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Research Article

Existence of Scalar Minimizers for 0-Nonconvex Autonomous Single Integrals with Relaxed Lagrangian at Zero Velocity Having No Isolated Local Minimum Points

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Abstract: We study the nonconvex integral $\int_a^b L(x(t),\ x'(t))\,dt$, defined in the class of the absolutely continuous functions $x:[a,\ b]\to\mathbb{R}$ having $x(a)=A$ & $x(b)=B$, using a superlinear $\mathscr{L}\otimes\mathscr{B}$-measurable nonconvex lagrangian $L:\mathbb{R}\times\mathbb{R}\to[0,\ \infty]$ freely taking ∞ values and having $L(s,\ \cdot)$ lower semicontinuous. Our aim is to look for weak hypotheses under which true minimizers still exist. In previous papers we have shown that 0-convexity $L^{**}(\cdot,\ 0)=L(\cdot,\ 0)$ suffices provided $L^{**}(\cdot,\cdot)$ is lower semicontinuous at velocity zero, namely lsc at $(s,\ 0)\ \forall s$. In this paper we present sufficient conditions for existence of true minimizers in the 0-nonconvex case instead, i.e. $L^{**}(\cdot,\ 0)<L(\cdot,\ 0)$. This is important because when a relaxed minimizer is not a true minimizer then there exists another relaxed minimizer $y(\cdot)$ which has a non-singleton constancy interval where $y(\cdot)\equiv s'$ with $L^{**}(s',\ 0)<L(s',\ 0)$. Our simplest hypothesis to avoid this is that sublevel sets of $L^{**}(\cdot,\ 0)$ contain no singletons, provided $L^{**}(\cdot,\cdot)$ and $(L-L^{**})(\cdot,\cdot)$ are both lsc at velocity zero. We also prove new necessary conditions.

Keywords: calculus of variations, optimal control, pointwise state and velocity constraints, general nonconvex lagrangians, Lipschitz regularity, DuBois-Reymond necessary condition

MSC: 49J05, 49J30, 49K05, 49K30

1. Introduction

We wish to find weak hypotheses under which the following single nonconvex integral, defined for scalar Absolutely Continuous (AC) functions $x(\cdot)$, still has true minimizers:

$$J(x(\cdot)):=\int_a^b L\left(x(t),x'(t)\right)\,dt \quad \text{on} \quad \mathscr{X}:=\left\{x(\cdot)\in W^{1,\,1}([a,\ b]):x(a)=A\ \ \&\ \ x(b)=B\right\}. \tag{1}$$

We allow $L(\cdot,\cdot)$ to take ∞ values freely and assume it to be a Basic Hypotheses function (BH-function), i.e. to satisfy our

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$$\text{Basic Hypotheses}: \; L : \mathbb{R} \times \mathbb{R} \to [0, \infty] \;\; is \;\; \mathscr{L} \otimes \mathscr{B} - \text{measurable}$$

$$\text{with} \;\; L(\cdot, 0) \;\; \text{Borel and} \;\; L(s, \cdot) \;\; \text{lsc} \;\; \text{(lower semicontinuous)} \; \forall s \tag{2}$$

$$\text{superlinear, i.e.} \;\; \frac{\inf L(\mathbb{R}, \xi)}{|\xi|} \to \infty \;\; \text{as} \;\; |\xi| \to \infty.$$

As usual, we define its bipolar $L^{**}(\cdot, \cdot)$ by $epi\, L^{**}(s, \cdot) := \overline{co}\; epi\, L(s, \cdot)\ \forall s$, where $\overline{co}\; epi$ denotes the closed-convex hull of the epigraph; and say that a function $y_c(\cdot)$ is a *relaxed minimizer* of the integral in (1) if it minimizes the *convexified integral*

$$\int_a^b L^{**}\left(x(t), x'(t)\right) dt \quad \text{on} \quad \mathscr{X}. \tag{3}$$

Our basic hypothesis (2) ensures Lebesgue-measurability of the integrands $L(x(\cdot), x'(\cdot))$ and $L^{**}(x(\cdot), x'(\cdot))$, so that the Lebesgue integrals in (1) and (3) always exist, with values in $[0, \infty]$. On the other hand, the growth condition in (2) allows one to apply Tonelli's direct method to prove existence of a relaxed minimizer $y_c(\cdot)$, using the results of [1, 2], which weakened Tonelli's lsc hypothesis on $L^{**}(\cdot, \cdot)$ in the scalar case. In [3] we further generalized these results, proving existence of relaxed minimizers whenever $L^{**}(\cdot, \cdot)$ is 0-lsc-convex, whose definition appears below, in (9). Three interesting special cases of this definition are the following, it suffices to have: either $L(\cdot, \cdot)$ lsc; or

$$L^{**}(\cdot, \cdot) \;\; \text{lsc at} \;\; \xi = 0 \quad \text{(or, more precisely, at} \;\; (s, 0) \;\; \forall s); \tag{4}$$

or else integrability in s of the slopes of $L^{**}(s, \cdot)$ near velocity zero, in the sense that

$$L^{**}(\cdot, 0) \;\text{is lsc and}\; \exists l, M : \mathbb{R} \to (0, \infty) \;\; \text{with} \;\; (l \times M)(\cdot) \in L^1_{loc}(\mathbb{R})$$

$$\text{such that}\; L^{**}(s, \xi) \le l(s) \;\; \forall |\xi| \le \frac{1}{M(s)} \;\; \forall s. \tag{5}$$

Once we ensure existence of a relaxed minimizer $y_c(\cdot)$, our aim is to smartly change $y_c(\cdot)$ in order to obtain a new relaxed minimizer $y(\cdot)$ which moreover truly minimizes the nonconvex integral in (1). A crucial hypothesis for this is

$$\text{0-convexity:} \quad L^{**}(\cdot, 0) = L(\cdot, 0). \tag{6}$$

Indeed, several papers (see the paragraph starting after (8)) have weakened the hypotheses on $L(\cdot, \cdot)$ through assuming (6).

However, desiring here to avoid imposing (6), we introduce the concept of 0-*relaxed minimizer* $y_0(\cdot)$ for the integral in (1). Its precise definition appears below (in (13), using (11), (12), (14)) but it means a minimizer of the 0-*convexified integral*, whose lagrangian $L^0(\cdot, \cdot)$ is $L(\cdot, \cdot)$ changed only at velocity zero, namely at the points $(s, 0)$, to equal $L^{**}(\cdot, \cdot)$ there. Clearly this is a true minimizer whenever (6) holds true. Moreover, by [3, theorem 1], such $y_0(\cdot)$ can always be taken *bimonotone*, which means, at least, that apart from an interval $(a',\ b') \subset [a,\ b]$ where $y_0(\cdot)$ has a constant value s', along the remaining subintervals $(a,\ a')$ and $(b',\ b)$ it is strictly monotone with derivative $y_0'(t) \neq 0$ a.e.. Both extreme cases $a' = b'$ and $(a',\ b') = (a,\ b)$ are possible.

So, why do we wish to avoid here the hypothesis (6)? Because while existence of a 0-relaxed minimizer $y_0(\cdot)$ is known under quite general hypotheses, such $y_0(\cdot)$ frequently is not a true minimizer because it stops at a point s' where (6) fails, namely $L^{**}(s',\ 0) < L(s',\ 0)$. Here is a simple intuitive example of this situation: $L(s,\ \xi) := s^2 + \left(\xi^2 - 1\right)^2$, in which, obviously, $L^{**}(s,\ \xi)$ turns out to equal $L(s,\ \xi)$ for $|\xi| \geq 1$ and be s^2 for $|\xi| \leq 1$ (why?), while $L^0(s,\ \xi)$ equals $L(s,\ \xi)$ for $\xi \neq 0$ (why?) and is s^2 at $\xi = 0$; then, with $A = 0 = B$, obviously $y_0(\cdot) \equiv 0$ is a relaxed minimizer and a 0-relaxed minimizer, but it is not a true minimizer, since $L^0(y_0(\cdot),\ y_0'(\cdot)) \equiv 0$ while $L(y_0(\cdot),\ y_0'(\cdot)) \equiv 1$. Indeed, it is well-known that true and relaxed minimizers give the same value to their respective integrals. In reality, there is no true minimizer, in this case (why?), however if we change s^2 to become the positive part of $s^2 - \varepsilon$, for a small $\varepsilon > 0$ then, as we show in Theorem 4 below, one can modify a 0-relaxed minimizer $y_0(\cdot)$ along the interval $(a',\ b')$ where it stops, so as to obtain a true minimizer $y(\cdot)$ which makes small oscillations there, instead of stopping.

We present (below, in (23)) a general extra hypothesis that allows such modification to be made. Provided $(L - L^{**})(\cdot,\ \cdot)$ is lsc at velocity zero, this extra hypothesis is satisfied whenever

$$\forall s' \in \mathbb{R}\ \ \exists s'' \neq s' : L^{**}(\cdot, 0)\ \text{ decreases along } co\left\{s', s''\right\} \text{as the distance from } s' \text{ increases.} \tag{7}$$

Intuitively, (7) means that sublevel sets of $L^{**}(\cdot,\ 0)$ have no singletons. Notice that the intuitive example above does not satisfy (7) at $s' = 0$ when $\varepsilon = 0$, but satisfies it when $\varepsilon > 0$ (why?).

Here is our simplest, yet powerful, result:

Theorem 1 Let $L(\cdot,\ \cdot)$ be a BH-function (as in (2)) with $L^{**}(\cdot,\ \cdot)$ and $(L - L^{**})(\cdot,\ \cdot)$ both lsc at $\xi = 0$. If (7) holds true then there exists a true minimizer for the fully nonconvex integral in (1).

This result is much simpler and stronger (by having much weaker hypotheses) than those existing in the literature. However, its hypotheses may still be much weakened. Indeed, as we show in section 3, our general extra hypothesis (23) is much weaker than (7). Indeed, (23) is much weaker than replacing the word "decreases" in (7) by "mean-decreases" and allows oscillations, as our final Example 2 shows. Moreover, the only points s' that matter in (7) are those satisfying the following:

$$L^{**}\left(s',\ 0\right) = \min L^{**}\left(co\ \left\{s',\ A,\ B\right\},\ 0\right) < L\left(s',\ 0\right).$$

Several authors have previously published results proving existence of true minimizers for the nonconvex integral (1). To begin with, [4–12] have considered sum-type lagrangians, namely

$$L(s,\ \xi) := \psi(s) + \rho(s)\, h(\xi) \tag{8}$$

without $\rho(\cdot)$, having $\psi(\cdot)$ lsc, $h(\cdot)$ lsc and with $\psi(\cdot)$ satisfying specific geometries. For example, [7] treated, more generally, the time-dependent vectorial case $\psi(t,\ s) + h(t,\ \xi)$ with $\psi(t,\ \cdot)$ concave; while [9] imposed (7) $\forall s' \in \mathbb{R}$ (but assuming strictly decrease in case $h^{**}(0) < h(0)$) and in [10] $\psi(\cdot)$ is assumed concave-monotone, namely concave (resp. monotone) along each interval of an open set $\mathscr{C}$ (resp. $\mathscr{M}$) with $\mathscr{C} \bigcup \mathscr{M} = \mathbb{R}$. On the other hand, [13, 14] treated the general case $L(s,\ \xi)$, imposing geometric constraints, at least (7).

Moreover, the papers [3, 9, 15–19] considered weaker hypotheses on $L(\cdot,\ \cdot)$ but under the extra hypothesis (6). While the first four papers dealt mainly with the sum case, [18] treated completely the affine case (8) with $\psi(\cdot)$, $\rho(\cdot)$,

$h(\cdot)$ all lsc, after preliminary results in [16]. Finally, [3] considered superlinear BH-functions $L(\cdot,\cdot)$ under (6) and (9) below (in particular under (4) or (5)). The paper [19] also considered Lipschitz minimizers.

For a broader mathematical audience, here are some classical books on relaxation and lower semicontinuity in variational problems: [20–22].

We finish the Introduction by explaining the organization of this paper. Its section 2 is devoted to present known definitions and results from our previous papers; and, using these, to prove a new result, Theorem 2, on existence of 0-relaxed minimizers. Section 3 is dedicated to our main result, Theorem 4, and to further considerations.

2. Basic definitions and existence of 0-relaxed minimizers

Here we recall definitions and results established in our previous papers [3, 17–19, 23, 24], we prove a new result, Theorem 2, on existence of 0-relaxed minimizers, and improve our DuBois-Reymond necessary condition.

Definition 1 Let $L(\cdot,\cdot)$ be a BH-function (as in (2)). We say that the function $L^{**}(\cdot,\cdot)$ is 0-lsc-convex whenever $L^{**}(\cdot,\cdot)$ is approximable by integrable slopes at zero, in the sense that these two conditions are satisfied:

$$\begin{aligned}&\forall n\in\mathbb{N}\quad \exists\varphi_n:\mathbb{R}\to[0,\,n]\quad \text{lsc with}\quad (\varphi_n(s))\nearrow L^{**}(s,\,0)\quad \forall s\\ &\exists m_n(\cdot)\in L^1_{loc}(\mathbb{R}):\quad L^{**}(s,\,\xi)\geq\varphi_n(s)+m_n(s)\,\xi\quad\forall s,\,\xi.\end{aligned}\tag{9}$$

Notice that any superlinear lsc function $L:\mathbb{R}\times\mathbb{R}\to[0,\,\infty]$ must have $L^{**}(\cdot,\cdot)$ 0-lsc-convex, because in such case $L^{**}(\cdot,\cdot)$ has to be lsc; indeed, more generally, for any BH-function $L(\cdot,\cdot)$, if $L^{**}(\cdot,\cdot)$ satisfies (4) or (5) then it satisfies also (9) (see [3, Theorem 1]). In particular (in contrast with (5), which is the special case of (9) for constant sequences) a 0-lsc-convex function $L^{**}(\cdot,\cdot)$ may have $L^{**}(s,\,0)=\infty$. Notice that there are 0-lsc-convex functions which are not lsc at $\xi=0$, an example appears below, in (26).

Proposition 1 (See [3, theorem 1]) Let $L(\cdot,\cdot)$ be a BH-function with $L^{**}(\cdot,\cdot)$ 0-lsc-convex. Then there exists a relaxed minimizer $y_c(\cdot)$, as defined before (3). Moreover, $y_c(\cdot)$ may be taken bimonotone, i.e. satisfying Definition 3 (b) below.

In what follows, considering the subdifferential $\partial L^{**}(s,\,\xi)$ of $L^{**}(s,\cdot)$ at ξ (see e.g. [25, p.20]), we define the 1-dim (or 0-dim, why?) faces of $epi\,L^{**}(s,\cdot)$ adjacent to $\xi=0$,

$$F(s):=(\partial L^{**}(s,\cdot))^{-1}(\partial L^{**}(s,\,0))=\{\xi\in\mathbb{R}:\ \partial L^{**}(s,\,\xi)\cap\partial L^{**}(s,\,0)\neq\emptyset\},$$

$$\begin{aligned}&\text{so that: } \{0\}\cup F(s)\ \text{is an interval}\ [\alpha(s),\,\beta(s)]\ \text{with}\ \alpha(s)\leq 0\leq\beta(s);\\ &L^{**}(s,\cdot)\ \text{is affine along each interval}\ [\alpha(s),\,0],\,[0,\,\beta(s)];\\ &L^{**}(s,\,\alpha(s))=L(s,\,\alpha(s)),\ L^{**}(s,\,\beta(s))=L(s,\,\beta(s));\\ &L^{**}(s,\,\cdot)\ \text{is affine along}\ [\alpha(s),\,\beta(s)]\ \text{at those } s \text{ where}\ L^{**}(s,0)<L(s,0).\end{aligned}\tag{10}$$

Definition 2 Considering any BH-function $L(\cdot,\cdot)$ and its bipolar $L^{**}(\cdot,\cdot)$, define its 0-*convexified* lagrangian by

$$L^0(s,\ \xi) := \begin{cases} L(s,\ \xi) & \text{for} \quad \xi \neq 0 \quad \forall s \\ L^{**}(s,\ 0) & \text{at} \quad \xi = 0 \quad \forall s. \end{cases} \tag{11}$$

Definition 3
(a) We call $y_c(\cdot)$ a relaxed minimizer provided $y_c(\cdot)$ minimizes (3).
(b) We call $y_c(\cdot)$ bimonotone whenever

(i) $\exists a',\ b'$ with $a \leq a' \leq b' \leq b$ and $y_c(\cdot)$ constant on $[a',\ b']$;

(ii) $y_c(\cdot)$ is strictly monotone along $[a,\ a']$ and is strictly monotone along $[b',\ b]$, with (12)

$y_c'(t) \notin \{0\} \cup (\alpha(y_c(t)),\ \beta(y_c(t)))$ for a.e. $t \in [a,\ a'] \cup [b',\ b]$.

We say that $y_c(\cdot)$ stops if $a' < b'$; otherwise we say that it does not stop. Notice that a bimonotone relaxed minimizer may increase strictly along one of the subintervals of non-constancy and decrease strictly along the other one.

(c) We call $y_0(\cdot)$ a 0-*relaxed minimizer* provided $y_0(\cdot)$ is a relaxed bimonotone minimizer satisfying

(i) $L^{**}\left(y_0(t),\ y_0'(t)\right) = L\left(y_0(t),\ y_0'(t)\right)$ a.e. on $[a,\ a']$ and on $[b',\ b]$;

(ii) $s' := y_0\left(a'\right) = y_0\left(b'\right) \implies L^{**}\left(s',\ 0\right) = \min L^{**}\left(y_0([a,\ b]),\ 0\right)$; (13)

(iii) $\int_a^b L^0\left(y_0(t),\ y_0'(t)\right)\ dt = \int_a^b L^{**}\left(y_0(t),\ y_0'(t)\right)\ dt$;

so that, in particular, $y_0(\cdot)$ also minimizes the 0-*convexified* integral

$$\int_a^b L^0\left(x(t),\ x'(t)\right)\ dt \quad \text{on} \quad \mathscr{X}. \tag{14}$$

Theorem 2 (There exist 0-relaxed minimizers)

Let $L(\cdot,\cdot)$ be a BH-function with $L^{**}(\cdot,\cdot)$ 0-lsc-convex. Then the nonconvex integral in (1) has a 0-relaxed minimizer $y_0(\cdot)$, as in Definition 3 (c) and (b).

Proof. From $L^{**}(\cdot,\cdot) \leq L^0(\cdot,\cdot) \leq L(\cdot,\cdot)$, it follows $L^{**}(\cdot,\cdot) = \left(L^0\right)^{**}(\cdot,\cdot) \leq L^0(\cdot,\cdot) \leq L(\cdot,\cdot)$; and since, from (11), $L^{**}(\cdot,\ 0) = L^0(\cdot,\ 0)$, one gets $L^{**}(\cdot,\ 0) = \left(L^0\right)^{**}(\cdot,\ 0) = L^0(\cdot,\ 0) \leq L(\cdot,\ 0)$. Clearly $L^0(\cdot,\cdot)$ is a 0-convex BH-function (as in (2)): it is $\mathscr{L} \otimes \mathscr{B}$-measurable, $L^0(s,\ \cdot)$ is lsc $\forall s$ (since $L^{**}(\cdot,\ 0) \leq L(\cdot,\ 0)$ and $L(s,\cdot)$ is lsc $\forall s$) and $L^0(\cdot,\cdot) \geq 0$ (since $L^{**}(\cdot,\cdot) \geq 0$). Moreover, $\left(L^0\right)^{**}(\cdot,\ 0) = L^0(\cdot,\ 0) = L^{**}(\cdot,\ 0)$ is lsc (by (9)) and $\left(L^0\right)^{**}(\cdot,\cdot) = L^{**}(\cdot,\cdot)$ is 0-lsc-convex. Therefore, Proposition 1 guarantees existence of a minimizer $y_c(\cdot)$ for the integral

$$\int_a^b \left(L^0\right)^{**}\left(x(t),\ x'(t)\right)\ dt \quad \text{on} \quad \mathscr{X}$$

satisfying Definition 3 (b). Notice that since $\left(L^0\right)^{**}(\cdot,\cdot) = L^{**}(\cdot,\cdot)$, such $y_c(\cdot)$ also minimizes (3), i.e. it is a relaxed minimizer, as in Definition 3 (a).

Assuming first that

$$\exists x(\cdot) \in \mathscr{X}: \quad \int_a^b L^{**}\left(x(t),\, x'(t)\right)\, dt < \infty, \tag{15}$$

let us prove that $y_c(\cdot)$ also satisfies (ii) in (13). In case $a' = b'$, we only need to prove it if $y_c(\cdot)$ is non-monotone (in which case a' is the point where $y_c(\cdot)$ changes from increasing to decreasing or vice-versa). Suppose, by contradiction, $L^{**}(s',\, 0) > \min L^{**}(y_c([a,\, b]),\, 0)$, let s'_0 be the point in $y_c([a,\, b])$ closer to s' where

$$L^{**}\left(s'_0,\, 0\right) = \min L^{**}(y_c([a,\, b]),\, 0);$$

and, just to fix ideas, let $y_c(\cdot)$ increase on $[a,\, a']$, decrease on $[a',\, b]$, with $s'_0 \geq \max\{A,\, B\}$, the other cases being similar. Set

$$a'_0 := \min\left\{t \in [a,\, b] : y_c(t) = s'_0\right\}, \quad b'_0 := \max\left\{t \in [a,\, b] : y_c(t) = s'_0\right\},$$

obtaining $a'_0 < b'_0$ and $L^{**}(y_c(t),\, 0) > L^{**}(s'_0,\, 0)\ \forall t \in (a'_0,\, b'_0)$. Defining

$$z(t) := \begin{cases} y_c(\cdot) \ \text{ for } \ t \in [a,\, a'_0] \cup [b'_0,\, b] \\ \\ s'_0 \ \text{ for } \ t \in [a'_0,\, b'_0], \end{cases}$$

by (9) we have (see [3, proof of Theorem 1])

$$\int_{a'_0}^{b'_0} L^{**}\left(y_c(t),\ y'_c(t)\right) dt \geq \int_{a'_0}^{b'_0} \varphi_n(y_c(t))\, dt + \int_{a'_0}^{b'_0} m_n(y_c(t))\, y'_c(t)\, dt = \int_{a'_0}^{b'_0} \varphi_n(y_c(t))\, dt,$$

hence, letting $n \to \infty$, by monotone convergence,

$$\int_{a'_0}^{b'_0} L^{**}\left(y_c(t),\ y'_c(t)\right) dt \geq \int_{a'_0}^{b'_0} L^{**}(y_c(t),\, 0)\, dt > \int_{a'_0}^{b'_0} L^{**}(z(t),\, 0)\, dt = \int_{a'_0}^{b'_0} L^{**}\left(z(t),\ z'(t)\right) dt.$$

Therefore

$$\int_a^b L^{**}\left(z(t),\ z'(t)\right) dt < \int_a^b L^{**}\left(y_c(t),\ y'_c(t)\right) dt. \tag{16}$$

Similarly, if $a' < b'$ and $L^{**}(s',\, 0) > \min L^{**}(y_c([a,\, b]),\, 0)$ then we may transform $y_c(\cdot)$ into a new function $z(\cdot) \in \mathscr{X}$ stopping at some $s'_z \in co\{A,\, B,\, s'\}$ satisfying $L^{**}\left(s'_z,\, 0\right) = \min L^{**}(z([a,\, b]),\, 0)$ and yielding (16).

Moreover, by [23, proposition 4], the relaxed minimizer $y_c(\cdot)$ can be changed so as to obtain another relaxed minimizer $y_0(\cdot)$ satisfying: (b) (i), with the same a', b', s' hence (b) (ii), (c) (ii) and

$$L^{**}\left(y_0(t),\ y_0'(t)\right) = \left(L^0\right)^{**}\left(y_0(t),\ y_0'(t)\right) = L^0\left(y_0(t),\ y_0'(t)\right) \quad \text{a.e. on } [a,\ b],$$

so that (c) of Definition 3 also holds true.

Assume, finally, that (15) is not true. Then clearly any function in $\mathscr{X}$ is a relaxed minimizer; and we redefine $y_c(\cdot)$ so as to become the affine function in $\mathscr{X}$. In case $A = B$ then clearly this affine $y_c(\cdot) \equiv A$ satisfies (ii) in (13). Moreover, since

$$\int_a^b L^{**}\left(y_c(t),\ y_c'(t)\right)\ dt = \int_a^b L^{**}(A,\ 0)\ dt = \infty,$$

we must have $L^{**}(A,\ 0) = \infty$ hence also $L(A,\ 0) = \infty$; so that this constant $y_c(\cdot)$ is a 0-relaxed and true minimizer for the fully nonconvex integral in (1).

On the other hand, if $A \neq B$ then the affine function $y_c(t) := \dfrac{B-A}{b-a}(t-a) + A$ is a monotone relaxed minimizer from which we can obtain, following the proof of [3, proposition 1] and using [23, proposition 4], a bimonotone relaxed minimizer which is a 0-relaxed and true minimizer. □

Similarly one proves the next

Corollary 1 (There exist monotone 0-relaxed minimizers)

Let $L(\cdot,\cdot)$ be a BH-function with $L^{**}(\cdot,\cdot)$ 0-lsc-convex and take a 0-relaxed minimizer $y_0(\cdot)$ given by Theorem 2.

If $\min L^{**}(y_0([a,\ b]),\ 0) = \min L^{**}(co\ \{A,\ B\},\ 0)$ then $y_0(\cdot)$ may be taken monotone.

This is true e.g. whenever either $y_0([a,\ b]) = co\ \{A,\ B\}$ or $\min L^{**}(co\ \{A,\ B\},\ 0) \leq L^{**}(s,\ 0)\ \forall s \in \mathbb{R}$.

Theorem 3 (Generalized DuBois-Reymond necessary condition)

Let $L(\cdot,\cdot)$ be a BH-function with $L^{**}(\cdot,\cdot)$ 0-lsc-convex satisfying (15). Consider a relaxed (in particular, a 0-relaxed) minimizer $y(\cdot)$ for the nonconvex integral in (1). Defining the domain of $L^{**}(s,\cdot)$ by

$$D_{L^{**}}(s) := \{\xi \in \mathbb{R} :\ L^{**}(s,\ \xi) < \infty\}$$

then clearly $y'(t) \in D_{L^{**}}(y(t))$ a.e. on $[a,\ b]$, by (15).

(a) In case

$$y'(t) \in \text{interior } D_{L^{**}}(y(t)) \quad \text{a.e. on } [a,\ b] \tag{17}$$

(e.g. if $D_{L^{**}}(y(t))$ is open a.e. on $[a,\ b]$) then surely $y(\cdot)$ satisfies the classical DuBois-Reymond (DB-R) inclusion, namely

$$\exists q \in \mathbb{R} :\ L^{**}\left(y(t),\ y'(t)\right)\ \in\ q + y'(t)\,\partial L^{**}\left(y(t),\ y'(t)\right) \quad \text{a.e. on } [a,\ b]. \tag{18}$$

(b) In case $y(\cdot)$ does not satisfy (17), $y(\cdot)$ may still satisfy (18), even in the extreme case where

$$y'(t) \in \text{boundary } D_{L^{**}}(y(t)) \quad \text{a.e. on } [a, b]. \tag{19}$$

(c) However, if $y(\cdot)$ does not satisfy (18) then $y'(t) \neq 0$ a.e. on $[a, b]$ and (19) must hold true (in particular, obviously,

$$L^{**}\left(y(t), y'(t)\right) = L\left(y(t), y'(t)\right) \quad \text{a.e. on } [a, b]$$

hence $y(\cdot)$ is a true minimizer of the nonconvex integral in (1)) while $y'(\cdot)$ must be either a.e. maximal or else a.e. minimal or, more precisely,

$$\begin{aligned} &\text{either } L^{**}\left(y(t), y'(t) \cdot r\right) = \infty \ \forall r \in (1, \infty) \quad \text{a.e. on } [a, b] \\ &\text{or else } L^{**}\left(y(t), y'(t) \cdot r\right) = \infty \ \forall r \in (0, 1) \quad \text{a.e. on } [a, b]. \end{aligned} \tag{20}$$

(d) Finally (recalling a', b', s' in (12) and (13)) if $y(\cdot)$ is bimonotone with $a' < b'$ then (18) must be satisfied; and if, moreover, $y(\cdot)$ is a 0-relaxed minimizer and 0 is on the boundary of $D_{L^{**}}(s')$ (so that $L^{**}(s', 0) = L(s', 0)$) then $y(\cdot)$ is a true minimizer for the nonconvex integral in (1).

Proof. In [24] we have extended previous results of [26, theorem 4] and of [27, theorem 6.10]. Following with attention the proof of [24, theorem 1], one notices that $y'(\cdot) \neq 0$ a.e. whenever (18) is not true. On the other hand, in [24] we have proved that, in such case, $y'(\cdot)$ is either a.e. maximal or else a.e. minimal, i.e. the alternative in (20). □

3. Existence of true minimizers

After the above preliminaries, finally in this section we present our main result (Theorem 4) together with one Remark and two Examples. Let us begin with a new definition, namely a modification of Definition 3 (b), see (12).

Definition 4 Recalling (10), we say that $y(\cdot) \in \mathscr{X}$ is a non-stopping *finitely-monotone* function provided $\exists N \in \mathbb{N}$ for which: the interval $[a, b)$ can be partitioned into N subintervals $[a_i, b_i)$ satisfying the following: $y(\cdot)$ is strictly monotone along each subinterval $[a_i, b_i)$, with derivative $y'(t) \notin \{0\} \cup (\alpha(y(t)), \beta(y(t)))$ a.e..

To present our main result, we also need the following definitions:

$$\begin{aligned} &\varphi(s) := \min L^{**}(co\{s, A, B\}, 0), \qquad S_{\min} := \{s \in \mathbb{R}: \ \varphi(s) = L^{**}(s, 0)\} \\ &S^{<} := \{s \in \mathbb{R}: \ L^{**}(s, 0) < L(s, 0)\}, \\ &\mu : \mathbb{R} \to (0, \infty], \qquad \mu(s) := \begin{cases} \dfrac{1}{|\alpha(s)|} + \dfrac{1}{\beta(s)} & \text{for } s \in S^{<} \\ \infty & \text{for } s \notin S^{<}. \end{cases} \end{aligned} \tag{21}$$

Moreover, for $s' \in S_{\min} \cap S^{<}$ and $s'' \neq s'$ having $\mu(\cdot) \in L^1(co\{s', s''\})$ we define the moving average $\overline{\varphi}$: $co\{s', s''\} \to [0, \infty]$,

$$\overline{\varphi}\left(s'\right) := L^{**}\left(s',\ 0\right),\ \overline{\varphi}\left(s\right) := \frac{\displaystyle\int_{co\left\{s',\ s\right\}} L^{**}\left(\sigma, 0\right)\mu\left(\sigma\right)\ d\sigma}{\displaystyle\int_{co\left\{s',\ s\right\}} \mu\left(\sigma\right)\ d\sigma} \quad \text{for } s \in co\left\{s',\ s''\right\}\setminus\left\{s'\right\}. \tag{22}$$

Theorem 4 (There exists a true minimizer)

Consider a BH-function $L(\cdot,\cdot)$ having $L^{**}(\cdot,\cdot)$ 0-lsc-convex and let $y_0(\cdot)$ be a 0-relaxed minimizer (as in Definition 3 (c), (b)) given by Theorem 2.

If $y_0(\cdot)$ is not a true minimizer of the nonconvex integral in (1) then certainly (15) holds true and we have $a' < b'$, $y_0([a',\ b']) = \{s'\}$, $s' \in S_{\min} \cap S^{<}$ (see (13) and (21)) and $y_0(\cdot)$ satisfies (18).

Moreover, if

$$\begin{aligned} &\text{either } \ (L - L^{**})(\cdot,\cdot) \ \text{ is lsc at } \ \left(s',\ 0\right) \\ &\text{or else } \ \exists s'' \neq s' \ \text{ with } \ \mu(\cdot) \in L^1\left(co\left\{s', s''\right\}\right); \\ &\text{and } \ \overline{\varphi}(\cdot) \ \text{ in } \ (22) \ \text{ satisfies } \ \overline{\varphi}(s) \le \overline{\varphi}\left(s'\right) \ \forall s \in co\left\{s', s''\right\} \end{aligned} \tag{23}$$

then we can modify $y_0(\cdot)$ along $(a',\ b')$ so as to reach a non-stopping finitely-monotone (as in Definition 4) true minimizer $y(\cdot)$ of the nonconvex integral in (1); and if $y(\cdot)$ does not satisfy (18) then it satisfies (19) hence (20).

In case $y_0(\cdot)$ is Lipschitz and $\alpha(\cdot)$, $\beta(\cdot)$ are both locally bounded then also $y(\cdot)$ is Lipschitz.

Remark 1 In case $y_0(\cdot)$ is not a true minimizer and one is unable to verify (23) by lack of knowledge about the point s' where $y_0(\cdot)$ stops then: in case we know that $y_0(\cdot)$ is monotone (see Corollary 1 above) it suffices to check that (23) is satisfied by some $s' \in S_{\min} \cap S^{<} \cap co\ \{A,\ B\}$; while if we know that $y_0(\cdot)$ is not monotone then one should ensure that (23) is satisfied by every $s' \in (S_{\min} \cap S^{<}) \setminus co\ \{A,\ B\}$. If one does not know whether $y_0(\cdot)$ is monotone or not then one should ensure both these conditions.

Notice that one way to guarantee the inequality in line 3 of (23) is when $L^{**}(\sigma,\ 0) \le L^{**}(s',\ 0)$ for a.e. $\sigma \in co\ \{s', s''\}$. Indeed, for $s \neq s'$, with $M(s) := \int_{co\{s',\ s\}} \mu(\sigma)\ d\sigma$,

$$\overline{\varphi}(s) = M(s)^{-1}\int_{co\{s',\ s\}} L^{**}(\sigma,\ 0)\mu(\sigma)\ d\sigma \le M(s)^{-1}\int_{co\{s',\ s\}} L^{**}\left(s',\ 0\right)\mu(\sigma)\ d\sigma = L^{**}\left(s',\ 0\right) = \overline{\varphi}\left(s'\right).$$

Clearly the difference $(L - L^{**})(\cdot,\cdot)$ is well-defined as soon as we set $(L - L^{**})(s,\ \xi) := 0$ wherever $L^{**}(s,\ \xi) = L(s,\ \xi) = \infty$, because then $(L - L^{**})(s,\ \xi) \in [0,\ \infty)$ where $0 \le L^{**}(s,\ \xi) \le L(s,\ \xi) < \infty$; while again $(L - L^{**})(s,\ \xi) = \infty$ is well-defined where $L^{**}(s,\ \xi) < L(s,\ \xi) = \infty$.

Notice that while Theorem 1 cannot be applied in our example (26) below with $1 < \delta < 2$, because $L^{**}(\cdot,\cdot)$ and $(L - L^{**})(\cdot,\cdot)$ are not lsc at velocity zero; on the contrary Theorem 4 can indeed be applied since lines 2 and 3 of (23) both hold true in this case. While we could have included in (23) only its lines 2 and 3, we feel that in explicit examples it is in general easier to check for the truth of its line 1, which anyway is much simpler to express, as we have done before (7).

In case $(L - L^{**})(\cdot,\ 0)$ is lsc at $s' \in S^{<}$, we may assume that $co\ \{s', s''\} \subset S^{<}$, because otherwise there would exist a sequence $(s_k) \subset co\ \{s', s''\}$ with $L^{**}(s_k,\ 0) = L(s_k,\ 0)$ and $(s_k) \to s'$, yielding an absurd: $0 < (L - L^{**})(s',\ 0) \le \liminf_{k\to\infty}(L - L^{**})(s_k,\ 0) = \liminf_{k\to\infty} 0 = 0$. Notice also the following: if $(L - L^{**})(\cdot,\ 0)$ is lsc at each $s' \in S^{<} \cap S_{\min}$ (so that each such $s' \in \text{interior}\, S^{<}$) and $\mu(\cdot) \in L^1_{loc}(S^{<})$ then line 2 of (23) holds true.

We claim that if in (23) $(L - L^{**})(\cdot, \cdot)$ is lsc at $(s', 0)$ then we can move s'' towards s' until reaching $\mu(\cdot) \in L^1(co\{s', s''\})$. To prove this, setting $\varepsilon(s) := \min\{|\alpha(s)|, \beta(s)\} \geq 0$ and $S_0 := co\{s', s''\} \setminus \{s'\}$, we show that

$$\exists s_0 \in S_0 : \varepsilon_0 := \inf\left\{\varepsilon(s) : s \in co\left\{s', s_0\right\}\right\} > 0, \text{ so that } \min\{|\alpha(s)|, \beta(s)\} \geq \varepsilon_0$$

$$\text{and } \mu(s) \leq 2/\varepsilon_0 \; \forall s \in co\left\{s', s_0\right\} \text{ hence } \mu(\cdot) \in L^1\left(co\left\{s', s_0\right\}\right).$$

Indeed, if such s_0 did not exist then it would $\exists (s_k) \subset S_0$ with $(s_k) \to s'$ for which $\varepsilon_k := \inf\{\varepsilon(s) : s \in co\{s', s_k\}\} = 0$ hence $\exists \sigma_k \in co\{s', s_k\} \setminus \{s'\}$ with $0 \leq \varepsilon(\sigma_k) \leq 1/k$, so that $(\sigma_k) \subset S_0$, $(\sigma_k) \to s'$, $(\varepsilon(\sigma_k)) \to 0$, in particular either $(\alpha(\sigma_k)) \to 0$ or $(\beta(\sigma_k)) \to 0$; and if e.g. $(\beta(\sigma_k)) \to 0$ hence $(\sigma_k, \beta(\sigma_k)) \to (s', 0)$, we would reach an absurd: $0 < (L - L^{**})(s', 0) \leq \liminf_{k \to \infty} (L - L^{**})(\sigma_k, \beta(\sigma_k)) = \liminf_{k \to \infty} 0 = 0$. This absurd proves the claim, showing that in the next proof, by moving s'' towards s', one may always use $\mu(\cdot) \in L^1(co\{s', s''\})$.

The reasonings in the three previous paragraphs show, in (23), how much more general is its line 2, relative to its line 1. Indeed, one may weaken the hypothesis in its line 1 to the point of allowing $(L - L^{**})(\cdot, 0)$ to be non-lsc at s', thus leaving open the possibility of a sequence $(s_k) \subset co\{s', s''\}$ to have $(s_k) \to s'$ with $(L - L^{**})(s_k, 0) = 0$, and $\alpha(s_k) = 0 = \beta(s_k)$ hence $\mu(s_k) = \infty$, $\forall k \in \mathbb{N}$; while still allowing satisfaction of line 2 of (23). But of course, such satisfaction implies that $\alpha(\cdot)$ and $\beta(\cdot)$ cannot be affine near anyone of the points s_k, on the contrary they must at least have infinite slope at each s_k; while $r(\cdot)$ in (24) below will be undefined at each s_k, which does not affect the construction of $y(\cdot)$ along (a', b'), in the proof of Theorem 4, because (s_k) has zero measure.

Since one may always assume $\mu(\cdot) \in L^1(co\{s', s''\})$ in (23), we have, for a.e. s in $co\{s', s''\}$, $\mu(s) \in (0, \infty)$ hence $\alpha(s) < 0 < \beta(s)$. Moreover the definitions of s' and $\overline{\varphi}(\cdot)$ in (22) and the inequality in (23) yield $0 \leq \overline{\varphi}(s) \leq \overline{\varphi}(s') < \infty$, i.e. $\overline{\varphi} : co\{s', s''\} \to [0, \infty)$.

Clearly the hypothesis (23) is satisfied whenever $\mu(\cdot) \in L^1(co\{s', s''\})$ and (7) holds true; but it cannot be satisfied e.g. at a unique global minimum point $s' \in S^<$ of $L^{**}(\cdot, 0)$. However, (23) is much more general than allowing the "decreases" in (7) to be replaced by "a.e. decreases" or "average-decreases". Indeed, please check our final Example 2 in which both $L^{**}(\cdot, 0)$ and $\overline{\varphi}(\cdot)$ oscillate frenetically near $s' = 0$ but still satisfy (23).

Proof. Take a 0-relaxed minimizer $y_0(\cdot)$ given by Theorem 2. Then: either $y(\cdot) := y_0(\cdot)$ is already a bimonotone true minimizer; or else we change $y_0(\cdot)$ so as to construct a non-stopping finitely-monotone true minimizer $y(\cdot)$.

Indeed, clearly $y_0(\cdot)$ satisfies (a), (b), (c) of Definition 3, in particular

$$\exists s' : \; L^{**}\left(s', 0\right) = \min L^{**}(y_0([a, b]), 0) = \varphi\left(s'\right) \text{ and } y_0([a, b]) = co\left\{s', A, B\right\}.$$

In case (15) is not true or $a' = b'$ or $L^{**}(s', 0) = L(s', 0)$ we may set $y(\cdot) := y_0(\cdot)$, thus obtaining a bimonotone true minimizer of the integral in (1).

Otherwise we have $a' < b'$ and $L^{**}(s', 0) < L(s', 0)$, so that $s' \in S_{\min} \cap S^<$ hence $\exists s'' \neq s'$ as in (23), according to Remark 1. In this case we construct a non-stopping finitely-monotone (see Definition 4) true minimizer $y(\cdot)$ for the integral in (1). We do it by changing $y_0(\cdot)$ only along its constancy interval, so as to oscillate there, instead of stopping (A simple intuitive example of a situation in which this construction could be applied appears above, in the paragraph preceding the paragraph containing (7)). Let us start by moving s'' in (23) towards s' until obtaining, for some $N \in \mathbb{N}$,

$$\frac{b' - a'}{\left|\int_{s'}^{s''} \mu(s)\, ds\right|} = N.$$

By Remark 1 we have, for a.e. s in $co\{s', s''\}$,

$$\alpha(s) < 0 < \beta(s), \quad (L-L^{**})(s,\ \alpha(s)) = 0, \quad (L-L^{**})(s,\ \beta(s)) = 0 \ \text{ and}$$
$$L^{**}(s,\ 0) = (1-r(s))\, L^{**}(s,\ \alpha(s)) + r(s)\, L^{**}(s,\ \beta(s)) \ \text{ with } \ r(s) := \frac{|\alpha(s)|}{|\alpha(s)|+\beta(s)}. \tag{24}$$

Let us assume, say $s' < s''$, just to simplify the following discussion. Set

$$\theta := \frac{N}{b'-a'}\int_{s'}^{s''} \frac{1}{|\alpha(s)|}\, ds, \quad \text{so that} \quad 1-\theta = \frac{N}{b'-a'}\int_{s'}^{s''} \frac{1}{\beta(s)}\, ds; \tag{25}$$

and define

$$\tau_+ : [s',\ s''] \longrightarrow \left[a',\ a' + (1-\theta)\,\frac{b'-a'}{N}\right], \quad \tau_+(s) := a' + \int_{s'}^{s} \frac{1}{\beta(\sigma)}\, d\sigma,$$

$$\tau_- : [s',\ s''] \longrightarrow \left[a' + (1-\theta)\,\frac{b'-a'}{N}, a' + \frac{b'-a'}{N}\right], \quad \tau_-(s) := a' + \frac{b'-a'}{N} + \int_{s'}^{s} \frac{1}{\alpha(\sigma)}\, d\sigma.$$

Since $1/\alpha(\cdot)$, $1/\beta(\cdot) \in L^1((s',\ s''))$, these functions $\tau_+(\cdot)$, $\tau_-(\cdot)$ are monotone AC with nonzero derivative a.e.. Indeed,

$$\tau_+(\cdot) \text{ increases with } \tau_+'(s) = 1/\beta(s) > 0 \ \text{ a.e. and } \ \tau_-(\cdot) \ \text{ decreases with } \ \tau_-'(s) = 1/\alpha(s) < 0 \text{ a.e..}$$

Moreover, by (25),

$$\tau_+(s') = a', \quad \tau_-(s') = a' + \frac{b'-a'}{N}, \quad \tau_-(s'') = a' + (1-\theta)\,\frac{b'-a'}{N} = \tau_+(s'').$$

The inverse functions of $\tau_+(\cdot)$, $\tau_-(\cdot)$, respectively

$$x_+ : \left[a',\ a' + (1-\theta)\,\frac{b'-a'}{N}\right] \to [s',\ s''], \quad x_- : \left[a' + (1-\theta)\,\frac{b'-a'}{N},\ a' + \frac{b'-a'}{N}\right] \to [s',\ s''],$$

are well-defined and are AC, $x_+(\cdot)$ increases and $x_-(\cdot)$ decreases (both with derivative $\neq 0$ a.e.); and

$$x_+(a') = s' = x_-\left(a' + \frac{b'-a'}{N}\right), \quad x_+\left(a' + (1-\theta)\,\frac{b'-a'}{N}\right) = s'' = x_-\left(a' + (1-\theta)\,\frac{b'-a'}{N}\right).$$

We may therefore define the function

$$x_1 : \left[a', \, a' + \frac{b' - a'}{N}\right] \longrightarrow [s', \, s''], \quad x_1(t) := \begin{cases} x_+(t) \ \text{ for } \ t \ \text{ on } \ \left[a', \, a' + (1-\theta)\, \dfrac{b' - a'}{N}\right] \\ \\ x_-(t) \ \text{ for } \ t \ \text{ on } \ \left[a' + (1-\theta)\, \dfrac{b' - a'}{N}, \, a' + \dfrac{b' - a'}{N}\right]. \end{cases}$$

Clearly $x_1(a') = s' = x_1\left(a' + \frac{b' - a'}{N}\right), x_1\left(a' + (1-\theta)\,\frac{b' - a'}{N}\right) = s''$ and

$$x_1'(t) = \begin{cases} \beta(x_1(t)) \text{ for a.e. } t \text{ on } \left[a', \, a' + (1-\theta)\, \dfrac{b' - a'}{N}\right] \\ \\ \alpha(x_1(t)) \text{ for a.e. } t \text{ on } \left[a' + (1-\theta)\, \dfrac{b' - a'}{N}, \, a' + \dfrac{b' - a'}{N}\right]. \end{cases}$$

Therefore, by (24),

$$\begin{aligned} \int_{s'}^{s''} L^{**}(s, \, 0) \ \mu(s) \ ds &= \int_{s'}^{s''} \left[L(s, \, \alpha(s))\, \frac{1}{|\alpha(s)|} + L(s, \, \beta(s))\, \frac{1}{\beta(s)}\right] \, ds \\ &= \int_{s'}^{s''} L(s, \, \beta(s))\, \frac{1}{\beta(s)} \, ds + \int_{s'}^{s''} L(s, \, \alpha(s))\, \frac{1}{|\alpha(s)|} \, ds \\ &= \int_{a'}^{a' + (1-\theta)\,\frac{b'-a'}{N}} L\left(x_+(t), x_+'(t)\right) \, dt + \int_{a' + (1-\theta)\,\frac{b'-a'}{N}}^{a' + \frac{b'-a'}{N}} L\left(x_-(t), x_-'(t)\right) \, dt \\ &= \int_{a'}^{a' + \frac{b'-a'}{N}} L\left(x_1(t), x_1'(t)\right) \, dt, \end{aligned}$$

using [3, proposition 3].

Repeating this construction $N-1$ times more we obtain

$$x_2(\cdot) \ \text{ on } \ \left[a' + \frac{b' - a'}{N}, \, a' + 2\frac{b' - a'}{N}\right], \, \ldots, \, x_N(\cdot) \ \text{ on } \ \left[a' + (N-1)\, \frac{b' - a'}{N}, \, b'\right]$$

in the same way as we did above for $x_1(\cdot)$ in $\left[a', \, a' + \frac{b' - a'}{N}\right]$. Then, glueing such N pieces we get an AC funtion

$$y:[a,\, b] \longrightarrow [s',\, s''],\ y(t) := \begin{cases} y_0(t) & \text{for } t \text{ on } [a,\, a'] \\ x_1(t) & \text{for } t \text{ on } \left[a',\, a' + \dfrac{b'-a'}{N}\right] \\ & \quad\vdots \\ x_N(t) & \text{for } t \text{ on } \left[a' + (N-1)\dfrac{b'-a'}{N},\, b'\right] \\ y_0(t) & \text{for } t \text{ on } [b',\, b] \end{cases}$$

having $y'(t) \neq 0$ a.e. and $y(a') = s' = y(b')$. For this $y(\cdot)$ we have (by (23)):

$$\int_{a'}^{b'} L\big(y(t), y'(t)\big)\ dt = N \int_{s'}^{s''} L^{**}(s,\, 0)\ \mu(s)\, ds \le \big(b' - a'\big) L^{**}\big(s',\, 0\big) = \int_{a'}^{b'} L^{**}\big(y_0(t),\ y_0'(t)\big)\ dt,$$

so that

$$\int_a^b L\big(y(t),\ y'(t)\big)\ dt \le \int_a^b L^{**}\big(y_0(t),\ y_0'(t)\big)\ dt.$$

But then such inequality $\le$ will have to be an equality, otherwise a contradiction would be reached, since $y_0(\cdot)$ is already a relaxed minimizer. Therefore $y(\cdot)$ is the desired non-stopping finitely-monotone true minimizer for the fully nonconvex integral in (1). □

Example 1 (Application to a wild lagrangian) Consider the BH-function

$$L(s,\ \xi) := \begin{cases} \left(1 + \xi\, s\, |s|^{-\delta}\right)^+ & \text{for } s \neq 0 \text{ and } \left(|\xi| = 1 \text{ or } |\xi| = \min\left\{1,\ |s|^{\delta-1}\right\}\right) \\ 1 & \text{for } s = 0 \text{ and } |\xi| = 1 \\ \infty & \text{for other } s,\, \xi, \end{cases} \tag{26}$$

where $(f(\cdot,\cdot))^+ := \max\{0,\ f(\cdot,\cdot)\}$. We have

$$L^{**}(s,\ \xi) = \begin{cases} \left(1 + \xi\, s\, |s|^{-\delta}\right)^+ & \text{for } s \neq 0 \text{ and } |\xi| \le 1 \\ 1 & \text{for } s = 0 \text{ and } |\xi| \le 1 \\ \infty & \text{for other } s,\, \xi. \end{cases}$$

Whenever $\delta < 2$, $L^{**}(\cdot,\cdot)$ is 0-lsc-convex. Indeed, $L^{**}(\cdot,\cdot)$ satisfies (9) with constant sequences

$$\varphi_n(\cdot) := L^{**}(\cdot,\, 0), \quad m_n(s) := \begin{cases} s\,|s|^{-\delta} & \text{for} \quad s \neq 0 \\ \\ 0 & \text{at} \quad s = 0. \end{cases}$$

Moreover $S^{<} = \mathbb{R}$ since $L^{**}(\cdot,\, 0) \equiv 1 < \infty \equiv L(\cdot,\, 0)$. Notice that if $1 < \delta < 2$ then $L^{**}(\cdot,\cdot)$ is not lsc at $\xi = 0$ (since $L^{**}(\cdot,\cdot)$ is not lsc at $(0,\, 0)$).

As an application of Theorem 4, let us show that for $\delta < 2$ there exists a true minimizer. Indeed, since

$$\alpha(s) = \begin{cases} -1 & \text{for} \quad \min\left\{1,\, |s|^{\delta-1}\right\} = 1 \ \text{ or } \ \left(\min\left\{1,\, |s|^{\delta-1}\right\} = |s|^{\delta-1} \quad \text{and} \quad s \leq 0\right) \\ \\ -|s|^{\delta-1} & \text{for} \quad \min\left\{1,\, |s|^{\delta-1}\right\} = |s|^{\delta-1} \quad \text{and} \quad s > 0 \end{cases}$$

$$\beta(s) = \begin{cases} 1 & \text{for} \quad \min\left\{1,\, |s|^{\delta-1}\right\} = 1 \ \text{ or } \ \left(\min\left\{1,\, |s|^{\delta-1}\right\} = |s|^{\delta-1} \quad \text{and} \quad s \geq 0\right) \\ \\ |s|^{\delta-1} & \text{for} \quad \min\left\{1,\, |s|^{\delta-1}\right\} = |s|^{\delta-1} \quad \text{and} \quad s < 0, \end{cases}$$

hence $\mu(\cdot) \in L^1_{loc}(\mathbb{R})$ in (21), (22), (23), all the hypotheses of Theorem 4 hold true in this case $\delta < 2$.

Notice that the graph of $L(s,\cdot)$ consist only of: either two points, over -1 and 1, in case $-1 = \alpha(s) < 0 < \beta(s) = 1$; or else four points, over -1, 1 and either $\pm\alpha(s)$ or $\pm\beta(s)$, in case either $\alpha(s)$ or $\beta(s)$ equal their second lines, more precisely in case either $-1 < \alpha(s) < 0$ or $0 < \beta(s) < 1$. This ensures that its bipolar $L^{**}(s,\cdot)$ is indeed given by the above expression. Notice also that for $1 < \delta < 2$ and $s \to 0$ we have either $\alpha(s) \to 0$ or $\beta(s) \to 0$, yielding zero-values of $L^{**}(s,\, \xi(s))$ with $\xi(s) \to 0$, thus contradicting lower semicontinuity at $(0,\, 0)$, since $L^{**}(0,\, 0) = 1$. Finally, we leave two questions to those readers who read carefully this example. Can one change, in the first line of (26), $(1 + \ldots)^+$ to $(\gamma + \ldots)^+$ with $\gamma \neq 1$? Can one apply Theorem 4 when $\delta \geq 2$?

Example 2 Let $L(s,\, \xi) = \psi(s) + h(\xi)$ with $\psi(s) := [s.\, \sin(1/s) + 1 - 2s]^+$ and $h : \mathbb{R} \to [0,\, \infty)$ superlinear lsc with $h^{**}(0) = 0 < h(0)$, $|\alpha(\cdot)| = \beta(\cdot) \equiv 2$, so that $\mu(\cdot) \equiv 1$, $L^{**}(\cdot,\, 0) = \psi(\cdot)$ and $\overline{\varphi}(s) = (s - s')^{-1} \int_{s'}^{s} \psi(\sigma)\, d\sigma$ for $s \neq s'$.

Clearly $\psi(\cdot)$ does not satisfy (7) with $s' = 0$ since, for any $s'' > 0$, $\psi(\cdot)$ does not decrease in $(s',\, s'')$. However, $\psi(\cdot)$ does satisfy (23) because $\psi(s) < \psi(s') = \psi(0) = 1$ $\forall s > 0$ so that $\overline{\varphi}(s) < \overline{\varphi}(s') = \psi(0) = 1$ $\forall s > 0$. Thus (23) is strictly weaker than (7). More precisely, as one easily checks, $\psi(s) \in co\,\{1 - 3s,\, 1 - s\}$ $\forall s \in \mathbb{R} \implies \overline{\varphi}(s) \in co\,\{1 - 3s/2,\, 1 - s/2\}$ $\forall s \in \mathbb{R}$.

Let us apply this information to answer two questions on the problem of minimizing the nonconvex integral in (1) with this $L(\cdot,\cdot)$ and $A = 0 = B$. First, could $y_0(\cdot) \equiv 0$ be a 0-relaxed minimizer? The answer is clearly no, since $\overline{\varphi}(s) \leq 1 - s/2 < \overline{\varphi}(s') = \psi(0) = 1$ for $s > 0$. Indeed, otherwise one could change $y_0(\cdot) \equiv 0$ along $(a',\, b')$ so as to reach a true minimizer $y(\cdot)$ of the nonconvex integral in (1) which oscilates (as in the proof of Theorem 4) and gives to the nonconvex integral in (1) a value strictly lower than the value $b - a$ given to the convexified integral by the supposed 0-relaxed minimizer $y_0(\cdot) \equiv 0$, absurd.

Second, notice that the set S_0 of local min points of $\psi(\cdot)$ in $(0,\, \infty)$ consists of a sequence $(s_k) \searrow 0$ (with $s_1 \approx 0.25$) plus an interval $[s_0,\, \infty) = \psi^{-1}(0)$ (with $s_0 \approx 0.90$). Certainly there exists a 0-relaxed non-monotone minimizer $y_0(\cdot)$, by Theorem 2; and if $y_0(\cdot)$ is not a true minimizer then (15) holds true and $y_0(\cdot)$ is a constant $s' > 0$ along some subinterval $(a',\, b')$. We claim that $s' \in S_0$. To prove this, notice first that if $s' \in \psi^{-1}(0)$ then clearly (23) holds true hence, by Theorem 4, there exists a true minimizer $y(\cdot)$. Otherwise $s' \in (0,\ s_0)$ and, to prove that $s' \in (s_k)$, set $s'_k :=$ $\min\{s \in (s_k,\ s_0):\ \psi(s) = \psi(s_k)\}$ and notice that $s' \in S_{\min} \cap (0,\ s_0) = \bigcup_{k=1}^{\infty} [s'_{k+1},\ s_k] \bigcup [s'_1,\ s_0)$.

But s' cannot be in $\bigcup_{k=0}^{\infty} [s'_{k+1},\ s_k)$ because otherwise one could apply Theorem 4 to prove existence of a true minimizer $y(\cdot)$ since, as one easily checks, $\forall k \in \mathbb{N}\ \exists s''_k \in (s'_k,\ s_{k-1}):\ \ \psi(s) < \psi(s'_k)\ \forall s \in (s'_k,\ s''_k)$; and this strict inequality would lead to an absurd, because $y(\cdot)$ would give to the nonconvex integral a value strictly lower than the value given by $y_0(\cdot)$ to the convexified integral.

We have thus proved our claim that the point s' where $y_0(\cdot)$ stops must belong to the set S_0 of local min points of $\psi(\cdot)$ in $(0,\ \infty)$. On the other hand, as one easily checks, if $s' \in \psi^{-1}(0)$ then there exists a true minimizer; while if $s' \in (s_k)$ then no true minimizer exists, unless $b-a$ has a very special value. However, one may easily enforce existence of true minimizers simply by flattening $\psi(\cdot)$ near each s_k, as we have done above, in the example presented in the paragraph before (7).

4. Conclusions

This paper deals with existence of scalar minimizers for nonconvex single integrals of the Calculus of Variations and with necessary conditions that these minimizers must satisfy, thus helping one to understand their behavior. Since our lagrangians are freely allowed to take ∞ values, the results here presented can easily be applied to minimization problems under pointwise state and/or velocity constraints, in particular to Optimal Control problems.

Our main result, Theorem 4, proves existence of true minimizers even without 0-convexity, namely allowing $L^{**}(\cdot,\ 0) < L(\cdot,\ 0)$. This is important because frequently relaxed minimizers fail to be true minimizers by remaining a constant s' along a subinterval $(a',\ b')$ with $L^{**}(s',\ 0) < L(s',\ 0)$. One sufficient condition for such existence is that the sublevel sets of $L^{**}(\cdot,\ 0)$ contain no singletons. This suffices if $L^{**}(\cdot,\cdot)$ and $(L-L^{**})(\cdot,\cdot)$ are both lsc at velocity zero. More generally, one may relax "sublevel" to "average-sublevel"; or, still more generally (23). Theorem 4 also presents a Lipschitz continuity result.

Concerning necessary conditions, we describe our previous results of bimonotonicity of minimizers in Definition 3; and prove in Theorem 3 a much more informative version of our own generalized DuBois-Reymond differential inclusion, as compared with the one appearing in our paper [24]. Namely: if a 0-relaxed minimizer $y_0(\cdot)$ does not satisfy the classical DuBois-Reymond necessary condition (18) then $y_0(\cdot)$ cannot stop (i.e. $a' = b'$ in (13)) hence it must be a true minimizer; and, moreover, its derivative must be either a.e. maximal or else a.e. minimal or, more precisely (20).

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Conflict of interest

The authors declare no competing financial interest.

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