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Research Article

An Analytical Algebraic Method for Solving Nonlinear Fractional Differential Equations with Conformable Fractional Derivatives

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Abstract: This paper presents a new and enhanced algebraic method for accurately solving nonlinear Fractional Differential Equations (FDEs). Using a complex fractional transformation, we translate nonlinear FDEs with Jumarie modified Riemann-Liouville derivatives into their corresponding ordinary differential equations. By applying this method to two nonlinear FDEs, including the time-fractional Bogoyavlenskii problem, we demonstrate its strength and adaptability. The method's effectiveness in solving various nonlinear FDEs is showcased, laying the groundwork for future developments in this rapidly evolving field. This research has significant implications for addressing complex issues in physics, engineering, and finance, providing a reliable and efficient approach to modeling and analyzing real-world systems. Furthermore, this study advances the field of fractional calculus, which has garnered attention for its ability to illustrate and explain intricate systems and processes.

Keywords: nonlinear fractional differential equation, algebraic method, fractional complex transformation, conformable fractional derivative, time-fractional bogoyavlenskii problem

MSC: 35C07, 35C08, 35C09

1. Introduction

Nonlinear Schrödinger Equations (NLSEs) have aroused interest among scientists in a wide range of fields, including fluid dynamics, plasma physics, and nonlinear optics. Many technical and scientific domains use these equations to model complex systems and processes [1–4]. The ability to reproduce several phenomena in light propagation in optical fibres, including four-wave mixing and self-phase modulation, makes them significant [5–7]. Research on Nonlinear Partial Differential Equations (NLPDEs) has accelerated dramatically because precise solutions are crucial for advancing our

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understanding of the theoretical and practical aspects of diverse processes [8, 9]. Solitons for the Nonlinear Schrödinger Equation (NLSE) are one area of special interest [10–12]. Stable waveforms known as soliton maintain their shape across extended distances. Due to their ability to maintain signal integrity over long distances, they are essential for applications in optical communication systems.

Therefore, in addition to advancing theoretical knowledge, the pursuit of precise solutions to NLPDEs propels developments in a variety of domains. One important kind of NLPDE with many uses is the Schrödinger equation, particularly in domains like plasma physics and nonlinear optics. It is used in nonlinear optics to explain how light behaves in nonlinear media, where the material's refractive index varies with intensity [13–15]. The nonlinear Schrödinger equation governs the resulting processes, which include self-focusing, soliton production, and harmonic frequency generation. The development of sophisticated optical technologies, such as lasers, waveguides, and optical communication systems, depends on these phenomena. Improving international communication networks, which are essential to continuous technological advancement, is one area in which Soliton solutions are very significant. Researchers have created advanced analytical and numerical methods to get soliton solutions for NLPDEs. These techniques offer vital insights into nonlinear wave phenomena and are necessary for precisely modelling and comprehending soliton dynamics.

We consider the time fractional Bogoyavlenskii equation as follows [16]:

$$\begin{cases} 4D_t^{\gamma}u + u_{xxy} - 4u^2u_y - 4u_xv = 0, \\ uu_y = v_x. \end{cases} \tag{1}$$

With the Bogoyavlenskii equation, a nonlinear partial differential equation, many physical processes have been simulated, including fluid dynamics, plasma physics, and optics. The ability of fractional-order generalisations of classical nonlinear equations to depict intricate and surprising physical system behaviours has attracted a lot of attention recently. This study examines the time-fractional Bogoyavlenskii equation, a fractional-order extension of the classical Bogoyavlenskii equation.

Researchers have shown a significant increase in interest in studying various real-life phenomena through NLDEs [17–19]. Recently, there has been a growing focus on nonlinear mathematical models aimed at understanding the transmission of pulses in optical fibers [20–22]. Equations of this kind are widely applied across many scientific fields and have been extensively studied from different viewpoints [23–26]. Various methodologies can be used to explore these soliton solutions [27–29]. An innovative approach involves using transformations to convert NLDEs into nonlinear Ordinary Differential Equations (ODEs). Solitons have the remarkable property of traveling long distances in optical fibers without losing their shape or undergoing distortion [30, 31]. This unique behavior results from the delicate balance between dispersion and nonlinearity in the fiber, which ensures that the soliton retains its form as it propagates [32–34].

1.1 *The definition of CFD*

Explicit solutions for NFPDEs can be obtained by utilising the advantages that Computational Fluid Dynamics (CFDs) have over other fractional derivative operators. Notably, alternate formulations of fractional derivatives are incompatible with the chain rule and cannot yield the soliton solutions of equations (2) [35, 36]. CFDs were so added to equation (2). The β-order CFDs operator is defined as follows [37]:

$$D_\eta^{\beta}w(\eta) = \lim_{\varepsilon\to 0}\frac{w(\varepsilon\eta^{1-\beta}+\eta)-w(\eta)}{\varepsilon}, \quad \beta \in (0,\ 1]. \tag{2}$$

The following properties of this derivative are utilized in this research:

$$D_\eta^\beta \eta^p = p\eta^{p-\beta}, \tag{3}$$

$$D_\eta^\beta(p_1\varpi(\eta) \pm p_2\pi(\eta)) = p_1 D_\eta^\beta(\varpi(\eta)) \pm p_2 D_\eta^\beta(\pi(\eta)), \tag{4}$$

$$D_\eta^\beta \chi[\zeta(\eta)] = \chi'_\zeta(\zeta(\eta)) D_\eta^\beta \zeta(\eta), \tag{5}$$

where $\varpi(\eta)$, $\pi(\eta)$, $\chi(\eta)$ & $\zeta(\eta)$ are functions that the can be differentiated arbitrarily, but p, p_1 & p_2 signify constants. Moreover, the following theorem shows that how CFD satisfies chain rule, a rule which is important in the solution of Caputo-Katugampola P-Laplacian Partial Equation (CKPPE).

Theorem 2.1 Let $\chi(\eta)$ & $\zeta(\eta)$ are arbitrary differentiable functions then

$$D_\eta^\beta \chi[\zeta(\eta)] = \chi'_\zeta(\zeta(\eta)) D_\eta^\beta \zeta(\eta).$$

Proof. If the function ζ is a constant in a neighbourhood η_0, then $D_\eta^\beta \chi(\zeta(\eta_0)) = 0$. However, we make the following assumption about non-constant function ζ in the vicinity of η_0. Here, we are able to find an $\varepsilon > 0 \ni \zeta(\eta_1) \neq \zeta(\eta_2)$ for any $\eta_1,\ \eta_2 \in (\eta_0 - \varepsilon_0,\ \eta_0 + \varepsilon_0)$. Thus, since the function ζ is continuous at η_0, for $\eta_0 > a$, $\eta_0^\beta \neq a$ (where $a \geq 0$), we acquire,

$$D_\eta^\beta(\chi \circ \zeta)(\eta_0) = \lim_{\varepsilon \to 0} \frac{\chi(\zeta(\eta_0 + \varepsilon \eta_0^{-\beta}(\eta_0 - a))) - \chi(\zeta(\eta_0))}{\varepsilon(1 - a\eta_0^{-\beta})}$$

$$= \lim_{\varepsilon \to 0} \frac{\chi(\zeta(\eta_0 + \varepsilon \eta_0^{-\beta}(\eta_0 - a))) - \chi(\zeta(\eta_0))}{\zeta(\eta_0 + \varepsilon \eta_0^{-\beta}(\eta_0 - a)) - \zeta(\eta_0)} \cdot \frac{\zeta(\eta_0 + \varepsilon \eta_0^{-\beta}(\eta_0 - a)) - \zeta(\eta_0)}{\varepsilon(1 - a\eta_0^{-\beta})}$$

$$= \lim_{\varepsilon_1 \to 0} \frac{\chi(\zeta(\eta_0) + \varepsilon_1) - \chi(\zeta(\eta_0))}{\varepsilon_1} \cdot \frac{\zeta(\eta_0 + \varepsilon \eta_0^{-\beta}(\eta_0 - a)) - \zeta(\eta_0)}{\varepsilon(1 - a\eta_0^{-\beta})}$$

$$= \chi'(\zeta(\eta_0)) D_\eta^\beta(\zeta)(\eta_0).$$

Thus CFD satisfies the chain rule.

2. Methodology of the modified extended direct algebraic method

In this section, the Extended Direct Algebraic Method (EDAM) approach is described. Examine the Fractional Partial Differential Equation (FPDE) with the following form [36, 37]:

$$P(w,\ \partial_t^\alpha w,\ \partial_{y_1}^\beta w,\ \partial_{y_2}^\gamma w,\ w^2,\ \ldots) = 0,\ \ 0 < \alpha,\ \beta,\ \gamma \leq 1, \tag{6}$$

where w is determined by $y_1,\ y_2,\ y_3,\ \ldots,\ y_r$, and t. (8) is solved using the following steps:

Step 1 Start by transforming the variables $w(y_1,\ y_2,\ y_3,\ \ldots,\ y_r)$ into $W(\Pi)$, where Π is specified in a variety of ways. (8) is changed by this transformation into a nonlinear ODE of the following form:

$$Q(W,\ W'W,\ W',\ \ldots) = 0, \tag{7}$$

where W in (9) has derivatives with respect to Π. The constant(s) of integration can be obtained by integrating (9) once or more times.

Step 2 Next, we presume that the solution to (9) is as follows:

$$W(\Pi) = \sum_{l=-M}^{M} d_l(\zeta(\Pi))^l, \tag{8}$$

where d_l $(l = -M,\ \ldots,\ 0,\ 1,\ 2,\ \ldots,\ M)$ are constants to be determined, and $\zeta(\Pi)$ is the general solution of the following ODE:

$$\zeta'(\Pi) = ln(\mho)(a + e\zeta(\Pi) + f(\zeta(\Pi))^2), \tag{9}$$

The constants $a,\ e$, and f are represented as $\mho \neq 0,\ 1$.

Step 3 By establishing the homogeneous balance between the highest order derivative and the largest nonlinear term in (9), the positive integer M in (10) is obtained. More specifically, the two equations provided [34] can be used to estimate the balance number:

$$D\left(\frac{d^kW}{d\xi^k}\right) = M + k \text{ and } D\left(W^j\left(\frac{d^kW}{d\xi^k}\right)^l\right) = Mj + l(k+M),$$

In this case, D represents the degree of $W(\xi)$ since $D[W(\xi)] = m$ and $j,\ k$, and l are whole numbers.

Step 4 Next, we integrate (9) to obtain (10), or the equation that results, and all of the terms of $\zeta(\Pi)$ are arranged in the same order.

The system of algebraic equations for d_l $(l = -M,\ \ldots,\ 0,\ 1,\ 2, \ldots,\ M)$ and other parameters is then obtained by setting all of the coefficients of the following polynomial to zero thereafter.

Step 5 Maple is used to solve this collection of algebraic equations.

Step 6 With the $\zeta(\Pi)$ solution of equation (11), the unknown values are then found and inserted into (10) to obtain the analytical answers to (8). It is possible to get the following families of solutions by using the generic solution of (11).

Family 1 Assuming that $f \neq 0$ and $\Lambda < 0$ are

$$\zeta_1(\Pi) = -\frac{e}{2f} + \frac{\sqrt{-\Lambda}\tan_{\mho}\left(\frac{1}{2}\sqrt{-\Lambda}\Pi\right)}{2f},$$

$$\zeta_2(\Pi) = -\frac{e}{2f} - \frac{\sqrt{-\Lambda}\cot_{\mho}\left(\frac{1}{2}\sqrt{-\Lambda}\Pi\right)}{2f},$$

$$\zeta_3(\Pi) = -\frac{e}{2f} + \frac{\sqrt{-\Lambda}\left(\tan_{\mho}\left(\sqrt{-\Lambda}\Pi\right) + \sec_{\mho}\left(\sqrt{-\Lambda}\Pi\right)\right)}{2f},$$

$$\zeta_4(\Pi) = -\frac{e}{2f} - \frac{\sqrt{-\Lambda}\left(\cot_{\mho}\left(\sqrt{-\Lambda}\Pi\right) + \csc_{\mho}\left(\sqrt{-\Lambda}\Pi\right)\right)}{2f},$$

and

$$\zeta_5(\Pi) = -\frac{e}{2f} + \frac{\sqrt{-\Lambda}\left(\tan_{\mho}\left(\frac{1}{4}\sqrt{-\Lambda}\Pi\right) - \cot_{\mho}\left(\frac{1}{4}\sqrt{-\Lambda}\Pi\right)\right)}{4f}.$$

Family 2 $f \neq 0$ and $\Lambda < 0$ are present.

$$\zeta_6(\Pi) = -\frac{e}{2f} - \frac{\sqrt{\Lambda}\tanh_{\mho}\left(\frac{1}{2}\sqrt{\Lambda}\Pi\right)}{2f},$$

$$\zeta_7(\Pi) = -\frac{e}{2f} - \frac{\sqrt{\Lambda}\coth_{\mho}\left(\frac{1}{2}\sqrt{\Lambda}\Pi\right)}{2f},$$

$$\zeta_8(\Pi) = -\frac{e}{2f} - \frac{\sqrt{\Lambda}\left(\tanh_{\mho}\left(\sqrt{\Lambda}\Pi\right) + i\,\mathrm{sech}_{\mho}\left(\sqrt{\Lambda}\Pi\right)\right)}{2f},$$

$$\zeta_9(\Pi) = -\frac{e}{2f} - \frac{\sqrt{\Lambda}\left(\coth_{\mho}\left(\sqrt{\Lambda}\Pi\right) + \mathrm{csch}_{\mho}\left(\sqrt{\Lambda}\Pi\right)\right)}{2f},$$

and

$$\zeta_{10}(\Pi) = -\frac{e}{2f} - \frac{\sqrt{\Lambda}\left(\tanh_{\mho}\left(\frac{1}{4}\sqrt{\Lambda}\Pi\right) - \coth_{\mho}\left(\frac{1}{4}\sqrt{\Lambda}\Pi\right)\right)}{4f}.$$

Family 3 For $af > 0$ and $e = 0$:

$$\zeta_{11}(\Pi)=\sqrt{\frac{a}{f}}\tan_{\mho}\left(\sqrt{af}\Pi\right),$$

$$\mu_{12}(\Pi)=-\sqrt{\frac{a}{f}}\cot_{\mho}\left(\sqrt{af}\Pi\right),$$

$$\zeta_{13}(\Pi)=\sqrt{\frac{a}{f}}\left(\tan_{\mho}\left(2\sqrt{df}\eta\right)+\sec_{\mho}\left(2\sqrt{af}\Pi\right)\right),$$

$$\zeta_{14}(\Pi)=-\sqrt{\frac{a}{f}}\left(\cot_{\mho}\left(2\sqrt{af}\Pi\right)+\csc_{\mho}\left(2\sqrt{af}\Pi\right)\right),$$

and

$$\zeta_{15}(\Pi)=\frac{1}{2}\sqrt{\frac{a}{f}}\left(\tan_{\mho}\left(\frac{1}{2}\sqrt{af}\Pi\right)-\cot_{\mho}\left(\frac{1}{2}\sqrt{af}\Pi\right)\right).$$

Family 4 For $af<0$ and $e=0$:

$$\zeta_{16}(\Pi)=-\sqrt{-\frac{a}{f}}\tanh_{\mho}\left(\sqrt{-af}\Pi\right),$$

$$\zeta_{17}(\Pi)=-\sqrt{-\frac{a}{f}}\coth_{\mho}\left(\sqrt{-af}\Pi\right),$$

$$\zeta_{18}(\Pi)=-\sqrt{-\frac{d}{f}}\left(\tanh_{\mho}\left(2\sqrt{-af}\Pi\right)+i\operatorname{sech}_{\mho}\left(2\sqrt{-af}\Pi\right)\right),$$

$$\zeta_{19}(\Pi)=-\sqrt{-\frac{a}{f}}\left(\coth_{\mho}\left(2\sqrt{-af}\Pi\right)+\operatorname{csch}_{\mho}\left(2\sqrt{-df}\Pi\right)\right),$$

and

$$\zeta_{20}(\Pi)=-\frac{1}{2}\sqrt{-\frac{a}{f}}\left(\tanh_{\mho}\left(\frac{1}{2}\sqrt{-af}\Pi\right)+\coth_{\mho}\left(\frac{1}{2}\sqrt{-af}\Pi\right)\right).$$

Family 5 For $f=a$ and $e=0$:

$$\zeta_{21}(\Pi)=\tan_{\mho}(a\Pi),$$

$$\zeta_{22}(\Pi) = -\cot_{\mho}(a\Pi),$$

$$\zeta_{23}(\Omega) = \tan_{\mho}(2a\Pi) + \sec_{\mho}(2d\Omega),$$

$$\zeta_{24}(\Pi) = -\cot_{\mho}(2a\Pi) + \csc_{\mho}(2a\Pi),$$

and

$$\zeta_{25}(\Pi) = \frac{1}{2}\tan_{\mho}\left(\frac{1}{2}a\Pi\right) - \frac{1}{2}\cot_{\mho}\left(\frac{1}{2}a\Pi\right).$$

Family 6 $f = -a$ and $e = 0$:

$$\zeta_{26}(\Pi) = -\tanh_{\mho}(a\Pi),$$

$$\zeta_{27}(\Pi) = -\coth_{\mho}(a\Pi),$$

$$\zeta_{28}(\Pi) = -\tanh_{\mho}(2a) + i\,\mathrm{sech}_{\mho}(2a\Pi),$$

$$\zeta_{29}(\Pi) = -\coth_{\mho}(2a\Pi) + \mathrm{csch}_{\mho}(2a\Pi),$$

and

$$\zeta_{30}(\Pi) = -\frac{1}{2}\tanh_{\mho}\left(\frac{1}{2}a\Pi\right) - \frac{1}{2}\coth_{\mho}\left(\frac{1}{2}a\Pi\right).$$

Family 7 $\Lambda = 0$:

$$\zeta_{31}(\Pi) = -2\,\frac{a\,(e\Omega\ln\mho + 2)}{e^2\ln(\mho)\Omega}.$$

Family 8 $f = 0$, $e = \varsigma$ and $a = n\varsigma$ (with $n \neq 0$).

$$\zeta_{32}(\Pi) = \mho^{\varsigma\Pi} - n.$$

Family 9 For $e = f = 0$:

$$\zeta_{33}(\Pi) = a\Omega\ln(\mho).$$

Family 10 $e = a = 0$:

$$\zeta_{34}(\Pi) = -\frac{1}{f\Pi\ln(\mho)}.$$

Family 11 $e \neq 0$, $f \neq 0$ and $a = 0$:

$$\zeta_{35}(\Pi) = -\frac{e}{f\left(\cosh_{\mho}(e\Pi) - \sinh_{\mho}(e\Pi) + 1\right)},$$

and

$$\zeta_{36}(\Pi) = -\frac{e\left(\cosh_{\mho}(e\Pi) + \sinh_{\mho}(e\Pi)\right)}{f\left(\cosh_{\mho}(e\Pi) + \sinh_{\mho}(e\Pi) + 1\right)}.$$

Family 12 $e = \varsigma$, $f = n\varsigma$ (with $n \neq 0$), and $a = 0$:

$$\zeta_{37}(\Pi) = \frac{\mho^{\varsigma\Pi}}{1 - n\mho^{\varsigma\Pi}}.$$

In the above solutions, $\Delta = e^2 - 4df$.

$$\sin_{\mho}(\Pi) = \frac{\mho^{i\Pi} - \mho^{-i\Pi}}{2i}, \quad \cos_{\mho}(\Pi) = \frac{\mho^{i\Pi} + \mho^{-i\Pi}}{2},$$

$$\sec_{\mho}(\Pi) = \frac{1}{\cos_{\mho}(\Pi)}, \quad \csc_{\mho}(\Pi) = \frac{1}{\sin_{\mho}(\Pi)},$$

$$\tan_{\mho}(\Pi) = \frac{\sin_{\mho}(\Pi)}{\cos_{\mho}(\Pi)}, \quad \cot_{\mho}(\Pi) = \frac{\cos_{\mho}(\Pi)}{\sin_{\mho}(\Pi)}.$$

Similarly,

$$\sinh_{\mho}(\Pi) = \frac{\mho^{\Pi} - \mho^{-\Pi}}{2}, \quad \cosh_{\mho}(\Pi) = \frac{\mho^{\Omega} + \mho^{-\Pi}}{2},$$

$$\mathrm{sech}_{\mho}(\Pi) = \frac{1}{\cosh_{\mho}(\Pi)}, \quad \mathrm{csch}_{\mho}(\Pi) = \frac{1}{\sinh_{\mho}(\Pi)},$$

$$\tanh_{\mho}(\Pi)=\frac{\sinh_{\mho}(\Pi)}{\cosh_{\mho}(\Pi)},\quad \coth_{\mho}(\Pi)=\frac{\cosh_{\mho}(\Pi)}{\sinh_{\mho}(\Pi)}.$$

3. Applications of the proposed method

By using the following transformations for (1), we have

$$\begin{cases} u(x,\ y,\ t)=K(\mu), \\ v(x,\ y,\ t)=W(\mu), \\ \mu=x+y-\dfrac{\beta t^{\gamma}}{\Gamma(1+\gamma)}, \end{cases} \tag{10}$$

we obtain

$$\begin{cases} -4WK'+K'''-4K^2K'-4K'W=0, \\ \dfrac{K^2}{2}=W. \end{cases} \tag{11}$$

The following ODE is produced by integrating the obtained result with respect to μ and setting the integration constant to zero after the second equation of (11) has been inserted into the first equation.

$$K''-2K^3-\beta K=0. \tag{12}$$

The balancing number $M=1$ is now obtained by balancing the highest order nonlinear term and the highest order derivative term. Therefore, the response is as follows:

$$K(\mu)=\sum_{i=-1}^{1} d_i(K(\mu))^i=d_{-1}(K(\mu))^{-1}+d_0+d_1(K(\mu))^1, \tag{13}$$

where the coefficients that will be found are d_{-1}, d_0, and d_1.

The terms with the same power are collected by (12) and (13) to produce a polynomial in $K(\mu)$. The polynomial's coefficients can be set to zero to produce a system of nonlinear algebraic equations. After using Maple to solve the system, we arrive at the two different solution scenarios listed below:

Case 1

$$\begin{aligned} &\beta=-\frac{1}{8}(\ln(\Delta))^2B^2+\frac{1}{2}(\ln(\Delta))^2AC,\ d_{-1}=\ln(\Delta)A, \\ &d_0=\frac{1}{2}\ln(\Delta)B,\ d_1=0. \end{aligned} \tag{14}$$

Case 2

$$\beta = -\frac{1}{8}(\ln(A))^2\mu^2 + \frac{1}{2}(\ln(A))^2\eta\ \nu,\ d_{-1} = 0,\ d_0 = \frac{1}{2}\ln(A)B,$$
$$d_1 = \ln(\Delta)C. \tag{15}$$

Under the assumption of scenario 1.

Group 1 The following solitary wave solutions for (1) are discovered using $\Psi < 0$ and $C \neq 0$, as well as (10) and (12):

$$K_1(x,\ y,\ t) = \ln(\Delta)A\left(-\frac{1}{2}\frac{B}{C} + \frac{1}{2}\frac{\sqrt{-\Psi}\tan\left(\frac{1}{2}\sqrt{-\Psi}\mu\right)}{C}\right)^{-1} + \frac{1}{2}\ln(\Delta)B, \tag{16}$$

$$W_1(x,\ y,\ t) = \frac{1}{2}\left(\ln(\Delta)A\left(-\frac{1}{2}\frac{B}{C} + \frac{1}{2}\frac{\sqrt{-\Psi}\tan\left(\nu\sqrt{-\Psi}\mu\right)}{C}\right)^{-1} + \frac{1}{2}\ln(\Delta)B\right)^2, \tag{17}$$

$$K_2(x,\ y,\ t) = \ln(\Delta)A\left(-\frac{1}{2}\frac{B}{C} - \frac{1}{2}\frac{\sqrt{-\Psi}\cot\left(\frac{1}{2}\sqrt{-\Psi}\mu\right)}{C}\right)^{-1} + \frac{1}{2}\ln(\Delta)B, \tag{18}$$

$$W_2(x,\ y,\ t) = \frac{1}{2}\left(\ln(\Delta)A\left(-\frac{1}{2}\frac{B}{C} - \frac{1}{2}\frac{\sqrt{-\Psi}\cot\left(\frac{1}{2}\sqrt{-\Psi}\mu\right)}{C}\right)^{-1} + \frac{1}{2}\ln(\Delta)B\right)^2, \tag{19}$$

$$K_3(x,\ y,\ t) = \ln(\Delta)A\left(-\frac{1}{2}\frac{B}{C} + \frac{1}{2}\frac{\sqrt{-\Psi}\left(\tan\left(\sqrt{-\Psi}\mu\right) \pm \left(\sec\left(\sqrt{-\Psi}\mu\right)\right)\right)}{C}\right)^{-1} + \frac{1}{2}\ln(\Delta)B, \tag{20}$$

$$W_3(x,\ y,\ t) = \frac{1}{2}\left(\ln(\Delta)A\left(-\frac{1}{2}\frac{B}{C} + \frac{1}{2}\frac{\sqrt{-\Psi}\left(\tan\left(\sqrt{-\Psi}\mu\right) \pm \left(\sec\left(\sqrt{-\Psi}\mu\right)\right)\right)}{C}\right)^{-1} + \frac{1}{2}\ln(\Delta)B\right)^2, \tag{21}$$

$$K_4(x,\ y,\ t) = \ln(\Delta)A\left(-\frac{1}{2}\frac{B}{C} + \frac{1}{2}\frac{\sqrt{-\Psi}\left(\cot\left(\sqrt{-\Psi}\mu\right) \pm \left(\csc\left(\sqrt{-\Psi}\mu\right)\right)\right)}{C}\right)^{-1} + \frac{1}{2}\ln(\Delta)B, \tag{22}$$

$$W_4(x,\ y,\ t) = \frac{1}{2}\left(\ln(\Delta)A\left(-\frac{1}{2}\frac{B}{C} + \frac{1}{2}\frac{\sqrt{-\Psi}\left(\cot\left(\sqrt{-\Psi}\mu\right) \pm \left(\csc\left(\sqrt{-\Psi}\mu\right)\right)\right)}{C}\right)^{-1} + \frac{1}{2}\ln(\Delta)B\right)^2, \tag{23}$$

and

$$K_5(x,\ y,\ t) = \ln(\Delta) A \left(-\frac{1}{2}\frac{B}{C} + \frac{1}{4}\frac{\sqrt{-\Psi}\left(\tan\left(\frac{1}{4}\sqrt{-\Psi}\mu\right) - \cot\left(\frac{1}{4}\sqrt{-\Psi}\mu\right)\right)}{C} \right)^{-1} + \frac{1}{2}\ln(\Delta) B, \tag{24}$$

$$W_5(x,\ y,\ t) = \frac{1}{2}\left(\ln(\Delta) A \left(-\frac{1}{2}\frac{B}{C} + \frac{1}{4}\frac{\sqrt{-\Psi}\left(\tan\left(\frac{1}{4}\sqrt{-\Psi}\mu\right) - \cot\left(\frac{1}{4}\sqrt{-\Psi}\mu\right)\right)}{C} \right)^{-1} + \frac{1}{2}\ln(\Delta) B \right)^2. \tag{25}$$

Group 2 The following solitary wave solutions are derived for (1) by using (10) and (12), which result in $\Psi > 0$ and $C \neq 0$:

$$K_6(x,\ y,\ t) = \ln(\Delta) A \left(-\frac{1}{2}\frac{B}{C} + \frac{1}{4}\frac{\sqrt{-\Psi}\left(\tan\left(\frac{1}{4}\sqrt{-\Psi}\mu\right) - \cot\left(\frac{1}{4}\sqrt{-\Psi}\mu\right)\right)}{C} \right)^{-1} + \frac{1}{2}\ln(\Delta) B, \tag{26}$$

$$W_6(x,\ y,\ t) = \frac{1}{2}\left(\ln(\Delta) A \left(-\frac{1}{2}\frac{B}{C} + \frac{1}{4}\frac{\sqrt{-\Psi}\left(\tan\left(\frac{1}{4}\sqrt{-\Psi}\mu\right) - \cot\left(\frac{1}{4}\sqrt{-\Psi}\mu\right)\right)}{C} \right)^{-1} + \frac{1}{2}\ln(\Delta) B \right)^2, \tag{27}$$

$$K_7(x,\ y,\ t) = \ln(\Delta) A \left(-\frac{1}{2}\frac{B}{C} - \frac{1}{2}\frac{\sqrt{\Psi}\coth\left(\frac{1}{2}\sqrt{-\Psi}\mu\right)}{C} \right)^{-1} + \frac{1}{2}\ln(\Delta) B, \tag{28}$$

$$W_7(x,\ y,\ t) = \frac{1}{2a_3 f_1}\left(\ln(\Delta) A \left(-\frac{1}{2}\frac{B}{C} - \frac{1}{2}\frac{\sqrt{\Psi}\coth\left(\frac{1}{2}\sqrt{-\Psi}\mu\right)}{C} \right)^{-1} + \frac{1}{2}\ln(\Delta) B \right)^2, \tag{29}$$

$$K_8(x,\ y,\ t) = \ln(\Delta) A \left(-\frac{1}{2}\frac{B}{C} - \frac{1}{2}\frac{\sqrt{\Psi}\left(\tanh\left(\sqrt{-\Psi}\mu\right) \pm \left(sech\left(\sqrt{-\Psi}\mu\right)\right)\right)}{C} \right)^{-1} + \frac{1}{2}\ln(\Delta) B, \tag{30}$$

$$W_8(x,\ y,\ t) = \frac{1}{2}\left(\ln(\Delta) A \left(-\frac{1}{2}\frac{B}{C} - \frac{1}{2}\frac{\sqrt{\Psi}\left(\tanh\left(\sqrt{-\Psi}\mu\right) \pm \left(sech\left(\sqrt{-\Psi}\mu\right)\right)\right)}{C} \right)^{-1} + \frac{1}{2}\ln(\Delta) B \right)^2, \tag{31}$$

$$K_9(x,\ y,\ t) = \ln(\Delta) A \left(-\frac{1}{2}\frac{B}{C} - \frac{1}{2}\frac{\sqrt{\Psi}\left(\coth\left(\sqrt{-\Psi}\mu\right) \pm \left(sech\left(\sqrt{-\Psi}\mu\right)\right)\right)}{C} \right)^{-1} + \frac{1}{2}\ln(\Delta) B, \tag{32}$$

$$W_9(x,\ y,\ t) = \frac{1}{2}\left(\ln(\Delta) A \left(-\frac{1}{2}\frac{B}{C} - \frac{1}{2}\frac{\sqrt{\Psi}\left(\coth\left(\sqrt{-\Psi}\mu\right) \pm \left(sech\left(\sqrt{-\Psi}\mu\right)\right)\right)}{C} \right)^{-1} + \frac{1}{2}\ln(\Delta) B \right)^2, \tag{33}$$

and

$$K_{10}(x,\ y,\ t) = \ln(\Delta) A \left(-\frac{1}{2}\frac{B}{C} - \frac{1}{4}\frac{\sqrt{\Psi}\left(\tanh\left(\frac{1}{4}\sqrt{\Psi}\mu\right) - \coth\left(\frac{1}{4}\sqrt{\Psi}\mu\right)\right)}{C} \right)^{-1} + \frac{1}{2}\ln(\Delta) B \tag{34}$$

$$W_{10}(x,\ y,\ t) = \frac{1}{2}\left(\ln(\Delta) A \left(-\frac{1}{2}\frac{B}{C} - \frac{1}{4}\frac{\sqrt{\Psi}\left(\tanh\left(\frac{1}{4}\sqrt{\Psi}\mu\right) - \coth\left(\frac{1}{4}\sqrt{\Psi}\mu\right)\right)}{C} \right)^{-1} + \frac{1}{2}\ln(\Delta) B \right)^2 \tag{35}$$

Group 3 Using (10) and (12), the following solitary wave solutions are derived for (1) for $CA > 0$ and $B = 0$:

$$K_{11}(x,\ y,\ t) = \ln(\Delta) A \frac{1}{\sqrt{\frac{A}{C}}} \left(\tan\left(\sqrt{CA}\mu\right)\right)^{-1}, \tag{36}$$

$$W_{11}(x,\ y,\ t) = \frac{1}{2}\left(\ln(\Delta) A \frac{1}{\sqrt{\frac{A}{C}}} \left(\tan\left(\sqrt{CA}\mu\right)\right)^{-1} \right)^2, \tag{37}$$

$$K_{12}(x,\ y,\ t) = -\ln(\Delta) A \frac{1}{\sqrt{\frac{A}{C}}} \left(\cot\left(\sqrt{CA}\mu\right)\right)^{-1}, \tag{38}$$

$$W_{12}(x,\ y,\ t) = \frac{1}{2}\left(-\ln(\Delta) A \frac{1}{\sqrt{\frac{A}{C}}} \left(\cot\left(\sqrt{CA}\mu\right)\right)^{-1} \right)^2, \tag{39}$$

$$K_{13}(x,\ y,\ t) = \ln(\Delta) A \frac{1}{\sqrt{\frac{A}{C}}} \left(\tan\left(2\sqrt{CA}\mu\right) \pm \left(\sec\left(2\sqrt{CA}\mu\right)\right)\right)^{-1}, \tag{40}$$

$$W_{13}(x,\ y,\ t)=\frac{1}{2}\left(\ln(\Delta)A\frac{1}{\sqrt{\frac{A}{C}}}\left(\tan\left(2\sqrt{CA}\mu\right)\pm\left(\sec\left(2\sqrt{CA}\mu\right)\right)\right)^{-1}\right)^{2},\tag{41}$$

$$K_{14}(x,\ y,\ t)=-\ln(\Delta)A\frac{1}{\sqrt{\frac{A}{C}}}\left(\cot\left(2\sqrt{CA}\mu\right)\pm\left(\csc\left(2\sqrt{CA}\mu\right)\right)\right)^{-1}\tag{42}$$

$$W_{14}(x,\ y,\ t)=\frac{1}{2}\left(-\ln(\Delta)A\frac{1}{\sqrt{\frac{A}{C}}}\left(\cot\left(2\sqrt{CA}\mu\right)\pm\left(\csc\left(2\sqrt{CA}\mu\right)\right)\right)^{-1}\right)^{2}\tag{43}$$

and

$$K_{15}(x,\ y,\ t)=2\ln(\Delta)A\frac{1}{\sqrt{\frac{A}{C}}}\left(\tan\left(\frac{1}{2}\sqrt{CA}\mu\right)-\cot\left(\frac{1}{2}\sqrt{CA}\mu\right)\right)^{-1},\tag{44}$$

$$W_{15}(x,\ y,\ t)=\frac{1}{2}\left(2\ln(\Delta)A\frac{1}{\sqrt{\frac{A}{C}}}\left(\tan\left(\frac{1}{2}\sqrt{CA}\mu\right)-\cot\left(\frac{1}{2}\sqrt{CA}\mu\right)\right)^{-1}\right)^{2}.\tag{45}$$

Group 4 Following the application of (10) and (12) to $CA<0$ and $B=0$, the solitary wave solutions for (1) are as follows:

$$K_{16}(x,\ y,\ t)=-\ln(\Delta)A\frac{1}{\sqrt{-\frac{A}{C}}}\left(\tanh\left(\sqrt{-CA}\mu\right)\right)^{-1},\tag{46}$$

$$W_{16}(x,\ y,\ t)=\frac{1}{2}\left(-\ln(\Delta)A\frac{1}{\sqrt{-\frac{A}{C}}}\left(\tanh\left(\sqrt{-CA}\mu\right)\right)^{-1}\right)^{2},\tag{47}$$

$$K_{17}(x,\ y,\ t)=-\ln(\Delta)A\frac{1}{\sqrt{-\frac{A}{C}}}\left(\coth\left(\sqrt{-CA}\mu\right)\right)^{-1},\tag{48}$$

$$W_{17}(x,\ y,\ t)=\frac{1}{2}\left(-\ln(\Delta)A\frac{1}{\sqrt{-\frac{A}{C}}}\left(\coth\left(\sqrt{-CA}\mu\right)\right)^{-1}\right)^{2}, \tag{49}$$

$$K_{18}(x,\ y,\ t)=-\ln(\Delta)A\frac{1}{\sqrt{-\frac{A}{C}}}\left(\tanh\left(2\sqrt{-CA}\mu\right)\pm\left(isech\left(2\sqrt{-CA}\mu\right)\right)\right)^{-1}, \tag{50}$$

$$W_{18}(x,\ y,\ t)=\frac{1}{2}\left(-\ln(\Delta)A\frac{1}{\sqrt{-\frac{A}{C}}}\left(\tanh\left(2\sqrt{-CA}\mu\right)\pm\left(isech\left(2\sqrt{-CA}\mu\right)\right)\right)^{-1}\right)^{2}, \tag{51}$$

$$K_{19}(x,\ y,\ t)=-\ln(\Delta)A\frac{1}{\sqrt{-\frac{A}{C}}}\left(\coth\left(2\sqrt{-CA}\mu\right)\pm\left(csch\left(2\sqrt{-CA}\mu\right)\right)\right)^{-1}, \tag{52}$$

$$W_{19}(x,\ y,\ t)=\frac{1}{2a_3f_1}\left(-\ln(\Delta)A\frac{1}{\sqrt{-\frac{A}{C}}}\left(\coth\left(2\sqrt{-CA}\mu\right)\pm\left(csch\left(2\sqrt{-CA}\mu\right)\right)\right)^{-1}\right)^{2}, \tag{53}$$

and

$$K_{20}(x,\ y,\ t)=-2\ln(\Delta)A\frac{1}{\sqrt{-\frac{A}{C}}}\left(\tanh\left(\frac{1}{2}\sqrt{-CA}\mu\right)+\coth\left(\frac{1}{2}\sqrt{-CA}\mu\right)\right)^{-1}. \tag{54}$$

$$W_{20}(x,\ y,\ t)=\frac{1}{2}\left(-2\ln(\Delta)A\frac{1}{\sqrt{-\frac{A}{C}}}\left(\tanh\left(\frac{1}{2}\sqrt{-CA}\mu\right)+\coth\left(\frac{1}{2}\sqrt{-CA}\mu\right)\right)^{-1}\right)^{2}. \tag{55}$$

Group 5 Assuming $C=A,\ B=0$, the following solitary wave solutions for (1) are derived by applying (10) and (12):

$$K_{21}(x,\ y,\ t)=\frac{\ln(\Delta)A}{\tan(A\mu)}, \tag{56}$$

$$W_{21}(x,\ y,\ t)=\frac{1}{2}\left(\frac{\ln(\Delta)A}{\tan(A\mu)}\right)^{2}, \tag{57}$$

$$K_{22}(x,\ y,\ t)=-\frac{\ln(\Delta)A}{\cot(A\mu)},\tag{58}$$

$$W_{22}(x,\ y,\ t)=\frac{1}{2}\left(-\frac{\ln(\Delta)A}{\cot(A\mu)}\right)^2\tag{59}$$

$$K_{23}(x,\ y,\ t)=\frac{\ln(\Delta)A}{\tan(2A\mu)\pm(\sec(2A\mu))},\tag{60}$$

$$W_{23}(x,\ y,\ t)=\frac{1}{2}\left(\frac{\ln(\Delta)A}{\tan(2A\mu)\pm(\sec(2A\mu))}\right)^2,\tag{61}$$

$$K_{24}(x,\ y,\ t)=\frac{\ln(\Delta)A}{-\cot(2A\mu)\pm(\csc(2A\mu))},\tag{62}$$

$$W_{24}(x,\ y,\ t)=\frac{1}{2}\left(\frac{\ln(\Delta)A}{-\cot(2A\mu)\pm(\csc(2A\mu))}\right)^2,\tag{63}$$

and

$$K_{25}(x,\ y,\ t)=\frac{\ln(\Delta)A}{\frac{1}{2}\tan\left(\frac{1}{2}A\mu\right)-\frac{1}{2}\cot\left(\frac{1}{2}A\mu\right)},\tag{64}$$

$$W_{25}(x,\ y,\ t)=\frac{1}{2}\left(\frac{\ln(\Delta)A}{\frac{1}{2}\tan\left(\frac{1}{2}A\mu\right)-\frac{1}{2}\cot\left(\frac{1}{2}A\mu\right)}\right)^2.\tag{65}$$

Group 6 When $C=-A,\ B=0$, as a result of utilizing (10) and (12) the subsequent solitary wave solutions are obtained for (1):

$$K_{26}(x,\ y,\ t)=\frac{\ln(\Delta)A}{\tanh(A^2)},\tag{66}$$

$$W_{26}(x,\ y,\ t)=\frac{1a_4s_1}{2}\left(\frac{\ln(\Delta)A}{\tanh(A^2)}\right)^2,\tag{67}$$

$$K_{27}(x,\ y,\ t)=-\frac{\ln(\Delta)A}{\coth(A\mu)},\tag{68}$$

$$W_{27}(x,\ y,\ t)=\frac{1}{2}\left(-\frac{\ln(\Delta)A}{\coth(A\mu)}\right)^2,\tag{69}$$

$$K_{28}(x,\ y,\ t) = \frac{\ln(\Delta) A}{-\tanh(2A\mu) \pm (isech(2A\mu))}, \tag{70}$$

$$W_{28}(x,\ y,\ t) = \frac{1}{2}\left(\frac{\ln(\Delta) A}{-\tanh(2A\mu) \pm (isech(2A\mu))}\right)^2, \tag{71}$$

$$K_{29}(x,\ y,\ t) = \frac{\ln(\Delta) A}{-\coth(2A\mu) \pm (cech(2A\mu))}, \tag{72}$$

$$W_{29}(x,\ y,\ t) = \frac{1}{2}\left(\frac{\ln(\Delta) A}{-\coth(2A\mu) \pm (cech(2A\mu))}\right)^2, \tag{73}$$

and

$$K_{30}(x,\ y,\ t) = \frac{\ln(\Delta) A}{-\frac{1}{2}\tanh\left(\frac{1}{2}A\mu\right) - \frac{1}{2}\coth\left(\frac{1}{2}A\mu\right)} + \frac{1}{2}\ln(\Delta) B, \tag{74}$$

$$W_{30}(x,\ y,\ t) = \frac{1}{2}\left(\frac{\ln(\Delta) A}{-\frac{1}{2}\tanh\left(\frac{1}{2}A\mu\right) - \frac{1}{2}\coth\left(\frac{1}{2}A\mu\right)} + \frac{1}{2}\ln(\Delta) B\right)^2. \tag{75}$$

Group 7 The following solitary wave solutions are produced for (1) when $\Psi = 0$ as a result of using (10) and (12):

$$K_{31}(x,\ y,\ t) = -\frac{1}{2}\frac{(\ln(\Delta))^2 B^2\mu}{B\mu\ \ln(\Delta) + 2} + \frac{1}{2}\ln(\Delta) B, \tag{76}$$

$$W_{31}(x,\ y,\ t) = \frac{1}{2}\left(-\frac{1}{2}\frac{(\ln(\Delta))^2 B^2\mu}{B\mu\ \ln(\Delta) + 2} + \frac{1}{2}\ln(\Delta) B\right)^2. \tag{77}$$

Group 8 Using (10) and (12), the following solitary wave solutions are obtained for (1) for $B = \rho,\ A = \pi\rho(\pi \neq 0),\ C = 0$.

$$K_{32}(x,\ y,\ t) = \frac{\ln(\Delta)\pi^2\rho}{\Delta^{\rho\mu} - \pi} + \frac{1}{2}\ln(\Delta)\rho \neq \frac{1}{2}\ln(\Delta)\rho, \tag{78}$$

$$W_{32}(x,\ y,\ t) = \frac{a_4 s_1}{a_3 f_1}\left(\frac{\ln(\Delta)\pi^2\rho}{\Delta^{\rho\mu} - \pi} + \frac{1}{2}\ln(\Delta)\rho \neq \frac{1}{2}\ln(\Delta)\rho\right)^2. \tag{79}$$

Group 9 Using (10) and (12), the following solitary wave solutions are obtained for (1) for $B = 0,\ C = 0$:

$$K_{33}(x,\ y,\ t) = \mu^{-1}, \tag{80}$$

$$W_{33}(x,\ y,\ t) = \frac{1}{2}\left(\mu^{-1}\right)^2. \tag{81}$$

Group 10 Using (10) and (12), the following solitary wave solutions are derived for (1) for $A := 0;\ B \neq 0;\ C \neq 0$:

$$K_{34}(x,\ y,\ t) = \frac{1}{2}\ln(\Delta) B, \tag{82}$$

$$W_{34}(x,\ y,\ t) = \frac{1}{2}\left(\frac{1}{2}\ln(\Delta) B\right)^2, \tag{83}$$

$$K_{35}(x,\ y,\ t) = \frac{1}{2}\ln(\Delta) B, \tag{84}$$

$$W_{35}(x,\ y,\ t) = \frac{1}{2}\left(\frac{1}{2}\ln(\Delta) B\right)^2. \tag{85}$$

Group 11 When $B = \rho,\ C = \pi\rho,\ A = 0$, the following solitary wave solutions are found for (1) since (10) and (12) are used:

$$K_{36}(x,\ y,\ t) = \frac{1}{2}\ln(\Delta)\rho \tag{86}$$

$$W_{36}(x,\ y,\ t) = \frac{1}{2}\left(\frac{1}{2}\ln(\Delta)\rho\right)^2. \tag{87}$$

The following sets of soliton solutions for align (1) are obtained by taking scenario number two into consideration:
Group 12 When $\Psi < 0,\ C \neq 0$, the following solitary wave solutions are derived for (1) by using (10) and (12):

$$K_{37}(x,\ y,\ t) = \ln(\Delta) C\left(-\frac{1}{2}\frac{B}{C} + \frac{1}{2}\frac{\sqrt{-\Psi}\tan\left(\frac{1}{2}\sqrt{-\Psi}\mu\right)}{C}\right) + \frac{1}{2}\ln(\Delta) B, \tag{88}$$

$$W_{37}(x,\ y,\ t) = \frac{1}{2}\left(\ln(\Delta) C\left(-\frac{1}{2}\frac{B}{C} + \frac{1}{2}\frac{\sqrt{-\Psi}\tan\left(\frac{1}{2}\sqrt{-\Psi}\mu\right)}{C}\right) + \frac{1}{2}\ln(\Delta) B\right)^2, \tag{89}$$

$$K_{38}(x,\ y,\ t)=\ln(\Delta)C\left(-\frac{1}{2}\frac{B}{C}-\frac{1}{2}\frac{\sqrt{-\Psi}\cot\left(\frac{1}{2}\sqrt{-\Psi}\mu\right)}{C}\right)+\frac{1}{2}\ln(\Delta)B, \tag{90}$$

$$W_{38}(x,\ y,\ t)=\frac{1}{2}\left(\ln(\Delta)C\left(-\frac{1}{2}\frac{B}{C}-\frac{1}{2}\frac{\sqrt{-\Psi}\cot\left(\frac{1}{2}\sqrt{-\Psi}\mu\right)}{C}\right)+\frac{1}{2}\ln(\Delta)B\right)^2, \tag{91}$$

$$K_{39}(x,\ y,\ t)=\ln(\Delta)C\left(-\frac{1}{2}\frac{B}{C}+\frac{1}{2}\frac{\sqrt{-\Psi}\left(\tan\left(\sqrt{-\Psi}\mu\right)\pm\left(\sec\left(\sqrt{-\Psi}\mu\right)\right)\right)}{C}\right)+\frac{1}{2}\ln(\Delta)B, \tag{92}$$

$$W_{39}(x,\ y,\ t)=\frac{1}{2}\left(\ln(\Delta)C\left(-\frac{1}{2}\frac{B}{C}+\frac{1}{2}\frac{\sqrt{-\Psi}\left(\tan\left(\sqrt{-\Psi}\mu\right)\pm\left(\sec\left(\sqrt{-\Psi}\mu\right)\right)\right)}{C}\right)+\frac{1}{2}\ln(\Delta)B\right)^2, \tag{93}$$

$$K_{40}(x,\ y,\ t)=\ln(\Delta)C\left(-\frac{1}{2}\frac{B}{C}+\frac{1}{2}\frac{\sqrt{-\Psi}\left(\cot\left(\sqrt{-\Psi}\mu\right)\pm\left(\csc\left(\sqrt{-\Psi}\mu\right)\right)\right)}{C}\right)+\frac{1}{2}\ln(\Delta)B, \tag{94}$$

$$W_{40}(x,\ y,\ t)=\frac{1}{2}\left(\ln(\Delta)C\left(-\frac{1}{2}\frac{B}{C}+\frac{1}{2}\frac{\sqrt{-\Psi}\left(\cot\left(\sqrt{-\Psi}\mu\right)\pm\left(\csc\left(\sqrt{-\Psi}\mu\right)\right)\right)}{C}\right)+\frac{1}{2}\ln(\Delta)B\right)^2, \tag{95}$$

and

$$K_{41}(x,\ y,\ t)=\ln(\Delta)C\left(-\frac{1}{2}\frac{B}{C}+\frac{1}{4}\frac{\sqrt{-\Psi}\left(\tan\left(\frac{1}{4}\sqrt{-\Psi}\mu\right)-\cot\left(\frac{1}{4}\sqrt{-\Psi}\mu\right)\right)}{C}\right)+\frac{1}{2}\ln(\Delta)B. \tag{96}$$

$$W_{41}(x,\ y,\ t)=\frac{1}{2}\left(\ln(\Delta)C\left(-\frac{1}{2}\frac{B}{C}+\frac{1}{4}\frac{\sqrt{-\Psi}\left(\tan\left(\frac{1}{4}\sqrt{-\Psi}\mu\right)-\cot\left(\frac{1}{4}\sqrt{-\Psi}\mu\right)\right)}{C}\right)+\frac{1}{2}\ln(\Delta)B\right)^2. \tag{97}$$

Group 13 When $\Psi>0,\ C\neq 0$, the following solitary wave solutions are derived for (1) by using (10) and (12):

$$K_{42}(x,\ y,\ t)=\ln(\Delta)C\left(-\frac{1}{2}\frac{B}{C}-\frac{1}{2}\frac{\sqrt{\Psi}\tanh\left(\frac{1}{2}\sqrt{-\Psi}\mu\right)}{C}\right)+\frac{1}{2}\ln(\Delta)B, \tag{98}$$

$$W_{42}(x,\ y,\ t)=\frac{1}{2}\left(\ln(\Delta)C\left(-\frac{1}{2}\frac{B}{C}-\frac{1}{2}\frac{\sqrt{\Psi}\tanh\left(\frac{1}{2}\sqrt{-\Psi}\mu\right)}{C}\right)+\frac{1}{2}\ln(\Delta)B\right)^{2}, \tag{99}$$

$$K_{43}(x,\ y,\ t)=\ln(\Delta)C\left(-\frac{1}{2}\frac{B}{C}-\frac{1}{2}\frac{\sqrt{\Psi}\coth\left(\frac{1}{2}\sqrt{-\Psi}\mu\right)}{C}\right)+\frac{1}{2}\ln(\Delta)B, \tag{100}$$

$$W_{43}(x,\ y,\ t)=\frac{1}{2}\left(\ln(\Delta)C\left(-\frac{1}{2}\frac{B}{C}-\frac{1}{2}\frac{\sqrt{\Psi}\coth\left(\frac{1}{2}\sqrt{-\Psi}\mu\right)}{C}\right)+\frac{1}{2}\ln(\Delta)B\right)^{2}, \tag{101}$$

$$K_{44}(x,\ y,\ t)=\ln(\Delta)C\left(-\frac{1}{2}\frac{B}{C}-\frac{1}{2}\frac{\sqrt{\Psi}\left(\tanh\left(\sqrt{-\Psi}\mu\right)\pm\left(sech\left(\sqrt{-\Psi}\mu\right)\right)\right)}{C}\right)+\frac{1}{2}\ln(\Delta)B, \tag{102}$$

$$W_{44}(x,\ y,\ t)=\frac{1}{2}\left(\ln(\Delta)C\left(-\frac{1}{2}\frac{B}{C}-\frac{1}{2}\frac{\sqrt{\Psi}\left(\tanh\left(\sqrt{-\Psi}\mu\right)\pm\left(sech\left(\sqrt{-\Psi}\mu\right)\right)\right)}{C}\right)+\frac{1}{2}\ln(\Delta)B\right)^{2}, \tag{103}$$

$$K_{45}(x,\ y,\ t)=\ln(\Delta)C\left(-\frac{1}{2}\frac{B}{C}-\frac{1}{2}\frac{\sqrt{\Psi}\left(\coth\left(\sqrt{-\Psi}\mu\right)\pm\left(sech\left(\sqrt{-\Psi}\mu\right)\right)\right)}{C}\right)+\frac{1}{2}\ln(\Delta)B \tag{104}$$

$$W_{45}(x,\ y,\ t)=\frac{a_4s_1}{a_3f_1}\left(\ln(\Delta)C\left(-\frac{1}{2}\frac{B}{C}-\frac{1}{2}\frac{\sqrt{\Psi}\left(\coth\left(\sqrt{-\Psi}\mu\right)\pm\left(sech\left(\sqrt{-\Psi}\mu\right)\right)\right)}{C}\right)+\frac{1}{2}\ln(\Delta)B\right)^{2} \tag{105}$$

and

$$K_{46}(x,\ y,\ t)=\ln(\Delta)C\left(-\frac{1}{2}\frac{B}{C}-\frac{1}{4}\frac{\sqrt{\Psi}\left(\tanh\left(\frac{1}{4}\sqrt{\Psi}\mu\right)-\coth\left(\frac{1}{4}\sqrt{\Psi}\mu\right)\right)}{C}\right)+\frac{1}{2}\ln(\Delta)B \tag{106}$$

$$W_{46}(x,\ y,\ t)=\frac{1}{2}\left(\ln(\Delta)C\left(-\frac{1}{2}\frac{B}{C}-\frac{1}{2}\frac{\sqrt{\Psi}\left(\tanh\left(\frac{1}{4}\sqrt{\Psi}\mu\right)-\coth\left(\frac{1}{4}\sqrt{\Psi}\mu\right)\right)}{C}\right)+\frac{1}{2}\ln(\Delta)B\right)^{2}. \tag{107}$$

Household 14 For (1), the following solitary wave solutions are found for $CA>0$ and $B=0$ due to the use of (10) and (12):

$$K_{47}(x,\ y,\ t)=\ln(\Delta)C\sqrt{\frac{A}{C}}\tan\left(\sqrt{CA}\mu\right), \tag{108}$$

$$W_{47}(x,\ y,\ t)=\frac{1}{2}\left(\ln(\Delta)C\sqrt{\frac{A}{C}}\tan\left(\sqrt{CA}\mu\right)\right)^2, \tag{109}$$

$$K_{48}(x,\ y,\ t)=-\ln(\Delta)C\sqrt{\frac{A}{C}}\cot\left(\sqrt{CA}\mu\right), \tag{110}$$

$$W_{48}(x,\ y,\ t)=\frac{1}{2}\left(-\ln(\Delta)C\sqrt{\frac{A}{C}}\cot\left(\sqrt{CA}\mu\right)\right)^2, \tag{111}$$

$$K_{49}(x,\ y,\ t)=\ln(\Delta)C\sqrt{\frac{A}{C}}\left(\tan\left(2\sqrt{CA}\mu\right)\pm\left(\sec\left(2\sqrt{CA}\mu\right)\right)\right), \tag{112}$$

$$W_{49}(x,\ y,\ t)=\frac{1}{2}\left(\ln(\Delta)C\sqrt{\frac{A}{C}}\left(\tan\left(2\sqrt{CA}\mu\right)\pm\left(\sec\left(2\sqrt{CA}\mu\right)\right)\right)\right)^2, \tag{113}$$

$$K_{50}(x,\ y,\ t)=-\ln(\Delta)C\sqrt{\frac{A}{C}}\left(\cot\left(2\sqrt{CA}\mu\right)\pm\left(\csc\left(2\sqrt{CA}\mu\right)\right)\right), \tag{114}$$

$$W_{50}(x,\ y,\ t)=\frac{1}{2}\left(-\ln(\Delta)C\sqrt{\frac{A}{C}}\left(\cot\left(2\sqrt{CA}\mu\right)\pm\left(\csc\left(2\sqrt{CA}\mu\right)\right)\right)\right)^2, \tag{115}$$

and

$$K_{51}(x,\ y,\ t)=\frac{1}{2}\ln(\Delta)C\sqrt{\frac{A}{C}}\left(\tan\left(\frac{1}{2}\sqrt{CA}\mu\right)-\cot\left(\frac{1}{2}\sqrt{CA}\mu\right)\right), \tag{116}$$

$$W_{51}(x,\ y,\ t)=\frac{1}{2}\left(\frac{1}{2}\ln(\Delta)C\sqrt{\frac{A}{C}}\left(\tan\left(\frac{1}{2}\sqrt{CA}\mu\right)-\cot\left(\frac{1}{2}\sqrt{CA}\mu\right)\right)\right)^2. \tag{117}$$

Household 15 Using (10) and (12), the following solitary wave solutions are derived for (1) where $B=0,\ AC<0$:

$$K_{52}(x,\ y,\ t)=-\ln(\Delta)C\sqrt{-\frac{A}{C}}\tanh\left(\sqrt{-CA}\mu\right), \tag{118}$$

$$W_{52}(x,\ y,\ t)=\frac{1}{2}\left(-\ln(\Delta)C\sqrt{-\frac{A}{C}}\tanh\left(\sqrt{-CA}\mu\right)\right)^2, \tag{119}$$

$$K_{53}(x,\ y,\ t) = -\ln(\Delta)C\sqrt{-\frac{A}{C}}\coth\left(\sqrt{-CA}\mu\right), \tag{120}$$

$$W_{53}(x,\ y,\ t) = \frac{1}{2}\left(-\ln(\Delta)C\sqrt{-\frac{A}{C}}\coth\left(\sqrt{-CA}\mu\right)\right)^2, \tag{121}$$

$$K_{54}(x,\ y,\ t) = -\ln(\Delta)C\sqrt{-\frac{A}{C}}\left(\tanh\left(2\sqrt{-CA}\mu\right) \pm \left(isech\left(2\sqrt{-CA}\mu\right)\right)\right), \tag{122}$$

$$W_{54}(x,\ y,\ t) = \frac{1}{2}\left(-\ln(\Delta)C\sqrt{-\frac{A}{C}}\left(\tanh\left(2\sqrt{-CA}\mu\right) \pm \left(isech\left(2\sqrt{-CA}\mu\right)\right)\right)\right)^2, \tag{123}$$

$$K_{55}(x,\ y,\ t) = -\ln(\Delta)C\sqrt{-\frac{A}{C}}\left(\coth\left(2\sqrt{-CA}\mu\right) \pm \left(csch\left(2\sqrt{-CA}\mu\right)\right)\right), \tag{124}$$

$$W_{55}(x,\ y,\ t) = \frac{1}{2}\left(-\ln(\Delta)C\sqrt{-\frac{A}{C}}\left(\coth\left(2\sqrt{-CA}\mu\right) \pm \left(csch\left(2\sqrt{-CA}\mu\right)\right)\right)\right)^2, \tag{125}$$

and

$$K_{56}(x,\ y,\ t) = -\frac{1}{2}\ln(\Delta)C\sqrt{-\frac{A}{C}}\left(\tanh\left(\frac{1}{2}\sqrt{-CA}\mu\right) + \coth\left(\frac{1}{2}\sqrt{-CA}\mu\right)\right) \tag{126}$$

$$W_{56}(x,\ y,\ t) = \frac{1}{2}\left(-\frac{1}{2}\ln(\Delta)C\sqrt{-\frac{A}{C}}\left(\tanh\left(\frac{1}{2}\sqrt{-CA}\mu\right) + \coth\left(\frac{1}{2}\sqrt{-CA}\mu\right)\right)\right)^2. \tag{127}$$

Household 16 Employing (10) and (12) results in the following solitary wave solutions for (1) for $VC = A,\ B = 0$:

$$K_{57}(x,\ y,\ t) = \ln(\Delta)C\tan(A\mu), \tag{128}$$

$$W_{57}(x,\ y,\ t) = \frac{1}{2}\left(\ln(\Delta)C\tan(A\mu)\right)^2, \tag{129}$$

$$K_{58}(x,\ y,\ t) = -\ln(\Delta)C\cot(A\mu), \tag{130}$$

$$W_{58}(x,\ y,\ t) = \frac{1}{2}\left(-\ln(\Delta)C\cot(A\mu)\right)^2, \tag{131}$$

$$K_{59} = \ln(\Delta)C\left(tan(2A\mu) \pm \left(sec(2A\mu)\right)\right), \tag{132}$$

$$W_{59} = \frac{1}{2}\left(\ln(\Delta) C (tan(2A\mu) \pm (sec(2A\mu)))\right)^2, \tag{133}$$

$$K_{60} = \ln(\Delta) C (-cot(2A\mu) \pm (csc(2A\mu))) \tag{134}$$

$$W_{60}(x,\ y,\ t) = \frac{1}{2}\left(\ln(\Delta) C (-\cot(2A\mu) \pm (\csc(2A\mu)))\right)^2 \tag{135}$$

and

$$K_{61}(x,\ y,\ t) = \ln(\Delta) C \left(\frac{1}{2}\tan\left(\frac{1}{2}A\mu\right) - \frac{1}{2}\cot\left(\frac{1}{2}A\mu\right)\right), \tag{136}$$

$$W_{61}(x,\ y,\ t) = \frac{1}{2}\left(\ln(\Delta) C \left(\frac{1}{2}\tan\left(\frac{1}{2}A\mu\right) - \frac{1}{2}\cot\left(\frac{1}{2}A\mu\right)\right)\right)^2. \tag{137}$$

Household 17 Using (10) and (12), the following solitary wave solutions are obtained for (1) for $C = -A,\ B = 0$:

$$K_{62}(x,\ y,\ t) = -\ln(\Delta) A \tanh\left(A^2\right), \tag{138}$$

$$W_{62}(x,\ y,\ t) = \frac{1}{2}\left(-\ln(\Delta) A \tanh\left(A^2\right)\right)^2, \tag{139}$$

$$K_{63}(x,\ y,\ t) = \ln(\Delta) A \coth(A\mu), \tag{140}$$

$$W_{63}(x,\ y,\ t) = \frac{1}{2}\left(\ln(\Delta) A \coth(A\mu)\right)^2, \tag{141}$$

$$K_{64}(x,\ y,\ t) = -\ln(\Delta) A (-\tanh(2A\mu) \pm (isech(2A\mu))), \tag{142}$$

$$W_{64}(x,\ y,\ t) = \frac{1}{2}\left(-\ln(\Delta) A (-\tanh(2A\mu) \pm (isech(2A\mu)))\right)^2, \tag{143}$$

$$K_{65}(x,\ y,\ t) = -\ln(\Delta) A (-\coth(2A\mu) \pm (cech(2A\mu))), \tag{144}$$

and

$$W_{65}(x,\ y,\ t) = \frac{1}{2}\left(-\ln(\Delta) A (-\coth(2A\mu) \pm (cech(2A\mu)))\right)^2. \tag{145}$$

$$K_{66}(x,\ y,\ t) = \ln(\Delta)\, C\left(-\frac{1}{2}\tanh\left(\frac{1}{2}A\mu\right) - \frac{1}{2}\coth\left(\frac{1}{2}A\mu\right)\right) + \frac{1}{2}\ln(\Delta)\, B, \tag{146}$$

and

$$W_{66}(x,\ y,\ t) = \ln(\Delta)\, C\left(-\frac{1}{2}\tanh\left(\frac{1}{2}A\mu\right) - \frac{1}{2}\coth\left(\frac{1}{2}A\mu\right)\right) + \frac{1}{2}\ln(\Delta)\, B\Bigg)^2. \tag{147}$$

Household 18 When $\Psi = 0$,

$$K_{67}(x,\ y,\ t) = -2\,\frac{CA\,(B\mu\,\ln(\Delta)+2)}{B^2\mu} + \frac{1}{2}\ln(\Delta)\, B, \tag{148}$$

$$W_{67}(x,\ y,\ t) = \frac{1}{2}\left(-2\,\frac{CA\,(B\mu\,\ln(\Delta)+2)}{B^2\mu} + \frac{1}{2}\ln(\Delta)\, B\right)^2. \tag{149}$$

Household 19 Assuming that $B = \rho,\ A = \pi\rho(\pi \neq 0),\ C = 0$,

$$K_{68}(x,\ y,\ t) = \frac{1}{2}\ln(\Delta)\,\rho, \tag{150}$$

$$W_{68}(x,\ y,\ t) = \frac{1}{2}\left(\frac{1}{2}\ln(\Delta)\,\rho\right)^2. \tag{151}$$

Household 20 The following solitary wave solutions for (1) for $B = 0,\ A = 0$ are derived using (10) and (12):

$$K_{69}(x,\ y,\ t) = -\mu^{-1} \tag{152}$$

$$W_{69}(x,\ y,\ t) = \frac{1}{2}\left(-\mu^{-1}\right)^2, \tag{153}$$

Household 21 The following solitary wave solutions are obtained for (1) for $A := 0;\ B \neq 0;\ C \neq 0$ due to the use of (10) and (12):

$$K_{70}(x,\ y,\ t) = -\frac{\ln(\Delta)\, B}{\cosh(B\mu) - \sinh(B\mu) + 1} + \frac{1}{2}\ln(\Delta)\, B, \tag{154}$$

$$W_{70}(x,\ y,\ t) = \frac{1}{2}\left(-\frac{\ln(\Delta)\, B}{\cosh(B\mu) - \sinh(B\mu) + 1} + \frac{1}{2}\ln(\Delta)\, B\right)^2. \tag{155}$$

$$K_{71}(x,\ y,\ t) = -\frac{1}{2}\ln(\Delta)\, B, \tag{156}$$

$$W_{71}(x,\ y,\ t) = \frac{1}{2}\left(-\frac{1}{2}\ln(\Delta)\,B\right)^2 \tag{157}$$

Household 22 As a consequence of using (10) and (12), the following solitary wave solutions are obtained for (1) for $B := \rho$; $C := \pi\rho$; $A := 0$.

$$K_{72}(x,\ y,\ t) = \frac{\ln(\Delta)\,\pi\,\rho\,\Delta^{\rho\,\mu}}{1-\pi\,\Delta^{\rho\,\mu}} + \frac{1}{2}\ln(\Delta)\,\rho, \tag{158}$$

$$W_{72}(x,\ y,\ t) = \frac{1}{2}\left(\frac{\ln(\Delta)\,\pi\,\rho\,\Delta^{\rho\,\mu}}{1-\pi\,\Delta^{\rho\,\mu}} + \frac{1}{2}\ln(\Delta)\,\rho\right)^2 \tag{159}$$

4. Results and discussion

The behavior of fractional nonlinear waves is examined in this novel work by applying the Modified Extended Direct Algebraic Method (MEDAM) to the time-fractional Bogoyavlenskii align. We discovered that MEDAM is highly reliable and effective in capturing accurate solutions that differentiate among many forms of nonlinear wave patterns, such as periodic waves, solitary waves, and rational solutions. Numerous physical scenarios, including fluid dynamics, plasma physics, and optics, could benefit from the use of these travelling wave solutions. The results demonstrate the effectiveness of the Modified Extended Direct Algebraic Method in solving nonlinear fractional differential equations. In particular, this approach has prven to be a useful tool for addressing difficult mathematical physics issues when used to fractional calculus. Fluid dynamics, plasma physics, nonlinear optics, and signal processing the Modified Extended Direct Algebraic Method is a very useful technique for studying the time-fractional Bogoyavlenskii equation and related nonlinear fractional models because it can accurately capture complex nonlinear dynamics. Finding these solutions is essential to comprehending how physical processes behave in systems governed by time-fractional Bogoyavlenskii equations, such as:

Figure 1a 3D surface plot showing wave-like behavior with sharp peaks and valleys. 1b Contour plot illustrating the wave's energy distribution. 1c Line plot showing the wave's intensity along the x-axis.

Figure 2a 3D plot displaying the wave's amplitude over x and y space. 2b Contour plot showing lines of equal wave strength. 2c Line plot showing the wave's strength along the x-axis.

Figure 3a 3D surface plot showing oscillating waves in both x and y directions. 3b Contour plot illustrating the wave's energy concentration. 3c Line plot showing the wave's intensity along the x-axis.

Figure 4a 3D surface plot showing two intersecting wave components. 4b Contour plot illustrating the wave's energy distribution. 4c Line plot showing the wave's intensity along the x-axis.

Figure 5 The figure shows a nonlinear wave system with localized energy regions, sharp intensity changes, and soliton-like behavior.

Figure 6 The figure illustrates a nonlinear wave system with localized energy structures, sharp intensity variations, and high-energy regions, characteristic of solitons or shock waves.

Figure 7 The figure shows a nonlinear wave system with localized energy and sudden changes, indicating soliton-like behavior.

Figure 8 The figure represents a nonlinear wave system with localized high-amplitude regions, sharp intensity spikes, and discontinuous intensity changes, supporting nonlinear phenomena like solitons or shock waves.

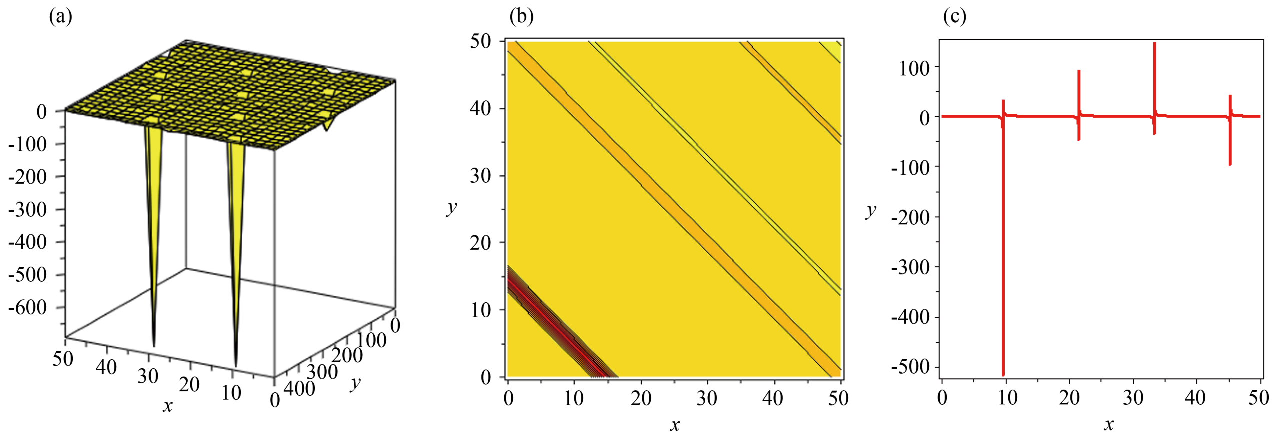

Figure 1. The 2D, three-dimensional and contour soliton solution soltion stated in (16) are plotted for $B = 0.4$, $C = 0.11$, $A = 1$, $\Delta = e$, $r = 1$, $t = 1$, $\beta = 0.3$

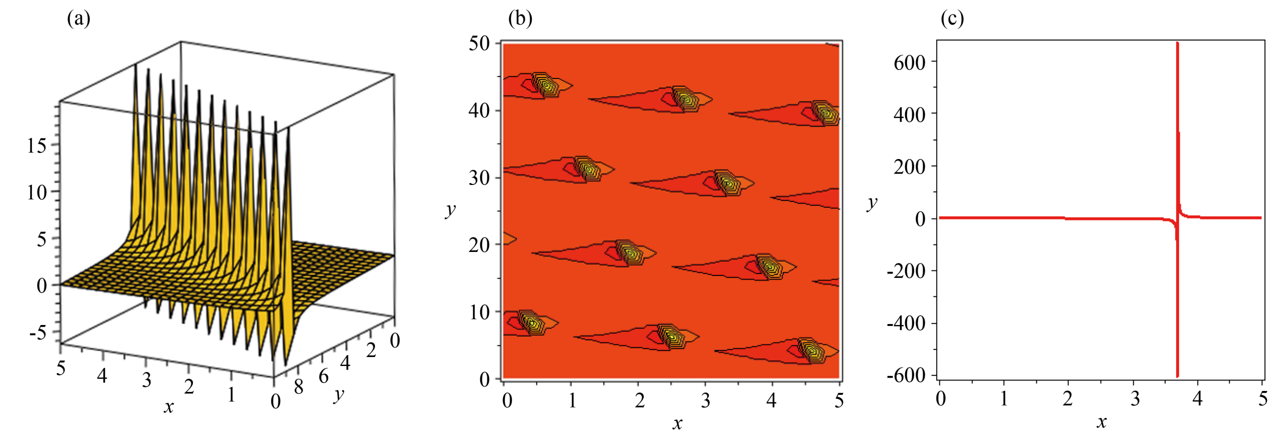

Figure 2. The 2D, three-dimensional and contour soliton solution stated in (18) are plotted for $B = 0.4$, $C = 0.11$, $A = 1$; $\Delta = e$, $r = 0.1$, $t = 1$, $\beta = 0.3$

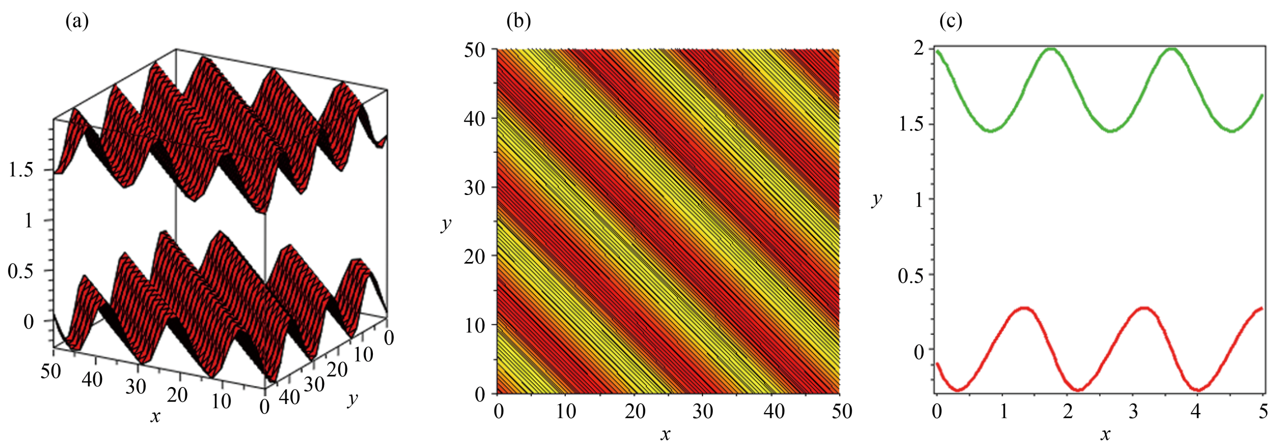

Figure 3. The 2D, three-dimensional and contour soliton solution stated in (26) are graphed for $B = 4$, $C = 11$, $A = 0.1$, $\Delta = e$, $r = 1$, $t = 1$, $\beta = 0.31$

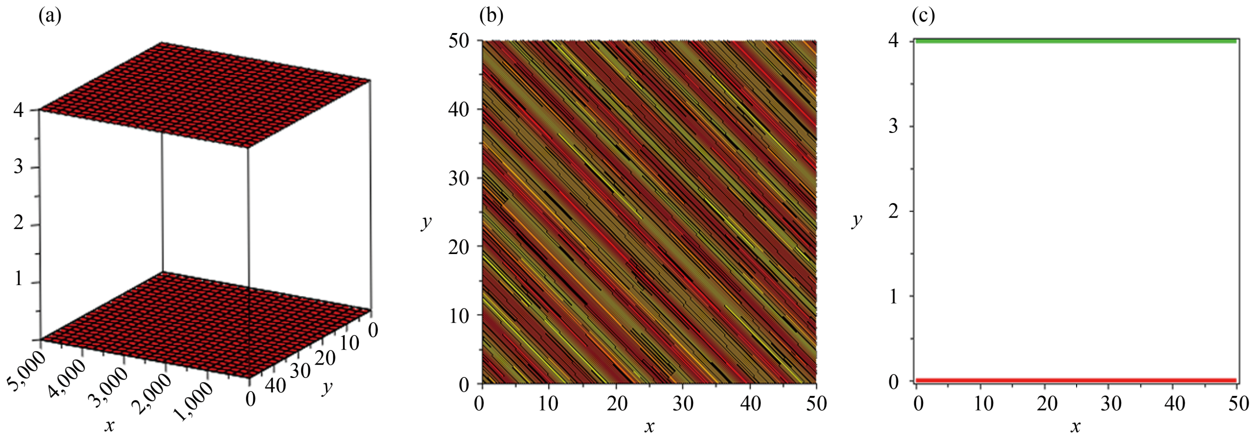

Figure 4. The 2D, three-dimenional and contur soliton stated in (28) are graphed for $B = 8,\ C = 0.1,\ A = 0.1,\ \Delta = e,\ r = 1,\ t = 0.1,\ \beta = 1$

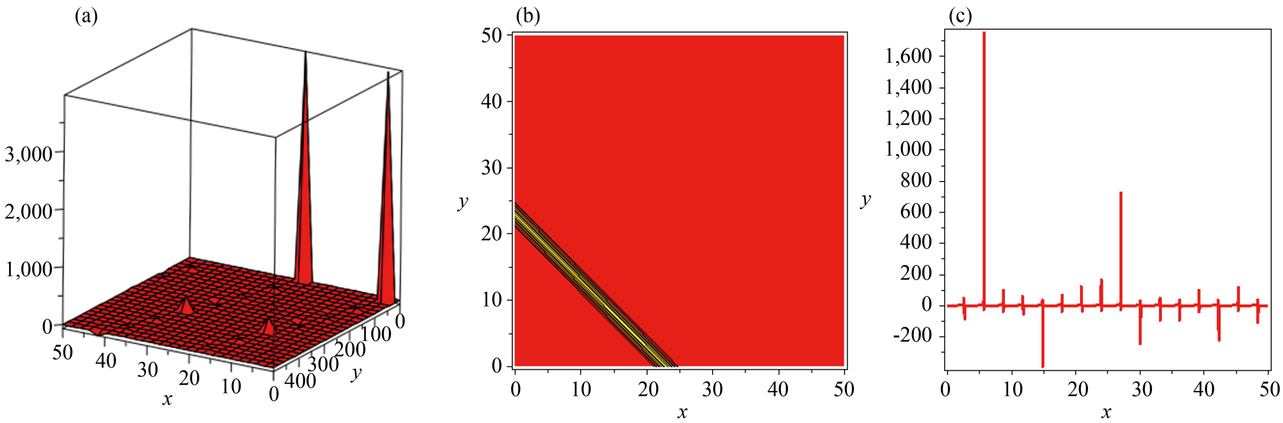

Figure 5. The 2D, three-dimensional and contour soliton solution stated in (88) are graphed for $B = 0.4,\ C = 1.1,\ A = 1,\ \Delta = e,\ r = 0.1,\ t = 1,\ \beta = 0.3$

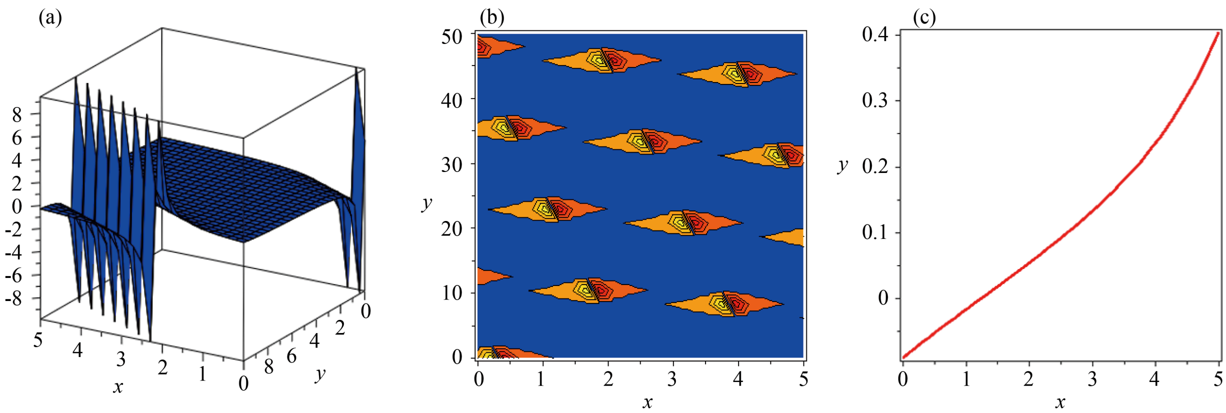

Figure 6. The 2D, three-dimensional and contour soliton stated in (90) are graphed for $B = 0.4,\ C = 0.11,\ A =,\ 1\Delta = e,\ r = 0.1,\ t = 1,\ \beta = 0.99$

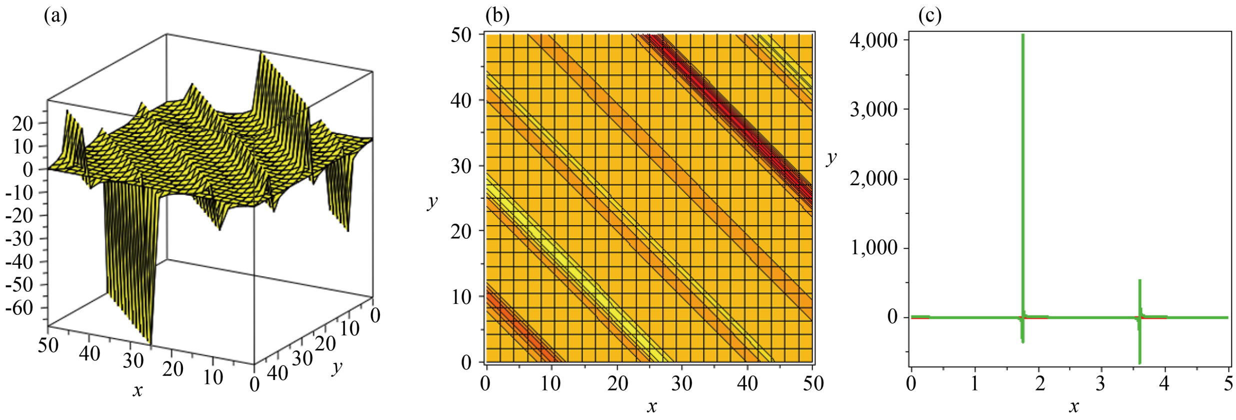

Figure 7. The 2D, three-dimensional and contour soliton stated in (98) are graphed for $B=4$, $C=11$, $A=0.1$, $\Delta=e$, $r=1$, $t=1$, $\beta=0.3$

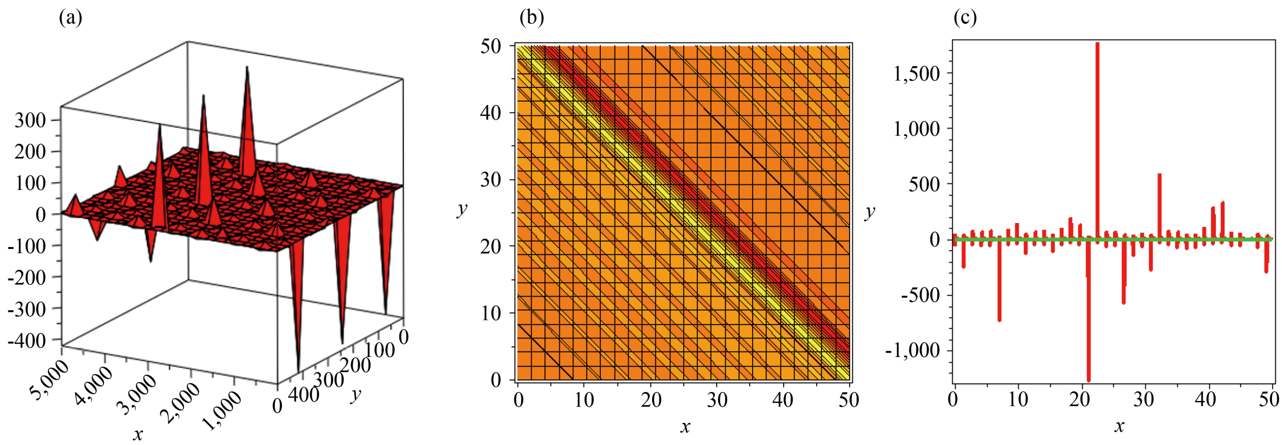

Figure 8. The 2D, three-dimensional and contour soliton solution stated in (100) are graphed for $B=8$, $C=1$, $A=11$, $\Delta=(e)^{(1)}$, $r=1$, $t=1$, $\beta=0.25$

5. Concluding remark

This study employed the Extended Direct Algebraic Method to obtain precise solutions of traveling waves for the time-fractional Bogoyavlenskii problem. The traveling wave solutions obtained in this study demonstrated great promise for use in fluid dynamics, plasma physics, optics, and other physical contexts. Solving these problems in the form of solitary waves was essential to understanding the behavior of complex physical systems. The results showed that the Modified Extended Direct Algebraic Method was effective in solving nonlinear fractional differential equations. This method, particularly when applied to fractional calculus, proved successful in tackling difficult mathematical physics issues. The exact solutions obtained in this work were used to validate numerical solutions and provided information about the physical behavior of the systems described by the time-fractional Bogoyavlenskii equation. The Modified Extended Direct Algebraic Method was shown to be a valuable addition to mathematical physics, and it can be applied to solve more nonlinear fractional differential equations.

Author contributions

Conceptualization, K.K.A., J.I., M.A.Y. and M.B.; Funding acquisition, D.B.; Investigation, J.I., M.B. and P.O.M.; Methodology, K.K.A., M.A.Y. and M.B.; Project administration, D.B. and K.K.A.; Software, M.A.Y.; Supervision, D.B. and P.O.M.; Writing-original draft, K.K.A., M.B. and J.I.; Writing-review & editing, M.A.Y. and P.O.M. All authors have read and agreed to the published version of the manuscript.

Data availability statement

Data is contained within the article or supplementary material.

Conflict of interest

The authors declare no conflict of interest.

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