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Research Article

Analysis of Nonlinear Coupled Jaulent-Miodek and Whitham-Broer-Kaup Equations Within Fractional Derivative

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Abstract: This paper investigates the nonlinear coupled Jaulent-Miodek (JM) and Whitham-Broer-Kaup (WBK) equations through the lens of fractional calculus, employing the Mohand Variational Iteration Method (MVIM) and q-Homotopy Mohand Transform Method (q-HMTM). These equations, pivotal in describing nonlinear wave phenomena and fluid dynamics, are studied in their fractional-order forms using the Caputo operator to extend traditional models. The proposed methods efficiently yield analytical and approximate solutions, showcasing their reliability and accuracy. The solutions derived are presented through numerical simulations and graphical depictions, revealing the intricate dependence of system behavior on fractional-order parameters. This sensitivity provides a mechanism for tuning physical phenomena modeled by JM and WBK equations, offering valuable insights into wave propagation, fluid dynamics, and other nonlinear coupled systems. The study establishes the efficacy of q-HMTM and MVIM in handling complex fractional-order systems, underlining their potential for broader applications in science and engineering. By bridging classical and fractional models, this work contributes to the ongoing development of advanced mathematical tools for analyzing nonlinear phenomena.

Keywords: nonlinear coupled Jaulent-Miodek (JM) equations, nonlinear coupled Whitham-Broer-Kaup (WBK) equations, caputo operator, fractional order differential equation, Mohand Variational Iteration Method (MVIM), q-Homotopy Mohand Transform Method (q-HMTM)

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1. Introduction

A simple and useful strategy for exactly solving a broad spectrum of problems is Fractional Calculus (FC). The primary significant Fractional Differential Equations (FDEs) generated by this dynamic area of mathematics extend the integer order to its fractional order and provide a wide class of mathematical contexts [1–3]. With significant consequences in many fields of science and engineering, recent years have provided a great research of various physical phenomena represented by fractional differential equations [4–7]. People include Liouville-Caputo, Hadamard, Atangana-Baleanu, Riemann-Liouville, Caputo-Fabrizio operators [8–11] contributed some fundamental concepts of fractional derivatives. The most widely used fractional derivative are the Riemann-Liouville Derivative (RLD) and Caputo Derivatives (CD). The RLD restricts itself to the Caputo Fractional Derivative (CFD), while the latter permits more intricate starting and

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boundary conditions. The CFD is used in mathematical modelling to provide insightful information about fractional-order dynamical systems, has been explained in a number of literature. Subjects of continuous interest and academic research across many fields, including bio mathematics and the study of memory events, the exploration and physical use of the derivative of fractional order remain fascinating [12]. Among the numerous commercial and scientific domains that have profited from non-linear models are astrophysics, hydrology, nuclear engineering, meteorology, and astrobiology [13, 14].

The Jaulent-Miodek (JM) equation is an evolution equation among many fields of physics including the field of optics fluid mechanics, and enhanced matter physics. Describing the energy-dependent Schrodinger ability [15] is usual practice using the JM equation. See [16–18] for the solutions to physical models in several fields of technology and science using these equation systems. The coupled fractional-order nonlinear Jaulent-Miodek equations are exclusively used in two scientific and technological domains: condensed matter physics [19, 20] and plasma physics [21]. In this study, we will investigate the nonlinear fractional JM equation of the form:

$$D_\omega^\delta \zeta_1(\varphi,\ \omega)+\frac{\partial^3\zeta_1(\varphi,\ \omega)}{\partial\varphi^3}+\frac{3}{2}\zeta_2(\varphi,\ \omega)\frac{\partial^3\zeta_2(\varphi,\ \omega)}{\partial\varphi^3}+\frac{9}{2}\frac{\partial\zeta_2(\varphi,\ \omega)}{\partial\varphi}\frac{\partial^2\zeta_2(\varphi,\ \omega)}{\partial\varphi^2}-6\zeta_1(\varphi,\ \omega)\frac{\partial\zeta_1(\varphi,\ \omega)}{\partial\varphi}$$

$$-\frac{3}{2}{\zeta_2}^2(\varphi,\ \omega)\frac{\partial\zeta_1(\varphi,\ \omega)}{\partial\varphi}-6\zeta_1(\varphi,\ \omega)\zeta_2(\varphi,\ \omega)\frac{\partial\zeta_2(\varphi,\ \omega)}{\partial\varphi}=0, \tag{1}$$

$$D_\omega^\delta \zeta_2(\varphi,\ \omega)+\frac{\partial^3\zeta_2(\varphi,\ \omega)}{\partial\varphi^3}-6\zeta_2(\varphi,\ \omega)\frac{\partial\zeta_1(\varphi,\ \omega)}{\partial\varphi}-6\zeta_1(\varphi,\ \omega)\frac{\partial\zeta_2(\varphi,\ \omega)}{\partial\varphi}-\frac{15}{2}{\zeta_2}^2(\varphi,\ \omega)\frac{\partial\zeta_2(\varphi,\ \omega)}{\partial\varphi}=0,$$

where $0<\delta\leq 1$.

First-order partial differential equations, both linear and nonlinear, may have their approximate solutions found using the following techniques: the fractional differential transformed multi-step method [22], the fractional natural decomposition method [23], the homotopy perturbation method [24–28], the variational iteration method [29], and the homotopy analysis method [30–35].

Shallow water propagation is described using several well-known integral models including the Boussinesq equation, KdV equation, WBK equation and K-P equation. Employing the Boussinesq approximation, Whitham, Broer, and Kaup constructed nonlinear WBK equations [36–38]:

$$D_\omega^\delta \zeta_1(\varphi,\ \omega)+\frac{\partial^2\zeta_1(\varphi,\ \omega)}{\partial\varphi^2}+\zeta_1(\varphi,\ \omega)\frac{\partial\zeta_1(\varphi,\ \omega)}{\partial\varphi}+\frac{\partial\zeta_2(\varphi,\ \omega)}{\partial\varphi}=0,$$

$$D_\omega^\delta \zeta_2(\varphi,\ \omega)+3\frac{\partial^3\zeta_1(\varphi,\ \omega)}{\partial\varphi^3}-\frac{\partial^2\zeta_2(\varphi,\ \omega)}{\partial\varphi^2}+\zeta_1(\varphi,\ \omega)\frac{\partial\zeta_2(\varphi,\ \omega)}{\partial\varphi}+\zeta_2(\varphi,\ \omega)\frac{\partial\zeta_1(\varphi,\ \omega)}{\partial\varphi}=0, \tag{2}$$

where $0<\delta\leq 1$.

where the fluids' heights and horizontal velocities, which deviate greatly from equilibrium, are represented by the values $\zeta_1(\varphi,\ \omega)$ and $\zeta_2(\varphi,\omega)$. Wang and Zheng [39] presented approximative solutions for the coupled fractional order equations of (WBK) by using an extended fractional Riccati subequation approach. El-Borai et al. [40] used the exponential function approach to solve coupled systems. Using the reduced differential transformation method, the author [41] obtained analytically approximate solutions to the model. Numerical solutions are also found using methods like the Variation Iteration Method (VIM), the coupled system 2, the Finite Difference Approach (FDA) [42], the Exponential-Function Method (EFM) [43], and others [44–46].

Recent developments related to fractional calculus have greatly enhanced the modeling ability for systems with nonlocality and memory effects in physics, engineering and finance. For instance, Luo et al. [47] has developed a stable second-order ADI Galerkin method for 3D nonlocal heat conduction modelling based on viscoelasticity capturing the fractional derivative effects in memory-dependent heat transfer problems. In the same vein, Qiu et al. [48] generalized tempered-type integro-differential equations and it was shown that the fractional equation was necessary to describe processes with non-uniform temporal scaling. Applying this concept in the field of financial mathematics, Nikan et al. [49] proposed a fractional Black-Scholes model of time derivative in fractional way for simulating market memory and volatility clustering that is required in order to price American and European options more accurately.

In the field of wave propagation and nonlinear dynamics, Liang et al. [50] reported detailed solutions that included kink-soliton and breather for a fractional Boiti-Leon-Manna-Pempinelli equation, showing the profoundness of wave interactions described by fractional operators in spaces of higher dimensions. Similarly, in [51], a variational principle with fractal derivative of the form for the nonlinear Schrodinger equation was formulated, emphasizing the importance of fractional structures in quantum mechanics wave functions. In addition, Wang et al. [52] developed an effective numerical approach in the analysis of circuits with local fractional derivatives that will be helpful for both understanding the fractional modeling and analyzing signal processing and circuit theory on a fractional basis. As a whole, these works demonstrate that fractional derivatives can be an effective tool to model real-world problems particularly those with nonlinearity and/or coupling. This supports our application of both the Caputo derivatives, and the Mohand Transform semi-analytical technique in solving nonlinear coupled fractional systems such as Jaulent-Miodek and Whitham-Broer-Kaup equations which are commonly found in fluid dynamics, nonlinear optics, and wave mechanics.

The Mohand transform is a new transform that we will utilize in this work to solve the fractional-order nonlinear coupled Jaulent-Miodek (JM) equations and the nonlinear coupled Whitham-Broer-Kaup (WBK) equations utilizing the q-Homotopy technique and variational iteration approach. He [53–59] developed the variational iteration technique, which has been useful in nonlinear poly crystalline solids [60–62], self-governing ordinary differential equations [58–71], and other fields.

Liao [72] proposed the Homotopy Analysis Method (HAM). From an initial assumption to an exact response, an endless mapping is generated after selection of an auxiliary linear operator. The auxiliary parameter confirm this convergence of the solution. Actually, when $n \geq 1$, and $q \in [0,\ 1/n]$, the q-HAM is an increase over $q \in [0,\ 1]$ in HAM. Including $(1/n)^m$ converges the answer more rapidly than the traditional HAM [73–76]. Semi-analytical methods with a suitable transform help to reduce the time required to search for solutions for nonlinear problems reflecting useful applications. The q-Homotopy Mohand Transform Technique (q-HMTM) combines the Mohand and q-HAM transforms. Its ability to adapt two powerful computational techniques for FDE problem makes it noteworthy. We can regulate the convergence area of solution series over a wide permitted domain by selecting an appropriate $\hbar$.

The originality of this work is in the sense that q-HMTM and MVIM are implemented for solution range-wise comparative studies to time-fractional nonlinear coupled fractional WBK and JM equations under Caputo framework. Traditional methods, which are based on classical integer-order models, are often applied separately to single equations without utilising the fractional dynamics approach. It is novel in three ways: (1) it is computationally efficient to derive high-accuracy approximate solutions; (2) it explains how fractional orders influence the emergence and properties of wave phenomena; and (3) it provides comparative example studies using the most advanced analytical results for coupled nonlinear systems. These developments not only enhance the analysis and evaluation of fractional-order models in theories but also further expand the field of application for techniques based on the Caputo-type derivatives in nonlinear wave dynamics.

The paper is organized as follows: Section 1 indicates the importance of fractional calculus and the leading reasons for considering the nonlinear JM and WBK equations. In section 2 we start by defining the basic definitions and operational rules of the Mohand Transform. Section 3 describes the proposed methods and sections 4 and 5, respectively, demonstrate both these methods to the time-fractional WBK equation and the time-fractional JM equation with their detailed derivations, numerical evidence which are validated in section 6 with outcomes. Finally, in section 7 we conclude the paper describing a summary of our findings and how accurate and efficient are the proposed algorithms.

2. Concepts of Mohand transform

We will first cover generally the Mohand Transform (MT) and related ideas in this part.

Definition 1 Suppose the function $\zeta(\omega)$ for which [77] specifies the MT as:

$$M[\zeta(\omega)] = R(s) = s^2 \int_0^{\omega} \zeta(\omega) e^{-s\omega} d\omega, \ k_1 \leq s \leq k_2.$$

The inverse Mohand Transformation (MIT) is represented as:

$$M^{-1}[R(s)] = \zeta(\omega),$$

Definition 2 [78]: The following is the definition of the MT fractional order derivative:

$$M[\zeta^{\delta}(\omega)] = s^{\delta} R(s) - \sum_{k=0}^{n-1} \frac{\zeta^{k}(0)}{s^{k-(\delta+1)}}, \ \ 0 < \delta \leq n$$

Definition 3 A few of the MT characteristics are as follows:

1. $M[\zeta'(\omega)] = sR(s) - s^2 R(0)$,
2. $M[\zeta''(\omega)] = s^2 R(s) - s^3 R(0) - s^2 R'(0)$,
3. $M[\zeta^{n}(\omega)] = s^{n} R(s) - s^{n+1} R(0) - s^{n} R'(0) - \cdots - s^{n} R^{n-1}(0)$.

Lemma 1 Suppose we have an exponentially ordered function $\zeta(\varphi, \ \omega)$. Then, the MT is describe as:

$$M[D_{\omega}^{r\delta} \zeta(\varphi, \ \omega)] = s^{r\delta} R(s) - \sum_{j=0}^{r-1} s^{\delta(r-j)-1} D_{\omega}^{j\delta} \zeta(\varphi, \ 0), \ 0 < \delta \leq 1, \tag{3}$$

where $\varphi = (\varphi_1, \ \varphi_2, \ \cdots, \ \varphi_{\delta}) \in \mathbb{R}^{\delta}$ and $\delta \in \mathbb{N}$ and $D_{\omega}^{r\delta} = D_{\omega}^{\delta}.D_{\omega}^{\delta}.\cdots.D_{\omega}^{\delta}(r - \text{times})$.

Proof. We will use the induction approach to show that Equation (3) is true. The consequent result is achieved by solving equation (3) using $r = 1$.

$$M[D_{\omega}^{2\delta} \zeta(\varphi, \ \omega)] = s^{2\delta} R(s) - s^{2\delta-1} \zeta(\varphi, \ 0) - s^{\delta-1} D_{\omega}^{\delta} \zeta(\varphi, \ 0).$$

For $r = 1$, Equation (3) is shown to be valid by definition 2. Equation (3) is modified by substituting $r = 2$ to get:

$$M[D_{r}^{2\delta} \zeta(\varphi, \omega)] = s^{2\delta} R(s) - s^{2\delta-1} \zeta(\varphi, \ 0) - s^{\delta-1} D_{\omega}^{\delta} \zeta(\varphi, \ 0). \tag{4}$$

Let's assume the Equation (4)'s left-hand side.

$$L.H.S = M[D_{\omega}^{2\delta} \zeta(\varphi, \ \omega)]. \tag{5}$$

The expression of Equation (5) may also be expressed in the following manner:

$$L.H.S = M[D_{\omega}^{\delta} D_{\omega}^{\delta} \zeta(\varphi,\ \omega)]. \tag{6}$$

Assume

$$z(\varphi,\ \omega) = D_{\omega}^{\delta} \zeta(\varphi,\ \omega). \tag{7}$$

Substitute Equation (7) into Equation (6), then solve to obtain the following.

$$L.H.S = M[D_{\omega}^{\delta} z(\varphi,\ \omega)]. \tag{8}$$

The CFD causes the subsequent variations to Equation (8):

$$L.H.S = M[J^{1-\delta} z'(\varphi,\ \omega)]. \tag{9}$$

The RL integral results in the subsequent modifications in Equation (9):

$$L.H.S = \frac{M[z'(\varphi, \omega)]}{s^{1-\delta}}. \tag{10}$$

These results are obtained by means of the MT derivative feature on Equation (10).

$$L.H.S = s^{\delta} Z(\varphi, s) - \frac{z(\varphi, 0)}{s^{1-\delta}}, \tag{11}$$

Equation (7) was employed to achieve this outcome.

$$Z(\varphi,\ s) = s^{\delta} R(s) - \frac{\zeta(\varphi,\ 0)}{s^{1-\delta}},$$

As, $M[z(\omega,\ \varphi)] = Z(\varphi,\ s).$ As a result, Equation (11) can also be expressed as follows:

$$L.H.S = s^{2\delta} R(s) - \frac{\zeta(\varphi, 0)}{s^{1-2\delta}} - \frac{D_{\omega}^{\delta} \zeta(\varphi, 0)}{s^{1-\delta}}, \tag{12}$$

Supposing that Equation (3) is valid for $r = K.$ Substitute $r = K$ into Equation (3).

$$M\left[D_{\omega}^{K\delta}\zeta(\varphi,\ \omega)\right]=s^{K\delta}R(s)-\sum_{j=0}^{K-1}s^{\delta(K-j)-1}D_{\omega}^{j\delta}D_{\omega}^{j\delta}\zeta(\varphi,\ 0),\ 0<\delta\leq 1. \tag{13}$$

Proving that Equation (3) is valid for $r=K+1$ is the next step. Now, solve Equation (3) by substituting $r=K+1$.

$$M\left[D_{\omega}^{(K+1)\delta}\zeta(\varphi,\ \omega)\right]=s^{(K+1)\delta}R(s)-\sum_{j=0}^{K}s^{\delta((K+1)-j)-1}D_{\omega}^{j\delta}\zeta(\varphi,\ 0). \tag{14}$$

The result can be derived using Equation (14).

$$L.H.S=M[D_{\omega}^{K\delta}(D_{\omega}^{K\delta})]. \tag{15}$$

Let

$$D_{\omega}^{K\delta}=g(\varphi,\ \omega).$$

Equation (15) becomes:

$$L.H.S=M[D_{\omega}^{\delta}g(\varphi,\ \omega)]. \tag{16}$$

Equation (16) is modified by employing both the RL integral and CFD.

$$L.H.S=s^{\delta}M[D_{\omega}^{K\delta}\zeta(\varphi,\ \omega)]-\frac{g(\varphi,\ 0)}{s^{1-\delta}}. \tag{17}$$

With the help of Equation (13), we may express Equation (17) as:

$$L.H.S=s^{r\delta}R(s)-\sum_{j=0}^{r-1}s^{\delta(r-j)-1}D_{\omega}^{j\delta}\zeta(\varphi,\ 0), \tag{18}$$

Another way of expressing Equation (18) is as follows:

$$L.H.S=M[D_{\omega}^{r\delta}\zeta(\varphi,\ 0)].$$

Mathematical induction demonstrates that Equation (3) is valid for $r=K+1$. It follows that Equation (3) is valid for any positive integer. □

Lemma 2 Let us assume that $\zeta(\varphi,\ \omega)$ has an exponential order. $M[\zeta(\varphi,\ \omega)]=R(s)$ denotes the MT of $\zeta(\varphi,\ \omega)$. Regarding the MT, the Fractional Multiple Power Series (FMPS) is expressed as follows:

$$R(s) = \sum_{r=0}^{\infty} \frac{\hbar_r(\varphi)}{s^{r\delta+1}},\ s > 0, \tag{19}$$

where, $\varphi = (s_1,\ \varphi_2,\ \cdots,\ \varphi_\delta) \in \mathbb{R}^\delta,\ \delta \in \mathbb{N}$.

Proof. To begin the proof, consider the following Taylor series form:

$$\zeta(\varphi,\ \omega) = \hbar_0(\varphi) + \hbar_1(\varphi)\frac{\omega^\delta}{\Gamma[\delta+1]} + \hbar_2(\varphi)\frac{\omega^{2\delta}}{\Gamma[2\delta+1]} + \cdots. \tag{20}$$

Applying MT on Equation (20), we get

$$M\left[\zeta(\varphi,\ \omega)\right] = M\left[\hbar_0(\varphi)\right] + M\left[\hbar_1(\varphi)\frac{\omega^\delta}{\Gamma[\delta+1]}\right] + M\left[\hbar_1(\varphi)\frac{\omega^{2\delta}}{\Gamma[2\delta+1]}\right] + \cdots$$

Utilize MT's characteristics to get the desired outcome.

$$M\left[\zeta(\varphi,\ \omega)\right] = \hbar_0(\varphi)\frac{1}{s} + \hbar_1(\varphi)\frac{\Gamma[\delta+1]}{\Gamma[\delta+1]}\frac{1}{s^{\delta+1}} + \hbar_2(\varphi)\frac{\Gamma[2\delta+1]}{\Gamma[2\delta+1]}\frac{1}{s^{2\delta+1}}\cdots$$

This results in an improved Taylor series. □

Lemma 3 Using $M[\zeta(\varphi,\ \omega)] = R(s)$ as the denotation for MT, the modified Taylor series in FMPS takes the form:

$$\hbar_0(\varphi) = \lim_{s\to\infty} sR(s) = \zeta(\varphi,\ 0). \tag{21}$$

Proof. To begin the proof, consider the following Taylor series form:

$$\hbar_0(\varphi) = sR(s) - \frac{\hbar_1(\varphi)}{s^\delta} - \frac{\hbar_2(\varphi)}{s^{2\delta}} - \cdots \tag{22}$$

Equation (22) is the result of computing and simplifying the limit found in Equation (21). □

Theorem 1 Consider $M[\zeta(\varphi,\ \omega)]$ represent a function. Then, $R(s)$ in FMPS form is represented as:

$$R(s) = \sum_{r=0}^{\infty} \frac{\hbar_r(\varphi)}{s^{r\delta+1}},\ s > 0,$$

where $\varphi = (\varphi_1,\ \varphi_2,\ \cdots,\ \varphi_\delta) \in \mathbb{R}^\delta$ and $\delta \in \mathbb{N}$. Then we have

$$\hbar_r(\varphi) = D_r^{r\delta}\zeta(\varphi,\ 0),$$

where, $D_{\omega}^{r\delta} = D_{\omega}^{\delta}.D_{\omega}^{\delta}.\cdots.D_{\omega}^{\delta}(r-\text{times})$.

Proof. To begin the proof, consider the following Taylor series form:

$$\hbar_1(\varphi) = s^{\delta+1}R(s) - s^{\delta}\hbar_0(\varphi) - \frac{\hbar_2(\varphi)}{s^{\delta}} - \frac{\hbar_3(\varphi)}{s^{2\delta}} - \cdots \tag{23}$$

The following outcome is obtained through the use of limit to Equation (23):

$$\hbar_1(\varphi) = \lim_{s\to\infty}(s^{\delta+1}R(s) - s^{\delta}\hbar_0(\varphi)) - \lim_{s\to\infty}\frac{\hbar_2(\varphi)}{s^{\delta}} - \lim_{s\to\infty}\frac{\hbar_3(\varphi)}{s^{2\delta}} - \cdots$$

Further simplifying the above equation.

$$\hbar_1(\varphi) = \lim_{s\to\infty}(s^{\delta+1}R(s) - s^{\delta}\hbar_0(\varphi)). \tag{24}$$

We formulate the following version of Equation (24) using the fundamental concepts given in Lemma 1:

$$\hbar_1(\varphi) = \lim_{s\to\infty}(sM[D_{\omega}^{\delta}\zeta(\varphi,\ \omega)](s)). \tag{25}$$

Lemma 2 serves as the basis for the derivation of Equation (25):

$$\hbar_1(\varphi) = D_{\omega}^{\delta}\zeta(\varphi,\ 0).$$

To obtain the next result, you must use the Taylor series and take $\lim_{s\to\infty}$ once more.

$$\hbar_2(\varphi) = s^{2\delta+1}R(s) - s^{2\delta}\hbar_0(\varphi) - s^{\delta}\hbar_1(\varphi) - \frac{\hbar_3(\varphi)}{s^{\delta}} - \cdots$$

The subsequent results are drawn from Lemma 2:

$$\hbar_2(\varphi) = \lim_{s\to\infty} s(s^{2\delta}R(s) - s^{2\delta-1}\hbar_0(\varphi) - s^{\delta-1}\hbar_1(\varphi)). \tag{26}$$

The following modifications to Equation (26) are made based on Lemmas 1 and 3:

$$\hbar_2(\varphi) = D_{\omega}^{2\delta}\zeta(\varphi,\ 0).$$

Using the same procedure, we obtain:

$$\hbar_3(\varphi) = \lim_{s\to\infty} s(M[D_\omega^{2\delta}\zeta(\varphi,\ \delta)](s)).$$

Lemma 3 is used to arrive at this ultimate result.

$$\hbar_3(\varphi) = D_\omega^{3\delta}\zeta(\varphi,\ 0).$$

Generally

$$\hbar_r(\varphi) = D_\omega^{r\delta}\zeta(\varphi,\ 0).$$

Proved. □

The improved Taylor's series convergence is defined and explained by the following theorem.

Theorem 2 The MFTS expression is characterized using Lemma 2 and is shown as follows: $M[\zeta(\omega,\ \varphi)] = R(s)$. When $|s^a M[D_\omega^{(K+1)\delta}\zeta(\varphi,\ \omega)]| \le T$, $0 < \delta \le 1\ \forall\, s > 0$, the residual $H_K(\varphi,\ s)$ of the new MFTS and validate the inequality:

$$|H_K(\varphi,\ s)| \le \frac{T}{s^{(K+1)\delta+1}},\ s > 0.$$

Proof. Let $M[D_\omega^{r\delta}\zeta(\varphi,\ \omega)](s)$ is defined on $s > 0$ for $r = 0,\ 1,\ 2,\ \cdots,\ K+1$ and assume $|sM[D_{\omega^{K+1}}\zeta(\varphi,\ \omega)]| \le T$. The improved Taylor series may be used to find the resultant relationship.

$$H_K(\varphi,\ s) = R(s) - \sum_{r=0}^{K} \frac{\hbar_r(\varphi)}{s^{r\delta+1}}. \tag{27}$$

Implementing the theorem 1 and Equation (27), we might obtain the following result:

$$H_K(\varphi, s) = R(s) - \sum_{r=0}^{K} \frac{D_\omega^{r\delta}\zeta(\varphi,\ 0)}{s^{r\delta+1}}. \tag{28}$$

Using Equation (28), and multiplying $s^{(K+1)a+1}$:

$$s^{(K+1)\delta+1}H_K(\varphi,\ s) = s(s^{(K+1)\delta}R(s) - \sum_{r=0}^{K} s^{(K+1-r)\delta-1}D_\omega^{r\delta}\zeta(\varphi,\ 0)). \tag{29}$$

Equation (29) takes the particular form when Lemma 1 is applied:

$$s^{(K+1)\delta+1}H_K(\varphi,\ s) = sM\left[D_\omega^{(K+1)\delta}\zeta(\varphi,\ \omega)\right]. \tag{30}$$

Take the absolute:

$$|s^{(K+1)\delta+1}H_K(\varphi,\ s)| = \left|sM\left[D_{\omega}^{(K+1)\delta}\zeta(\varphi,\ \omega)\right]\right|. \tag{31}$$

To obtain the intended outcome, the condition of Equation (31) must be utilized.

$$\frac{-T}{s^{(K+1)\delta+1}} \leq H_K(\varphi,\ s) \leq \frac{T}{s^{(K+1)\delta+1}}. \tag{32}$$

Another way to express Equation (32) result is as follows:

$$|H_K(\varphi,\ s)| \leq \frac{T}{s^{(K+1)\delta+1}}.$$

Hence, a unique condition for the series' convergence is obtained. □

3. Methodologies

3.1 *q-HMTM methodology*

Examine the subsequent nonlinear, non-homogeneous fractional order PDE:

$$D_{\omega}^{\delta}\zeta(\varphi,\ \omega) + \mathscr{R}\zeta(\varphi,\ \omega) + \mathscr{N}\zeta(\varphi,\ \omega) = \mathbb{H}(\varphi,\ \omega),\ n-1 < \delta \leq n. \tag{33}$$

The CFD is denoted by $D_{\mathscr{R}}^{\delta}\zeta(\varphi,\ \omega)$, whereas the source is $\mathbb{H}(\varphi,\ \omega)$. The linear and nonlinear operators are represented by $\mathscr{R}$ and $\mathscr{N}$, respectively.

To Equation (33), apply the Mohand transform.

$$\mathscr{M}[\zeta(\varphi,\ \omega)] - \frac{1}{s^{\delta}}\sum_{k=0}^{n-1} s^{\delta-k-1}\zeta^{k}(\varphi,\ 0) + \frac{1}{s^{\delta}}[\mathscr{M}[\mathscr{R}\zeta(\varphi,\ \omega)] + \mathscr{M}[\mathscr{N}\zeta(\varphi,\ \omega)] - \mathscr{M}[\mathbb{H}(\varphi,\ \omega)]] = 0, \tag{34}$$

The non-linear operator has the following definition:

$$\begin{aligned} N[\psi(\varphi,\ \omega;\ q)] = \mathscr{M}[\psi(\varphi,\ \omega;\ q)] - \frac{1}{s^{\delta}}\sum_{k=0}^{n-1} s^{\delta-k-1}\psi^{k}(\varphi,\ \omega;\ q)(0^{+}) + \frac{1}{s^{\delta}}[\mathscr{M}[\mathscr{R}\psi(\varphi,\ \omega;\ q)] \\ + \mathscr{M}[\mathscr{N}\psi(\varphi,\ \omega;\ q)] - \mathscr{M}[\mathbb{H}(\varphi,\ \omega)]], \end{aligned} \tag{35}$$

The real-valued function in this case is $\psi(\varphi,\ \omega;\ q))$ with respect to φ, ω, and $q \in \left[0,\ \frac{1}{n}\right]$. We can define a homotopy as follows:

$$(1-nq)\mathscr{M}[\psi(\varphi,\ \omega;\ q)-\zeta_0(\varphi,\ \omega)]=\hbar q\mathfrak{h}(\varphi,\ \omega)\mathscr{N}[\psi(\varphi,\ \omega;\ q)], \tag{36}$$

Within the equation that has been shown above, the starting condition is represented by ζ_0, while the auxiliary parameter is represented by $\hbar = 0$.

The result that follows is valid for both 0 and $\frac{1}{n}$.

$$\psi(\varphi,\ \omega;\ 0)=\zeta_0(\varphi,\ \omega),\ \psi\left(\varphi,\ \omega;\ \frac{1}{n}\right)=\zeta(\varphi,\ \omega), \tag{37}$$

The solution $\psi(\varphi,\ \omega;\ q)$ differs from the first estimate $\zeta_0(\varphi,\ \omega)$ to $\zeta(\varphi,\ \omega)$ due of the intensification of q. We may deduce the following by using the Taylor theorem on $\psi(\varphi,\ \omega;\ q)$ with respect to q:

$$\psi(\varphi,\ \omega;\ q)=\zeta_0(\varphi,\ \omega)+\sum_{m=1}^{\infty}\zeta_m(\varphi,\ \omega)q^m, \tag{38}$$

where

$$\zeta_m=\frac{1}{m!}\frac{\partial^m\psi(\varphi,\ \omega;\ q)}{\partial q^m}|_{q=0}, \tag{39}$$

For the suitable values of $\zeta_0(\varphi,\ \omega)$, n, and $\hbar$, the series 36 converges at $q=\frac{1}{n}$. Thus,

$$\psi(\varphi,\ \omega;\ q)=\zeta_0(\varphi,\ \omega)+\sum_{m=1}^{\infty}\zeta_m(\varphi,\ \omega)\left(\frac{1}{n}\right)^m, \tag{40}$$

With respect to the embedding parameter q, the derivative of Equation (36) may be obtained by keeping $q = 0$, dividing by $m!$, and computing the derivative.

$$\mathscr{M}[\zeta_m(\varphi,\ \omega)-k_m\zeta_{m-1}(\varphi,\ \omega)]=\hbar\mathfrak{h}(\varphi,\ \omega)\mathscr{R}_m\left(\overrightarrow{\zeta}_{m-1}\right), \tag{41}$$

The auxiliary parameter $\hbar \neq 0$ and the vectors are specified as follows:

$$\overrightarrow{\zeta}_m=[\zeta_0(\varphi,\ \omega),\ \zeta_1(\varphi,\ \omega),\ \cdots,\ \zeta_m(\varphi,\ \omega)], \tag{42}$$

When Eq (41) is subjected to the inverse Mohand transform, the following is the result:

$$\zeta_m(\varphi,\ \omega)=k_m\zeta_{m-1}(\varphi,\ \omega)+\hbar\mathscr{M}^{-1}\left[\mathfrak{h}(\varphi,\ \omega)\mathscr{R}_m\left(\overrightarrow{\zeta}_{m-1}\right)\right], \tag{43}$$

$$\mathscr{R}_m(\overrightarrow{\mathbb{K}}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathscr{N}[\psi(\varphi,\ \omega;\ q)]}{\partial q^{m-1}}\ |_{q=0},$$

$$k_m = \begin{cases} 0 & if\ \ m \leq 1, \\ 1 & if\ \ m > 1, \end{cases} \tag{44}$$

Equation (43) must be solved in order to identify the components of the q-HMTM solution.

3.2 *MVIM methodology*

Examine the subsequent nonlinear, nonhomogeneous fractional order PDE:

$$D_\omega^\delta \zeta(\varphi,\ \omega) = \mathscr{R}\zeta(\varphi,\ \omega) + \mathscr{N}\zeta(\varphi,\ \omega) + \mathbb{H}(\varphi,\ \omega),\ n-1 < \delta \leq n. \tag{45}$$

Initial condition

$$\zeta(\varphi,\ 0) = \zeta_0(\varphi). \tag{46}$$

As a result of implementing the Mohand transform to Equation (45), the subsequent result is produced.

$$\mathscr{M}[D_\omega^\delta \zeta(\varphi,\ \omega)] = \mathscr{M}[\mathscr{R}\zeta(\varphi,\ \omega) + \mathscr{N}\zeta(\varphi,\ \omega) + \mathbb{H}(\varphi,\ \omega)], \tag{47}$$

Through the use of the iteration feature of transform, we are able to deduce the following outcome:

$$\mathscr{M}[\zeta(\varphi,\ \omega)] - \sum_{k=0}^{m-1} s^{\delta-k-1}\frac{\partial^k \zeta(\varphi,\ \omega)}{\partial \omega^k}|_{\omega=0} = \mathscr{M}[\mathscr{R}\zeta(\varphi,\ \omega) + \mathscr{N}\zeta(\varphi,\ \omega) + \mathbb{H}(\varphi,\ \omega)], \tag{48}$$

Through the utilization of an iterative technique that incorporates the Lagrange multiplier $(-\lambda(s))$,

$$\mathscr{M}[\zeta_{n+1}(\varphi,\ \omega)] = \mathscr{M}[\zeta_n(\varphi,\ \omega)] - \lambda(s)\left[\mathscr{M}[\zeta_n(\varphi,\ \omega)] - \sum_{k=0}^{m-1} s^{\delta-k-1}\frac{\partial^k \zeta(\varphi,\ 0)}{\partial \omega^k}\right], \tag{49}$$

Where $\lambda(s) = -\frac{1}{s^\delta}$ and Equation (49) are inserted into Equation (48), resulting in the following expression:

$$\begin{aligned} \mathscr{M}[\zeta_{n+1}(\varphi,\ \omega)] = \mathscr{M}[\zeta_n(\varphi,\ \omega)] - \lambda(s)\Big[\mathscr{M}[\zeta_n(\varphi,\ \omega)] - \sum_{k=0}^{m-1} s^{\delta-k-1}\frac{\partial^k \zeta(\varphi,\ 0)}{\partial \omega^k} \\ + \mathscr{M}[\mathscr{R}\zeta(\varphi,\ \omega) + \mathscr{N}\zeta(\varphi,\ \omega) + \mathbb{H}(\varphi,\ \omega)]\Big], \end{aligned} \tag{50}$$

We are able to deduce the following by applying the Mohand inverse transform to Equation (50):

$$\zeta_{n+1}(\varphi,\ \omega) = \zeta_n(\varphi,\ \omega) + \mathscr{M}^{-1}\left[\frac{1}{s^{\delta}}\sum_{k=0}^{m-1} s^{\delta-k-1}\frac{\partial^k \zeta(\varphi,\ 0)}{\partial \omega^k} + \mathscr{M}[\mathscr{R}\zeta(\varphi,\ \omega) + \mathscr{N}\zeta(\varphi,\ \omega) + \mathbb{H}(\varphi,\ \omega)]\right], \tag{51}$$

The initial condition is given as:

$$\zeta_0(\varphi,\ \omega) = \mathscr{M}^{-1}\left[\frac{1}{s^{\delta}}\sum_{k=0}^{m-1} s^{\delta-k-1}\frac{\partial^k \zeta(\varphi,\ 0)}{\partial \omega^k}\right], \tag{52}$$

The iterative scheme is given as:

$$\zeta_{n+1}(\varphi,\ \omega) = \zeta_n(\varphi,\ \omega) + \mathscr{M}^{-1}\left[\frac{1}{s^{\delta}}\sum_{k=0}^{m-1} s^{\delta-k-1}\frac{\partial^k \zeta(\varphi,\ 0)}{\partial \omega^k} + \mathscr{M}[\mathscr{R}\zeta(\varphi,\ \omega) + \mathscr{N}\zeta(\varphi,\ \omega) + \mathbb{H}(\varphi,\ \omega)]\right], \tag{53}$$

4. Example 1

4.1 *Implementation of q-HMTM*

Examining the time-fractional nonlinear coupled WBK equations:

$$\begin{aligned} &D_{\omega}^{\delta}\zeta_1(\varphi,\ \omega) + \frac{\partial^2 \zeta_1(\varphi,\ \omega)}{\partial \varphi^2} + \zeta_1(\varphi,\ \omega)\frac{\partial \zeta_1(\varphi,\ \omega)}{\partial \varphi} + \frac{\partial \zeta_2(\varphi,\ \omega)}{\partial \varphi} = 0,\\ &D_{\omega}^{\delta}\zeta_2(\varphi,\ \omega) + 3\frac{\partial^3 \zeta_1(\varphi,\ \omega)}{\partial \varphi^3} - \frac{\partial^2 \zeta_2(\varphi,\ \omega)}{\partial \varphi^2} + \zeta_1(\varphi,\ \omega)\frac{\partial \zeta_2(\varphi,\ \omega)}{\partial \varphi} + \zeta_2(\varphi,\ \omega)\frac{\partial \zeta_1(\varphi,\ \omega)}{\partial \varphi} = 0, \end{aligned} \tag{54}$$

where $0 < \delta \leq 1$.

Initial conditions:

$$\begin{aligned} &\zeta_1(\varphi,\ 0) = \frac{1}{2}(1 - 16\tanh(-2\varphi)),\\ &\zeta_2(\varphi,\ 0) = 16\left(1 - \tanh^2(-2\varphi)\right), \end{aligned} \tag{55}$$

$$\begin{aligned} &\mathscr{M}[\zeta_1(\varphi,\ \omega)] + s\left(\frac{1}{2}(1 - 16\tanh(-2\varphi))\right) + \frac{1}{s^{\delta}}\mathscr{M}\left[\frac{\partial^2 \zeta_1(\varphi,\ \omega)}{\partial \varphi^2} + \zeta_1(\varphi,\ \omega)\frac{\partial \zeta_1(\varphi,\ \omega)}{\partial \varphi} + \frac{\partial \zeta_2(\varphi,\ \omega)}{\partial \varphi}\right] = 0,\\ &\mathscr{M}[\zeta_2(\varphi,\ \omega)] + s\left(16\left(1 - \tanh^2(-2\varphi)\right)\right) + \frac{1}{s^{\delta}}\mathscr{M}\left[3\frac{\partial^3 \zeta_1(\varphi,\ \omega)}{\partial \varphi^3} - \frac{\partial^2 \zeta_2(\varphi,\ \omega)}{\partial \varphi^2} + \zeta_1(\varphi,\ \omega)\frac{\partial \zeta_2(\varphi,\ \omega)}{\partial \varphi}\right.\\ &\left. + \zeta_2(\varphi,\ \omega)\frac{\partial \zeta_1(\varphi,\ \omega)}{\partial \varphi}\right] = 0. \end{aligned} \tag{56}$$

We define the nonlinear operators as:

$$\mathcal{N}^2[\psi_1(\varphi,\ \omega;\ q),\ \psi_2(\varphi,\ \omega;\ q)] = \mathscr{M}[\psi_1(\varphi,\ \omega;\ q)] + s\left(\frac{1}{2}(1-16\tanh(-2\varphi))\right)$$

$$+\frac{1}{s^\delta}\mathscr{M}\left[\frac{\partial^2\psi_1(\varphi,\ \omega;\ q)}{\partial\varphi^2} + \psi_1(\varphi,\ \omega;\ q)\frac{\partial\psi_1(\varphi,\ \omega;\ q)}{\partial\varphi} + \frac{\partial\psi_2(\varphi,\ \omega;\ q)}{\partial\varphi}\right],$$

$$\mathcal{N}^3[\psi_1(\varphi,\ \omega;\ q),\ \psi_2(\varphi,\ \omega;\ q)] = \mathscr{M}[\psi_2(\varphi,\ \omega;\ q)] + s\left(16\left(1-\tanh^2(-2\varphi)\right)\right) \tag{57}$$

$$+\frac{1}{s^\delta}\mathscr{M}\left[3\frac{\partial^3\psi_1(\varphi,\ \omega;\ q)}{\partial\varphi^3} - \frac{\partial^2\psi_2(\varphi,\ \omega;\ q)}{\partial\varphi^2}\right.$$

$$\left.+\psi_1(\varphi,\ \omega;\ q)\frac{\partial\psi_2(\varphi,\ \omega;\ q)}{\partial\varphi} + \psi_2(\varphi,\ \omega;\ q)\frac{\partial\psi_1(\varphi,\ \omega;\ q)}{\partial\varphi}\right].$$

The Mohand operators is written as:

$$\mathscr{M}[\zeta_{1m}(\varphi,\ \omega) - k_m\zeta_{1m-1}(\varphi,\ \omega)] = \hbar\mathfrak{h}(\varphi,\ \omega)\mathscr{R}_{1,\ m}\left[\overrightarrow{\zeta}_{1m-1},\ \overrightarrow{\zeta}_{2m-1}\right],$$

$$\mathscr{M}[\zeta_{2m}(\varphi,\ \omega) - k_m\zeta_{2m-1}(\varphi,\ \omega)] = \hbar\mathfrak{h}(\varphi,\ \omega)\mathscr{R}_{2,\ m}\left[\overrightarrow{\zeta}_{1m-1},\ \overrightarrow{\zeta}_{2m-1}\right]. \tag{58}$$

Here,

$$\mathscr{R}_{1,\ m}\left[\overrightarrow{\zeta}_{1m-1},\ \overrightarrow{\zeta}_{2m-1}\right] = \mathscr{M}[\zeta_{1m-1}(\varphi,\ \omega)] + s\left(1-\frac{k_m}{n}\right)\left(\frac{1}{2}(1-16\tanh(-2\varphi))\right) + \frac{1}{s^\delta}\mathscr{M}\left[\frac{\partial^2\zeta_{1m-1}(\varphi,\ \omega)}{\partial\varphi^2}\right.$$

$$\left.+\sum_{i=0}^{m-1}\zeta_{1i}(\varphi,\ \omega)\frac{\partial\zeta_{1m-1-i}(\varphi,\ \omega)}{\partial\varphi} + \frac{\partial\zeta_{2m-1}(\varphi,\ \omega)}{\partial\varphi}\right],$$

$$\mathscr{R}_{2,\ m}\left[\overrightarrow{\zeta}_{1m-1},\ \overrightarrow{\zeta}_{2m-1}\right] = \mathscr{M}[\zeta_{2m-1}(\varphi,\ \omega)] + s\left(1-\frac{k_m}{n}\right)\left(16\left(1-\tanh^2(-2\varphi)\right)\right) + \frac{1}{s^\delta}\mathscr{M}\left[3\frac{\partial^3\zeta_{1m-1}(\varphi,\ \omega)}{\partial\varphi^3}\right.$$

$$-\frac{\partial^2\zeta_{2m-1}(\varphi,\ \omega)}{\partial\varphi^2} + \sum_{i=0}^{m-1}\zeta_{1i}(\varphi,\ \omega)\frac{\partial\zeta_{2m-1-i}(\varphi,\ \omega)}{\partial\varphi}$$

$$\left.+\sum_{i=0}^{m-1}\zeta_{2i}(\varphi,\ \omega)\frac{\partial\zeta_{1m-1-i}(\varphi,\ \omega)}{\partial\varphi}\right], \tag{59}$$

$$\zeta_{1m}(\varphi,\ \omega) = k_m \zeta_{1m-1}(\varphi,\ \omega) + \hbar \mathscr{M}^{-1}[\mathfrak{h}(\varphi,\ \omega)\mathscr{R}_{1,\ m}(\overrightarrow{\zeta}_{1m-1},\ \overrightarrow{\zeta}_{2m-1}),$$
$$\zeta_{2m}(\varphi,\ \omega) = k_m \zeta_{2m-1}(\varphi,\ \omega) + \hbar \mathscr{M}^{-1}[\mathfrak{h}(\varphi,\ \omega)\mathscr{R}_{2,\ m}(\overrightarrow{\zeta}_{1m-1},\ \overrightarrow{\zeta}_{2m-1}). \tag{60}$$

The use of initial conditions (60) lead us to the subsequent result.

$$\zeta_{1_1}(\varphi,\ \omega) = \frac{8\hbar\ \text{sech}^2(2\varphi)\omega^\delta}{\Gamma(\delta+1)},$$
$$\zeta_{2_1}(\varphi,\ \omega) = -\frac{32\hbar \tanh(2\varphi)\text{sech}^2(2\varphi)\omega^\delta}{\Gamma(\delta+1)}, \tag{61}$$

$$\zeta_{1_2}(\varphi,\ \omega) = 8\hbar\text{sech}^2(2\varphi)\omega^\delta \left(\frac{n+\hbar}{\Gamma(\delta+1)} - \frac{2\hbar\tanh(2\varphi)\omega^\delta}{\Gamma(2\delta+1)}\right),$$
$$\zeta_{2_2}(\varphi,\ \omega) = 16\hbar\text{sech}^4(2\varphi)\omega^\delta \left(\frac{2\hbar(\cosh(4\varphi)-2)\omega^\delta}{\Gamma(2\delta+1)} - \frac{\sinh(4\varphi)(n+\hbar)}{\Gamma(\delta+1)}\right), \tag{62}$$

and so on.

The other terms of the solution obtain in this manner. Equation (54) q-HATM solution is determined as follows:

$$\zeta_1(\varphi,\ \omega) = \zeta_{1_0} + \sum_{m=1}^{\infty} \zeta_{1_m}\left(\frac{1}{n}\right)^m,$$
$$\zeta_2(\varphi,\ \omega) = \zeta_{2_0} + \sum_{m=1}^{\infty} \zeta_{2_m}\left(\frac{1}{n}\right)^m. \tag{63}$$

For $\delta = 1$, $\hbar = -1$ and $n = 1$ solutions $\sum_{m=1}^{N} \zeta_{1_m}\left(\frac{1}{n}\right)^m$ and $\sum_{m=1}^{N} \zeta_{2_m}\left(\frac{1}{n}\right)^m$ converges to the exact solutions as $N \to \infty$.

$$\zeta_1(\varphi,\ \omega) = \frac{1}{2}(1-16\tanh(-2\varphi)) + \frac{8\hbar\ \text{sech}^2(2\varphi)\omega^\delta}{\Gamma(\delta+1)} + 8\hbar\text{sech}^2(2\varphi)\omega^\delta\left(\frac{n+\hbar}{\Gamma(\delta+1)} - \frac{2\hbar\tanh(2\varphi)\omega^\delta}{\Gamma(2\delta+1)}\right) + \cdots,$$

$$\zeta_2(\varphi,\ \omega) = 16\left(1-\tanh^2(-2\varphi)\right) - \frac{32\hbar\tanh(2\varphi)\text{sech}^2(2\varphi)\omega^\delta}{\Gamma(\delta+1)} + 16\hbar\text{sech}^4(2\varphi)\omega^\delta\left(\frac{2\hbar(\cosh(4\varphi)-2)\omega^\delta}{\Gamma(2\delta+1)}\right.$$
$$\left. - \frac{\sinh(4\varphi)(n+\hbar)}{\Gamma(\delta+1)}\right) + \cdots. \tag{64}$$

4.2 *Implementation of MVIM*

Examining the time-fractional nonlinear coupled Whitham-Broer-Kaup (WBK) equations:

$$
\begin{aligned}
D_{\omega}^{\delta}\zeta_1(\varphi,\ \omega) &= -\frac{\partial^2\zeta_1(\varphi,\ \omega)}{\partial\varphi^2} - \zeta_1(\varphi,\ \omega)\frac{\partial\zeta_1(\varphi,\ \omega)}{\partial\varphi} - \frac{\partial\zeta_2(\varphi,\ \omega)}{\partial\varphi},\\
D_{\omega}^{\delta}\zeta_2(\varphi,\ \omega) &= -3\frac{\partial^3\zeta_1(\varphi,\ \omega)}{\partial\varphi^3} + \frac{\partial^2\zeta_2(\varphi,\ \omega)}{\partial\varphi^2} - \zeta_1(\varphi,\ \omega)\frac{\partial\zeta_2(\varphi,\ \omega)}{\partial\varphi} - \zeta_2(\varphi,\ \omega)\frac{\partial\zeta_1(\varphi,\ \omega)}{\partial\varphi},
\end{aligned} \tag{65}
$$

where $0 < \delta \leq 1$.

Initial conditions:

$$
\begin{aligned}
\zeta_1(\varphi,\ 0) &= \frac{1}{2}(1 - 16\tanh(-2\varphi)),\\
\zeta_2(\varphi,\ 0) &= 16\left(1 - \tanh^2(-2\varphi)\right),
\end{aligned} \tag{66}
$$

Utilizing the recursive formula described in (53),

$$
\begin{aligned}
\zeta_{1_{n+1}}(\varphi,\ \omega) &= \zeta_{1_n}(\varphi,\ \omega) + \mathscr{M}^{-1}\left[\frac{1}{s^{\delta}}\sum_{k=0}^{m-1}s^{\delta-k-1}\frac{\partial^k\zeta_1(\varphi,\ 0)}{\partial\omega^k} + \mathscr{M}\left[-\frac{\partial^2\zeta_{1_n}(\varphi,\ \omega)}{\partial\varphi^2} - \zeta_{1_n}(\varphi,\ \omega)\frac{\partial\zeta_{1_n}(\varphi,\ \omega)}{\partial\varphi}\right.\right.\\
&\left.\left.\quad -\frac{\partial\zeta_{2_n}(\varphi,\ \omega)}{\partial\varphi}\right]\right],\\
\zeta_{2_{n+1}}(\varphi,\ \omega) &= \zeta_{2_n}(\varphi,\ \omega) + \mathscr{M}^{-1}\left[\frac{1}{s^{\delta}}\sum_{k=0}^{m-1}s^{\delta-k-1}\frac{\partial^k\zeta_2(\varphi,\ 0)}{\partial\omega^k} + \mathscr{M}\left[-3\frac{\partial^3\zeta_{1_n}(\varphi,\ \omega)}{\partial\varphi^3} + \frac{\partial^2\zeta_{2_n}(\varphi,\ \omega)}{\partial\varphi^2}\right.\right.\\
&\left.\left.\quad -\zeta_{1_n}(\varphi,\ \omega)\frac{\partial\zeta_{2_n}(\varphi,\ \omega)}{\partial\varphi} - \zeta_{2_n}(\varphi,\ \omega)\frac{\partial\zeta_{1_n}(\varphi,\ \omega)}{\partial\varphi}\right]\right].
\end{aligned} \tag{67}
$$

Inserting $n = 0$ into the equation above results in the second approximation:

$$
\begin{aligned}
\zeta_{1_1}(\varphi,\ \omega) &= \zeta_{1_0}(\varphi,\ \omega) + \mathscr{M}^{-1}\left[\frac{1}{s^{\delta}}\sum_{k=0}^{m-1}s^{\delta-k-1}\frac{\partial^k\zeta_1(\varphi,\ 0)}{\partial\omega^k} + \mathscr{M}\left[-\frac{\partial^2\zeta_{1_0}(\varphi,\ \omega)}{\partial\varphi^2} - \zeta_{1_0}(\varphi,\ \omega)\frac{\partial\zeta_{1_0}(\varphi,\ \omega)}{\partial\varphi}\right.\right.\\
&\left.\left.\quad -\frac{\partial\zeta_{2_0}(\varphi,\ \omega)}{\partial\varphi}\right]\right],
\end{aligned}
$$

$$\zeta_{21}(\varphi,\ \omega) = \zeta_{20}(\varphi,\ \omega) + \mathscr{M}^{-1}\left[\frac{1}{s^{\delta}}\sum_{k=0}^{m-1} s^{\delta-k-1}\frac{\partial^{k}\zeta_{2}(\varphi,\ 0)}{\partial\omega^{k}} + \mathscr{M}\left[-3\frac{\partial^{3}\zeta_{10}(\varphi,\ \omega)}{\partial\varphi^{3}} + \frac{\partial^{2}\zeta_{20}(\varphi,\ \omega)}{\partial\varphi^{2}}\right.\right.$$
$$\left.\left. -\zeta_{10}(\varphi,\ \omega)\frac{\partial\zeta_{20}(\varphi,\ \omega)}{\partial\varphi} - \zeta_{20}(\varphi,\ \omega)\frac{\partial\zeta_{10}(\varphi,\ \omega)}{\partial\varphi}\right]\right], \tag{68}$$

by simplification, we get

$$\begin{aligned} \zeta_{11}(\varphi,\ \omega) &= \frac{1}{2} + 8\tanh(2\varphi) - \frac{8\mathrm{sech}^{2}(2\varphi)\omega^{\delta}}{\Gamma(\delta+1)}, \\ \zeta_{21}(\varphi,\ \omega) &= \frac{16\mathrm{sech}^{2}(2\varphi)\left(2\tanh(2\varphi)\omega^{\delta} + \Gamma(\delta+1)\right)}{\Gamma(\delta+1)}, \end{aligned} \tag{69}$$

Put $n = 1$ in Equation (67), we have

$$\begin{aligned} \zeta_{12}(\varphi,\ \omega) &= \zeta_{11}(\varphi,\ \omega) + \mathscr{M}^{-1}\left[\frac{1}{s^{\delta}}\sum_{k=0}^{m-1} s^{\delta-k-1}\frac{\partial^{k}\zeta_{1}(\varphi,\ 0)}{\partial\omega^{k}} + \mathscr{M}\left[-\frac{\partial^{2}\zeta_{11}(\varphi,\ \omega)}{\partial\varphi^{2}} - \zeta_{11}(\varphi,\ \omega)\frac{\partial\zeta_{11}(\varphi,\ \omega)}{\partial\varphi}\right.\right. \\ &\quad \left.\left. -\frac{\partial\zeta_{21}(\varphi,\ \omega)}{\partial\varphi}\right]\right], \\ \zeta_{22}(\varphi,\ \omega) &= \zeta_{21}(\varphi,\ \omega) + \mathscr{M}^{-1}\left[\frac{1}{s^{\delta}}\sum_{k=0}^{m-1} s^{\delta-k-1}\frac{\partial^{k}\zeta_{2}(\varphi,\ 0)}{\partial\omega^{k}} + \mathscr{M}\left[-3\frac{\partial^{3}\zeta_{11}(\varphi,\ \omega)}{\partial\varphi^{3}} + \frac{\partial^{2}\zeta_{21}(\varphi,\ \omega)}{\partial\varphi^{2}}\right.\right. \\ &\quad \left.\left. -\zeta_{11}(\varphi,\ \omega)\frac{\partial\zeta_{21}(\varphi,\ \omega)}{\partial\varphi} - \zeta_{21}(\varphi,\ \omega)\frac{\partial\zeta_{11}(\varphi,\ \omega)}{\partial\varphi}\right]\right]. \end{aligned} \tag{70}$$

The final outcome is as follows as a consequence of simplifying the expression.

$$\begin{aligned} \zeta_{12}(\varphi,\ \omega) &= 1/2 + 8\tanh(2\varphi) - \frac{8\mathrm{sech}^{2}(2\varphi)\omega^{\delta}}{\Gamma(\delta+1)} - \frac{16\tanh(2\varphi)\mathrm{sech}^{2}(2\varphi)\omega^{2\delta}}{\Gamma(2\delta+1)} \\ &\quad + \frac{256\tanh(2\varphi)\mathrm{sech}^{4}(2\varphi)\omega^{3\delta}\Gamma(2\delta+1)}{\Gamma(\delta+1)^{2}\Gamma(3\delta+1)} + \cdots, \\ \zeta_{22}(\varphi,\ \omega) &= 16\mathrm{sech}^{2}(2\varphi) + \frac{32\tanh(2\varphi)\mathrm{sech}^{2}(2\varphi)\omega^{\delta}}{\Gamma(\delta+1)} - \frac{32\mathrm{sech}^{4}(2\varphi)\omega^{2\delta}}{\Gamma(2\delta+1)} + \frac{64\tanh^{2}(2\varphi)\mathrm{sech}^{2}(2\varphi)\omega^{2\delta}}{\Gamma(2\delta+1)} \\ &\quad + \frac{512\mathrm{sech}^{6}(2\varphi)\omega^{3\delta}\Gamma(2\delta+1)}{\Gamma(\delta+1)^{2}\Gamma(3\delta+1)} - \frac{2,048\tanh^{2}(2\varphi)\mathrm{sech}^{4}(2\varphi)\omega^{3\delta}\Gamma(2\delta+1)}{\Gamma(\delta+1)^{2}\Gamma(3\delta+1)}, \end{aligned} \tag{71}$$

5. Example 2

5.1 *Implementation of q-HMTM*

Examining the time-fractional nonlinear coupled JM equations:

$$D_{\omega}^{\delta}\zeta_1(\varphi,\ \omega)+\frac{\partial^3\zeta_1(\varphi,\ \omega)}{\partial\varphi^3}+\frac{3}{2}\zeta_2(\varphi,\ \omega)\frac{\partial^3\zeta_2(\varphi,\ \omega)}{\partial\varphi^3}+\frac{9}{2}\frac{\partial\zeta_2(\varphi,\ \omega)}{\partial\varphi}\frac{\partial^2\zeta_2(\varphi,\ \omega)}{\partial\varphi^2}-6\zeta_1(\varphi,\ \omega)\frac{\partial\zeta_1(\varphi,\ \omega)}{\partial\varphi}$$

$$-\frac{3}{2}{\zeta_2}^2(\varphi,\ \omega)\frac{\partial\zeta_1(\varphi,\ \omega)}{\partial\varphi}-6\zeta_1(\varphi,\ \omega)\zeta_2(\varphi,\ \omega)\frac{\partial\zeta_2(\varphi,\ \omega)}{\partial\varphi}=0, \tag{72}$$

$$D_{\omega}^{\delta}\zeta_2(\varphi,\ \omega)+\frac{\partial^3\zeta_2(\varphi,\ \omega)}{\partial\varphi^3}-6\zeta_2(\varphi,\ \omega)\frac{\partial\zeta_1(\varphi,\ \omega)}{\partial\varphi}-6\zeta_1(\varphi,\ \omega)\frac{\partial\zeta_2(\varphi,\ \omega)}{\partial\varphi}-\frac{15}{2}{\zeta_2}^2(\varphi,\ \omega)\frac{\partial\zeta_2(\varphi,\ \omega)}{\partial\varphi}=0,$$

where $0<\delta\leq 1$.

Initial condition:

$$\zeta_1(\varphi,\ 0)=\frac{1}{8}c^2\left(1-4\mathrm{sech}^2\left(\frac{c\varphi}{2}\right)\right), \tag{73}$$

$$\zeta_2(\varphi,\ 0)=c\ \mathrm{sech}\left(\frac{c\varphi}{2}\right),$$

$$\mathscr{M}[\zeta_1(\varphi,\ \omega)]+s\left(\frac{1}{8}c^2\left(1-4\mathrm{sech}^2\left(\frac{c\varphi}{2}\right)\right)\right)+\frac{1}{s^\delta}\mathscr{M}\left[\frac{\partial^3\zeta_1(\varphi,\ \omega)}{\partial\varphi^3}+\frac{3}{2}\zeta_2(\varphi,\ \omega)\frac{\partial^3\zeta_2(\varphi,\ \omega)}{\partial\varphi^3}\right.$$

$$+\frac{9}{2}\frac{\partial\zeta_2(\varphi,\ \omega)}{\partial\varphi}\frac{\partial^2\zeta_2(\varphi,\ \omega)}{\partial\varphi^2}-6\zeta_1(\varphi,\ \omega)\frac{\partial\zeta_1(\varphi,\ \omega)}{\partial\varphi}-\frac{3}{2}{\zeta_2}^2(\varphi,\ \omega)\frac{\partial\zeta_1(\varphi,\ \omega)}{\partial\varphi}$$

$$\left.-6\zeta_1(\varphi,\ \omega)\zeta_2(\varphi,\ \omega)\frac{\partial\zeta_2(\varphi,\ \omega)}{\partial\varphi}\right]=0,$$

$$\mathscr{M}[\zeta_2(\varphi,\ \omega)]+s\left(c\ \mathrm{sech}\left(\frac{c\varphi}{2}\right)\right)+\frac{1}{s^\delta}\mathscr{M}\left[\frac{\partial^3\zeta_2(\varphi,\ \omega)}{\partial\varphi^3}-6\zeta_2(\varphi,\ \omega)\frac{\partial\zeta_1(\varphi,\ \omega)}{\partial\varphi}-6\zeta_1(\varphi,\ \omega)\frac{\partial\zeta_2(\varphi,\ \omega)}{\partial\varphi}\right.$$

$$\left.-\frac{15}{2}{\zeta_2}^2(\varphi,\ \omega)\frac{\partial\zeta_2(\varphi,\ \omega)}{\partial\varphi}\right]=0. \tag{74}$$

We define the nonlinear operators as:

$$\mathcal{N}^2[\psi_1(\varphi,\ \omega;\ q),\ \psi_2(\varphi,\ \omega;\ q)] = \mathcal{M}[\psi_1(\varphi,\ \omega;\ q)] + s\left(\frac{1}{8}c^2\left(1-4\mathrm{sech}^2\left(\frac{c\varphi}{2}\right)\right)\right) + \frac{1}{s^\delta}\mathcal{M}\left[\frac{\partial^3\psi_1(\varphi,\ \omega;\ q)}{\partial\varphi^3}\right.$$

$$+\frac{3}{2}\psi_2(\varphi,\ \omega;\ q)\frac{\partial^3\psi_2(\varphi,\ \omega;\ q)}{\partial\varphi^3} + \frac{9}{2}\frac{\partial\psi_2(\varphi,\ \omega;\ q)}{\partial\varphi}\frac{\partial^2\psi_2(\varphi,\ \omega;\ q)}{\partial\varphi^2}$$

$$-6\psi_1(\varphi,\ \omega;\ q)\frac{\partial\psi_1(\varphi,\ \omega;\ q)}{\partial\varphi} - \frac{3}{2}\psi_2^2(\varphi,\ \omega;\ q)\frac{\partial\psi_1(\varphi,\ \omega;\ q)}{\partial\varphi}$$

$$\left.-6\psi_1(\varphi,\ \omega;\ q)\psi_2(\varphi,\ \omega;\ q)\frac{\partial\psi_2(\varphi,\ \omega;\ q)}{\partial\varphi}\right],$$

$$\mathcal{N}^3[\psi_1(\varphi,\ \omega;\ q),\ \psi_2(\varphi,\ \omega;\ q)] = \mathcal{M}[\psi_2(\varphi,\ \omega;\ q)] + s\left(c\ \mathrm{sech}\left(\frac{c\varphi}{2}\right)\right)$$

$$+\frac{1}{s^\delta}\mathcal{M}\left[\frac{\partial^3\psi_2(\varphi,\ \omega;\ q)}{\partial\varphi^3} - 6\psi_2(\varphi,\ \omega;\ q)\frac{\partial\psi_1(\varphi,\ \omega;\ q)}{\partial\varphi}\right.$$

$$\left.-6\psi_1(\varphi,\ \omega;\ q)\frac{\partial\psi_2(\varphi,\ \omega;\ q)}{\partial\varphi} - \frac{15}{2}\psi_2^2(\varphi,\ \omega;\ q)\frac{\partial\psi_2(\varphi,\ \omega;\ q)}{\partial\varphi}\right]. \tag{75}$$

The Mohand operators is written as:

$$\begin{aligned}
&\mathcal{M}[\zeta_{1m}(\varphi,\ \omega) - k_m\zeta_{1m-1}(\varphi,\ \omega)] = \hbar\mathfrak{h}(\varphi,\ \omega)\mathcal{R}_{1,\ m}\left[\overrightarrow{\zeta}_{1m-1},\ \overrightarrow{\zeta}_{2m-1}\right],\\
&\mathcal{M}[\zeta_{2m}(\varphi,\ \omega) - k_m\zeta_{2m-1}(\varphi,\ \omega)] = \hbar\mathfrak{h}(\varphi,\ \omega)\mathcal{R}_{2,\ m}\left[\overrightarrow{\zeta}_{1m-1},\ \overrightarrow{\zeta}_{2m-1}\right].
\end{aligned} \tag{76}$$

Here,

$$\mathcal{R}_{1,\ m}\left[\overrightarrow{\zeta}_{1m-1},\ \overrightarrow{\zeta}_{2m-1}\right] = \mathcal{M}[\zeta_{1m-1}(\varphi,\ \omega)] + s\left(1-\frac{k_m}{n}\right)\left(\frac{1}{8}c^2\left(1-4\mathrm{sech}^2\left(\frac{c\varphi}{2}\right)\right)\right)$$

$$+\frac{1}{s^\delta}\mathcal{M}\left[\frac{\partial^3\zeta_{1m-1}(\varphi,\ \omega)}{\partial\varphi^3} + \frac{3}{2}\sum_{i=0}^{m-1}\zeta_{2i}(\varphi,\ \omega)\frac{\partial^3\zeta_{2m-1-i}(\varphi,\ \omega)}{\partial\varphi^3}\right.$$

$$+\frac{9}{2}\sum_{i=0}^{m-1}\frac{\partial\zeta_{2i}(\varphi,\ \omega)}{\partial\varphi}\frac{\partial^2\zeta_{2m-1-i}(\varphi,\ \omega)}{\partial\varphi^2} - 6\sum_{i=0}^{m-1}\zeta_{1i}(\varphi,\ \omega)\frac{\partial\zeta_{1m-1-i}(\varphi,\ \omega)}{\partial\varphi}$$

$$-\frac{3}{2}\sum_{r=0}^{m-1}\sum_{i=0}^{m-1-r}\zeta_{2r}(\varphi,\ \omega)\zeta_{2i}(\varphi,\ \omega)\frac{\partial\zeta_{1m-1-r-i}(\varphi,\ \omega)}{\partial\varphi}$$

$$-6\sum_{r=0}^{m-1}\sum_{i=0}^{m-1-r}\zeta_{1r}(\varphi,\ \omega)\zeta_{2i}(\varphi,\ \omega)\frac{\partial\zeta_{2m-1-r-i}(\varphi,\ \omega)}{\partial\varphi}\Bigg],$$

$$\mathscr{R}_{2,\ m}\left[\overrightarrow{\zeta}_{1m-1},\ \overrightarrow{\zeta}_{2m-1}\right]=\mathscr{M}[\zeta_{2m-1}(\varphi,\ \omega)]+s\left(1-\frac{k_m}{n}\right)\left(c\ \mathrm{sech}\left(\frac{c\varphi}{2}\right)\right)$$

$$+\frac{1}{s^\delta}\mathscr{M}\left[\frac{\partial^3\zeta_{2m-1}(\varphi,\ \omega)}{\partial\varphi^3}-6\sum_{i=0}^{m-1}\zeta_{2i}(\varphi,\ \omega)\frac{\partial\zeta_{1m-1-i}(\varphi,\ \omega)}{\partial\varphi}-6\sum_{i=0}^{m-1}\zeta_{1i}(\varphi,\ \omega)\right.$$

$$\left.\frac{\partial\zeta_{2m-1-i}(\varphi,\ \omega)}{\partial\varphi}-\frac{15}{2}\sum_{r=0}^{m-1}\sum_{i=0}^{m-1-r}\zeta_{2r}(\varphi,\ \omega)\zeta_{2i}(\varphi,\ \omega)\times\frac{\partial\zeta_{2m-1-r-1}(\varphi,\ \omega)}{\partial\varphi}\right], \tag{77}$$

$$\zeta_{1m}(\varphi,\ \omega)=k_m\zeta_{1m-1}(\varphi,\ \omega)+\hbar\mathscr{M}^{-1}[\mathfrak{h}(\varphi,\ \omega)\mathscr{R}_{1,\ m}\left(\overrightarrow{\zeta}_{1m-1},\ \overrightarrow{\zeta}_{2m-1}\right),$$

$$\zeta_{2m}(\varphi,\ \omega)=k_m\zeta_{2m-1}(\varphi,\ \omega)+\hbar\mathscr{M}^{-1}[\mathfrak{h}(\varphi,\ \omega)\mathscr{R}_{2,\ m}\left(\overrightarrow{\zeta}_{1m-1},\ \overrightarrow{\zeta}_{2m-1}\right). \tag{78}$$

The use of initial conditions (78) lead us to the subsequent result.

$$\begin{aligned}\zeta_{11}(\varphi,\ \omega)&=-\frac{2c^5\hbar\omega^\delta\sinh^4\left(\frac{c\varphi}{2}\right)\mathrm{csch}^3(c\varphi)}{\Gamma(\delta+1)},\\ \zeta_{21}(\varphi,\ \omega)&=\frac{c^4\hbar\omega^\delta\sinh^3\left(\frac{c\varphi}{2}\right)\mathrm{csch}^2(c\varphi)}{\Gamma(\delta+1)},\end{aligned} \tag{79}$$

$$\begin{aligned}\zeta_{12}(\varphi,\ \omega)&=\frac{1}{16}c^5\hbar\omega^\delta\mathrm{sech}^4\left(\frac{c\varphi}{2}\right)\left(-\frac{c^3\hbar\omega^\delta(\cosh(c\varphi)-2)}{\Gamma(2\delta+1)}-\frac{2(n+\hbar)\sinh(c\varphi)}{\Gamma(\delta+1)}\right),\\ \zeta_{22}(\varphi,\ \omega)&=\frac{1}{32}c^4\hbar\omega^\delta\mathrm{sech}^3\left(\frac{c\varphi}{2}\right)\left(\frac{c^3\hbar\omega^\delta(\cosh(c\varphi)-3)}{\Gamma(2\delta+1)}+\frac{4(n+\hbar)\sinh(c\varphi)}{\Gamma(\delta+1)}\right),\end{aligned} \tag{80}$$

and so on.

The other terms of the solution obtain in this manner. Equation (72) q-HATM solution is determined as follows:

$$\begin{aligned}\zeta_1(\varphi,\ \omega)&=\zeta_{10}+\sum_{m=1}^{\infty}\zeta_{1m}\left(\frac{1}{n}\right)^m,\\ \zeta_2(\varphi,\ \omega)&=\zeta_{20}+\sum_{m=1}^{\infty}\zeta_{2m}\left(\frac{1}{n}\right)^m.\end{aligned} \tag{81}$$

For $\delta = 1$, $\hbar = -1$ and $n = 1$ solutions $\sum_{m=1}^{N} \zeta_{1m}\left(\frac{1}{n}\right)^m$ and $\sum_{m=1}^{N} \zeta_{2m}\left(\frac{1}{n}\right)^m$ converges to the exact solutions as $N \to \infty$.

$$\zeta_1(\varphi,\ \omega) = \frac{1}{8}c^2\left(1 - 4\text{sech}^2\left(\frac{c\varphi}{2}\right)\right) - \frac{2c^5\hbar\omega^\delta \sinh^4\left(\frac{c\varphi}{2}\right)\text{csch}^3(c\varphi)}{\Gamma(\delta+1)}$$

$$+\frac{1}{16}c^5\hbar\omega^\delta\text{sech}^4\left(\frac{c\varphi}{2}\right)\left(-\frac{c^3\hbar\omega^\delta(\cosh(c\varphi)-2)}{\Gamma(2\delta+1)} - \frac{2(n+\hbar)\sinh(c\varphi)}{\Gamma(\delta+1)}\right) + \cdots,$$

$$\zeta_2(\varphi,\ \omega) = c\text{sech}\left(\frac{c\varphi}{2}\right) + \frac{c^4\hbar\omega^\delta \sinh^3\left(\frac{c\varphi}{2}\right)\text{csch}^2(c\varphi)}{\Gamma(\delta+1)}$$

$$+\frac{1}{32}c^4\hbar\omega^\delta\text{sech}^3\left(\frac{c\varphi}{2}\right)\left(\frac{c^3\hbar\omega^\delta(\cosh(c\varphi)-3)}{\Gamma(2\delta+1)} + \frac{4(n+\hbar)\sinh(c\varphi)}{\Gamma(\delta+1)}\right) + \cdots. \tag{82}$$

5.2 *Implementation of MVIM*

Examining the time-fractional nonlinear coupled Jaulent-Miodek (JM) equations:

$$D_\omega^\delta\zeta_1(\varphi,\ \omega) = -\frac{\partial^3\zeta_1(\varphi,\ \omega)}{\partial\varphi^3} - \frac{3}{2}\zeta_2(\varphi,\ \omega)\frac{\partial^3\zeta_2(\varphi,\ \omega)}{\partial\varphi^3} - \frac{9}{2}\frac{\partial\zeta_2(\varphi,\ \omega)}{\partial\varphi}\frac{\partial^2\zeta_2(\varphi,\ \omega)}{\partial\varphi^2} + 6\zeta_1(\varphi,\ \omega)\frac{\partial\zeta_1(\varphi,\ \omega)}{\partial\varphi}$$

$$+\frac{3}{2}{\zeta_2}^2(\varphi,\ \omega)\frac{\partial\zeta_1(\varphi,\ \omega)}{\partial\varphi} + 6\zeta_1(\varphi,\ \omega)\zeta_2(\varphi,\ \omega)\frac{\partial\zeta_2(\varphi,\ \omega)}{\partial\varphi}, \tag{83}$$

$$D_\omega^\delta\zeta_2(\varphi,\ \omega) = -\frac{\partial^3\zeta_2(\varphi,\ \omega)}{\partial\varphi^3} + 6\zeta_2(\varphi,\ \omega)\frac{\partial\zeta_1(\varphi,\ \omega)}{\partial\varphi} + 6\zeta_1(\varphi,\ \omega)\frac{\partial\zeta_2(\varphi,\ \omega)}{\partial\varphi} + \frac{15}{2}{\zeta_2}^2(\varphi,\ \omega)\frac{\partial\zeta_2(\varphi,\ \omega)}{\partial\varphi},$$

where $0 < \delta \leq 1$.

Initial conditions:

$$\zeta_1(\varphi,\ 0) = \frac{1}{8}c^2\left(1 - 4\text{sech}^2\left(\frac{c\varphi}{2}\right)\right),$$

$$\zeta_2(\varphi,\ 0) = c\ \text{sech}\left(\frac{c\varphi}{2}\right), \tag{84}$$

Utilizing the recursive formula described in (53),

$$\zeta_{1_{n+1}}(\varphi,\ \omega)=\zeta_{1_n}(\varphi,\ \omega)+\mathscr{M}^{-1}\left[\frac{1}{s^\delta}\sum_{k=0}^{m-1}s^{\delta-k-1}\frac{\partial^k\zeta_1(\varphi,\ 0)}{\partial\omega^k}+\mathscr{M}\left[-\frac{\partial^3\zeta_{1_n}(\varphi,\ \omega)}{\partial\varphi^3}-\frac{3}{2}\zeta_{2_n}(\varphi,\ \omega)\frac{\partial^3\zeta_{2_n}(\varphi,\ \omega)}{\partial\varphi^3}\right.\right.$$

$$-\frac{9}{2}\frac{\partial\zeta_{2_n}(\varphi,\ \omega)}{\partial\varphi}\frac{\partial^2\zeta_{2_n}(\varphi,\ \omega)}{\partial\varphi^2}+6\zeta_{1_n}(\varphi,\ \omega)\frac{\partial\zeta_{1_n}(\varphi,\ \omega)}{\partial\varphi}$$

$$\left.\left.+\frac{3}{2}\zeta_{2_n}^2(\varphi,\ \omega)\frac{\partial\zeta_{1_n}(\varphi,\ \omega)}{\partial\varphi}+6\zeta_{1_n}(\varphi,\ \omega)\zeta_{2_n}(\varphi,\ \omega)\frac{\partial\zeta_{2_n}(\varphi,\ \omega)}{\partial\varphi}\right]\right],$$

$$\zeta_{2_{n+1}}(\varphi,\ \omega)=\zeta_{2_n}(\varphi,\ \omega)+\mathscr{M}^{-1}\left[\frac{1}{s^\delta}\sum_{k=0}^{m-1}s^{\delta-k-1}\frac{\partial^k\zeta_2(\varphi,\ 0)}{\partial\omega^k}+\mathscr{M}\left[-\frac{\partial^3\zeta_{2_n}(\varphi,\ \omega)}{\partial\varphi^3}+6\zeta_{2_n}(\varphi,\ \omega)\frac{\partial\zeta_{1_n}(\varphi,\ \omega)}{\partial\varphi}\right.\right.$$

$$\left.\left.+6\zeta_{1_n}(\varphi,\ \omega)\frac{\partial\zeta_{2_n}(\varphi,\ \omega)}{\partial\varphi}+\frac{15}{2}\zeta_{2_n}^2(\varphi,\ \omega)\frac{\partial\zeta_{2_n}(\varphi,\ \omega)}{\partial\varphi}\right]\right]. \tag{85}$$

Inserting $n=0$ into the equation above results in the second approximation:

$$\zeta_{1_1}(\varphi,\ \omega)=\zeta_{1_0}(\varphi,\ \omega)+\mathscr{M}^{-1}\left[\frac{1}{s^\delta}\sum_{k=0}^{m-1}s^{\delta-k-1}\frac{\partial^k\zeta_1(\varphi,\ 0)}{\partial\omega^k}+\mathscr{M}\left[-\frac{\partial^3\zeta_{1_0}(\varphi,\ \omega)}{\partial\varphi^3}-\frac{3}{2}\zeta_{2_0}(\varphi,\ \omega)\frac{\partial^3\zeta_{2_0}(\varphi,\ \omega)}{\partial\varphi^3}\right.\right.$$

$$-\frac{9}{2}\frac{\partial\zeta_{2_0}(\varphi,\ \omega)}{\partial\varphi}\frac{\partial^2\zeta_{2_0}(\varphi,\ \omega)}{\partial\varphi^2}+6\zeta_{1_0}(\varphi,\ \omega)\frac{\partial\zeta_{1_0}(\varphi,\ \omega)}{\partial\varphi}$$

$$\left.\left.+\frac{3}{2}\zeta_{2_0}^2(\varphi,\ \omega)\frac{\partial\zeta_{1_0}(\varphi,\ \omega)}{\partial\varphi}+6\zeta_{1_0}(\varphi,\ \omega)\zeta_{2_0}(\varphi,\ \omega)\frac{\partial\zeta_{2_0}(\varphi,\ \omega)}{\partial\varphi}\right]\right],$$

$$\zeta_{2_1}(\varphi,\ \omega)=\zeta_{2_0}(\varphi,\ \omega)+\mathscr{M}^{-1}\left[\frac{1}{s^\delta}\sum_{k=0}^{m-1}s^{\delta-k-1}\frac{\partial^k\zeta_2(\varphi,\ 0)}{\partial\omega^k}+\mathscr{M}\left[-\frac{\partial^3\zeta_{2_0}(\varphi,\ \omega)}{\partial\varphi^3}+6\zeta_{2_0}(\varphi,\ \omega)\frac{\partial\zeta_{1_0}(\varphi,\ \omega)}{\partial\varphi}\right.\right.$$

$$\left.\left.+6\zeta_{1_0}(\varphi,\ \omega)\frac{\partial\zeta_{2_0}(\varphi,\ \omega)}{\partial\varphi}+\frac{15}{2}\zeta_{2_0}^2(\varphi,\ \omega)\frac{\partial\zeta_{2_0}(\varphi,\ \omega)}{\partial\varphi}\right]\right]. \tag{86}$$

by simplification, we get

$$\begin{aligned}
\zeta_{1_1}(\varphi,\ \omega)&=\frac{1}{8}c^2\left(-4\operatorname{sech}^2\left(\frac{c\varphi}{2}\right)+1+\frac{2c^3\omega^\delta\tanh\left(\frac{c\varphi}{2}\right)\operatorname{sech}^2\left(\frac{c\varphi}{2}\right)}{\Gamma(\delta+1)}\right),\\
\zeta_{2_1}(\varphi,\ \omega)&=-\frac{c\operatorname{sech}\left(\frac{c\varphi}{2}\right)\left(c^3\omega^\delta\tanh\left(\frac{c\varphi}{2}\right)-4\Gamma(\delta+1)\right)}{4\Gamma(\delta+1)},
\end{aligned} \tag{87}$$

Put $n = 1$ in Equation (85), we have

$$\zeta_{1_2}(\varphi,\ \omega) = \zeta_{1_1}(\varphi,\ \omega) + \mathscr{M}^{-1}\left[\frac{1}{s^{\delta}}\sum_{k=0}^{m-1} s^{\delta-k-1}\frac{\partial^k \zeta_1(\varphi,\ 0)}{\partial \omega^k} + \mathscr{M}\left[-\frac{\partial^3 \zeta_{1_1}(\varphi,\ \omega)}{\partial \varphi^3} - \frac{3}{2}\zeta_{2_1}(\varphi,\ \omega)\frac{\partial^3 \zeta_{2_1}(\varphi,\ \omega)}{\partial \varphi^3}\right.\right.$$

$$-\frac{9}{2}\frac{\partial \zeta_{2_1}(\varphi,\ \omega)}{\partial \varphi}\frac{\partial^2 \zeta_{2_1}(\varphi,\ \omega)}{\partial \varphi^2} + 6\zeta_{1_1}(\varphi,\ \omega)\frac{\partial \zeta_{1_1}(\varphi,\ \omega)}{\partial \varphi}$$

$$\left.\left.+\frac{3}{2}\zeta_{2_1}^2(\varphi,\ \omega)\frac{\partial \zeta_{1_1}(\varphi,\ \omega)}{\partial \varphi} + 6\zeta_{1_1}(\varphi,\ \omega)\zeta_{2_1}(\varphi,\ \omega)\frac{\partial \zeta_{2_1}(\varphi,\ \omega)}{\partial \varphi}\right]\right],$$

$$\zeta_{2_2}(\varphi,\ \omega) = \zeta_{2_1}(\varphi,\ \omega) + \mathscr{M}^{-1}\left[\frac{1}{s^{\delta}}\sum_{k=0}^{m-1} s^{\delta-k-1}\frac{\partial^k \zeta_2(\varphi,\ 0)}{\partial \omega^k} + \mathscr{M}\left[-\frac{\partial^3 \zeta_{2_1}(\varphi,\ \omega)}{\partial \varphi^3} + 6\zeta_{2_1}(\varphi,\ \omega)\frac{\partial \zeta_{1_1}(\varphi,\ \omega)}{\partial \varphi}\right.\right.$$

$$\left.\left.+6\zeta_{1_1}(\varphi,\ \omega)\frac{\partial \zeta_{2_1}(\varphi,\ \omega)}{\partial \varphi} + \frac{15}{2}\zeta_{2_1}^2(\varphi,\ \omega)\frac{\partial \zeta_{2_1}(\varphi,\ \omega)}{\partial \varphi}\right]\right]. \tag{88}$$

The final outcome is as follows as a consequence of simplifying the expression.

$$\zeta_{1_2}(\varphi,\ \omega) = \frac{c^2 \operatorname{sech}^8\left(\frac{c\varphi}{2}\right)}{4{,}096\Gamma(\delta+1)^4}\Big(-48c^{12}\omega^{4\delta}\sinh^2\left(\frac{c\varphi}{2}\right)(3\cosh(c\varphi)-8) + 3c^9\omega^{3\delta}\Gamma(\delta+1)(85\sinh(c\varphi)$$

$$-20\sinh(2c\varphi)+\sinh(3c\varphi)) + 16c^6\omega^{2\delta}\Gamma(\delta+1)^2(9\cosh(c\varphi)-\cosh(3c\varphi)+8)$$

$$+32c^3\omega^{\delta}\Gamma(\delta+1)^3 \times \sinh^5(c\varphi)\operatorname{csch}^4\left(\frac{c\varphi}{2}\right) + 256\Gamma(\delta+1)^4\cosh^6\left(\frac{c\varphi}{2}\right)(\cosh(c\varphi)-7)\Big) + \cdots,$$

$$\zeta_{2_2}(\varphi,\ \omega) = \frac{c \operatorname{sech}^7\left(\frac{c\varphi}{2}\right)}{512\Gamma(\delta+1)^4}\Big(15c^{12}\omega^{4\delta}\sinh^2\left(\frac{c\varphi}{2}\right)(\cosh(c\varphi)-3) - 6c^9\omega^{3\delta}\Gamma(\delta+1)\sinh(c\varphi)(3\cosh(c\varphi)-7)$$

$$+16c^6\omega^{2\delta}\Gamma(\delta+1)^2\cosh^4\left(\frac{c\varphi}{2}\right)(\cosh(c\varphi)-3) - 4c^3\omega^{\delta}\Gamma(\delta+1)^3\sinh^5(c\varphi)\operatorname{csch}^4\left(\frac{c\varphi}{2}\right)$$

$$+512\Gamma(\delta+1)^4 \times \cosh^6\left(\frac{c\varphi}{2}\right)\Big), \tag{89}$$

6. Results and discussion

In this section, we present the numerical simulation results for the Caputo fractional derivative of the nonlinear coupled WBK and JM Equations by using q-HMTM and MVIM. It is shown that both proposed schemes are efficient in terms of computation cost for the fractional order δ and reliable with respect to various wave effects.

Table 1. Absolute error for q-HMTM and MVIM solution $\zeta_1(\varphi,\ \omega)$

ω	φ	$\text{MVIM}_{\delta=0.4}$	$\text{MVIM}_{\delta=0.6}$	$\text{MVIM}_{\delta=1.0}$	$q\text{-HMTM}_{\delta=1.0}$	Exact	$\text{MVIM Error}_{\delta=1.0}$	$q\text{-HMTM Error}_{\delta=1.0}$
0.01	1.0	8.08714	8.16863	8.20651	8.20651	8.20651	7.489296×10^{-7}	3.383072×10^{-7}
	1.2	8.31051	8.34932	8.36678	8.36678	8.36678	2.531222×10^{-7}	1.650933×10^{-7}
	1.4	8.41402	8.43193	8.43987	8.43987	8.43987	9.521649×10^{-8}	7.695583×10^{-8}
	1.6	8.46119	8.46934	8.47292	8.47292	8.47292	3.888325×10^{-8}	3.515124×10^{-8}
	1.8	8.48253	8.48621	8.48782	8.48782	8.48782	1.666889×10^{-8}	1.591127×10^{-8}
	2.0	8.49214	8.49380	8.49453	8.49453	8.49453	7.326425×10^{-9}	7.173085×10^{-9}
0.10	1.0	7.85664	8.00699	8.15066	8.15025	8.14990	7.621910×10^{-4}	3.515680×10^{-4}
	1.2	8.18432	8.27083	8.34103	8.34094	8.34077	2.602920×10^{-4}	1.722640×10^{-4}
	1.4	8.35238	8.39539	8.42816	8.42814	8.42806	9.870270×10^{-5}	8.044200×10^{-5}
	1.6	8.43246	8.45266	8.46764	8.46763	8.46759	4.050500×10^{-5}	3.677300×10^{-5}
	1.8	8.46941	8.47866	8.48544	8.48544	8.48542	1.740890×10^{-5}	1.665130×10^{-5}
	2.0	8.48620	8.49040	8.49345	8.49345	8.49345	7.661240×10^{-6}	7.507901×10^{-6}

Table 2. Absolute error for q-HMTM and MVIM solution $\zeta_2(\varphi,\ \omega)$

ω	φ	$\text{MVIM}_{\delta=0.4}$	$\text{MVIM}_{\delta=0.6}$	$\text{MVIM}_{\delta=1.0}$	$q\text{-HMTM}_{\delta=1.0}$	Exact	$\text{MVIM Error}_{\delta=1.0}$	$q\text{-HMTM Error}_{\delta=1.0}$
0.01	1.0	1.58891	1.29700	1.15241	1.15241	1.15241	4.255482×10^{-6}	1.148858×10^{-6}
	1.2	0.74459	0.59682	0.52843	0.52843	0.52843	1.303142×10^{-6}	6.162036×10^{-7}
	1.4	0.34116	0.27108	0.23963	0.23963	0.23963	4.430964×10^{-7}	2.986277×10^{-7}
	1.6	0.15466	0.12240	0.10812	0.10812	0.10812	1.684300×10^{-7}	1.387226×10^{-7}
	1.8	0.06977	0.05512	0.04867	0.04867	0.04867	6.931011×10^{-8}	6.326269×10^{-8}
	2.0	0.03140	0.02479	0.02188	0.02188	0.02188	2.984039×10^{-8}	2.861491×10^{-8}
0.10	1.0	2.16479	1.85303	1.36547	1.36858	1.36976	4.289980×10^{-3}	1.183360×10^{-3}
	1.2	1.17144	0.89074	0.62924	0.62993	0.63057	1.327650×10^{-3}	6.407130×10^{-4}
	1.4	0.57120	0.41300	0.28601	0.28615	0.28646	4.561570×10^{-4}	3.116890×10^{-4}
	1.6	0.26616	0.18825	0.12918	0.12921	0.12935	1.747340×10^{-4}	1.450270×10^{-4}
	1.8	0.12156	0.08513	0.05818	0.05818	0.05825	7.223300×10^{-5}	6.618560×10^{-5}
	2.0	0.05502	0.03836	0.02617	0.02617	0.02620	3.117210×10^{-5}	2.994660×10^{-5}

In Table 1, we present the absolute error for $\zeta_1(\varphi,\ \omega)$ obtained from q-HMTM and MVIM at various values of δ. It is observed that the error decreases considerably with increasing δ from 0.4 to 1. The MVIM solutions and the q-HMTM solution for both $\omega = 0.01$ and $\omega = 0.1$ converge to the exact solution, with the q-HMTM demonstrating better accuracy and high precision. Table 2 provides similar information for $\zeta_2(\varphi,\ \omega)$. The general pattern of decreasing error with larger δ holds. This observation is consistent with the idea that systems at low δ tend to have more memory and slower diffusion, while $\delta \to 1$ creates dynamics closer to that of the classical system. In the physical terms, it means that the fractional order is a degree of freedom provided for modeling the wave properties such as speed, amplitude and decay in the control sense which makes this model more realistic within applications over fluid flows and dispersive media.

As we can see in Figures 1 and 2, it has been represented that how δ impact the shape of both profile of ζ_1 and ζ_2. The profiles are steeper and their peaks sharper with increasing δ, i.e., diffusivity is reduced, non-linearity increased. The trend is identified from subplots (a)-(c) for $\delta = \{0.4,\ 0.6,\ 1.0\}$ respectively. For $\omega = 0.1$ and 2D comparisons of the results under both methods are shown in Figure 1d and Figure 2d that present a very good level of agreement endorsing accuracy represented in Tables 1 and 2. The side-by-side comparisons of the methods are presented in Figures 3 and 4, where a little variation in the solutions is noticed implying that both q-HMTM, as well as MVIM, are capable of solving such fractional systems.

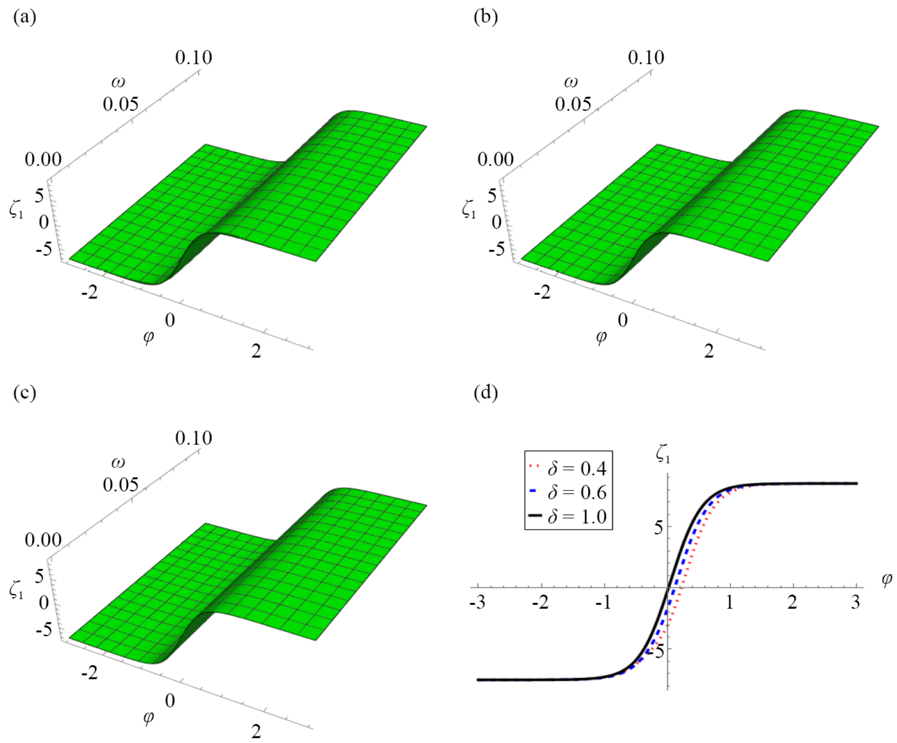

Figure 1. (a) determine the fractional order effect at $\delta = 0.4$ on $\zeta_1(\varphi, \omega)$, (b) determine the fractional order effect at $\delta = 0.6$ on $\zeta_1(\varphi, \omega)$, (c) determine fractional order effect at $\delta = 1.0$ on $\zeta_1(\varphi, \omega)$, and (d) 2D comparison of the fractional order for the solution $\zeta_1(\varphi, \omega)$ using the proposed methods for $\omega = 0.1$

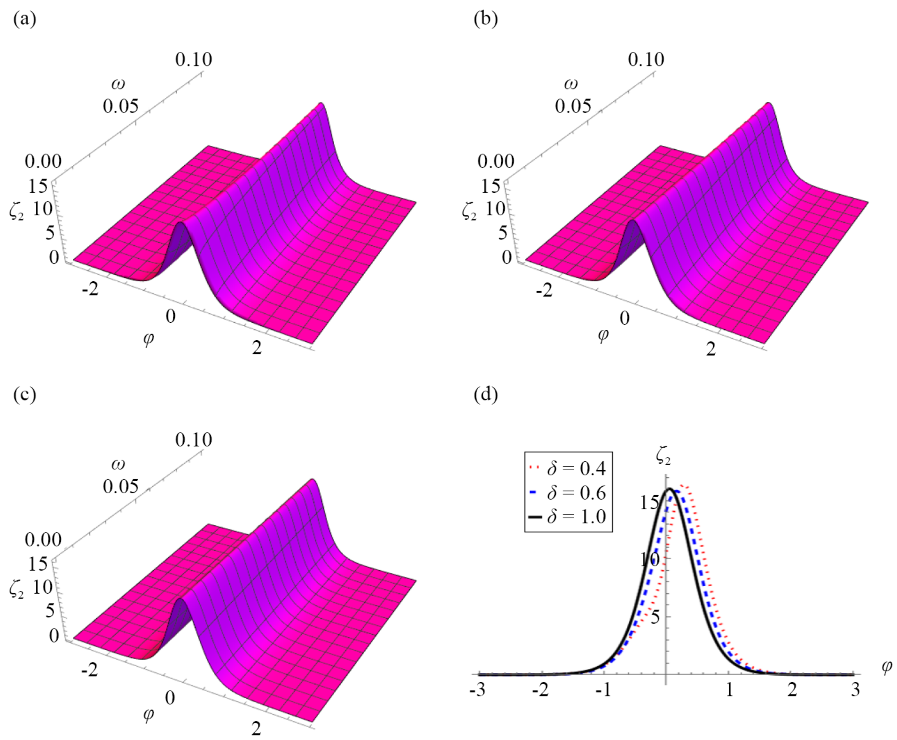

Figure 2. (a) determine the fractional order effect at $\delta = 0.4$ on $\zeta_2(\varphi, \omega)$, (b) determine the fractional order effect at $\delta = 0.6$ on $\zeta_2(\varphi, \omega)$, (c) determine fractional order effect at $\delta = 1.0$ on $\zeta_2(\varphi, \omega)$, and (d) 2D comparison of the fractional order for the solution $\zeta_2(\varphi, \omega)$ using the proposed methods for $\omega = 0.1$

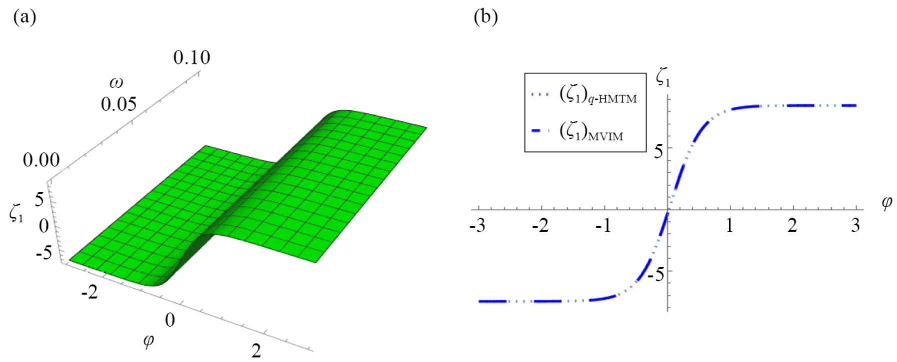

Figure 3. The approximate solution comparison of the proposed methods

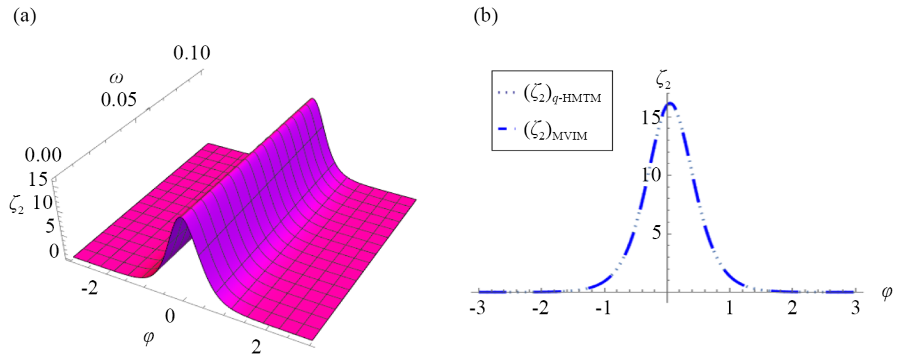

Figure 4. The approximate solution comparison of the proposed methods

The results are qualitatively the same for the JM system. Table 3 shows the error analysis for $\zeta_1(\varphi,\ \omega)$ at $c = 0.5$, both the methods are converging to exact solution. The q-HMTM achieves extremely low error margins showing excellent convergence properties and strong applicability to highly nonlinear systems with memory. In Table 4 we show the error of $\zeta_2(\varphi,\ \omega)$ under the same parameters. The trend of decrease in error with increase in δ is consistent. This is in agreement with the belief that fractional order models are tunable in such a way to reproduce physical systems closer and where nonlocality and memory effects play important roles.

Table 3. Absolute error for q-HMTM and MVIM solution $\zeta_1(\varphi,\ \omega)$ for $c = 0.5$

ω	φ	MVIM$_{\delta=0.4}$	MVIM$_{\delta=0.6}$	MVIM$_{\delta=1.0}$	q-HMTM$_{\delta=1.0}$	Exact	MVIM Error$_{\delta=1.0}$	q-HMTM Error$_{\delta=1.0}$
0.01	1.0	-0.0859245	-0.0861239	-0.0862339	-0.0862339	-0.0862339	9.418372×10^{-9}	2.131697×10^{-12}
	1.2	-0.0827647	-0.0829942	-0.0831213	-0.0831213	-0.0831213	8.336172×10^{-9}	2.366995×10^{-12}
	1.4	-0.0791855	-0.0794411	-0.0795831	-0.0795831	-0.0795831	7.160297×10^{-9}	2.519734×10^{-12}
	1.6	-0.0752473	-0.0755249	-0.0756794	-0.0756794	-0.0756794	5.930337×10^{-9}	2.590996×10^{-12}
	1.8	-0.0710132	-0.0713084	-0.0714731	-0.0714731	-0.0714731	4.684408×10^{-9}	2.585848×10^{-12}
	2.0	-0.0665466	-0.0668551	-0.0670276	-0.0670276	-0.0670276	3.457391×10^{-9}	2.512587×10^{-12}
0.10	1.0	-0.0854063	-0.0857312	-0.0860701	-0.0860711	-0.0860711	9.495355×10^{-7}	2.135560×10^{-9}
	1.2	-0.0821735	-0.0825433	-0.0829322	-0.0829330	-0.0829330	8.418361×10^{-7}	2.369713×10^{-9}
	1.4	-0.0785313	-0.0789398	-0.0793719	-0.0793727	-0.0793727	7.243785×10^{-7}	2.521302×10^{-9}
	1.6	-0.0745408	-0.0749813	-0.0754497	-0.0754503	-0.0754503	6.011613×10^{-7}	2.591443×10^{-9}
	1.8	-0.0702651	-0.0707310	-0.0712283	-0.0712287	-0.0712287	4.760566×10^{-7}	2.585269×10^{-9}
	2.0	-0.0657676	-0.0662522	-0.0667713	-0.0667717	-0.0667717	3.526254×10^{-7}	2.511119×10^{-9}

Table 4. Absolute error for q-HMTM and MVIM solution $\zeta_2(\varphi,\ \omega)$ for $c = 0.5$

ω	φ	$\text{MVIM}_{\delta=0.4}$	$\text{MVIM}_{\delta=0.6}$	$\text{MVIM}_{\delta=1.0}$	$q\text{-HMTM}_{\delta=1.0}$	Exact	MVIM $\text{Error}_{\delta=1.0}$	q-HMTM $\text{Error}_{\delta=1.0}$
0.01	1.0	0.484096	0.484508	0.484735	0.484735	0.484735	2.081617×10^{-8}	2.802758×10^{-12}
	1.2	0.477524	0.478005	0.478270	0.478270	0.478270	1.937598×10^{-8}	3.183231×10^{-12}
	1.4	0.469969	0.470513	0.470814	0.470814	0.470814	1.777356×10^{-8}	3.481437×10^{-12}
	1.6	0.461513	0.462114	0.462449	0.462449	0.462449	1.604902×10^{-8}	3.695044×10^{-12}
	1.8	0.452245	0.452898	0.453262	0.453262	0.453262	1.424274×10^{-8}	3.825106×10^{-12}
	2.0	0.442258	0.442956	0.443345	0.443345	0.443345	1.239380×10^{-8}	3.875677×10^{-12}
0.10	1.0	0.483025	0.483696	0.484397	0.484399	0.484399	2.068545×10^{-6}	2.808652×10^{-9}
	1.2	0.476284	0.477060	0.477875	0.477877	0.477877	1.923956×10^{-6}	3.188008×10^{-9}
	1.4	0.468573	0.469445	0.470365	0.470367	0.470367	1.763944×10^{-6}	3.485064×10^{-9}
	1.6	0.459977	0.460935	0.461951	0.461953	0.461953	1.592451×10^{-6}	3.697474×10^{-9}
	1.8	0.450584	0.451619	0.452721	0.452722	0.452722	1.413399×10^{-6}	3.826351×10^{-9}
	2.0	0.440488	0.441590	0.442767	0.442768	0.442768	1.230550×10^{-6}	3.875845×10^{-9}

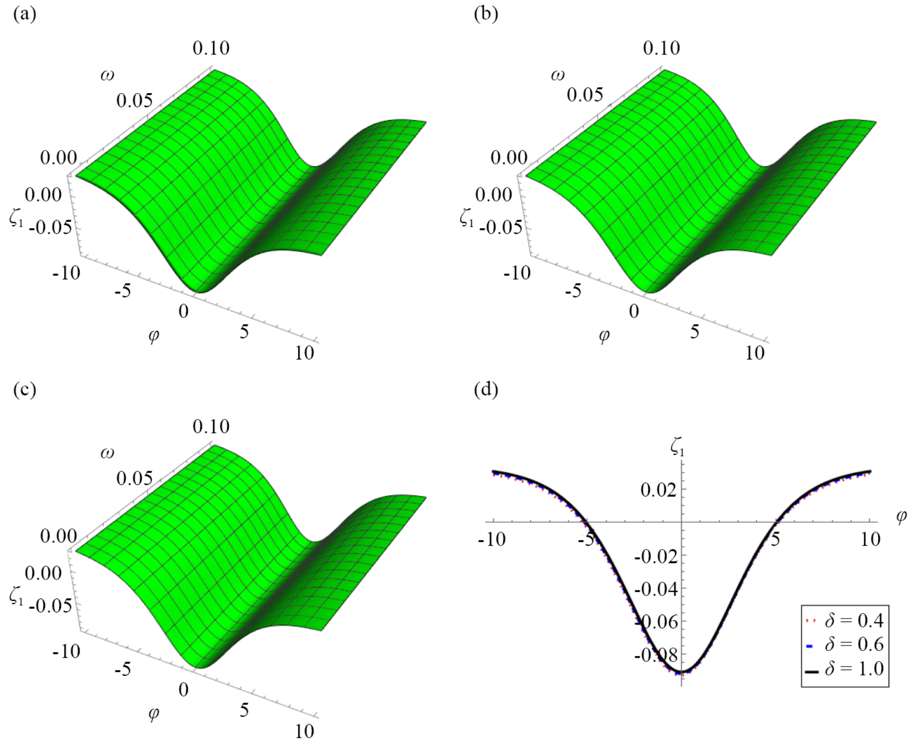

Figure 5. (a) determine the fractional order effect at $\delta = 0.4$ on $\zeta_1(\varphi,\ \omega)$, (b) determine the fractional order effect at $\delta = 0.6$ on $\zeta_1(\varphi,\ \omega)$, (c) determine fractional order effect at $\delta = 1.0$ on $\zeta_1(\varphi,\ \omega)$, and (d) 2D comparison of the fractional order for the solution $\zeta_1(\varphi,\ \omega)$ using the proposed methods for $\omega = 0.1$ and $c = 0.5$

The Figure 5 and Figure 6 show the variation of solutions of JM system with respect to fractional order. Surface plots in (a)-(c) of the subfigures for varying δ reflect a change in flatness of solution and scatter strength, while subfigure (d) show agreement between methods on a 2D domain. Again, in Figures 7 and 8, comparison to the solution between q-HMTM and MVIM for $\zeta_1$and ζ_2 is depicted. These plots essentially confirm that both approaches are capable of accurately monitoring the dynamics of solutions in the case of nonlinear fractional models.

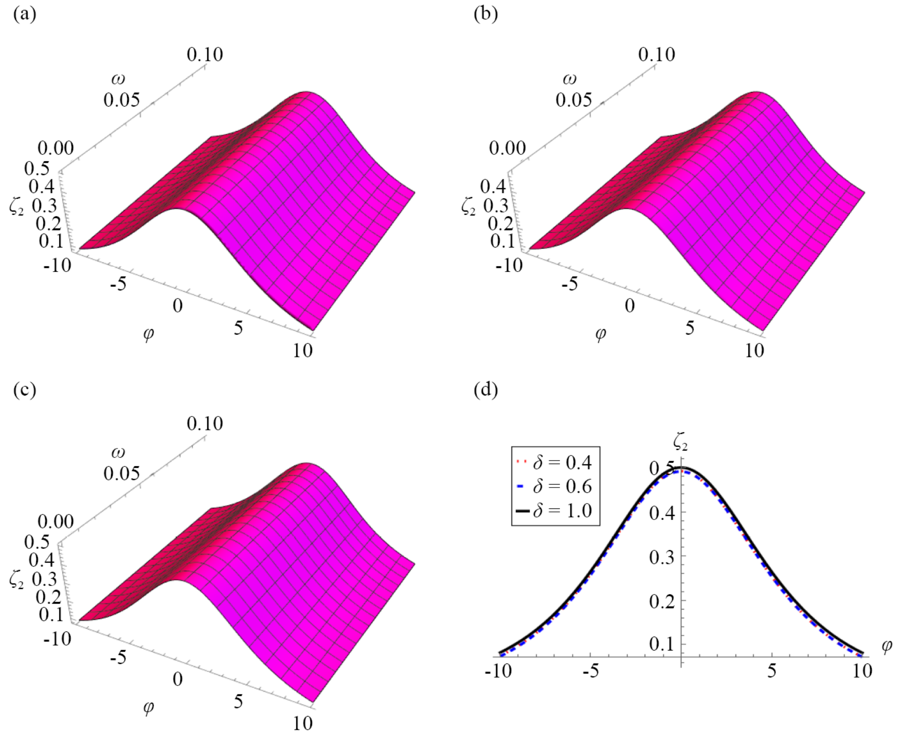

Figure 6. (a) determine the fractional order effect at $\delta = 0.4$ on $\zeta_2(\varphi, \omega)$, (b) determine the fractional order effect at $\delta = 0.6$ on $\zeta_2(\varphi, \omega)$, (c) determine fractional order effect at $\delta = 1.0$ on $\zeta_2(\varphi, \omega)$, and (d) 2D comparison of the fractional order for the solution $\zeta_2(\varphi, \omega)$ using the proposed methods for $\omega = 0.1$ and $c = 0.5$

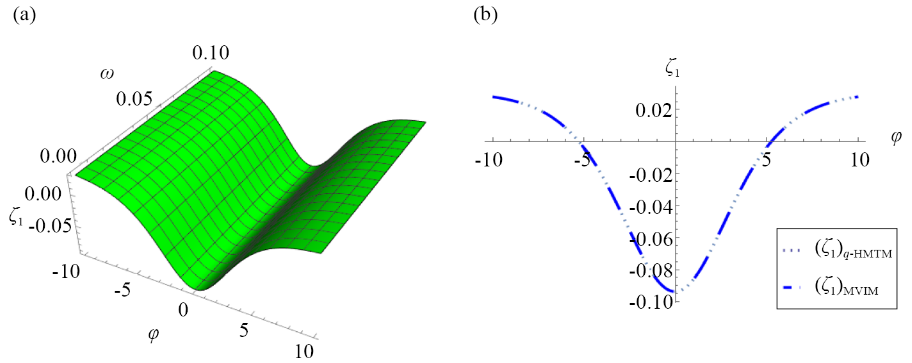

Figure 7. The approximate solution comparison of the proposed methods

The Caputo fractional derivative was chosen over other definitions (such as the local fractional derivative, He's fractional derivative, Beta Fractional Derivative, Atangana-Baleanu fractional derivative, conformable fractional derivative, M-truncated derivative) for several key reasons:

• Caputo derivatives can handle initial conditions in the classical (integer-order) sense, which is more easily understood physically and makes simulation easier.

• The Caputo derivative allows us to model processes that have a memory which starts in an initial time, and this is completely compatible with physical process like heat conduction, fluid flow or wave propagation for instance.

• Caputo's form is appropriate to combine q-HMTM and MVIM methods together in a manner that it guarantees both analytical mineability and numerical solvability.

This study reveals that the fractional models described by Caputo derivatives and solved by means of proposed q-HMTM and MVIM are accurate, efficient, and physically appealing. They provide an important basis for modeling complex non-linear systems in which memory plays a vital role.

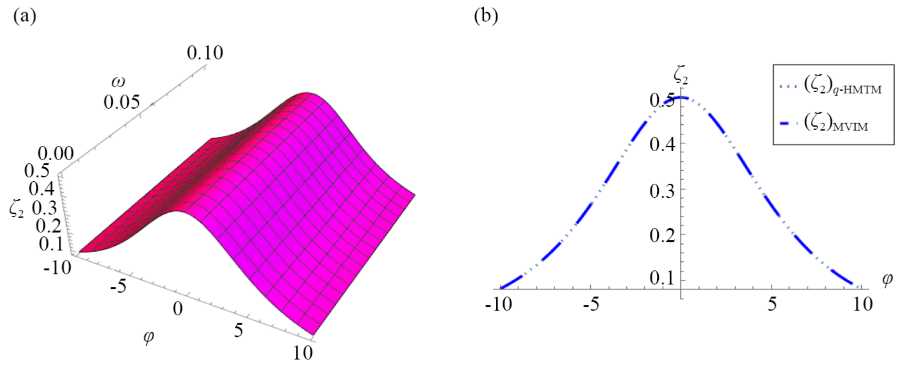

Figure 8. The approximate solution comparison of the proposed methods

7. Conclusion

In this study, we analyzed the nonlinear coupled JM and WBK equations within the framework of fractional calculus using the MVIM and the q-HMTM. These advanced techniques proved to be highly effective in handling the complexities of fractional-order differential equations, particularly when coupled with the Caputo operator. By employing these methods, we derived analytical and approximate solutions that exhibit the intricate dynamics of fractional systems. Our findings revealed the sensitivity of the solutions to fractional-order parameters, demonstrating the significant role of the fractional derivative in controlling the system's behavior. The results, validated through numerical simulations and graphical representations, provide deeper insights into the JM and WBK models, which have applications in fluid dynamics, wave propagation, and other areas involving nonlinear coupled systems. The use of q-HMTM and MVIM not only offers a robust framework for solving nonlinear fractional equations but also underscores their applicability to other complex fractional models. This work contributes to the growing body of knowledge on fractional-order systems and highlights the potential of these methods for advancing mathematical modeling and analysis in engineering and scientific fields. Future studies may extend these methodologies to explore higher-dimensional and multi-fractional systems for broader applications.

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Conflict of interest

The authors declare no conflict of interest.

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