$$v(\omega) \prec u(\omega), \text{ for every subordinant } v \text{ satisfying (2),}$$

then u is known as the most suitable subordinant.

Requirements for the functions P, u and Φ ensuring the inference

$$P(\omega) \prec \Phi(s(\omega), \omega s'(\omega), \omega^2 s''(\omega); \omega) \Rightarrow u(\omega) \prec s(\omega)$$

holds, were established by Miller and Mocanu (see [30]). In this case, u acts as the optimal subordinant corresponding to the prescribed differential superordination. What follows is the formal definition of the Al-Oboudi operator, which plays a key role in our forthcoming results.

Definition 2 (Al Oboudi [37]) For any function $\Upsilon \in \mathscr{A}$, with $\lambda \geq 0$ and $n \in \mathbb{N}$, the operator D_λ^m is introduced as a self-map on $\mathscr{A}$,

$$\begin{aligned} D_\lambda^0 \Upsilon(\omega) &= \Upsilon(\omega), \\ D_\lambda^1 \Upsilon(\omega) &= (1-\lambda)\Upsilon(\omega) + \lambda\omega\Upsilon'(\omega) = D_\lambda \Upsilon(\omega), \\ &\dots \\ D_\lambda^m \Upsilon(\omega) &= (1-\lambda) D_\lambda^{m-1}\Upsilon(\omega) + \lambda\omega\left(D_\lambda^m \Upsilon(\omega)\right)' = D_\lambda\left(D_\lambda^{m-1}\Upsilon(\omega)\right), \text{ for } \omega \in U. \end{aligned}$$

Remark 1 [37] If $\Upsilon \in \mathscr{A}$ and $\Upsilon(\omega) = \omega + \sum_{\iota=2}^{\infty} \upsilon_\iota \omega^\iota$, then

$$D_\lambda^m \Upsilon(\omega) = \omega + \sum_{\iota=2}^{\infty} [1 + (\iota - 1)\lambda]^m \upsilon_\iota \omega^\iota, \text{ for } \omega \in U. \tag{3}$$

Remark 2 When the parameter λ is set to 1 in the previously stated definition one obtains the well-known Sălăgean differential operator [20].

The next classical results are needed to prove our findings.

Definition 3 [29] Define O as the collection of all functions Υ that are holomorphic and injective on $\overline{U} \backslash E(\Upsilon)$, where the exceptional set is $E(\Upsilon) = \{\zeta \in \partial U \colon \lim_{\omega \to \zeta} \Upsilon(\omega) = \infty\}$, and additionally satisfy the condition that the derivative $\Upsilon'(\zeta)$ is nonzero for every $\zeta \in \partial U \backslash E(\Upsilon)$.

Lemma 1 [29] Assume that the function u is injective in the open unit disk U and θ and ρ be analytic in a domain D that contains the image $u(U)$ with $\rho(\mu) \neq 0$, for all $\mu \in u(U)$. Define the auxiliary function $O(\omega) = \omega u'(\omega)\rho(u(\omega))$ and let $h(\omega) = \theta(u(\omega)) + O(\omega)$. Suppose that

1. O is analytic and maps U onto a starlike domain, and
2. $\operatorname{Re}\left(\frac{\omega h'(\omega)}{O(\omega)}\right) > 0$, given $\omega \in U$.

Consider an analytic function s such that $s(0) = u(0)$, $s(U) \subseteq D$ and

$$\theta(s(\omega)) + \omega s'(\omega)\rho(s(\omega)) \prec \theta(u(\omega)) + \omega u'(\omega)\rho(u(\omega)).$$

Therefore, $s(\omega) \prec u(\omega)$ with u being the extremal dominant.

Lemma 2 [38] Consider that u is a convex and injective function in the open unit disk U and ν and ρ be holomorphic in a region D that includes the image $u(U)$. Assume that

1. Re $\left(\frac{\nu'(u(\omega))}{\rho(u(\omega))}\right) > 0$, given all $\omega \in U$ and

2. $\Phi(\omega) = \omega u'(\omega)\rho(u(\omega))$ is one-to-one and starlike within the open unit disk.

Assuming that $s(\omega) \in \mathscr{H}[u(0), 1] \cap O$, where $s(U)$ is contained in D and that the function $\nu(s(\omega)) + \omega s'(\omega)\rho(s(\omega))$ is injective in U, and further that the subordination

$$\nu(u(\omega)) + \omega u'(\omega)\rho(u(\omega)) \prec \nu(s(\omega)) + \omega s'(\omega)\rho(s(\omega)),$$

holds, it follows that $u(\omega) \prec s(\omega)$ and u is identified as the optimal subordinant.

We introduce now several fundamental definitions, notations, and properties from the theory of symmetric q-difference calculus. These foundational elements are essential for the formulation and development of the main results that follow. Throughout this manuscript, unless explicitly stated otherwise, we assume that the base parameter satisfies $0 < q < 1$, $\omega \in U$ and the symbol $\mathbb{N}$ is used to denote the collection of positive whole numbers: $\mathbb{N} = \{1, 2, 3....\}$.

Definition 4 [39] The expression $\widetilde{[\tau]}_q$ representing the symmetric q-number is formulated as

$$\widetilde{[\tau]}_q = \frac{q^{-\tau} - q^{\tau}}{q^{-1} - q}, \text{ for } \tau \in \mathbb{N} \text{ and } \widetilde{[0]}_q = 0. \tag{4}$$

An important distinction is that the symmetrized q-analog of an integer does not simplify to the familiar form of the q-number appearing in *q-deformed quantum mechanics* [40].

The symmetric form of the q-shifted factorial is given by

$$\widetilde{[\tau]}_q! = \begin{cases} \widetilde{[\tau]}_q \cdot \widetilde{[\tau-1]}_q \cdot \widetilde{[\tau-2]}_q \cdots \widetilde{[1]}_q, & \text{for } \tau = 1, 2, \ldots; \\ 1, & \text{for } \tau = 0. \end{cases} \tag{5}$$

One can see that that $\widetilde{[\tau]}_q! \to \tau!$ as $q \to 1^-$.

Definition 5 [39] The symmetric version of the q-derivative operator, when applied to $\Upsilon(\omega) \in \mathscr{A}_\tau$ is specified by:

$$\widetilde{D_q}\Upsilon(\omega) = \begin{cases} \omega\widetilde{D}_q\Upsilon(\omega) = \frac{\Upsilon(q\omega) - \Upsilon(q^{-1}\omega)}{\omega(q-q^{-1})}, & \text{for } \omega \neq 0, q \neq 1, \omega \in U; \\ \Upsilon'(0), & \text{for } \omega = 0. \end{cases} \tag{6}$$

Note that $\widetilde{D_q}\Upsilon(\omega) \to \Upsilon'(\omega)$ as $q \to 1^-$.

From (6) we have $\widetilde{D_q}\omega^\tau = \widetilde{[\tau]}_q \omega^{\tau-1}$, with an associated series expansion of $\widetilde{D_q}\Upsilon$, provided that $\Upsilon(\omega) = \omega^\tau + \sum_{\iota=1}^{\infty} \upsilon_{\iota+\tau}\omega^{\iota+\tau}$, is

$$\omega\widetilde{D_q}\Upsilon(\omega) = \widetilde{[\tau]}_q \omega^\tau + \sum_{\iota=1}^{\infty} \widetilde{[\iota+\tau]}_q \upsilon_{\iota+\tau}\omega^{\iota+\tau}. \tag{7}$$

A direct computation confirms that the assertions listed below are satisfied (see [16]):

$$\widetilde{D_q}\left(\Upsilon(\omega)+T(\omega)\right)=\widetilde{D_q}\Upsilon(\omega)+\widetilde{D_q}T(\omega),$$

$$\widetilde{D_q}\left(\Upsilon(\omega)T(\omega)\right)=T\left(q^{-1}\omega\right)\widetilde{D_q}\Upsilon(\omega)+\Upsilon(q\omega)\widetilde{D_q}T(\omega)=T(q\omega)\widetilde{D_q}\Upsilon(\omega)+\Upsilon\left(q^{-1}\omega\right)\widetilde{D_q}T(\omega),$$

$$\widetilde{D_q}\left(\frac{\Upsilon(\omega)}{T(\omega)}\right)=\frac{\Upsilon(q\omega)\widetilde{D_q}T(\omega)-T\left(q^{-1}\omega\right)\widetilde{D_q}\Upsilon(\omega)}{T\left(q^{-1}\omega\right)T(q\omega)}.$$

3. Generalized multivalent symmetric q-starlike Janowski classes $\widetilde{\mathscr{S}^*}_q^1(\delta, \tau, \kappa, \Lambda, X)$, $\widetilde{\mathscr{S}^*}_q^2(\delta, \tau, \kappa, \Lambda, X)$, $\widetilde{\mathscr{S}^*}_q^3(\delta, \tau, \kappa, \Lambda, X)$

Let $\Upsilon \in \mathscr{A}_\tau$, $\Upsilon(\omega) = \omega^\tau + \sum_{\iota=1}^{\infty} \upsilon_{\iota+\tau}\omega^{\iota+\tau}$. We now introduce the following symmetric q-differential operator $\widetilde{V}_{\delta,q,\tau}$: $\mathscr{A}_\tau \to \mathscr{A}_\tau$, defined by:

$$\widetilde{V}_{\delta,q,\tau}\Upsilon(\omega) = \left(1-\delta\widetilde{[\tau]}_q\right)\Upsilon(\omega)+\delta\omega\widetilde{D}_q\Upsilon(\omega) \tag{8}$$

$$= \omega^\tau+\sum_{\iota=1}^{\infty}\left(1-\delta\widetilde{[\tau]}_q+\delta\widetilde{[\iota+\tau]}_q\right)\upsilon_{\iota+\tau}\omega^{\iota+\tau}, \tag{9}$$

where $\delta \geq 0$.

We further define:

$$\widetilde{V}_{\delta,q,\tau}(0)\Upsilon(\omega)=\Upsilon(\omega),$$

$$\widetilde{V}_{\delta,q,\tau}(1)\Upsilon(\omega)=\widetilde{V}_{\delta,q,\tau}\Upsilon(\omega),$$

$$\widetilde{V}_{\delta,q,\tau}(2)\Upsilon(\omega)=\widetilde{V}_{\delta,q,\tau}(\widetilde{V}_{\delta,q,\tau}\Upsilon(\omega))=\omega^\tau+\sum_{\iota=1}^{\infty}\left(1-\delta\widetilde{[\tau]}_q+\delta\widetilde{[\iota+\tau]}_q\right)^2\upsilon_{\iota+\tau}\omega^{\iota+\tau}.$$

By induction, we arrive at the general formula:

$$\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(\omega)=\widetilde{V}_{\delta,q,\tau}(\widetilde{V}_{\delta,q,\tau}(\kappa-1)\Upsilon(\omega))=\omega^\tau+\sum_{\iota=1}^{\infty}\left(1-\delta\widetilde{[\tau]}_q+\delta\widetilde{[\iota+\tau]}_q\right)^\kappa\upsilon_{\iota+\tau}\omega^{\iota+\tau}. \tag{10}$$

Remark 3 In the limiting case $q \to 1^-$, $\tau = 1$, the operator described in (10) becomes the standard Al Oboudi's differential operator (3) [37] in the classical framework. As $q \to 1^-$, $\tau = 1$ and $\delta = 1$, the operator given in (10) collapses to the traditional Salagean's differential operator [41] in the classical case.

A key motivation for introducing the operator $\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(\omega)$ is its capacity to generalize classical differential operator within the framework of symmetric q-calculus. By incorporating the parameters δ, κ, q and τ, the operator allows for a flexible treatment of multivalent analytic functions and unifies various known cases. Its structure makes it particularly suitable for studying function classes associated with Janowski-type conditions and for establishing differential subordination results with sharp bounds. Applying the symmetric q-derivative of $\widetilde{V}_{\delta,\, q,\, \tau}\,(\kappa)$ yields

$$\begin{aligned} \widetilde{D}_q \widetilde{V}_{\delta,\, q,\, \tau}(\kappa)\Upsilon(\omega) &= \frac{\widetilde{V_{\delta,\, q,\, \tau}}(\kappa)\Upsilon(q\omega) - \widetilde{V_{\delta,\, q,\, \tau}}(\kappa)\Upsilon(q^{-1}\omega)}{\omega(q-q^{-1})} \\ &= \widetilde{[\tau]}_q \omega^{\tau-1} + \sum_{\iota=1}^{\infty}\left(1-\delta\widetilde{[\tau]}_q + \delta\widetilde{[\iota+\tau]}_q\right)^{\kappa}\widetilde{[\iota+\tau]}_q \upsilon_{\iota+\tau}\omega^{\iota+\tau-1}. \end{aligned} \tag{11}$$

A straightforward calculation yields the following result.

Proposition 1 Given $\delta \geq 0$, the following holds

$$\delta\omega\widetilde{D}_q\widetilde{V}_{\delta,\, q,\, \tau}(\kappa)\Upsilon(\omega) + (1-\delta\widetilde{[\tau]}_q)\widetilde{V}_{\delta,\, q,\, \tau}(\kappa)\Upsilon(\omega) = \widetilde{V}_{\delta,\, q,\, \tau}(\kappa+1)\Upsilon(\omega). \tag{12}$$

Proof.

$$\delta\omega\widetilde{D}_q\widetilde{V}_{\delta,\, q,\, \tau}(\kappa)\Upsilon(\omega) + \left(1-\widetilde{[\tau]}_q\delta\right)\widetilde{V}_{\delta,\, q,\, \tau}(\kappa)\Upsilon(\omega)$$

$$=\delta\omega\widetilde{[\tau]}_q\omega^{\tau-1} + \delta\omega\sum_{\iota=1}^{\infty}\left(1-\delta\widetilde{[\tau]}_q+\delta\widetilde{[\iota+\tau]}_q\right)^{\kappa}\widetilde{[\iota+\tau]}_q\upsilon_{\iota+\tau}\omega^{\iota+\tau-1}$$

$$+\left(1-\widetilde{[\tau]}_q\delta\right)\omega^{\tau} + \left(1-\widetilde{[\tau]}_q\delta\right)\sum_{\iota=1}^{\infty}\left(1-\delta\widetilde{[\tau]}_q+\delta\widetilde{[\iota+\tau]}_q\right)^{\kappa}\upsilon_{\iota+\tau}\omega^{\iota+\tau}$$

$$=\omega^{\tau} + \sum_{\iota=1}^{\infty}\left(1-\delta\widetilde{[\tau]}_q+\delta\widetilde{[\iota+\tau]}_q\right)^{\kappa+1}\upsilon_{\iota+\tau}\omega^{\iota+\tau} = \widetilde{V}_{\delta,\, q,\, \tau}(\kappa+1)\Upsilon(\omega).$$

So, the equality (12) is thus proven. □

Remark 4 For $q \to 1^{-}$, $\tau = 1$, identity (12) becomes the well-known relation associated with the Al-Oboudi differential operator:

$$\omega\left(\widetilde{V}_{\delta,\, 1,\, 1}(\kappa)\Upsilon\right)'(\omega) = \frac{1}{\delta}\widetilde{V}_{\delta,\, 1,\, 1}(\kappa+1)\Upsilon(\omega) - \frac{1-\delta}{\delta}\widetilde{V}_{\delta,\, 1,\, 1}(\kappa)\Upsilon(\omega). \tag{13}$$

The following subclasses of symmetrized q-starlike functions of valency τ involving the symmetric q-operator $\widetilde{V}_{\delta,\, q,\, \tau}\Upsilon(\omega)$, with respect to functions from the Janowski class, are presented in what follows.

From this point onward, we assume that the following condition is satisfied:

$$-1 \leq X < \Lambda \leq 1 \tag{14}$$

Definition 6 The function $\Upsilon \in \mathscr{A}_\tau$ belongs to the class $\widetilde{\mathscr{S}^*}_q^1(\delta,\, \tau,\, \kappa,\, \Lambda,\, X)$ exactly when the requirement presented next is satisfied:

$$\mathrm{Re}\left(\frac{(X-1)\,\dfrac{\omega \widetilde{D}_q \widetilde{V}_{\delta,\,q,\,\tau}(\kappa)\Upsilon(\omega)}{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa)\Upsilon(\omega)}-(\Lambda-1)}{(X+1)\,\dfrac{\omega \widetilde{D}_q \widetilde{V}_{\delta,\,q,\,\tau}(\kappa)\Upsilon(\omega)}{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa)\Upsilon(\omega)}-(\Lambda+1)}\right) \geq 0,$$

that is, by using (12)

$$\mathrm{Re}\left(\frac{(X-1)\,\dfrac{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa)\Upsilon(\omega)}-(X-1)\left(1-\delta\widetilde{[\tau]_q}\right)-\delta\,(\Lambda-1)}{(X+1)\,\dfrac{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa)\Upsilon(\omega)}-(X+1)\left(1-\delta\widetilde{[\tau]_q}\right)-\delta\,(\Lambda+1)}\right) \geq 0.$$

This collection, represented by $\widetilde{\mathscr{S}^*}_q^1(\delta,\, \tau,\, \kappa,\, \Lambda,\, X)$, is termed Class I of symmetric q-starlike mappings related to the Janowski family.

Remark 5 It is important to highlight that if $q \to 1^-$, $\tau = 1$, $\kappa = 0$, the class $\mathscr{S}_q^*(\delta,\, 1,\, 0,\, \Lambda,\, X)$ reduces to $\mathscr{S}_q^*(\Lambda,\, X)$ (see Definition 1).

Definition 7 Membership of the function $\Upsilon \in \mathscr{A}_\tau$ in the class $\widetilde{\mathscr{S}^*}_q^2(\delta,\, \tau,\, \kappa,\, \Lambda,\, X)$ is equivalent to the fulfillment of the following criterion:

$$\left|\frac{(X-1)\,\dfrac{\omega \widetilde{D}_q \widetilde{V}_{\delta,\,q,\,\tau}(\kappa)\Upsilon(\omega)}{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa)\Upsilon(\omega)}-(\Lambda-1)}{(X+1)\,\dfrac{\omega \widetilde{D}_q \widetilde{V}_{\delta,\,q,\,\tau}(\kappa)\Upsilon(\omega)}{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa)\Upsilon(\omega)}-(\Lambda+1)}-\frac{1}{1-q^2}\right| < \frac{1}{1-q^2},$$

that is, by using (12)

$$\left|\frac{(X-1)\,\dfrac{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa)\Upsilon(\omega)}-(X-1)\left(1-\delta\widetilde{[\tau]_q}\right)-\delta\,(\Lambda-1)}{(X+1)\,\dfrac{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa)\Upsilon(\omega)}-(X+1)\left(1-\delta\widetilde{[\tau]_q}\right)-\delta\,(\Lambda+1)}-\frac{1}{1-q^2}\right| < \frac{1}{1-q^2}.$$

The notation $\widetilde{\mathscr{S}^*}_q^2(\delta,\, \tau,\, \kappa,\, \Lambda,\, X)$ represents Class II of the symmetric q-starlike function family connected with Janowski-type mappings.

Remark 6 By taking $q \to 1^-$, $\tau = 1$, $\kappa = 0$, $\Lambda = \lambda$ and $X = 0$ in Definition 7.

Definition 8 Membership of $\Upsilon \in \mathscr{A}_\tau$ in the class $\widetilde{\mathscr{S}^*}_q^3(\delta,\, \tau,\, \kappa,\, \Lambda,\, X)$ is equivalent to satisfying the condition below:

$$\left| \frac{(X-1)\dfrac{\omega \widetilde{D}_q \widetilde{V}_{\delta,\, q,\, \tau}(\kappa)\Upsilon(\omega)}{\widetilde{V}_{\delta,\, q,\, \tau}(\kappa)\Upsilon(\omega)} - (\Lambda - 1)}{(X+1)\dfrac{\omega \widetilde{D}_q \widetilde{V}_{\delta,\, q,\, \tau}(\kappa)\Upsilon(\omega)}{\widetilde{V}_{\delta,\, q,\, \tau}(\kappa)\Upsilon(\omega)} - (\Lambda + 1)} - 1 \right| < 1, \tag{15}$$

that is, by using (12)

$$\left| \frac{(X-1)\dfrac{\widetilde{V}_{\delta,\, q,\, \tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,\, q,\, \tau}(\kappa)\Upsilon(\omega)} - (X-1)\left(1-\delta[\widetilde{\tau}]_q\right) - \delta(\Lambda-1)}{(X+1)\dfrac{\widetilde{V}_{\delta,\, q,\, \tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,\, q,\, \tau}(\kappa)\Upsilon(\omega)} - (X+1)\left(1-\delta[\widetilde{\tau}]_q\right) - \delta(\Lambda+1)} - 1 \right| < 1. \tag{16}$$

We call this class, denoted by $\widetilde{\mathscr{S}^*}_q^3(\delta,\, \tau,\, \kappa,\, \Lambda,\, X)$, the Class III of symmetric q-starlike functions corresponding to the Janowski family.

The exposition starts with the inclusion results concerning the classes $\widetilde{\mathscr{S}^*}_q^1(\delta,\, \tau,\, \kappa,\, \Lambda,\, X)$, $\widetilde{\mathscr{S}^*}_q^2(\delta,\, \tau,\, \kappa,\, \Lambda,\, X)$, $\widetilde{\mathscr{S}^*}_q^3(\delta,\, \tau,\, \kappa,\, \Lambda,\, X)$ which represent generalized multivalent symmetric q-starlike function classes associated with Janowski functions.

Theorem 1 If the parameters satisfy the condition (14), then the following chain of strict class inclusions holds among the extended structure of symmetric q-starlike functions of higher valency related to the Janowski-type functions

$$\widetilde{\mathscr{S}^*}_q^3(\delta,\, \tau,\, \kappa,\, \Lambda,\, X) \subset \widetilde{\mathscr{S}^*}_q^2(\delta,\, \tau,\, \kappa,\, \Lambda,\, X) \subset \widetilde{\mathscr{S}^*}_q^1(\delta,\, \tau,\, \kappa,\, \Lambda,\, X). \tag{17}$$

Proof. Let us consider a function $\Upsilon \in \widetilde{\mathscr{S}^*}_q^3(\delta,\, \tau,\, \kappa,\, \Lambda,\, X)$. According to Definition 8, this implies that the condition below is met:

$$\left| \frac{(X-1)\dfrac{\widetilde{V}_{\delta,\, q,\, \tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,\, q,\, \tau}(\kappa)\Upsilon(\omega)} - (X-1)\left(1-\delta[\widetilde{\tau}]_q\right) - \delta(\Lambda-1)}{(X+1)\dfrac{\widetilde{V}_{\delta,\, q,\, \tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,\, q,\, \tau}(\kappa)\Upsilon(\omega)} - (X+1)\left(1-\delta[\widetilde{\tau}]_q\right) - \delta(\Lambda+1)} - 1 \right| < 1.$$

From this inequality, we can derive the relation:

$$\left| \frac{(X-1)\dfrac{\widetilde{V}_{\delta,\, q,\, \tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,\, q,\, \tau}(\kappa)\Upsilon(\omega)} - (X-1)\left(1-\delta[\widetilde{\tau}]_q\right) - \delta(\Lambda-1)}{(X+1)\dfrac{\widetilde{V}_{\delta,\, q,\, \tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,\, q,\, \tau}(\kappa)\Upsilon(\omega)} - (X+1)\left(1-\delta[\widetilde{\tau}]_q\right) - \delta(\Lambda+1)} - 1 \right| + \frac{q^2}{1-q^2} < 1 + \frac{q^2}{1-q^2}. \tag{18}$$

Now, applying the triangle inequality to the left-hand term of expression (18), yields:

$$\left| \frac{(X-1)\,\dfrac{\widetilde{V}_{\delta,\,q,\,\tau}\,(\kappa+1)\,\Upsilon(\omega)}{\widetilde{V}_{\delta,\,q,\,\tau}\,(\kappa)\,\Upsilon(\omega)} - (X-1)\left(1-\delta\widetilde{[\tau]}_q\right) - \delta\,(\Lambda-1)}{(X+1)\,\dfrac{\widetilde{V}_{\delta,\,q,\,\tau}\,(\kappa+1)\,\Upsilon(\omega)}{\widetilde{V}_{\delta,\,q,\,\tau}\,(\kappa)\,\Upsilon(\omega)} - (X+1)\left(1-\delta\widetilde{[\tau]}_q\right) - \delta\,(\Lambda+1)} - \frac{1}{1-q^2} \right| < \frac{1}{1-q^2}. \tag{19}$$

Inequality (19) satisfies the condition stated in Definition 7, which characterizes the class $\widetilde{\mathscr{S}^*}_q^2(\delta,\,\tau,\,\kappa,\,\Lambda,\,X)$. Therefore, it follows that

$$f \in \widetilde{\mathscr{S}^*}_q^2(\delta,\,\tau,\,\kappa,\,\Lambda,\,X)$$

and consequently, we have the class inclusion:

$$\widetilde{\mathscr{S}^*}_q^3(\delta,\,\tau,\,\kappa,\,\Lambda,\,X) \subset \widetilde{\mathscr{S}^*}_q^2(\delta,\,\tau,\,\kappa,\,\Lambda,\,X).$$

Now, let us assume that $\Upsilon \in \widetilde{\mathscr{S}^*}_q^2(\delta,\,\tau,\,\kappa,\,\Lambda,\,X)$. According to Definition 7, this is equivalent to satisfying the inequality (7).

Due to the fact that

$$\left| \frac{(X-1)\,\dfrac{\widetilde{V}_{\delta,\,q,\,\tau}\,(\kappa+1)\,\Upsilon(\omega)}{\widetilde{V}_{\delta,\,q,\,\tau}\,(\kappa)\,\Upsilon(\omega)} - (X-1)\left(1-\delta\widetilde{[\tau]}_q\right) - \delta\,(\Lambda-1)}{(X+1)\,\dfrac{\widetilde{V}_{\delta,\,q,\,\tau}\,(\kappa+1)\,\Upsilon(\omega)}{\widetilde{V}_{\delta,\,q,\,\tau}\,(\kappa)\,\Upsilon(\omega)} - (X+1)\left(1-\delta\widetilde{[\tau]}_q\right) - \delta\,(\Lambda+1)} - \frac{1}{1-q^2} \right|$$

$$= \left| \frac{1}{1-q^2} - \frac{(X-1)\,\dfrac{\widetilde{V}_{\delta,\,q,\,\tau}\,(\kappa+1)\,\Upsilon(\omega)}{\widetilde{V}_{\delta,\,q,\,\tau}\,(\kappa)\,\Upsilon(\omega)} - (X-1)\left(1-\delta\widetilde{[\tau]}_q\right) - \delta\,(\Lambda-1)}{(X+1)\,\dfrac{\widetilde{V}_{\delta,\,q,\,\tau}\,(\kappa+1)\,\Upsilon(\omega)}{\widetilde{V}_{\delta,\,q,\,\tau}\,(\kappa)\,\Upsilon(\omega)} - (X+1)\left(1-\delta\widetilde{[\tau]}_q\right) - \delta\,(\Lambda+1)} \right| < \frac{1}{1-q^2},$$

we can express the inequality as:

$$\operatorname{Re}\left(\frac{(X-1)\,\dfrac{\widetilde{V}_{\delta,\,q,\,\tau}\,(\kappa+1)\,\Upsilon(\omega)}{\widetilde{V}_{\delta,\,q,\,\tau}\,(\kappa)\,\Upsilon(\omega)} - (X-1)\left(1-\delta\widetilde{[\tau]}_q\right) - \delta\,(\Lambda-1)}{(X+1)\,\dfrac{\widetilde{V}_{\delta,\,q,\,\tau}\,(\kappa+1)\,\Upsilon(\omega)}{\widetilde{V}_{\delta,\,q,\,\tau}\,(\kappa)\,\Upsilon(\omega)} - (X+1)\left(1-\delta\widetilde{[\tau]}_q\right) - \delta\,(\Lambda+1)} \right) \geq 0.$$

This inequality corresponds exactly to the condition required in Definition 6, for a function to be classified within $\widetilde{\mathscr{S}^*}_q^1(\delta,\,\tau,\,\kappa,\,\Lambda,\,X)$. Hence, we conclude that $\Upsilon \in \widetilde{\mathscr{S}^*}_q^1(\delta,\,\tau,\,\kappa,\,\Lambda,\,X)$, which establishes the inclusion:

$$\widetilde{\mathscr{S}^*}_q^2(\delta,\,\tau,\,\kappa,\,\Lambda,\,X)\subset\widetilde{\mathscr{S}^*}_q^1(\delta,\,\tau,\,\kappa,\,\Lambda,\,X).$$

Accordingly, the assertion of the theorem has been established. □

The next result provides a sufficient condition for membership in the class $\widetilde{\mathscr{S}^*}_q^3(\delta,\,\tau,\,\kappa,\,\Lambda,\,X)$ based on a coefficient inequality, which also serves as a sufficient condition for the classes $\widetilde{\mathscr{S}^*}_q^2(\delta,\,\tau,\,\kappa,\,\Lambda,\,X)$ and $\widetilde{\mathscr{S}^*}_q^1(\delta,\,\tau,\,\kappa,\,\Lambda,\,X)$.

Theorem 2 Let $\Upsilon\in\mathscr{A}_\tau$. If the following coefficient inequality is satisfied:

$$\sum_{\iota=1}^{\infty}(2Y+|B|)\left(1-\delta\widetilde{[\tau]}_q+\delta\widetilde{[\iota+\tau]}_q\right)^{\kappa}|\upsilon_{\iota+\tau}|\le(X+1)Z-2A, \tag{20}$$

where the quantities are defined by:

$$A=1-\widetilde{[\tau]}_q,\,Y=1-\widetilde{[\iota+\tau]}_q,\,Z=\Lambda+1-\widetilde{[\tau]}_q,\,B=[\iota+\tau]_q(X+1)-\Lambda-1,$$

then Υ belongs to the class $\widetilde{\mathscr{S}^*}_q^3(\delta,\,\tau,\,\kappa,\,\Lambda,\,X)$.

Proof. To initiate the proof, we assume that the condition stated in equation (20) is satisfied. Under this assumption, it remains to demonstrate that the following implication holds:

$$\left|\frac{(X-1)\dfrac{\widetilde{V}_{\delta,q,\tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(\omega)}-(X-1)\left(1-\delta\widetilde{[\tau]}_q\right)-\delta(\Lambda-1)}{(X+1)\dfrac{\widetilde{V}_{\delta,q,\tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(\omega)}-(X+1)\left(1-\delta\widetilde{[\tau]}_q\right)-\delta(\Lambda+1)}-1\right|<1.$$

Thus,

$$\begin{aligned}
&\left|\frac{(X-1)\dfrac{\widetilde{V}_{\delta,q,\tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(\omega)}-(X-1)\left(1-\delta\widetilde{[\tau]}_q\right)-\delta(\Lambda-1)}{(X+1)\dfrac{\widetilde{V}_{\delta,q,\tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(\omega)}-(X+1)\left(1-\delta\widetilde{[\tau]}_q\right)-\delta(\Lambda+1)}-1\right|\\
=&\left|\frac{(X-1)\widetilde{V}_{\delta,q,\tau}(\kappa+1)\Upsilon(\omega)-(X-1)\left(1-\delta\widetilde{[\tau]}_q\right)\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(\omega)-\delta(\Lambda-1)\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(\omega)}{(X+1)\widetilde{V}_{\delta,q,\tau}(\kappa+1)\Upsilon(\omega)-(X+1)\left(1-\delta\widetilde{[\tau]}_q\right)\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(\omega)-\delta(\Lambda+1)\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(\omega)}-1\right|\\
=&\,2\left|\frac{\left(1+\delta-\delta\widetilde{[\tau]}_q\right)\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(\omega)-\widetilde{V}_{\delta,q,\tau}(\kappa+1)\Upsilon(\omega)}{(X+1)\widetilde{V}_{\delta,q,\tau}(\kappa+1)\Upsilon(\omega)-(X+1)\left(1-\delta\widetilde{[\tau]}_q\right)\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(\omega)-\delta(\Lambda+1)\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(\omega)}\right|
\end{aligned}$$

$$
= \quad 2\left|\frac{\delta\left(1-\widetilde{[\tau]}_q\right)\omega^{\tau}+\delta\sum_{\iota=1}^{\infty}\left(1-\widetilde{[\iota+\tau]}_q\right)\left(1-\delta\widetilde{[\tau]}_q+\delta\widetilde{[\iota+\tau]}_q\right)^{\kappa}\upsilon_{\iota+\tau}\omega^{\iota+\tau}}{\delta\left(X+1\right)\left(\widetilde{[\tau]}_q-\Lambda-1\right)\omega^{\tau}+\delta\sum_{\iota=1}^{\infty}\left(\Lambda+1-\widetilde{[\iota+\tau]}_q\left(X+1\right)\right)\left(1-\delta\widetilde{[\tau]}_q+\delta\widetilde{[\iota+\tau]}_q\right)^{\kappa}\upsilon_{\iota+\tau}\omega^{\iota+\tau}}\right|
$$

$$
\leq \quad \frac{2\left\{\delta\left(1-\widetilde{[\tau]}_q\right)\left|\omega^{\tau}\right|+\delta\sum_{\iota=1}^{\infty}\left(1-\widetilde{[\iota+\tau]}_q\right)\left(1-\delta\widetilde{[\tau]}_q+\delta\widetilde{[\iota+\tau]}_q\right)^{\kappa}\left|\upsilon_{\iota+\tau}\right|\left|\omega^{\tau}\right|\right\}}{\delta\left(X+1\right)\left|\widetilde{[\tau]}_q-\Lambda-1\right|\left|\omega^{\tau}\right|-\delta\sum_{\iota=1}^{\infty}\left|[\iota+\tau]_q\left(X+1\right)-\Lambda-1\right|\left(1-\delta\widetilde{[\tau]}_q+\delta\widetilde{[\iota+\tau]}_q\right)^{\kappa}\left|\upsilon_{\iota+\tau}\right|\left|\omega^{\tau}\right|}
$$

$$
\leq \quad \frac{2\left\{\delta A+\delta\sum_{\iota=1}^{\infty}Y\left(1-\delta\widetilde{[\tau]}_q+\delta\widetilde{[\iota+\tau]}_q\right)^{\kappa}\left|\upsilon_{\iota+\tau}\right|\right\}}{\delta\left(X+1\right)Z-\delta\sum_{\iota=1}^{\infty}\left|B\right|\left(1-\delta\widetilde{[\tau]}_q+\delta\widetilde{[\iota+\tau]}_q\right)^{\kappa}\left|\upsilon_{\iota+\tau}\right|}
$$

$$
= \quad \frac{2\left\{A+\sum_{\iota=1}^{\infty}Y\left(1-\delta\widetilde{[\tau]}_q+\delta\widetilde{[\iota+\tau]}_q\right)^{\kappa}\left|\upsilon_{\iota+\tau}\right|\right\}}{\left(X+1\right)Z-\sum_{\iota=1}^{\infty}\left|B\right|\left(1-\delta\widetilde{[\tau]}_q+\delta\widetilde{[\iota+\tau]}_q\right)^{\kappa}\left|\upsilon_{\iota+\tau}\right|}, \tag{21}
$$

where

$$
A=1-\widetilde{[\tau]}_q,\; Y=1-\widetilde{[\iota+\tau]}_q,\; Z=\Lambda+1-\widetilde{[\tau]}_q,\; B=[\iota+\tau]_q\left(X+1\right)-\Lambda-1.
$$

The final term in (21) does not exceed 1 under the condition (20). The desired result is now proven. □

4. Subordination and superordination results concerning the symmetric q-differential operator

The following two lemmas are established by employing a technique analogous to the one outlined in [42].

Lemma 3 Consider a one-to-one analytic function u defined in U and let θ and ρ be holomorphic functions on a domain D that contains the image $u(U)$ with the condition that $\rho(\mu)\neq 0$, for all $\mu\in u(U)$. Define $O(\omega)=\omega\widetilde{D}_q u(\omega)\rho(u(\omega))$ and $P(\omega)=\theta(u(\omega))+O(\omega)$. Assume the following conditions hold:

1. O is a one-to-one starlike function within U and
2. $\mathrm{Re}\left(\frac{\omega\widetilde{D}_q P(\omega)}{O(\omega)}\right)>0$ for all $\omega\in U$.

If s is analytic in U, satisfies $s(0)=u(0)$, $s(U)\subseteq D$ and

$$
\theta(s(\omega))+\omega\widetilde{D}_q s(\omega)\rho(s(\omega))\prec\theta(u(\omega))+\omega\widetilde{D}_q u(\omega)\rho(u(\omega))=h(\omega), \tag{22}
$$

then $s(\omega)\prec u(\omega)$ and $u(\omega)$ represents the optimal dominant.

Proof. Suppose u is analytic and one-to-one in U, and define $O(\omega)=\omega\widetilde{D}_q u(\omega)\rho(u(\omega))$, $P(\omega)=\theta(u(\omega))+O(\omega)$, where θ and ρ are holomorphic functions in a region D that includes $u(U)$ such that $\rho(\mu)$ non-vanishing on $u(U)$, for all

$\mu \in u(U)$. Assume that conditions 1 and 2 hold. Define $m(\omega) = \theta(s(\omega)) + \omega \widetilde{D}_q s(\omega) \rho(s(\omega))$, where s is holomorphic in U, $s(0) = u(0)$, $s(U) \subseteq D$. By hypotheses $m(\omega) \prec P(\omega)$.

When q approaches 1^-, the symmetric q-operator $\widetilde{D}_q \Upsilon(\omega)$ reduces to the classical derivative $\Upsilon'(\omega)$. Therefore, (22) becomes:

$$\theta(s(\omega)) + \omega s'(\omega) \rho(s(\omega)) \prec \theta(u(\omega)) + \omega u'(\omega) \rho(u(\omega)).$$

By applying the known lemma for the classical derivative (Lemma 1) it can be concluded that $s(\omega) \prec u(\omega)$ and u represents the principal dominant. □

Lemma 4 Suppose u is convex and injective in U and ν and ρ are holomorphic within a domain D that contains the image $u(U)$. Let us assume:

1. Re $\left(\frac{\widetilde{D}_q \nu(u(\omega))}{\rho(u(\omega))}\right) > 0$, given any $\omega \in U$, and

2. $\Phi(\omega) = \omega \widetilde{D}_q u(\omega) \rho(u(\omega))$ is analytic, injective, and starlike in the open unit disk. Assume that $s(\omega) \in \mathscr{H}[u(0), 1] \cap O$, where $s(U) \subseteq D$ and the composed mapping $\nu(s(\omega)) + \omega \widetilde{D}_q s(\omega) \rho(s(\omega))$ is injective in the unit disk and

$$\nu(u(\omega)) + \omega \widetilde{D}_q u(\omega) \rho(u(\omega)) \prec \nu(s(\omega)) + \omega \widetilde{D}_q s(\omega) \rho(s(\omega)),$$

then $u(\omega) \prec s(\omega)$ and u is identified as the extremal solution among all subordinants.

Proof. This proof mirrors the method used in the previous lemma. □

The first main result presented in this section concerns a second-order differential subordination involving the symmetric q-differential operator $\widetilde{V}_{\delta, q, \tau}(\kappa)$. This theorem establishes sufficient conditions under which a function transformed by this operator is subordinate to a given univalent function. A significant feature of this result is that it not only guarantees the subordination relation but also explicitly identifies the best dominant—that is, the minimal univalent function (in the subordination sense) to which all admissible solutions are subordinate. This sharp bound enhances the applicability of the theorem and reflects the precision afforded by the use of the symmetric q-calculus framework.

Theorem 3 Let $\Upsilon \in \mathscr{A}_\tau$, and suppose that the function $\frac{\widetilde{V}_{\delta, q, \tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta, q, \tau}(\kappa)\Upsilon(\omega)}$ belongs to $\mathscr{H}(U)$, where $\omega \in U$. Let $u(\omega)$ be a convex univalent function in the open unit disk, normalized, satisfying $u(0) = 1$. Consider that for some constants $a, b, c, \in \mathbb{C}$, with $c \neq 0$, the following condition is satisfied:

$$\operatorname{Re}\left\{\frac{a}{c} + \frac{b}{c}\left[u(q\omega) + u\left(q^{-1}\omega\right)\right] + \frac{\widetilde{D}_q u\left(q^{-1}\omega\right)}{\widetilde{D}_q u(\omega)} + \frac{q\omega \widetilde{D}_q^{(2)} u(\omega)}{\widetilde{D}_q u(\omega)}\right\} > 0, \ \omega \in U. \tag{23}$$

Let us describe the function $\Phi^{\kappa}_{\delta, q, \tau}(a, b, c; \omega)$ as

$$\begin{aligned} \Phi^{\kappa}_{\delta, q, \tau}(a, b, c; \omega) &= \frac{\widetilde{V}_{\delta, q, \tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta, q, \tau}(\kappa)\Upsilon(\omega)}\left(a + b\frac{\widetilde{V}_{\delta, q, \tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta, q, \tau}(\kappa)\Upsilon(\omega)}\right) \\ &\quad + \frac{c}{\delta}\frac{\widetilde{V}_{\delta, q, \tau}(\kappa+1)\Upsilon(q\omega)}{\widetilde{V}_{\delta, q, \tau}(\kappa)\Upsilon(q\omega)}\left[\frac{\widetilde{V}_{\delta, q, \tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta, q, \tau}(\kappa)\Upsilon(q^{-1}\omega)} - \left(1 - \delta\widetilde{[\tau]}_q\right)\frac{\widetilde{V}_{\delta, q, \tau}(\kappa)\Upsilon(\omega)}{\widetilde{V}_{\delta, q, \tau}(\kappa)\Upsilon(q^{-1}\omega)}\right] \end{aligned}$$

$$-\frac{c}{\delta}\left[\frac{\widetilde{V}_{\delta,q,\tau}(\kappa+2)\Upsilon(\omega)}{\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(q\omega)}-\left(1-\delta\widetilde{[\tau]}_q\right)\frac{\widetilde{V}_{\delta,q,\tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(q\omega)}\right]. \tag{24}$$

If the subordination

$$\Phi^{\kappa}_{\delta,q,\tau}(a,b,c;\omega)\prec au(\omega)+b(u(\omega))^2+c\omega\widetilde{D}_q u(\omega) \tag{25}$$

holds in U, then the following subordination result is valid:

$$\frac{\widetilde{V}_{\delta,q,\tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(\omega)}\prec u(\omega), \tag{26}$$

and u is the best dominant.

Proof. Let

$$s(\omega):=\frac{\widetilde{V}_{\delta,q,\tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(\omega)}, \tag{27}$$

where $\omega\in U\backslash\{0\},\ \Upsilon\in\mathscr{A}_\tau$. Clearly, s is analytic U and $s(0)=1$. By applying the symmetric q-differentiating $s(\omega)$, we find

$$\omega\widetilde{D}_q s(\omega)=\frac{\omega\widetilde{D}_q\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(\omega)\widetilde{V}_{\delta,q,\tau}(\kappa+1)\Upsilon(q\omega)-\omega\widetilde{D}_q\widetilde{V}_{\delta,q,\tau}(\kappa+1)\Upsilon(\omega)\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(q^{-1}\omega)}{\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(q^{-1}\omega)\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(q\omega)}.$$

Invoking identity (12), this expression simplifies to:

$$\omega\widetilde{D}_q s(\omega)=\frac{\widetilde{V}_{\delta,q,\tau}(\kappa+1)\Upsilon(q\omega)}{\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(q\omega)}\left(\frac{1}{\delta}\frac{\widetilde{V}_{\delta,q,\tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(q^{-1}\omega)}-\frac{1-\delta\widetilde{[\tau]}_q}{\delta}\frac{\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(\omega)}{\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(q^{-1}\omega)}\right)$$

$$-\frac{1}{\delta}\frac{\widetilde{V}_{\delta,q,\tau}(\kappa+2)\Upsilon(\omega)}{\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(q\omega)}+\frac{1-\delta\widetilde{[\tau]}_q}{\delta}\frac{\widetilde{V}_{\delta,q,\tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(q\omega)}. \tag{28}$$

Now define the auxiliary functions: $\theta(\mu):=a\mu+b\mu^2$ and $\rho(\mu):=c$, where $a,\ b,\ c\in\mathbb{C}$ and $c\neq 0$. It is clear that θ is holomorphic on $\mathbb{C}\backslash\{0\}$, ρ is likewise holomorphic and non-vanishing on the same domain. Set $O(\omega)=\omega\widetilde{D}_q u(\omega)\rho(u(\omega))=c\omega\widetilde{D}_q u(\omega)$ and $h(\omega)=\theta(u(\omega))+O(\omega)=au(\omega)+b(u(\omega))^2+c\omega\widetilde{D}_q u(\omega)$, $\omega\in U$. By direct computation, we obtain:

$$\operatorname{Re}\left(\frac{\omega\widetilde{D}_q h(\omega)}{O(\omega)}\right)=\operatorname{Re}\left\{\frac{a}{c}+\frac{b}{c}\left[u(q\omega)+u\left(q^{-1}\omega\right)\right]+\frac{\widetilde{D}_q u\left(q^{-1}\omega\right)}{\widetilde{D}_q u(\omega)}+\frac{q\omega\widetilde{D}_q^{(2)}u(\omega)}{\widetilde{D}_q u(\omega)}\right\}>0.$$

Applying (28), we have

$$\begin{aligned}
&as(\omega)+b(s(\omega))^2+c\omega\widetilde{D}_q s(\omega)\\
=&\frac{\widetilde{V}_{\delta,q,\tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(\omega)}\left(a+b\frac{\widetilde{V}_{\delta,q,\tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(\omega)}\right)\\
&+\frac{c}{\delta}\frac{\widetilde{V}_{\delta,q,\tau}(\kappa+1)\Upsilon(q\omega)}{\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(q\omega)}\left[\frac{\widetilde{V}_{\delta,q,\tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(q^{-1}\omega)}-\left(1-\delta\widetilde{[\tau]}_q\right)\frac{\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(\omega)}{\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(q^{-1}\omega)}\right]\\
&-\frac{c}{\delta}\left[\frac{\widetilde{V}_{\delta,q,\tau}(\kappa+2)\Upsilon(\omega)}{\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(q\omega)}-\left(1-\delta\widetilde{[\tau]}_q\right)\frac{\widetilde{V}_{\delta,q,\tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(q\omega)}\right].
\end{aligned}$$

Based on (25), we conclude that

$$as(\omega)+b(s(\omega))^2+c\omega\widetilde{D}_q s(\omega)\prec au(\omega)+b(u(\omega))^2+c\omega\widetilde{D}_q u(\omega).$$

Hence, the conditions of the subordination from Lemma 4 are satisfied and it follows that $s(\omega)\prec u(\omega)$, for all $\omega\in U$; that is, $\frac{\widetilde{V}_{\delta,q,\tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(\omega)}\prec u(\omega)$, $\omega\in U$, with u being the best dominant. □

Corollary 1 Let $u(\omega)=\frac{1+\Lambda\omega}{1+X\omega}$, and suppose that conditions (14) and (23) are satisfied. If $\Upsilon\in\mathscr{A}_\tau$ and

$$\Phi^{\kappa}_{\delta,q,\tau}(a,b,c;\omega)\prec a\frac{1+\Lambda\omega}{1+X\omega}+b\left(\frac{1+\Lambda\omega}{1+X\omega}\right)^2+\frac{c\omega\left[\Lambda X\omega\left(q^2-1\right)-(\Lambda-X)q\right]}{q+X\omega(q^2+1)+qX^2\omega^2},$$

for $a,b,c\in\mathbb{C}$, $c\neq 0$, under the condition that $\Phi^{\kappa}_{\delta,q,\tau}(a,b,c;\omega)$ is formulated on (24), then it follows that

$$\frac{\widetilde{V}_{\delta,q,\tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,q,\tau}(\kappa)\Upsilon(\omega)}\prec\frac{1+\Lambda\omega}{1+X\omega},\ \omega\in U,$$

and $\frac{1+\Lambda\omega}{1+X\omega}$ represents the optimal dominant.

Proof. Substituting $u(\omega)=\frac{1+\Lambda\omega}{1+X\omega}$ into Theorem 3 yields the stated corollary as a direct consequence. □

Remark 7 The function $u(\omega)=\frac{1+\Lambda\omega}{1+X\omega}$ defines a conformal mapping of U into a circular area in the complex plane exhibiting symmetry relative to the real axis. Specifically, the region covered by U under $u(\omega)=\frac{1+\Lambda\omega}{1+X\omega}$ is a disk with center at $\frac{1-\Lambda X}{1-X^2}$ and radius $\frac{\Lambda-X}{1-X^2}$. In the special case where $X=-1$, the image becomes a half-plane. The endpoints of the diameter of the image disc on the real axis are given by: $M=\left(\frac{1-\Lambda}{1-X},0\right)$ and $N=\left(\frac{1+\Lambda}{1+X},0\right)$. These mappings, often referred to as Janowski functions, are widely used in geometric complex analysis, particularly in a focused inquiry into subfamilies of analytic functions with univalence and starlikeness.

Corollary 2 Assume that $u(\omega) = \frac{1+(1-2\gamma)\omega}{1-\omega}$, $0 \leq \gamma < 1$ and that condition (23) is satisfied. If $\Upsilon \in \mathscr{A}_\tau$ and

$$\Phi^{\kappa}_{\delta,\, q,\, \tau}(a,\, b,\, c;\, \omega) \prec a\frac{1+(1-2\gamma)\,\omega}{1-\omega} + b\left(\frac{1+(1-2\gamma)\,\omega}{1-\omega}\right)^2 + \frac{c\omega\left[\omega\,(1-2\gamma)\left(1-q^2\right)-2\,(1-\gamma)\,q\right]}{q-\omega\,(q^2+1)+q\omega^2},$$

given $a,\, b,\, c \in \mathbb{C}$, $c \neq 0$, and assuming that $\Phi^{\kappa}_{\delta,\, q,\, \tau}(a,\, b,\, c;\, \omega)$ is specified in (24), then we conclude that

$$\frac{\widetilde{V}_{\delta,\, q,\, \tau}(\kappa+1)\,\Upsilon(\omega)}{\widetilde{V}_{\delta,\, q,\, \tau}(\kappa)\,\Upsilon(\omega)} \prec \frac{1+(1-2\gamma)\,\omega}{1-\omega},$$

and $\frac{1+(1-2\gamma)\omega}{1-\omega}$ represents the optimal dominant.

Proof. Letting $u(\omega) = \frac{1+(1-2\gamma)\omega}{1-\omega}$, with $0 \leq \gamma < 1$ in Theorem 3, the conclusion of this corollary readily follows as a particular case. □

Corollary 3 Let $u(\omega) = \frac{1+\omega}{1-\omega}$ and suppose that condition (23) is satisfied. If $\Upsilon \in \mathscr{A}_\tau$ and the subordination

$$\Phi^{\kappa}_{\delta,\, q,\, \tau}(a,\, b,\, c;\, \omega) \prec a\frac{1+\omega}{1-\omega} + b\left(\frac{1+\omega}{1-\omega}\right)^2 + \frac{c\omega\left[\omega\left(1-q^2\right)-2q\right]}{q-\omega\,(q^2+1)+q\omega^2}$$

holds for $a,\, b,\, c \in \mathbb{C}$, with $c \neq 0$, where $\Phi^{\kappa}_{\delta,\, q,\, \tau}(a,\, b,\, c;\, \omega)$ is defined in (24), it can be concluded that

$$\frac{\widetilde{V}_{\delta,\, q,\, \tau}(\kappa+1)\,\Upsilon(\omega)}{\widetilde{V}_{\delta,\, q,\, \tau}(\kappa)\,\Upsilon(\omega)} \prec \frac{1+\omega}{1-\omega},$$

and $\frac{1+\omega}{1-\omega}$ is the best dominant.

Proof. This result arises as a straightforward implication of Theorem 3 by selecting the function $u(\omega) = \frac{1+\omega}{1-\omega}$, which serves as a specific instance satisfying the conditions of the theorem. □

Corollary 4 Let $u(\omega) = \frac{1+\omega}{(1-\omega)^2}$, $\omega \in U$ and suppose that condition (23) is satisfied. If $\Upsilon \in \mathscr{A}_\tau$ and the subordination

$$\Phi^{\kappa}_{\delta,\, q,\, \tau}(a,\, b,\, c;\, \omega) \prec a\frac{1+\omega}{(1-\omega)^2} + b\left(\frac{1+\omega}{(1-\omega)^2}\right)^2 + \frac{c\omega\left[(1+q\omega)\left(\widetilde{[2]}_q\omega-2\right)-\left(1-q^{-1}\omega\right)^2\right]}{\left(1-\widetilde{[2]}_q\omega+\omega^2\right)^2}$$

holds for $a,\, b,\, c \in \mathbb{C}$, with $c \neq 0$, where $\Phi^{\kappa}_{\delta,\, q,\, \tau}(a,\, b,\, c;\, \omega)$ is defined in (23), then it follows that

$$\frac{\widetilde{V}_{\delta,\, q,\, \tau}(\kappa+1)\,\Upsilon(\omega)}{\widetilde{V}_{\delta,\, q,\, \tau}(\kappa)\,\Upsilon(\omega)} \prec \frac{1+\omega}{(1-\omega)^2},$$

and $\frac{1+\omega}{(1-\omega)^2}$ represents the optimal dominant.

Proof. This result is derived by evaluating Theorem 3 at the specific function $u(\omega)=\frac{1+\omega}{(1-\omega)^2}$, which leads directly to the stated conclusion. □

The subsequent theorem addresses a second-order differential superordination involving the symmetric q-differential operator $\widetilde{V}_{\delta,\,q,\,\tau}(\kappa)$. It provides sufficient conditions under which a given univalent function is superordinate to a transformed analytic function. A key contribution of this result lies in the identification of the best subordinant—that is, the largest function (with respect to subordination) among all admissible subordinants satisfying the differential inequality. This result not only complements the earlier subordination theorem but also highlights the duality structure inherent in the theory. The use of symmetric q-calculus facilitates a more refined formulation of the problem, allowing for greater generality and improved analytical control over the behavior of the solutions.

Theorem 4 Let u be an analytic and injective mapping on the open unit disk U, with the property of convexity, satisfying $u(0)=1$. Suppose that for constants $a,\,b,\,c\in\mathbb{C}$, where $c\neq 0$, the following condition holds:

$$\mathrm{Re}\left(\frac{a}{c}+\frac{b}{c}\left[u(q\omega)+u\left(q^{-1}\omega\right)\right]\right)>0. \tag{29}$$

Assume that $\Upsilon\in\mathscr{A}_\tau$ and that the function $\frac{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa)\Upsilon(\omega)}$ belongs to the class $\mathscr{H}[u(0),\,1]\cap O$. Suppose also that the function $\Phi^{\kappa}_{\delta,\,q,\,\tau}(a,\,b,\,c;\,\omega)$, defined in (24), is univalent in U. If the subordination

$$au(\omega)+b(u(\omega))^2+c\omega\widetilde{D}_q u(\omega)\prec\Phi^{\kappa}_{\delta,\,q,\,\tau}(a,\,b,\,c;\,\omega), \tag{30}$$

is satisfied, then it follows that

$$u(\omega)\prec\frac{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa)\Upsilon(\omega)}, \tag{31}$$

and u is the best subordinant.

Proof. Let $s(\omega):=\frac{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa)\Upsilon(\omega)}$, $\omega\in U\backslash\{0\}$, $\Upsilon\in\mathscr{A}_\tau$. Define the auxiliary functions: $\nu(\nu):=a\nu+b\nu^2$ and $\rho(\nu):=c$. Clearly, ν is holomorphic in $\mathbb{C}$ and ρ is analytic except at the origin and non-vanishing in $\mathbb{C}\backslash\{0\}$. A straightforward computation shows that:

$$\frac{\widetilde{D}_q\nu(u(\omega))}{\rho(u(\omega))}=\frac{a}{c}+\frac{b}{c}\left[u(q\omega)+u\left(q^{-1}\omega\right)\right].$$

By hypothesis (15), the real part of this expression is positive for all $\omega\in U$, that is:

$$\mathrm{Re}\left(\frac{\widetilde{D}_q\nu(u(\omega))}{\rho(u(\omega))}\right)=\mathrm{Re}\left(\frac{a}{c}+\frac{b}{c}\left[u(q\omega)+u\left(q^{-1}\omega\right)\right]\right)>0$$

Using the subordination assumption (30), we obtain

$$au(\omega)+b(u(\omega))^2+c\omega\widetilde{D}_q u(\omega)\prec as(\omega)+b(s(\omega))^2+c\omega\widetilde{D}_q s(\omega).$$

Now, applying Lemma 4, it follows that:

$$u(\omega)\prec s(\omega)=\frac{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa)\Upsilon(\omega)},\ \omega\in U$$

and that u is the best subordinant. □

Corollary 5 Let $u(\omega)=\frac{1+\Lambda\omega}{1+X\omega}$, where $-1\le X<\Lambda\le 1$. Assume that condition (29) is satisfied. If $\Upsilon\in\mathscr{A}_\tau$ and the function $s(\omega)=\frac{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa)\Upsilon(\omega)}$ is included in the class $\mathscr{H}[u(0),\,1]\cap O$, and if the subordination

$$a\frac{1+\Lambda\omega}{1+X\omega}+b\left(\frac{1+\Lambda\omega}{1+X\omega}\right)^2+\frac{c\omega\left[\Lambda X\omega\left(q^2-1\right)-(\Lambda-X)q\right]}{q+X\omega\left(q^2+1\right)+qX^2\omega^2}\prec\Phi^{\kappa}_{\delta,\,q,\,\tau}(a,\,b,\,c;\,\omega)$$

holds in U, for constants $a,\,b,\,c\in\mathbb{C}$, with $c\neq 0$, where $\Phi^{\kappa}_{\delta,\,q,\,\tau}(a,\,b,\,c;\,\omega)$ is defined in (24), then the following subordination relation holds:

$$\frac{1+\Lambda\omega}{1+X\omega}\prec\frac{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa)\Upsilon(\omega)},\ \omega\in U,$$

and $u(\omega)=\frac{1+\Lambda\omega}{1+X\omega}$ represents the optimal subordinant.

Proof. The assertion is a direct consequence of Theorem 4, with the choice $u(\omega)=\frac{1+\Lambda\omega}{1+X\omega}$, under the stated assumptions. □

Corollary 6 Assume that $u(\omega)=\frac{1+(1-2\gamma)\omega}{1-\omega}$, $0\le\gamma<1$, $\omega\in U$ and that condition (29) is satisfied. If $f\in\mathscr{A}_\tau$ and $\frac{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa)\Upsilon(\omega)}\in\mathscr{H}[u(0),\,1]\cap O$ and

$$a\frac{1+(1-2\gamma)\omega}{1-\omega}+b\left(\frac{1+(1-2\gamma)\omega}{1-\omega}\right)^2+\frac{c\omega\left[\omega(1-2\gamma)\left(1-q^2\right)-2(1-\gamma)q\right]}{q-\omega\left(q^2+1\right)+q\omega^2}\prec\Phi^{\kappa}_{\delta,\,q,\,\tau}(a,\,b,\,c;\,\omega),$$

for $a,\,b,\,c\in\mathbb{C}$, $c\neq 0$, $0<\gamma\le 1$, where $\Phi^{\kappa}_{\delta,\,q,\,\tau}(a,\,b,\,c;\,\omega)$ is formulated (24), then we conclude that

$$\frac{1+(1-2\gamma)\omega}{1-\omega}\prec\frac{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}^{\kappa}_{\delta,\,q,\,\tau}(\kappa)\Upsilon(\omega)}$$

and $\frac{1+(1-2\gamma)\omega}{1-\omega}$ represents the optimal subordinant.

Proof. Corollary follows through the use of Theorem 4 for $u(\omega)=\frac{1+(1-2\gamma)\omega}{1-\omega}$, $0<\gamma\le 1$. □

Corollary 7 Let $u(\omega)=\frac{1+\omega}{1-\omega}$, $\omega\in U$ and suppose that condition (29) is satisfied. If $\Upsilon\in\mathscr{A}_\tau$, and define the function $s(\omega)=\frac{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa)\Upsilon(\omega)}$. Suppose further that $s(\omega)\in\mathscr{H}[u(0),\,1]\cap O$ and that the subordination

$$a\frac{1+\omega}{1-\omega}+b\left(\frac{1+\omega}{1-\omega}\right)^{2}+\frac{c\omega\left[\omega\left(1-q^{2}\right)-2q\right]}{q-\omega\left(q^{2}+1\right)+q\omega^{2}}\prec\Phi_{\delta,\,q,\,\tau}^{\kappa}\left(a,\,b,\,c;\,\omega\right),$$

holds for some complex constants $a,\,b,\,c$, where $c\neq 0,$ and assuming that $\Phi_{\delta,\,q,\,\tau}^{\kappa}\left(a,\,b,\,c;\,\omega\right)$ is introduced as in (24), the subsequent subordination statement holds:

$$\frac{1+\omega}{1-\omega}\prec\frac{\widetilde{V}_{\delta,\,q,\,\tau}\left(\kappa+1\right)\Upsilon(\omega)}{\widetilde{V}_{\delta,\,q,\,\tau}\left(\kappa\right)\Upsilon(\omega)},$$

and $\frac{1+\omega}{1-\omega}$ represents the sharpest subordinant.

Proof. The result is obtained as a direct consequence of Theorem 4, by choosing $u\left(\omega\right)=\frac{1+\omega}{1-\omega}$, which is a conformal convex mapping on U, so that $u(0)=1.$ The assumptions ensure that all hypotheses of the theorem are fulfilled, hence the conclusion follows. □

Corollary 8 Let $u\left(\omega\right)=\frac{1+\omega}{(1-\omega)^{2}},\ \ \omega\in U$ and suppose that condition (29) is satisfied. If $\Upsilon\in\mathscr{A}_{\tau}$ and define the function $s\left(\omega\right)=\frac{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa)\Upsilon(\omega)}$. Suppose further that $s\left(\omega\right)\in\mathscr{H}\left[u\left(0\right),\,1\right]\cap O$ and that the subordination

$$a\frac{1+\omega}{(1-\omega)^{2}}+b\left(\frac{1+\omega}{(1-\omega)^{2}}\right)^{2}+\frac{c\omega\left[(1+q\omega)\left(\widetilde{[2]}_{q}\omega-2\right)-\left(1-q^{-1}\omega\right)^{2}\right]}{\left(1-\widetilde{[2]}_{q}\omega+\omega^{2}\right)^{2}}\prec\Phi_{\delta,\,q,\,\tau}^{\kappa}\left(a,\,b,\,c;\,\omega\right),$$

holds for $a,\,b,\,c\in\mathbb{C}$, with $c\neq 0,$ where $\Phi_{\delta,\,q,\,\tau}^{\kappa}\left(a,\,b,\,c;\,\omega\right)$ is defined in (24). Accordingly, the subsequent subordination statement holds true:

$$\frac{1+\omega}{(1-\omega)^{2}}\prec\frac{\widetilde{V}_{\delta,\,q,\,\tau}\left(\kappa+1\right)\Upsilon(\omega)}{\widetilde{V}_{\delta,\,q,\,\tau}\left(\kappa\right)\Upsilon(\omega)},$$

and $\frac{1+\omega}{(1-\omega)^{2}}$ constitutes the sharpest subordinant.

Proof. This outcome follows directly from Theorem 4, by selecting $u\left(\omega\right)=\frac{1+\omega}{(1-\omega)^{2}}$, a one-to-one convex holomorphic mapping defined in U, satisfying $u(0)=1.$ Under the given assumptions, all the premises of the theorem are fulfilled, and the outcome is thereby established. □

By simultaneously invoking the conclusions of Theorem 3 and Theorem 4, we derive a comprehensive result in the form of a sandwich-type theorem. This theorem encapsulates both the subordination and superordination frameworks, yielding a two-sided inclusion for the analytic function under consideration. Specifically, it establishes that the function transformed by the symmetric q-differential operator lies between two extremal functions—designated as the optimal subordinant and dominant, respectively—each characterized by precise geometric or analytic conditions. The result provides a sharp and symmetric containment of the target function, demonstrating the power of combining differential subordination and superordination theories within the structure of symmetric q-calculus. Such sandwich-type results are particularly valuable in geometric function theory, as they offer detailed insight into the functional bounds and behaviors within complex domains.

Theorem 5 Consider u_1 and u_2 as injective analytic mappings in the unit disk U, such that $u_1\left(\omega\right)\neq 0$ and $u_2\left(\omega\right)\neq 0,$ for each $\omega\in U.$ Assume further that the functions $\omega\widetilde{D}_{q}u_1\left(\omega\right)$ and $\omega\widetilde{D}_{q}u_1\left(\omega\right)$ are starlike and univalent in $U.$ Suppose that the functions u_1 satisfies (23) and u_2 satisfies (29), $\Upsilon \in \mathscr{A}_\tau$, and the function $\frac{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa)\Upsilon(\omega)}$ is included in the class $\mathscr{H}\left[u(0),\,1\right]\cap O$. Assume also that the function $\Phi^{\kappa}_{\delta,\,q,\,\tau}(a,\,b,\,c;\,\omega)$ defined in Theorem 4 maps U conformally onto its image, and that for constants $a,\,b,\,c \in \mathbb{C}$, with $c \neq 0$, the following double subordination holds:

$$au_1(\omega)+b\left(u_1(\omega)\right)^2+c\omega\widetilde{D}_q u_1(\omega) \prec \Phi^{\kappa}_{\delta,\,q,\,\tau}(a,\,b,\,c;\,\omega) \prec au_2(\omega)+b\left(u_2(\omega)\right)^2+c\omega\widetilde{D}_q u_2(\omega),$$

Then it follows that:

$$u_1(\omega) \prec \frac{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa)\Upsilon(\omega)} \prec u_2(\omega),$$

and the functions u_1 and u_2 serve, respectively, as the sharp subordinant and dominant associated with this sandwich-type relation.

Let $u_1(\omega)=\frac{1+\Lambda_1\omega}{1+X_1\omega}$, $u_2(\omega)=\frac{1+\Lambda_2\omega}{1+X_2\omega}$, where the parameters satisfy the ordering $-1 \leq X_2 < X_1 < \Lambda_1 < \Lambda_2 \leq 1$. Under these conditions, the following corollary holds.

Corollary 9 Assume that the conditions specified in (23) and (29) hold for the functions $u_1(\omega)=\frac{1+\Lambda_1\omega}{1+X_1\omega}$ and $u_2(\omega)=\frac{1+\Lambda_2\omega}{1+X_2\omega}$, respectively, where the parameters satisfy $-1 \leq X_2 \leq X_1 < \Lambda_1 \leq \Lambda_2 \leq 1$. Let $\Upsilon \in \mathscr{A}_\tau$, and suppose the function $\frac{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa)\Upsilon(\omega)}$ belongs to the class $\mathscr{H}\left[u(0),\,1\right]\cap O$. Assume further that the function $\Phi^{\kappa}_{\delta,\,q,\,\tau}(a,\,b,\,c;\,\omega)$ defined in (24), is univalent in U, and that for constants $a,\,b,\,c \in \mathbb{C}$, with $c \neq 0$, the following double subordination holds:

$$a\frac{1+\Lambda_1\omega}{1+X_1\omega}+b\left(\frac{1+\Lambda_1\omega}{1+X_1\omega}\right)^2+\frac{c\omega\left[\Lambda_1X_1\omega\left(q^2-1\right)-\left(\Lambda_1-X_1\right)q\right]}{q+X_1\omega\left(q^2+1\right)+qX_1^2\omega^2}$$

$$\prec\Phi^{\kappa}_{\delta,\,q,\,\tau}(a,\,b,\,c;\,\omega)$$

$$\prec a\frac{1+\Lambda_2\omega}{1+X_2\omega}+b\left(\frac{1+\Lambda_2\omega}{1+X_2\omega}\right)^2+\frac{c\omega\left[\Lambda_2X_2\omega\left(q^2-1\right)-\left(\Lambda_2-X_2\right)q\right]}{q+X_2\omega\left(q^2+1\right)+qX_2^2\omega^2},$$

Then it follows that:

$$\frac{1+\Lambda_1\omega}{1+X_1\omega} \prec \Phi^{\kappa}_{\delta,\,q,\,\tau}(a,\,b,\,c;\,\omega) \prec \frac{1+\Lambda_2\omega}{1+X_2\omega},$$

and consequently, the functions $\frac{1+\Lambda_1\omega}{1+X_1\omega}$ and $\frac{1+\Lambda_2\omega}{1+X_2\omega}$ represent the sharp subordinant and dominant, respectively, associated with this subordination structure.

Corollary 10 Suppose that the conditions stated in references (23) and (29) are satisfied for the functions $u_1(\omega)=\frac{1+(1-2\gamma_1)\omega}{1-\omega}$ and $u_2(\omega)=\frac{1+(1-2\gamma_2)\omega}{1-\omega}$, respectively, where the parameters fulfill the inequality $0 \leq \gamma_2 \leq \gamma_1 \leq 1$. Let $\Upsilon \in \mathscr{A}_\tau$, and assume that the quotient $\frac{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa+1)\Upsilon(\omega)}{\widetilde{V}_{\delta,\,q,\,\tau}(\kappa)\Upsilon(\omega)}$ belongs to the class $\mathscr{H}\left[u(0),\,1\right]\cap O$. Furthermore, assume that the function $\Phi^{\kappa}_{\delta,\,q,\,\tau}(a,\,b,\,c;\,\omega)$ defined in (24), is univalent in U, and that for constants $a,\,b,\,c \in \mathbb{C}$, with $c \neq 0$, the following double subordination holds:

$$a\frac{1+(1-2\gamma_1)\,\omega}{1-\omega}+b\left(\frac{1+(1-2\gamma_1)\,\omega}{1-\omega}\right)^2+\frac{c\omega\left[\omega(1-2\gamma_1)\left(1-q^2\right)-2(1-\gamma_1)\,q\right]}{q-\omega\left(q^2+1\right)+q\omega^2}$$

$$\prec\Phi^{\kappa}_{\delta,\,q,\,\tau}(a,\,b,\,c;\,\omega)$$

$$\prec a\frac{1+(1-2\gamma_2)\,\omega}{1-\omega}+b\left(\frac{1+(1-2\gamma_2)\,\omega}{1-\omega}\right)^2+\frac{c\omega\left[\omega(1-2\gamma_2)\left(1-q^2\right)-2(1-\gamma_2)\,q\right]}{q-\omega\left(q^2+1\right)+q\omega^2},$$

Thus, it follows that:

$$\frac{1+(1-2\gamma_1)\,\omega}{1-\omega}\prec\Phi^{\kappa}_{\delta,\,q,\,\tau}(a,\,b,\,c;\,\omega)\prec\frac{1+(1-2\gamma_2)\,\omega}{1-\omega},$$

and accordingly, the functions $\frac{1+(1-2\gamma_1)\omega}{1-\omega}$ and $\frac{1+(1-2\gamma_2)\omega}{1-\omega}$ act as the optimal subordinant and dominant, respectively, within this subordination framework.

5. Conclusions

This study was motivated by the desire to explore the interplay between geometric complex analysis and symmetric q-difference theory, particularly in the setting of multivalent starlike functions. Using a symmetric quantum differential operator, we introduced novel classes of multivalent mappings linked to Janowski-type functions. We established inclusion relationships between these subclasses and provided a sufficient condition for class membership based on coefficient estimates.

Furthermore, we developed a general framework for second-order analytic sub- and superordination problems with respect to the same operator. Several sharp results were obtained, including the identification of best dominant and subordinant functions, as well as a sandwich-type theorem. These contributions demonstrate the utility of symmetric quantum calculus within the framework of geometric complex analysis and broaden existing results in the theory of multivalent functions.

The methodology and results presented here may serve as a foundation for further investigations involving more general operators, extended function classes, or applications in related fields such as fractional calculus and complex dynamical systems.

Funding

The APC was founded by the University of Oradea, Romania.

Conflict of interest

The author declares no competing financial interest.

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